

**MOSAIC SOLUTIONS AND SPATIAL ENTROPY
FOR A CLASS OF NEURAL NETWORK MODELS**

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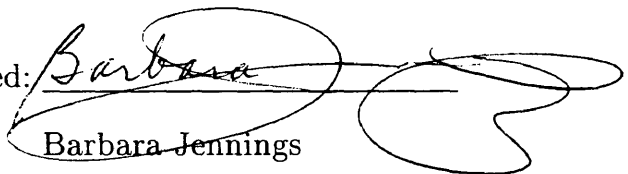
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Golden, Colorado

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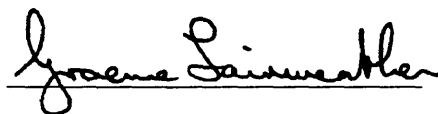
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ABSTRACT

In this thesis, we present a lattice differential equation model for a class of neural networks. We define a subset of the equilibrium solutions we call mosaic equilibrium solutions. Existence and stability theorems are proved for mosaic equilibrium solutions. Regions of stability are defined and spatial entropy calculations, as a measure of the complexity of the system, are presented that give insights in to the effects of spatial coupling.

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Chapter 1

Introduction

Neural networks are computational models characterized by patterns of weighted interconnections between neurons or cells. The method of determining the weights is called a training algorithm which resets the weights in accordance with some activation function. The result is a system which trains itself to recognize patterns or emulate functions. Traditional nets such as the Hopfield Net and the standard Backpropagation Neural Network have been intriguing to many disciplines. Although training can be slow, the resulting network can be very useful in real-time situations. Rigorous analysis of some neural network models has been done to verify the convergence of such systems [14] and to find appropriate learning algorithms to meet the needs of varied disciplines [11]. Results verifying the existence of stable equilibria have been generally accepted. However, problems persist in many areas in terms of convergence rate and misclassification. Some researchers have approached the problem through statistical methods and probability theory [2] or methods in electrical engineering theory [13]. We would like to take another look at the underlying mathematical model of various networks and evaluate the computational model in terms of spatial entropy, which may be considered a measure of the system's complexity.

Typically, results and comparisons with reaction-diffusion models leave the impression that chaotic behavior is possible, if not predictable. Hopfield nets have been modeled with reaction-diffusion equations and backpropagation networks have been found to be sensitive to initial conditions and to behave in accordance with the existence of strange attractors. Since lattice differential equations have been studied and theorems for their existence and uniqueness proved, the use of techniques associated with these systems seems well suited to the task of developing a general neural network model. This enables our analysis to include spatial entropy, as a measure of the system complexity, which can be calculated. This allows the effect of spatial coupling to be explored.

In this thesis we consider a general model of a class of neural networks. The model we consider is

$$\dot{x}(\eta) = \alpha_\eta + \beta_\eta x(\eta) + \sum_{\eta' \in \Lambda(\eta)} \beta_{\eta'} x(\eta') + \mu_\eta f(x(\eta)) + \sum_{\eta' \in \Lambda(\eta)} \mu_{\eta'} f(x(\eta')), \text{ for } \eta \in \mathcal{L} \quad (1.1)$$

with $\eta \in \mathcal{L}$ representing the spatial variable. We take the function f in (1.1) to be

$$f(x(\eta)) = \begin{cases} \gamma x(\eta) & |x(\eta)| < 1 \\ \gamma & x(\eta) \in [1, \infty) \\ -\gamma & x(\eta) \in (-\infty, -1] \end{cases} \quad (1.2)$$

We let \mathcal{L} represent the set of discrete points comprising the lattice, with η the independent spatial variable. A neighborhood of a specific neuron is given by all $\eta' \in \Lambda(\eta)$ defined as a subset of \mathcal{L} . Then x is the state of a neuron, and \dot{x} is the derivative with respect to time, $\frac{dx}{dt}$. We consider $\beta_{\eta'}$ to be spatial coupling coefficients. $f(x(\eta'))$ is the activation on the neuron and $\mu_{\eta'}$ the weights on the network connections. The

value γ is an arbitrary threshold value. In neural networks it is typically 1. In some networks we can have a range from 0 to 1. Here we want to keep the model as general as we can. The bias term for each neuron is given by α_η . This model contains arbitrary couplings through both x and $f(x)$ dependent on space.

We prove the existence of equilibrium solutions called mosaic solutions. Mosaic solutions are equilibrium solutions that belong to a specific, enumerated set. These solutions correspond to $x_\eta \in \{(-\infty, -1], 0, [1, \infty)\}$ which give $f(x_\eta) \in \{-\gamma, 0, \gamma\}$. Through the establishment of stability criteria we will be able to define regions of stability. We will then define these regions for a general one-dimensional system, where we take $\Lambda(\eta) = \{\eta - 1, \eta + 1\}$. We will calculate spatial entropy to get a measure of the complexity of the parameter space of (1.1) and (1.2).

Our contribution is to provide conditions for existence and stability of mosaic equilibrium solutions of (1.1) and (1.2) and calculate the spatial entropy of the general one-dimensional system. Using the existence and stability criteria we calculate the spatial entropy of the resulting general system. This model, which represents a general class of neural networks, will allow us to see the effect of spatial coupling coefficients in such systems. This will give us some insight into the effects of parameter choices in general and allow us to examine specific cases. We also look at the symmetric one-dimensional case to determine the effect of symmetric coupling coefficients by using the existence and stability criteria to calculate the spatial entropy of the symmetric one-dimensional system. Since neural networks are well-known as gradient systems and their convergence to stable points has been well established [14], we hope to add an understanding of the role played by the parameters in determining system

complexity. We want to examine the effect of the coupling parameters β_η and μ_η .

The model, (1.1) and (1.2), may be considered a lattice differential equation. We will use the techniques of [3] to analyze the model and determine the spatial entropy of the system, which can be thought of as the measure of the size and complexity of a nonempty set of mosaic solutions. The work of Chow, Mallet-Paret and Van Vleck gives us a basis for this analysis. In [3], they describe the time evolution of such a system as a lattice differential equation. Given \mathcal{L} is the set of discrete points comprising the lattice, we take $\eta \in \mathcal{L}$ such that u_η is an infinite vector representing the state of the dynamical system. This system has the form $\dot{u} = g(u)$, where the \dot{u} is taken to be the derivative with respect to time. Specifically, in [3] they examine systems

$$\dot{u}_i = -\beta\Delta u_i - f(u_i)$$

and

$$\dot{u}_{i,j} = -\beta^+ \Delta^+ u_{i,j} - \beta^\times \Delta^\times u_{i,j} - f(u_{i,j})$$

for a one-dimensional lattice and a two-dimensional lattice with Δ denoting the discrete Laplace operator. The existence and stability of mosaic solutions is given and an approximation of the basins of attractions made through a calculation of the spatial entropy. equations [3]. In [1] a similar method has been used to examine Cahn-Hilliard type equations.

Historically, neural networks have been described as dynamical systems by many authors [9, 12, 19, 4] and empirical studies have reported chaotic behavior [15]. Other work deals with the neuronal activity itself as being modeled by a diffusion approximation [14, 7, 18, 16].

Hopfield [14] gave a formulation of such a system as

$$\dot{u}_i = -\frac{1}{R_i}u_i + \sum_{j=1}^n T_{i,j}v_j + I_i \quad (1.3)$$

where R_i represents transmembrane resistance, I_i is any other fixed input current to neuron i and $T_{i,j}$ is the synapse efficacy. This model is based on continuous variations and responses, yet retains significant behaviors of his earlier discrete model. Through Lyapunov stability analysis, he determines that for a symmetric $T_{i,j}$, the time evolution of the system finds minima in the error function. Later work extends this to non-symmetric cases as well.

In [9], Edwards approximates neural network dynamics with a reaction-diffusion equation of the form

$$v_t = \frac{1}{G'(v)} \left[\gamma \epsilon^2 \frac{\tau}{2} \Delta v + (\gamma \tau_0 - \alpha G(v)) \right].$$

Then in [10], he explores the stable patterns in spatially discrete forms. His conclusion is that, for neural network equations, existing stable patterns are of high energy or remain stable. However, random initial conditions and smaller step size of the discretization could lead to a loss in the number of the stable patterns.

The Cellular Neural Network (CNN) was originally formulated by Chua and Yang in [4]. In [19], Thiran presents a thorough analysis of the stability of the CNN. Thiran presents the CNN to be the set of coupled ordinary differential equations represented by

$$\dot{x}_{i,j}(t) = -x_{i,j}(t) + \sum_{C_{k,l} \in \mathcal{N}_{i,j}} A_{i,j,k,l} y_{k,l}(t) + I_{i,j}$$

where $I_{i,j}$ is a time-invariant bias, and where \dot{x} stands for $\frac{dx}{dt}$. $C_{i,j}$ is the cell under investigation and $\mathcal{N}_{i,j}$ is the set of all cells forming $C_{i,j}$'s neighborhood. Also, $x_{i,j}$ is

the state of a cell at time t and $y_{i,j}$ is the output of the cell such that

$$y_{i,j}(t) = f(x_{i,j}(t))$$

where $f(x) = \frac{1}{2}(|x + 1| - |x - 1|)$. Thiran concludes that the CNN can reproduce the behavior of more complex non-linear systems, even those where the nonlinearity is decidedly different. He also asserts that the system seems to depend more on the type of connections and couplings than the nonlinear function used to produce the output of the cell. More recently, this examination of coupling parameters has been explored by Chua, as well. In [5] the basic definition given by Thiran and Chua [6] is expanded to reveal a sum of inputs to the neighboring cells.

Empirically, Kolen and Pollack [15] have demonstrated the effect of the selection of initial weights on a feed-forward network. Extreme sensitivity to initial weight configuration was observed in deterministic experiments. Other investigators, including desJardins [8], explore the criteria for selection of bias values. How and when to apply the networks and the choice of weights, bias and learning rate is a continual topic of discussion.

We seek to add an analysis of the spatial entropy of a system representing a class of neural networks in general terms to this existing work. We provide a basis for exploration of the stability in non-symmetric cases as well as the symmetric case. By making a general model, we hope to lay a foundation for studying the spatial entropy of a large class of neural networks. In this context we hope to explore the effect of the spatial coupling on the dynamics of the system.

This paper includes existence and stability theorems and their proofs, a general one-dimensional analysis of spatial entropy, application to specific neural networks

and conclusions as to our results. In Chapter 2 we define mosaic solutions, and present and prove theorems that establish the existence and stability of mosaic solutions. In Chapter 3 we define spatial entropy and give a basis for the calculations in the one-dimensional case. In Chapter 4 we calculate general results for a system defined by (1.1) and (1.2). Finally, in Chapter 5 we draw conclusions from our exploration and give possible future directions.

Chapter 2

Existence and Stability Theorems

In the model we consider, (1.1) and (1.2), no initial assumptions of symmetry or relative magnitude of the β_η or μ_η is made to allow the examination of many neural networks. In this chapter we define mosaic solutions and prove their existence and establish stability criteria.

We define **mosaic solutions** to be equilibrium solutions of (1.1) and (1.2) where $x(\eta) \in \{(-\infty, -1], 0, [1, \infty)\}$ for each η in \mathcal{L} . These will yield $f(x(\eta)) \in \{-\gamma, 0, \gamma\}$.

To help simplify many equations, we introduce some additional notation. First, let

$$\xi_\eta = -[\alpha_\eta + \mu_\eta f(x(\eta)) + \sum_{\eta' \in \Lambda(\eta)} \mu_{\eta'} f(x(\eta'))].$$

This term contains all the function values and the constant term. This generally is the activation of the neurons and bias term. Also, let

$$m_\eta = \beta_\eta x(\eta) + \sum_{\eta' \in \Lambda(\eta)} \beta_{\eta'} x(\eta')$$

This term has all the references to $x(\eta)$ alone. This is the initial state of the neurons.

Now, we state a theorem to establish the existence of a mosaic solution.

Theorem 2.0.1 *A solution $x(\eta) \in (-\infty, -1], 0, [1, \infty)$ is a mosaic equilibrium solution of (1.1) and (1.2) iff*

$$\xi_\eta = m_\eta \quad (2.1)$$

Proof: For a solution to exist,

$$0 = \dot{x}(\eta) = \alpha_\eta + \beta_\eta x(\eta) + \sum_{\eta' \in \Lambda(\eta)} \beta_{\eta'} x(\eta') + \mu_\eta f(x(\eta)) + \sum_{\eta' \in \Lambda(\eta)} \mu_{\eta'} f(x(\eta')).$$

That is,

$$-[\alpha_\eta + \mu_\eta f(x(\eta)) + \sum_{\eta' \in \Lambda(\eta)} \mu_{\eta'} f(x(\eta'))] = \beta_\eta x(\eta) + \sum_{\eta' \in \Lambda(\eta)} \beta_{\eta'} x(\eta')$$

or simply, $\xi_\eta = m_\eta$. ■

The variables, ξ_η and m_η , have three basic shapes based on the value of $x_\eta \in \{(-\infty, -1], 0, [1, \infty)\}$.

Using

$$L(k) = \begin{cases} 0 & \text{if } x_k = 0 \\ 1 & \text{if } x_k \in [1, \infty) \\ -1 & \text{if } x_k \in (-\infty, -1] \end{cases}$$

we have

Form 1: For $x(\eta) = 0$, we have $\xi_\eta = -[\alpha_\eta + \sum_{\eta' \in \Lambda(\eta)} \mu_{\eta'} L(\eta') \gamma]$ and $m_\eta = \sum_{\eta' \in \Lambda(\eta)} \beta_{\eta'} x(\eta')$,

Form 2: For $x(\eta) \geq 1$, we have $\xi_\eta = -[\alpha_\eta + \gamma \mu_\eta + \sum_{\eta' \in \Lambda(\eta)} \mu_{\eta'} L(\eta') \gamma]$ and

$$m_\eta = \beta_\eta + \sum_{\eta' \in \Lambda(\eta)} \beta_{\eta'} x(\eta'),$$

Form 3: For $x(\eta) \leq -1$, we have $\xi_\eta = -[\alpha_\eta - \gamma\mu_\eta + \sum_{\eta' \in \Lambda(\eta)} \mu_{\eta'} L(\eta') \gamma]$ and

$$m_\eta = \beta_\eta + \sum_{\eta' \in \Lambda(\eta)} \beta_{\eta'} x(\eta').$$

We will use ξ_η and m_η to represent these values in all three cases.

The next results characterize stable mosaic equilibria. We use the terms $\overline{M}_{\eta'} = \max\{x(\eta') + \delta, -1\}$ and $\underline{M}_{\eta'} = \min\{x(\eta') - \delta, 1\}$ in the statement of the theorem and more fully define and bound them in the proof. Consider the sets, $Z(\eta) = \{\eta' \in \Lambda(\eta) | x(\eta') = 0\}$, $P(\eta) = \{\eta' \in \Lambda(\eta) | x(\eta') \geq 1\}$, and $N(\eta) = \{\eta' \in \Lambda(\eta) | x(\eta') \leq -1\}$. For simplicity, we will use $(N + P)(\eta) = N(\eta) \cup P(\eta)$, $P = P(\eta)$ and $N = N(\eta)$.

Theorem 2.0.2 *If $x(\eta)$ is a mosaic equilibrium solution of (1.1) and (1.2) and the following inequality is satisfied when $|x(\eta)| \geq 1$:*

$$\theta \sum_{Z(\eta)} (|\beta_{\eta'}| + \gamma|\mu_{\eta'}|) + \delta(\beta_\eta + \sum_{(N+P)(\eta)} |\beta_{\eta'}|) \leq \overline{\Omega} \quad (2.2)$$

and

$$\theta \sum_{Z(\eta)} (|\beta_{\eta'}| + \gamma|\mu_{\eta'}|) + \delta(\beta_\eta + \sum_{(N+P)(\eta)} |\beta_{\eta'}|) \leq -\underline{\Omega} \quad (2.3)$$

or if the mosaic equilibrium satisfies the following inequality when $x(\eta) = 0$:

$$\theta(\beta_\eta + \mu_\eta \gamma + \sum_{Z(\eta)} (|\beta_{\eta'}| + \gamma|\mu_{\eta'}|)) + \delta \sum_{(N+P)(\eta)} |\beta_{\eta'}| \leq \overline{\Omega} \quad (2.4)$$

and

$$\theta(\beta_\eta + \mu_\eta \gamma + \sum_{Z(\eta)} (|\beta_{\eta'}| + \gamma|\mu_{\eta'}|)) + \delta \sum_{(N+P)(\eta)} |\beta_{\eta'}| \leq -\underline{\Omega} \quad (2.5)$$

where

$$\begin{aligned} \overline{\Omega} &= -\gamma \left[\sum_{P \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \sum_{N \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \right] \leq 0 \\ \underline{\Omega} &= -\gamma \left[\sum_{P \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \sum_{N \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \right] \geq 0 \end{aligned} \quad (2.6)$$

Proof:

Assume that $x(\eta)$ is a mosaic equilibrium solution. Suppose that $0 < \theta < 1$ and $0 < \delta < 1$. Then we will take a $v \in \mathcal{N}(x, \theta, \delta)$. That is, $v_\eta \in (x_\eta - \delta, x_\eta + \delta)$ for $|x_\eta| \geq 1$ and $v_\eta \in (x_\eta - \theta, x_\eta + \theta)$ when $x_\eta = 0$.

Then for any $v \in \mathcal{N}(x, \theta, \delta)$ the following is defined

$$\dot{v}(\eta) = \alpha_\eta + \beta_\eta v(\eta) + \sum_{\eta' \in \Lambda(\eta)} \beta_{\eta'} v(\eta') + \mu_\eta f(v(\eta)) + \sum_{\eta' \in \Lambda(\eta)} \mu_{\eta'} f(v(\eta'))$$

where f is defined by formula (1.2).

Then we have

$$-\theta \sum_{\eta' \in Z(\eta)} |\beta_{\eta'}| \leq \sum_{\eta' \in Z(\eta)} \beta_{\eta'} v(\eta') \leq \theta \sum_{\eta' \in Z(\eta)} |\beta_{\eta'}| \quad (2.7)$$

and

$$-\theta\gamma \sum_{\eta' \in Z(\eta)} |\mu_{\eta'}| \leq \sum_{\eta' \in Z(\eta)} \mu_{\eta'} f(v(\eta')) \leq \theta\gamma \sum_{\eta' \in Z(\eta)} |\mu_{\eta'}| \quad (2.8)$$

for the case where $x(\eta)$ is zero.

The case where $|x(\eta)| \geq \pm 1$ is more complicated. We can first bound the terms with β .

$$\begin{aligned} -\delta \sum_{\eta' \in (N+P)(\eta)} |\beta_{\eta'}| + \sum_{\eta' \in (N+P)(\eta)} \beta_{\eta'} x(\eta') &\leq \sum_{\eta' \in (N+P)(\eta)} \beta_{\eta'} v(\eta') \\ &\leq \delta \sum_{\eta' \in (N+P)(\eta)} |\beta_{\eta'}| + \sum_{\eta' \in (N+P)(\eta)} \beta_{\eta'} x(\eta') \end{aligned} \quad (2.9)$$

Now for a δ -neighborhood of $x(\eta')$, we want to bound $f(x(\eta') \pm \delta)$ as a value at the right endpoint or a value at the left endpoint depending on whether x_η is in the set $[1, \infty)$ or in the set $(-\infty, -1]$. This will give us a bound on the function value $f(v(\eta'))$. The right endpoint value for a δ -neighborhood involving a mosaic solution

in $(-\infty, -1]$ is given by

$$-1 \leq \overline{M_{\eta'}} = \max\{x(\eta') + \delta, -1\} \leq -1 + \delta$$

while the left endpoint value for a δ -neighborhood involving a mosaic solution in $[1, \infty)$ will be noted as

$$1 - \delta \leq \underline{M_{\eta'}} = \min\{x(\eta') - \delta, 1\} \leq 1$$

Then we can bound the μ_{η} terms in sets where the mosaic solution is in the positive range $x(\eta) \in [0, \infty)$ or in the negative range $x(\eta) \in (-\infty, -1]$. We have

$$\begin{aligned} \gamma \sum_{P \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} \underline{M_{\eta'}} + \gamma \sum_{P \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} &\leq \sum_P \mu_{\eta'} f(v(\eta')) \leq \gamma \sum_{P \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} + \gamma \sum_{P \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} \overline{M_{\eta'}}, \\ \gamma \sum_{P \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\underline{M_{\eta'}} - 1) + \gamma \sum_P \mu_{\eta'} &\leq \sum_P \mu_{\eta'} f(v(\eta')) \leq \gamma \sum_P \mu_{\eta'} + \gamma \sum_{P \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\underline{M_{\eta'}} - 1), \end{aligned} \quad (2.10)$$

$$\begin{aligned} -\gamma \sum_{N \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} - \gamma \sum_{N \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} \overline{M_{\eta'}} &\leq \sum_N \mu_{\eta'} f(v(\eta')) \leq -\gamma \sum_{N \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} - \gamma \sum_{N \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} \overline{M_{\eta'}}, \\ -\gamma \sum_N \mu_{\eta'} - \gamma \sum_{N \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\overline{M_{\eta'}} + 1) &\leq \sum_N \mu_{\eta'} f(v(\eta')) \leq -\gamma \sum_N \mu_{\eta'} - \gamma \sum_{N \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\overline{M_{\eta'}} + 1), \end{aligned} \quad (2.11)$$

With these inequalities established, we consider three cases: $x(\eta) = 0$, $x(\eta) \geq 1$, and $x(\eta) \leq -1$. We will consider each case by examining the value of the derivative of v at each endpoint. We expect the value at the right endpoint to give us a negative derivative value and the value at the left endpoint to give us a positive derivative

value. This will guarantee that the values in the δ or θ neighborhoods will remain within the prescribed neighborhood indicating stability.

Case 1a: $x(\eta) = 0$ and $v(\eta) = \theta$

$$\begin{aligned}
\dot{v}(\eta) &= \alpha_\eta + \beta_\eta v(\eta) + \sum \beta_{\eta'} v(\eta') + \mu_\eta f(v(\eta)) + \sum \mu_{\eta'} f(v(\eta')) \\
&\leq \alpha_\eta + \beta_\eta \theta + \mu_\eta \gamma \theta + \theta \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|) + \sum_{N+P} \beta_{\eta'} x(\eta') + \delta \sum_{N+P} |\beta_{\eta'}| \\
&\quad + \gamma \sum_P \mu_{\eta'} + \gamma \sum_{P \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \gamma \sum_N \mu_{\eta'} - \gamma \sum_{N \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \\
&= \alpha_\eta + \sum_P \mu_{\eta'} - \gamma \sum_N \mu_{\eta'} + \sum_{N+P} \beta_{\eta'} x(\eta') + \theta (\beta_\eta + \mu_\eta \gamma + \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|)) \\
&\quad + \delta \sum_{N+P} |\beta_{\eta'}| + \gamma \left[\sum_{P \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \sum_{N \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \right] \\
&= -\xi_\eta + m_\eta + \theta (\beta_\eta + \mu_\eta \gamma + \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|)) + \delta \sum_{N+P} |\beta_{\eta'}| \\
&\quad + \gamma \left[\sum_{P \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \sum_{N \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \right] \\
&= \theta (\beta_\eta + \mu_\eta \gamma + \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|)) + \delta \left(\sum_{N+P} |\beta_{\eta'}| \right) - \overline{\Omega} \\
&\leq 0
\end{aligned}$$

using (2.7), (2.8), (2.9), (2.10), (2.1), and (2.4).

Case 1b: $x(\eta) = 0$ and $v(\eta) = -\theta$

$$\begin{aligned}
\dot{v}(\eta) &= \alpha_\eta + \beta_\eta v(\eta) + \sum \beta_{\eta'} v(\eta') + \mu_\eta f(v(\eta)) + \sum \mu_{\eta'} f(v(\eta')) \\
&\geq \alpha_\eta - \beta_\eta \theta - \mu_\eta \gamma \theta - \theta \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|) + \sum_{N+P} \beta_{\eta'} x(\eta') - \delta \sum_{N+P} |\beta_{\eta'}|
\end{aligned}$$

$$\begin{aligned}
& +\gamma \sum_P \mu_{\eta'} + \gamma \sum_{P \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \gamma \sum_N \mu_{\eta'} - \gamma \sum_{N \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \\
& = \alpha_\eta + \sum_P \mu_{\eta'} - \gamma \sum_N \mu_{\eta'} + \sum_{N+P} \beta_{\eta'} x(\eta') - \theta(\beta_\eta + \mu_\eta \gamma + \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|)) \\
& \quad - \delta \sum_{N+P} |\beta_{\eta'}| + \gamma \left[\sum_{P \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \sum_{N \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \right] \\
& = -\xi_\eta + m_\eta - \theta(\beta_\eta + \mu_\eta \gamma + \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|)) - \delta \sum_{N+P} |\beta_{\eta'}| \\
& \quad + \gamma \left[\sum_{P \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \sum_{N \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \right] \\
& = -\theta(\beta_\eta + \mu_\eta \gamma + \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|)) - \delta \left(\sum_{N+P} |\beta_{\eta'}| \right) - \underline{\Omega} \\
& \geq 0
\end{aligned}$$

using (2.7), (2.8), (2.9), (2.11), (2.1), and (2.5).

Case 2a: $x(\eta) \geq 1$ and $v(\eta) = x(\eta) + \delta$

$$\begin{aligned}
\dot{v}(\eta) & = \alpha_\eta + \beta_\eta v(\eta) + \sum \beta_{\eta'} v(\eta') + \mu_\eta f(v(\eta)) + \sum \mu_{\eta'} f(v(\eta')) \\
& \leq \alpha_\eta + \beta_\eta x(\eta) + \delta \beta_\eta + \mu_\eta \gamma + \theta \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|) + \sum_{N+P} \beta_{\eta'} x(\eta') + \delta \sum_{N+P} |\beta_{\eta'}| \\
& \quad + \gamma \sum_P \mu_{\eta'} + \gamma \sum_{P \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \gamma \sum_N \mu_{\eta'} - \gamma \sum_{N \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \\
& = \alpha_\eta + \mu_\eta \gamma + \gamma \sum_P \mu_{\eta'} - \gamma \sum_N \mu_{\eta'} + \beta_\eta x(\eta) + \sum_{N+P} \beta_{\eta'} x(\eta') + \theta \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|) \\
& \quad + \delta(\beta_\eta + \sum_{N+P} |\beta_{\eta'}|) + \gamma \sum_{P \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \gamma \sum_{N \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \\
& = -\xi_\eta + m_\eta + \theta \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|) + \delta(\beta_\eta + \sum_{N+P} |\beta_{\eta'}|)
\end{aligned}$$

$$\begin{aligned}
& +\gamma \sum_{P \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \gamma \sum_{N \cap \{\mu_{\eta'} < 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \\
= & \theta \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|) + \delta (\beta_\eta + \sum_{N+P} |\beta_{\eta'}|) - \underline{\Omega} \\
\leq & 0
\end{aligned} \tag{2.12}$$

using (2.7), (2.8), (2.9), (2.10), (2.1), and (2.2).

Case 2b: $x(\eta) \geq 1$ and $v(\eta) = x(\eta) - \delta$

$$\begin{aligned}
\dot{v}(\eta) & = \alpha_\eta + \beta_\eta v(\eta) + \sum \beta_{\eta'} v(\eta') + \mu_\eta f(v(\eta)) + \sum \mu_{\eta'} f(v(\eta')) \\
\leq & \alpha_\eta + \beta_\eta x(\eta) - \delta \beta_\eta + \mu_\eta \gamma - \theta \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|) + \sum_{N+P} \beta_{\eta'} x(\eta') - \delta \sum_{N+P} |\beta_{\eta'}| \\
& + \gamma \sum_P \mu_{\eta'} + \gamma \sum_{P \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \gamma \sum_N \mu_{\eta'} - \gamma \sum_{N \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \\
= & \alpha_\eta + \mu_\eta \gamma + \gamma \sum_P \mu_{\eta'} - \gamma \sum_N \mu_{\eta'} + \beta_\eta x(\eta) + \sum_{N+P} \beta_{\eta'} x(\eta') - \theta \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|) \\
& - \delta (\beta_\eta + \sum_{N+P} |\beta_{\eta'}|) + \gamma \sum_{P \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \gamma \sum_{N \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \\
= & -\xi_\eta + m_\eta - \theta \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|) - \delta (\beta_\eta + \sum_{N+P} |\beta_{\eta'}|) \\
& + \gamma \sum_{P \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\underline{M}_{\eta'} - 1) - \gamma \sum_{N \cap \{\mu_{\eta'} \geq 0\}} \mu_{\eta'} (\overline{M}_{\eta'} + 1) \\
= & -\theta \sum_Z (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|) - \delta (\beta_\eta + \sum_{N+P} |\beta_{\eta'}|) - \underline{\Omega} \\
\geq & 0
\end{aligned} \tag{2.13}$$

using (2.7), (2.8), (2.9), (2.10), (2.1), and (2.4).

The two cases where $x(\eta) \leq -1$, represented by 3a and 3b, follow in the same

manner as 2a and 2b. The only difference is the term $-\gamma \sum_N \mu_{\eta'}$ which was positive for 2a and 2b. This is one of the variations of ξ_η and disappears when the existence criterion is applied. ■

We note here that when $|x_{\eta'}| > 1, \forall \eta'$ such that the $\underline{M}_{\eta'} = 1$ or $\overline{M}_{\eta'} = -1$, the terms $\underline{\Omega}$ and $\overline{\Omega}$ are zero for $\delta > 0$ sufficiently small. We can then state simpler stability criteria in a Corollary 2.14.

Corollary 2.0.1 *If $x(\eta)$ is an equilibrium solution of (1.1) and (1.2) and the following inequality is satisfied when $|x(\eta)| \geq 1$:*

$$\beta_\eta + \sum_{(N+P)(\eta)} |\beta_{\eta'}| \leq 0 \quad (2.14)$$

or if the equilibrium satisfies the following inequality when $x(\eta) = 0$:

$$\beta_\eta + \mu_\eta \gamma + \sum_{Z(\eta)} (|\beta_{\eta'}| + \gamma |\mu_{\eta'}|) \leq 0 \quad (2.15)$$

then the equilibrium solution is stable.

This will ensure that we have the stability conditions met when $|x_{\eta'}| > 1$, since we can take θ sufficiently small for (2.14) and δ sufficiently small for (2.15) to meet (2.2), (2.3), (2.4) and (2.5).

Chapter 3

Spatial Entropy

Spatial entropy is a measure of the spatial disorder of a set. We describe the spatial ordering as the ability to exhibit pattern formation as opposed to a spatially disordered structure. The spatial entropy, $h(U) \geq 0$, measures the number of different patterns discernible among the elements of U in finite subsets of the lattice on which the set is defined. Larger values of $h(U)$ imply greater spatial complexity in U . In [3], the spatial entropy is calculated by a technique based on a Markov shift, using a transfer matrix method to compute the entropy of the Markov shift.

In general, the spatial entropy of $h(U)$ can be derived from a construction of successive stages of a partition [3, 17]. Let \mathcal{A} be a set of d elements and $D \geq 1$ be an integer representing the lattice dimension. Consider any nonempty subset $U \subseteq \mathcal{A}^{\mathbb{Z}^D}$. Here, $\mathcal{A}^{\mathbb{Z}^D}$ will represent the mosaic solutions. The set U is said to be **translation invariant** if $S_k(U) = U$ for each $1 \leq k \leq D$, where $S_k : \mathcal{A}^{\mathbb{Z}^D} \rightarrow \mathcal{A}^{\mathbb{Z}^D}$ is the shift operator. Assume U is translation invariant. Given any D -tuple $N = \{N_1, N_2, \dots, N_D\}$ of positive integers, let

$$E_N = \{(i_1, i_2, i_3, \dots, i_D) \in \mathbb{Z}^D \mid 0 \leq i_k \leq N_k - 1 \text{ for } 1 \leq k \leq D\}.$$

Then there is a natural projection

$$\pi_N : \mathcal{A}^{\mathbb{Z}^D} \rightarrow \mathcal{A}^{E_N}$$

which restricts any $u \in \mathcal{A}^{\mathbb{Z}^D}$ to a finite subset of the lattice. Then let

$$\Gamma_N(U) = \text{card}(\pi_N(U))$$

So, $\Gamma_N(U)$ counts the number of distinct patterns among the elements of U .

The spatial entropy $h(U)$ of the set U is defined as the limit

$$h(U) = \lim_{N \rightarrow \infty} \frac{1}{N_1 N_2 \cdots N_D} \log \Gamma_N(U).$$

To calculate the entropy, we consider the transfer of one stable mosaic equilibrium to another while going from one lattice point to its adjacent lattice points. If this transformation again results in a stable mosaic equilibrium, the entropy of the region is higher than if there were not another stable point. We use a method of **Markov Shifts**, which depends on building boolean matrices, that is matrices whose elements are 0 or 1. These **transfer matrices** indicate the possibility of a transfer from one equilibrium mosaic solution to another. For eligible points a sequence of overlap values is obtained and serves as the row and column definitions. The element value 1 marks a valid transfer from one equilibrium configuration to another. The zero indicates no such possibility. In the one-dimensional case the process is very well-studied and an explicit calculation of the entropy is well defined. Using the transfer matrix M_1 ,

$$h(U(M_1)) = \text{the logarithm of the largest eigenvalue } \lambda \geq 0 \text{ of the matrix } M_1.$$

gives the entropy of the region under consideration. An undefined entropy value indicates that no pattern formation occurs. An entropy value of 0 denotes pattern formation while an entropy value greater than 0 indicates spatial chaos.

Chapter 4

Spatial Entropy in One Space dimension

In one space dimension, for $\mathcal{L} = \mathbb{Z}$, the general model (1.1) becomes the differential equation

$$\dot{x}_\eta = -\alpha_\eta + \mu_{\eta-1}f(x_{\eta-1}) + \mu_\eta f(x_\eta) + \mu_{\eta+1}f(x_{\eta+1}) + \beta_{\eta-1}x_{\eta-1} + \beta_\eta x_\eta + \beta_{\eta+1}x_{\eta+1} \quad (4.1)$$

We define $U = S_\eta(\beta_\eta, \mu_\eta)$ to be the set of stable mosaic solutions of (4.1) and (1.2) for the one-dimensional case. We are interested in how the parameters (β_η, μ_η) affect the spatial entropy. We are looking for dependences that give entropy values of zero as opposed to those where the entropy value is positive. As defined by [3], our system exhibits **spatial chaos** when $h(S_\eta(\beta_\eta, \mu_\eta)) > 0$ and exhibits **pattern formation** when $h(S_\eta(\beta_\eta, \mu_\eta)) = 0$.

The following tables relate the existence conditions and the stability criteria to the specific triples. They are separated into three categories by the similarity of the values contained within each triple. Table 4.1 contains the triples where the magnitude of all three values is one. Table 4.2 contains triples where the zeros are first or last. Table 4.3 contains triples with zeros only in the middle position.

For the first table we take

$$m_\eta = \beta_{\eta-1}x_{\eta-1} + \beta_\eta x_\eta + \beta_{\eta+1}x_{\eta+1}$$

for each instance of the existence condition.

Triple	Existence	Stability
(+1, +1, +1)	$-\alpha_\eta - \gamma\mu_{\eta-1} - \gamma\mu_\eta - \gamma\mu_{\eta+1} = m_\eta$	$\beta_\eta < - \beta_{\eta-1} - \beta_{\eta+1} $
(-1, -1, -1)	$-\alpha_\eta + \gamma\mu_{\eta-1} + \gamma\mu_\eta + \gamma\mu_{\eta+1} = m_\eta$	$\beta_\eta < - \beta_{\eta-1} - \beta_{\eta+1} $
(+1, +1, -1)	$-\alpha_\eta - \gamma\mu_{\eta-1} - \gamma\mu_\eta + \gamma\mu_{\eta+1} = m_\eta$	$\beta_\eta < - \beta_{\eta-1} - \beta_{\eta+1} $
(-1, -1, +1)	$-\alpha_\eta + \gamma\mu_{\eta-1} + \gamma\mu_\eta - \gamma\mu_{\eta+1} = m_\eta$	$\beta_\eta < - \beta_{\eta-1} - \beta_{\eta+1} $
(+1, -1, -1)	$-\alpha_\eta - \gamma\mu_{\eta-1} + \gamma\mu_\eta + \gamma\mu_{\eta+1} = m_\eta$	$\beta_\eta < - \beta_{\eta-1} - \beta_{\eta+1} $
(-1, +1, +1)	$-\alpha_\eta + \gamma\mu_{\eta-1} - \gamma\mu_\eta - \gamma\mu_{\eta+1} = m_\eta$	$\beta_\eta < - \beta_{\eta-1} - \beta_{\eta+1} $
(+1, -1, +1)	$-\alpha_\eta - \gamma\mu_{\eta-1} + \gamma\mu_\eta - \gamma\mu_{\eta+1} = m_\eta$	$\beta_\eta < - \beta_{\eta-1} - \beta_{\eta+1} $
(-1, +1, -1)	$-\alpha_\eta + \gamma\mu_{\eta-1} - \gamma\mu_\eta + \gamma\mu_{\eta+1} = m_\eta$	$\beta_\eta < - \beta_{\eta-1} - \beta_{\eta+1} $

Table 4.1: No Zeros

4.1 The General One-dimensional Case

By analyzing the information in Tables 4.1 - 4.3, we obtain a general picture of the stability regions in the general one-dimensional case. We define transfer between available triples by taking two such triples. Call these triples, where transfer exists, $T_1(x_{\eta-1}^1, x_\eta^1, x_{\eta+1}^1)$ and $T_2(x_{\eta-1}^2, x_\eta^2, x_{\eta+1}^2)$. Suppose both satisfy existence and stability criteria. Furthermore, $x_{\eta+1}^1 = x_{\eta-1}^2$, $\beta_{\eta+1}^1 = \beta_{\eta-1}^2$, and $\mu_{\eta+1}^1 = \mu_{\eta-1}^2$. We then

Triple	Existence	Stability
$(0, +1, 0)$	$-\alpha_\eta - \gamma\mu_\eta = \beta_\eta x_\eta$	$\beta_\eta < 0$
$(0, -1, 0)$	$-\alpha_\eta + \gamma\mu_\eta = \beta_\eta x_\eta$	$\beta_\eta < 0$
$(+1, +1, 0)$	$-\alpha_\eta - \gamma\mu_{\eta-1} - \gamma\mu_\eta = \beta_{\eta-1}x_{\eta-1} + \beta_\eta x_\eta$	$\beta_\eta < - \beta_{\eta-1} $
$(-1, -1, 0)$	$-\alpha_\eta + \gamma\mu_{\eta-1} + \gamma\mu_\eta = \beta_{\eta-1}x_{\eta-1} + \beta_\eta x_\eta$	$\beta_\eta < - \beta_{\eta-1} $
$(0, -1, -1)$	$-\alpha_\eta + \gamma\mu_\eta + \gamma\mu_{\eta+1} = \beta_\eta x_\eta + \beta_{\eta+1}x_{\eta+1}$	$\beta_\eta < - \beta_{\eta+1} $
$(0, +1, +1)$	$-\alpha_\eta - \gamma\mu_\eta - \gamma\mu_{\eta+1} = \beta_\eta x_\eta + \beta_{\eta+1}x_{\eta+1}$	$\beta_\eta < - \beta_{\eta+1} $
$(+1, -1, 0)$	$-\alpha_\eta - \gamma\mu_{\eta-1} + \gamma\mu_\eta = \beta_{\eta-1}x_{\eta-1} + \beta_\eta x_\eta$	$\beta_\eta < - \beta_{\eta-1} $
$(-1, +1, 0)$	$-\alpha_\eta + \gamma\mu_{\eta-1} - \gamma\mu_\eta = \beta_{\eta-1}x_{\eta-1} + \beta_\eta x_\eta$	$\beta_\eta < - \beta_{\eta-1} $
$(0, +1, -1)$	$-\alpha_\eta - \gamma\mu_\eta + \gamma\mu_{\eta+1} = \beta_\eta x_\eta + \beta_{\eta+1}x_{\eta+1}$	$\beta_\eta < - \beta_{\eta+1} $
$(0, -1, +1)$	$-\alpha_\eta + \gamma\mu_\eta - \gamma\mu_{\eta+1} = \beta_\eta x_\eta + \beta_{\eta+1}x_{\eta+1}$	$\beta_\eta < - \beta_{\eta+1} $

Table 4.2: No Middle Zeros

Triple	Existence	Stability
(+1, 0, +1)	$-\alpha_\eta - \gamma\mu_{\eta-1} - \gamma\mu_{\eta+1} = \beta_{\eta-1}x_{\eta-1} + \beta_{\eta+1}x_{\eta+1}$	$\beta_\eta < -\gamma\mu_\eta$
(-1, 0, -1)	$-\alpha_\eta + \gamma\mu_{\eta-1} + \gamma\mu_\eta + \gamma\mu_{\eta+1} = \beta_{\eta-1}x_{\eta-1} + \beta_{\eta+1}x_{\eta+1}$	$\beta_\eta < -\gamma\mu_\eta$
(+1, 0, -1)	$-\alpha_\eta - \gamma\mu_{\eta-1} + \gamma\mu_{\eta+1} = \beta_{\eta-1}x_{\eta-1} + \beta_{\eta+1}x_{\eta+1}$	$\beta_\eta < -\gamma\mu_\eta$
(-1, 0, +1)	$-\alpha_\eta + \gamma\mu_{\eta-1} - \gamma\mu_{\eta+1} = \beta_{\eta-1}x_{\eta-1} + \beta_{\eta+1}x_{\eta+1}$	$\beta_\eta < -\gamma\mu_\eta$
(+1, 0, 0)	$-\alpha_\eta - \gamma\mu_{\eta-1} = \beta_{\eta-1}x_{\eta-1}$	$\beta_\eta < -\gamma\mu_\eta - \beta_{\eta+1} - \gamma \mu_{\eta+1} $
(-1, 0, 0)	$-\alpha_\eta + \gamma\mu_{\eta-1} = \beta_{\eta-1}x_{\eta-1}$	$\beta_\eta < -\gamma\mu_\eta - \beta_{\eta+1} - \gamma \mu_{\eta+1} $
(0, 0, +1)	$-\alpha_\eta - \gamma\mu_{\eta+1} = \beta_{\eta+1}x_{\eta+1}$	$\beta_\eta < -\gamma\mu_\eta - \beta_{\eta-1} - \gamma \mu_{\eta-1} $
(0, 0, -1)	$-\alpha_\eta + \gamma\mu_{\eta+1} = \beta_{\eta+1}x_{\eta+1}$	$\beta_\eta < -\gamma\mu_\eta - \beta_{\eta-1} - \gamma \mu_{\eta-1} $
(0, 0, 0)	$-\alpha_\eta = 0$	$\beta_\eta < -\gamma\mu_\eta - \beta_{\eta-1} - \beta_{\eta+1} - \gamma \mu_\eta - 1 - \gamma \mu_\eta + 1 $

Table 4.3: Middle Zeros

characterize the conditions in general that allows a transfer through:

$$-\alpha_\eta^1 \pm \gamma\mu_{\eta-1}^1 \pm \gamma\mu_\eta^1 - \beta_{\eta-1}^1 - \beta_\eta^1 = -\alpha_\eta^2 \pm \gamma\mu_\eta^2 \pm \gamma\mu_{\eta+1}^2 - \beta_\eta^2 - \beta_{\eta+1}^2 \quad (4.2)$$

For each possible triple in each stability region, the transfer condition 4.2 is specific to a triple pair that is compatible to the transfer. There are eight regions of stability defined for specific groups of triples. By compiling the stability criteria from Tables 4.1 - 4.3, we identify eight regions in the $(\beta_\eta, \beta_{\eta-1}, \beta_{\eta+1}, \mu_\eta, \mu_{\eta-1}, \mu_{\eta+1})$ parameter space.

$$\mathbf{U1:} \quad \beta_\eta < -|\beta_{\eta-1}| - |\beta_{\eta+1}|$$

$$\mathbf{U2:} \quad \beta_\eta < 0$$

$$\mathbf{U3:} \quad \beta_\eta < -|\beta_{\eta-1}|$$

$$\mathbf{U4:} \quad \beta_\eta < -|\beta_{\eta+1}|$$

$$\mathbf{U5:} \quad \beta_\eta < -\gamma\mu_\eta$$

$$\mathbf{U6:} \quad \beta_\eta < -\gamma\mu_\eta - |\beta_{\eta-1}| - \gamma|\mu_{\eta-1}|$$

$$\mathbf{U7:} \quad \beta_\eta < -\gamma\mu_\eta - |\beta_{\eta+1}| - \gamma|\mu_{\eta+1}|$$

$$\mathbf{U8:} \quad \beta_\eta < -\gamma\mu_\eta - |\beta_{\eta-1}| - \gamma|\mu_{\eta-1}| - |\beta_{\eta+1}| - \gamma|\mu_{\eta+1}|$$

For transfer to occur, the last term of the first triple, T_1 , must be equal to the first term in the second triple, T_2 . These eight regions defined in Tables 4.1, 4.2, and 4.3, contain specific sets of triples. The stability regions from T_1 and T_2 combine to give

	U1	U2	U3	U4	U5	U6	U7	U8
U1	•		•		•		•	
U2		•		•		•		•
U3		•		•		•		•
U4	•		•		•		•	
U5	•		•		•		•	
U6	•		•		•		•	
U7		•		•		•		•
U8		•		•		•		•

Table 4.4: Possible Transfers for Triple1 to Triple2

transfer conditions when triples are compatible. Table 4.4 shows the compatibility possibilities between eligible triples in their stability regions.

We determine the composite stability regions by correlating the region from the left with a column designated above. The dots represent the compatibility of the third element of the triple $T_1 \in Un^1$ (along the left) and the first element of the triple $T_2 \in Um^2$ (from above). If transfer is possible we find the composite region $Un^1 \rightarrow Um^2$ which will be called Rp .

$$U1^1 \rightarrow U1^2 \text{ or } R1: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta-1}^{(1)}| - |\beta_{\eta-1}^{(2)}| - |\beta_{\eta+1}^{(1)}| - |\beta_{\eta+1}^{(2)}|$$

$$U1^1 \rightarrow U3^2 \text{ or } R2: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta-1}^{(1)}| - |\beta_{\eta-1}^{(2)}| - |\beta_{\eta+1}^{(1)}|$$

$$U1^1 \rightarrow U5^2 \text{ or } R3: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta-1}^{(1)}| - |\beta_{\eta+1}^{(1)}| - |\beta_{\eta+1}^{(2)}|$$

$$U1^1 \rightarrow U7^2 \text{ or } R4: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta-1}^{(1)}| - |\beta_{\eta+1}^{(1)}| - \gamma(\mu_\eta^{(2)} + |\mu_{\eta+1}^{(2)}|)$$

$$U2^1 \rightarrow U2^2 \text{ or } R5: \quad \beta_\eta^1 + \beta_\eta^2 < 0$$

$$U2^1 \rightarrow U4^2 \text{ or } R6: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta+1}^{(2)}|$$

$$U2^1 \rightarrow U6^2 \text{ or } R7: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta-1}^{(2)}| - \gamma(\mu_\eta^{(2)} + |\mu_{\eta+1}^{(2)}|)$$

$$U2^1 \rightarrow U8^2 \text{ or } R8: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta-1}^{(2)}| - |\beta_{\eta+1}^{(2)}| - \gamma(\mu_\eta^{(2)} + |\mu_{\eta-1}^{(2)}| + |\mu_{\eta+1}^{(2)}|)$$

$$U3^1 \rightarrow U2^2 \text{ or } R9: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta-1}^{(1)}|$$

$$U3^1 \rightarrow U4^2 \text{ or } R10: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta-1}^{(1)}| - |\beta_{\eta+1}^{(2)}|$$

$$U3^1 \rightarrow U6^2 \text{ or } R11: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta-1}^{(1)}| - |\beta_{\eta-1}^{(2)}| - \gamma(\mu_\eta^{(2)} + |\mu_{\eta-1}^{(2)}|)$$

$$U3^1 \rightarrow U8^2 \text{ or } R12: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta-1}^{(1)}| - |\beta_{\eta-1}^{(2)}| - |\beta_{\eta+1}^{(2)}| - \gamma(\mu_\eta^{(2)} + |\mu_{\eta-1}^{(2)}| + |\mu_{\eta+1}^{(2)}|)$$

$$U4^1 \rightarrow U1^2 \text{ or } R13: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta-1}^{(2)}| - |\beta_{\eta+1}^{(1)}| - |\beta_{\eta+1}^{(2)}|$$

$$U4^1 \rightarrow U3^2 \text{ or } R14: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta-1}^{(2)}| - |\beta_{\eta+1}^{(1)}|$$

$$U4^1 \rightarrow U5^2 \text{ or } R15: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta+1}^{(1)}| - \gamma\mu_\eta^{(2)}$$

$$U4^1 \rightarrow U7^2 \text{ or } R16: \quad \beta_\eta^1 + \beta_\eta^2 < -|\beta_{\eta+1}^{(1)}| - |\beta_{\eta+1}^{(2)}| - \gamma(\mu_\eta^{(2)} + |\mu_{\eta+1}^{(2)}|)$$

$$U5^1 \rightarrow U1^2 \text{ or } R17: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma\mu_\eta^{(1)} - |\beta_{\eta-1}^{(2)}| - |\beta_{\eta+1}^{(2)}|$$

$$U5^1 \rightarrow U3^2 \text{ or } R18: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma\mu_\eta^{(1)} - |\beta_{\eta-1}^{(2)}|$$

$$U5^1 \rightarrow U5^2 \text{ or } R19: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + \mu_\eta^{(2)})$$

$$U5^1 \rightarrow U7^2 \text{ or } R20: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + \mu_\eta^{(2)} + |\mu_{\eta+1}^{(2)}|) - |\beta_{\eta+1}^{(2)}|$$

$$U6^1 \rightarrow U1^2 \text{ or } R21: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + |\mu_{\eta-1}^{(1)}|) - |\beta_{\eta-1}^{(1)}| - |\beta_{\eta-1}^{(2)}| + -|\beta_{\eta+1}^{(2)}|$$

$$U6^1 \rightarrow U3^2 \text{ or } R22 : \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + |\mu_{\eta-1}^{(1)}|) - |\beta_{\eta-1}^{(1)}| - |\beta_{\eta-1}^{(2)}|$$

$$U6^1 \rightarrow U5^2 \text{ or } R23: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + |\mu_{\eta-1}^{(1)}|) - |\beta_{\eta-1}^{(1)}| - \gamma\mu_\eta^{(2)}$$

$$U6^1 \rightarrow U7^2 \text{ or } R24: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + |\mu_{\eta-1}^{(1)}|) - |\beta_{\eta-1}^{(1)}| - \gamma(\mu_\eta^{(2)} + |\mu_{\eta+1}^{(2)}|) - |\beta_{\eta+1}^{(2)}|$$

$$U7^1 \rightarrow U2^2 \text{ or } R25: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + |\mu_{\eta+1}^{(1)}|) - |\beta_{\eta+1}^{(1)}|$$

$$U7^1 \rightarrow U4^2 \text{ or } R26: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + |\mu_{\eta+1}^{(1)}|) - |\beta_{\eta+1}^{(1)}| - |\beta_{\eta+1}^{(2)}|$$

$$U7^1 \rightarrow U6^2 \text{ or } R27: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + |\mu_{\eta+1}^{(1)}|) - |\beta_{\eta+1}^{(1)}| - |\beta_{\eta-1}^{(2)}| - \gamma(\mu_\eta^{(2)} + |\mu_{\eta-1}^{(2)}|)$$

$$U7^1 \rightarrow U8^2 \text{ or } R28: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + |\mu_{\eta+1}^{(1)}|) - |\beta_{\eta+1}^{(1)}| - |\beta_{\eta+1}^{(2)}| - |\beta_{\eta-1}^{(2)}| - \gamma(\mu_\eta^{(2)} + |\mu_{\eta-1}^{(2)}| + |\mu_{\eta+1}^{(2)}|)$$

$$U8^1 \rightarrow U2^2 \text{ or } R29: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + |\mu_{\eta+1}^{(1)}| + |\mu_{\eta-1}^{(1)}|) - |\beta_{\eta-1}^{(1)}| - |\beta_{\eta+1}^{(1)}|$$

$$U8^1 \rightarrow U4^2 \text{ or } R30: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + |\mu_{\eta+1}^{(1)}| + |\mu_{\eta-1}^{(1)}|) - |\beta_{\eta-1}^{(1)}| - |\beta_{\eta+1}^{(1)}| - |\beta_{\eta+1}^{(2)}|$$

$$U8^1 \rightarrow U6^2 \text{ or } R31: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + |\mu_{\eta+1}^{(1)}| + |\mu_{\eta-1}^{(1)}|) - |\beta_{\eta-1}^{(1)}| - |\beta_{\eta+1}^{(1)}| - |\beta_{\eta-1}^{(2)}| - \gamma(\mu_\eta^{(2)} + |\mu_{\eta-1}^{(2)}|)$$

$$U8^1 \rightarrow U8^2 \text{ or } R32: \quad \beta_\eta^1 + \beta_\eta^2 < -\gamma(\mu_\eta^{(1)} + |\mu_{\eta+1}^{(1)}| + |\mu_{\eta-1}^{(1)}|) - |\beta_{\eta-1}^{(1)}| - |\beta_{\eta+1}^{(1)}| - |\beta_{\eta-1}^{(2)}| - |\beta_{\eta+1}^{(2)}| - \gamma(\mu_\eta^{(2)} + |\mu_{\eta-1}^{(2)}| + |\mu_{\eta+1}^{(2)}|)$$

The 32 regions described above, then, have specific triples that have existence and stability properties. By taking the eligible triples, we can extract the doubles that represent the overlap between those triples where the transfer conditions hold. These doubles can be ordered to form transfer matrices. The ordered list of doubles serves as the row names and column names to construct the transfer matrix. An association

of the row double and a compatible column double yields a 1 in the associated entry of the matrix. That is, the pattern must be possible and the result of the type represented by the stability region from which the original triples were extracted. If the row double is not compatible with the column double or the resulting triple is not admissible to the region, the associated matrix entry is 0. The eigenvalue of such a matrix is then calculated.

We identify the possible triples to make a transfer possible in R1.

$$T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)}) \text{ gives } (\pm 1, \pm 1, \pm 1) \rightarrow (\pm 1, \pm 1, \pm 1).$$

This pair of possible triples has eligible doubles in order to serve as rows and columns of a matrix:

$(1, 1), (-1, 1), (-1, -1), (1, -1)$. The transfer matrix is then

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The largest eigenvalue of this matrix is 2. Thus the entropy for this region is $\ln(2)$.

Appendix A contains the list of doubles and transfer matrices for the other 31 regions. The entropies for the regions are summarized in the Table 4.5. When the eigenvalue of the matrix is 0 we cannot calculate $\ln(0)$, hence we say the entropy is undefined. Frequently in this case the transfer matrix might contain all zeros, since the pattern formed by the overlap does not exist in the region under consideration.

There are three regions in this general case where the entropy indicates the existence of spatial chaos. There are also three regions of pattern formation.

Entropy value	Regions
0	R28, R31, R32
$\ln(2)$	R1, R2, R13
undefined	all others

Table 4.5: Entropy and Regions

4.2 The Symmetric One-dimensional Case

In the symmetric case, we take $\beta_{\eta-1} = \beta_{\eta+1}$ and $\mu_{\eta-1} = \mu_{\eta+1}$. We will use the basic regions defined by Tables 4.1 - 4.3. The stability regions defined by these inequalities are given in Figure 4.1. We represent these inequalities in the $\beta_{\eta} \times \beta_{\eta-1}$ plane. The three different views are the result of the influence of the term $-\gamma\mu_{\eta}$. In Figure 4.1 (a) $-\gamma\mu_{\eta} > 0$, in Figure 4.1 (b) $-\gamma\mu_{\eta} = 0$, and in Figure 4.1 (c) $-\gamma\mu_{\eta} < 0$. Seventeen regions are defined in the symmetric case when $|\gamma\mu_{\eta}| > 0$. When $\gamma\mu_{\eta} = 0$ there are eight regions.

With our restrictions for the symmetric one-dimensional case the existence criteria can be represented as lines in the $\beta_{\eta} \times \beta_{\eta-1}$ plane. The existence lines can then be examined as they cross the stability regions. It is clear that for existence and stability to coexist, the intercepts of the existence lines must be less than zero. That is, $1/x_{\eta}\xi_{\eta} < 0$. We'll calculate the entropy for the stability regions formed when $-\gamma\mu_{\eta} < 0$. Furthermore, we'll consider the case where $1/x_{\eta}\xi_{\eta} < -\gamma\mu_{\eta}$. Given these further constraints, the admissible triples for each region S_n can be determined. Each

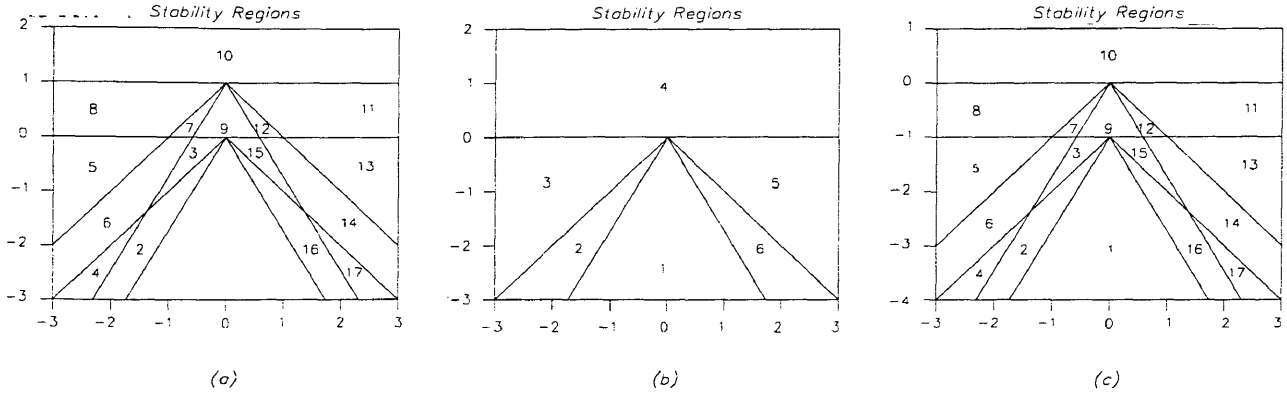


Figure 4.1: Symmetric Case Stability Regions

existence line represents several triples in terms of ξ_η .

It is to be noted that when $-\gamma\mu_\eta > 0$, the configuration of regions remains the same, but Regions S7, S8, S9, S11, S12 are not regions of stability.

The seventeen regions formed by the symmetric case are as follows

1. **Region S1** is defined by $\beta_\eta < -\gamma\mu_\eta - 2|\beta_{\eta-1}|$. Existence lines crossing the region include

line1: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

line2: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

line3: $\beta_\eta x_\eta = \xi_\eta$

line4: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1}x_{\eta-1}$

line5: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1} x_{\eta-1}$

line6: $\beta_{\eta-1}(x_{\eta-1} + x_{\eta+1}) = \xi_\eta$

line7: $\beta_{\eta-1} x_{\eta-1} = \xi_\eta$

This allows the triples

$(1, 1, 1), (-1, 1, 1), (1, 1, -1), (-1, 1, -1)$ from line1

$(1, -1, 1), (-1, -1, 1), (1, -1, -1), (-1, -1, -1)$ from line2

$(0, 1, 0), (0, -1, 0)$ from line3

$(0, 1, 1), (0, 1, -1), (1, 1, 0), (-1, 1, 0)$ from line4

$(0, -1, 1), (0, -1, -1), (1, -1, 0), (-1, -1, 0)$ from line5

$(1, 0, 1), (1, 0, -1), (-1, 0, 1), (-1, 0, -1)$ from line6

$(1, 0, 0), (-1, 0, 0), (0, 0, 1), (0, 0, -1)$ from line7

2. **Region S2** is defined by $\beta_\eta > -\gamma\mu_\eta - 2|\beta_{\eta-1}|$, $\beta_\eta < -\gamma\mu_\eta - |\beta_{\eta-1}|$, $\beta_\eta < -2|\beta_{\eta-1}|$. The possible existence lines include

line1: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

line3: $\beta_\eta x_\eta = \xi_\eta$

line4: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1} x_{\eta-1}$

line5: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1} x_{\eta-1}$

line6: $\beta_{\eta-1}(x_{\eta-1} + x_{\eta+1}) = \xi_\eta$

line7: $\beta_{\eta-1} x_{\eta-1} = \xi_\eta$

This allows triples

$(1, 1, 1), (-1, 1, 1), (1, 1, -1), (-1, 1, -1)$ from line1

$(0, 1, 0), (0, -1, 0)$ from line3

$(0, 1, 1), (0, 1, -1), (1, 1, 0), (-1, 1, 0)$ from line4

$(0, -1, 1), (0, -1, -1), (1, -1, 0), (-1, -1, 0)$ from line5

$(1, 0, 1), (1, 0, -1), (-1, 0, 1), (-1, 0, -1)$ from line6

$(1, 0, 0), (-1, 0, 0), (0, 0, 1), (0, 0, -1)$ from line7

3. **Region S3** is defined by $\beta_\eta < -2|\beta_{\eta-1}|$, $\beta_\eta > -\gamma\mu_\eta - |\beta_{\eta-1}|$, $\beta_\eta < -\gamma\mu_\eta$. The possible existence lines include

line1: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

line4: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1}x_{\eta-1}$

This allows triples

$(1, 1, 1), (-1, 1, 1), (1, 1, -1), (-1, 1, -1)$ from line1

$(0, 1, 1), (0, 1, -1), (1, 1, 0), (-1, 1, 0)$ from line4

4. **Region S4** is defined by $\beta_\eta > -2|\beta_{\eta-1}|$, $\beta_\eta < -\gamma\mu_\eta - |\beta_{\eta-1}|$. The possible existence lines include

line5: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}x_\eta$

line6: $\beta_{\eta-1}(x_{\eta-1} + x_{\eta+1}) = \xi_\eta$

This allows triples

$(0, -1, 1), (0, -1, -1), (1, -1, 0), (-1, -1, 0)$ from line5

$(1, 0, 1), (1, 0, -1), (-1, 0, 1), (-1, 0, -1)$ from line6

5. **Region S5** is defined by $\beta_\eta > -|\beta_{\eta-1}|, \beta_\eta < -\gamma\mu_\eta$. The possible existence lines include

line4: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1} x_{\eta-1}$

This allows triples

$(0, 1, 1), (0, 1, -1), (1, 1, 0), (-1, 1, 0)$ from line4

6. **Region S6** is defined by $\beta_\eta > -2|\beta_{\eta-1}|, \beta_\eta > -\gamma\mu_\eta - |\beta_{\eta-1}|, \beta_\eta < -|\beta_{\eta-1}|$. The possible existence lines include

line1: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

line3: $\beta_\eta x_\eta = \xi_\eta$

line4: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1} x_{\eta-1}$

line6: $\beta_{\eta-1}(x_{\eta-1} + x_{\eta+1}) = \xi_\eta$

line7: $\beta_{\eta-1} x_{\eta-1} = \xi_\eta$

This allows triples

$(1, 1, 1), (-1, 1, 1), (1, 1, -1), (-1, 1, -1)$ from line1

$(0, 1, 0), (0, -1, 0)$ from line3

$(0, 1, 1), (0, 1, -1), (1, 1, 0), (-1, 1, 0)$ from line4

$(1, 0, 1), (1, 0, -1), (-1, 0, 1), (-1, 0, -1)$ from line6

$(1, 0, 0), (-1, 0, 0), (0, 0, 1), (0, 0, -1)$ from line7

7. **Region S7** is defined by $\beta_\eta > -2|\beta_{\eta-1}|, \beta_\eta < -|\beta_{\eta-1}|$. The possible existence lines include

line1: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

This allows triples

$(1, 1, 1), (-1, 1, 1), (1, 1, -1), (-1, 1, -1)$ from line1

8. **Region S8** is defined by $\beta_\eta < 0, \beta_\eta > -\gamma\mu_\eta, \beta_\eta < -|\beta_{\eta-1}|$. The possible existence lines include

line1: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

line4: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1}x_{\eta-1}$

line6: $\beta_{\eta-1}(x_{\eta-1} + x_{\eta+1}) = \xi_\eta$

line7: $\beta_{\eta-1}x_{\eta-1} = \xi_\eta$

This allows triples

$(1, 1, 1), (-1, 1, 1), (1, 1, -1), (-1, 1, -1)$ from line1

$(0, 1, 1), (0, 1, -1), (1, 1, 0), (-1, 1, 0)$ from line4

$(1, 0, 1), (1, 0, -1), (-1, 0, 1), (-1, 0, -1)$ from line6

$(1, 0, 0), (-1, 0, 0), (0, 0, 1), (0, 0, -1)$ from line7

9. **Region S9** is defined by $\beta_\eta > -\gamma\mu_\eta, \beta_\eta < -2|\beta_{\eta-1}|$. Existence lines crossing the region include

line1: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

line2: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

line3: $\beta_\eta x_\eta = \xi_\eta$

line4: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1}x_{\eta-1}$

line5: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}x_{\eta-1}$

This allows the triples

$(1, 1, 1), (-1, 1, 1), (1, 1, -1), (-1, 1, -1)$ from line1

$(1, -1, 1), (-1, -1, 1), (1, -1, -1), (-1, -1, -1)$ from line2

$(0, 1, 0), (0, -1, 0)$ from line3

$(0, 1, 1), (0, 1, -1), (1, 1, 0), (-1, 1, 0)$ from line4

$(0, -1, 1), (0, -1, -1), (1, -1, 0), (-1, -1, 0)$ from line5

10. **Region S10** is defined by $\beta_\eta > 0$. There are no regions of stability here.

11. **Region S11** is defined by $\beta_\eta < 0$, $\beta_\eta > -\gamma\mu_\eta$, $\beta_\eta < -|\beta_{\eta-1}|$. Existence lines crossing the region include

line2: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

line5: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}x_\eta$

This allows triples

$(1, -1, 1), (-1, -1, 1), (1, -1, -1), (-1, -1, -1)$ from line2

$(0, -1, 1), (0, -1, -1), (1, -1, 0), (-1, -1, 0)$ from line5

12. **Region S12** is described by $\beta_\eta > -2|\beta_{\eta-1}|$, $\beta_\eta < -|\beta_{\eta-1}|$. The possible existence lines include

line2: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

This allows triples

$(1, -1, 1), (-1, -1, 1), (1, -1, -1), (-1, -1, -1)$ from line2

13. **Region S13** is described by $\beta_\eta > -|\beta_{\eta-1}|$, $\beta_\eta = -\gamma\mu_\eta$. The possible existence lines include

line5: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}x_{\eta-1}$

This allows triples

$(0, -1, 1), (0, -1, -1), (1, -1, 0), (-1, -1, 0)$ from line5

14. **Region S14** is described by $\beta_\eta > -2|\beta_{\eta-1}|$, $\beta_\eta > -\gamma\mu_\eta - |\beta_{\eta-1}|$, $\beta_\eta < -|\beta_{\eta-1}|$.

Existence lines crossing the region include

line2: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

line3: $\beta_\eta x_\eta = \xi_\eta$

line5: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}x_\eta$

This allows triples

$(1, -1, 1), (-1, -1, 1), (1, -1, -1), (-1, -1, -1)$ from line2

$(0, 1, 0), (0, -1, 0)$ from line3

$(0, -1, 1), (0, -1, -1), (1, -1, 0), (-1, -1, 0)$ from line5

15. **Region S15** is defined by $\beta_\eta < -2|\beta_{\eta-1}|$, $\beta_\eta > -\gamma\mu_\eta - |\beta_{\eta-1}|$, $\beta_\eta < -\gamma\mu_\eta$. The possible existence lines include

line2: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

line5: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}x_{\eta-1}$

This allows triples

$(1, -1, 1), (-1, -1, 1), (1, -1, -1), (-1, -1, -1)$ from line2

$(0, -1, 1), (0, -1, -1), (1, -1, 0), (-1, -1, 0)$ from line5

16. **Region S16** is defined by $\beta_\eta > -\gamma\mu_\eta - 2|\beta_{\eta-1}|$, $\beta_\eta < -\gamma\mu_\eta - |\beta_{\eta-1}|$, $\beta_\eta < -2|\beta_{\eta-1}|$. The possible existence lines include

line2: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}(x_{\eta-1} + x_{\eta+1})$

line3: $\beta_\eta x_\eta = \xi_\eta$

line4: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1}x_{\eta-1}$

line5: $\beta_\eta x_\eta = \xi_\eta + \beta_{\eta-1}x_{\eta-1}$

This allows triples

$(1, -1, 1), (-1, -1, 1), (1, -1, -1), (-1, -1, -1)$ from line2

$(0, 1, 0), (0, -1, 0)$ from line3

$(0, 1, 1), (0, 1, -1), (1, 1, 0), (-1, 1, 0)$ from line4

$(0, -1, 1), (0, -1, -1), (1, -1, 0), (-1, -1, 0)$ from line5

17. **Region S17** is defined by $\beta_\eta > -2|\beta_{\eta-1}|$, $\beta_\eta < -\gamma\mu_\eta - |\beta_{\eta-1}|$. The possible existence lines include

line4: $\beta_\eta x_\eta = \xi_\eta - \beta_{\eta-1} x_{\eta-1}$

This allows triples

$(0, 1, 1), (0, 1, -1), (1, 1, 0), (-1, 1, 0)$ from line4

To calculate the entropy we consider the transfer of one stable equilibrium to another while going from one lattice point to its adjacent lattice points. If the transformation again yields an existence point (a possible triple in the region under consideration), then there is another point of stability and the entropy of the region is higher than if there were not another possible stable point. Using a Markov Shift we must first construct a transfer matrix. This is a matrix of boolean values representing the possibility of a transfer of stable points. The log of the largest eigenvalue of the transfer matrix is the entropy of the region.

Complete calculations are in Appendix B. We summarize the results in Table 4.6. If the eigenvalue of the transfer matrix is 0, the resulting entropy value is undefined.

Five regions show a high entropy value indicating the presence of spatial chaos. Seven show regions of pattern formation.

Entropy value	Regions
0	S3, S7, S8, S11, S12, S15, S16
$\ln(1.32)$	S4
$\ln(1.618)$	S6
$\ln(2)$	S2, S9
$\ln(2.89)$	S1
undefined	all others

Table 4.6: Entropy and Symmetric Regions

4.3 Comparison of the General and Symmetric One-dimensional Cases

When we compare the entropies of the regions in the symmetric case with those of the general case some significant correlations appear. The regions S2 and S9 in the symmetric case have the same entropy level as R1, R2, and R13 in the general case. The form of the stability criteria is very similar.

$$\begin{aligned}
R1 \quad \beta_\eta^1 + \beta_\eta^2 &< -|\beta_{\eta-1}^1| - |\beta_{\eta+1}^1| - |\beta_{\eta-1}^2| - |\beta_{\eta+1}^2| \\
R2 \quad \beta_\eta^1 + \beta_\eta^2 &< -|\beta_{\eta-1}^1| - |\beta_{\eta+1}^1| - |\beta_{\eta-1}^2| \\
R13 \quad \beta_\eta^1 + \beta_\eta^2 &< -|\beta_{\eta+1}^1| - |\beta_{\eta-1}^2| - |\beta_{\eta+1}^2| \\
S2 \quad \beta_\eta &< -2|\beta_{\eta-1}| \text{ and } \beta_\eta > -\gamma\mu_\eta - |\beta_{\eta-1}| \\
S9 \quad \beta_\eta &< -2|\beta_{\eta-1}| \text{ and } \beta_\eta > -\gamma\mu_\eta
\end{aligned} \tag{4.3}$$

$$\tag{4.4}$$

The general form of the boundaries are very similar. The restrictions in the symmetric case reduce to a single region by virtue of very strict assumptions. For this initial comparison we have restricted the intercept of existence lines, $\xi_\eta < -\gamma\mu_\eta$. In the symmetric case the inequality may be seen as representing the overlap of the general cases. Since in the general case the same number of admissible triples is originally present, the entropy in the symmetric case is not any higher than in the general case.

The symmetric case reflects a dependence on the value of $\frac{1}{x_\eta}\xi_\eta$ in relation to the value of $-\gamma\mu_\eta$. Region S9 is a region on the edge of chaos. Depending on the value of $-\gamma\mu_\eta$ alone, it has relatively high entropy or is a region of no pattern formation.

However, if we look at those regions in the symmetric case of high entropy which were not evident in the general case, it is possible to form a reasonable composite region that could be present in a symmetric or otherwise restricted case. Consider S1 in the symmetric case. The stability region is given by $\beta_\eta < -\gamma\mu_\eta - 2|\beta_{\eta-1}|$. We find the resulting entropy to be $\ln(2.89)$, very close to the maximum expected for a solution set of three elements. In building a composite region from the general regions, we look for those elements that would naturally lead to a similar stability form. R1, R10, R11, R12 all have references to the $|\beta_{\eta-1}|$ -type terms. R19, R20, R23, R24, R27, R28, R29, R31, and R32 all have the sum of the two $-\gamma\mu_\eta$ -type terms. The resulting transfer matrix has entropy $\ln(3)$. This indicates that it is possible for the symmetric case to effectively merge general regions into composite regions with higher entropy.

Chapter 5

Conclusion

We can conclude from the previous analysis, that there is a basis for describing chaotic behavior. Regions in a general one-dimensional parameter space have been identified that have a spatial entropy value in excess of zero. These regions of spatial chaos tend to leave the neural network model sensitive to initial conditions. Coupling coefficients and connectivity do play a part in the behavior of the time-evolutionary system. In the general one-dimensional case we find 32 regions defined, 3 regions have entropy greater than zero, the value calculated as $\ln(2)$. For mosaic solution sets of 3 values the largest entropy is $\ln(3)$ making those values of $\ln(2)$ an indicator of spatial chaos. Since it's possible for these regions to overlap, the possibility for even larger entropy values exists.

Symmetric coefficients increase the chances of chaotic behavior by raising the entropy calculation in predictable regions. We found 17 distinct regions where the largest entropy value is $\ln(2.89)$ in one region. Four other regions have entropy greater than 0. A comparison of the general and symmetric regions leads us to suspect an overlapping condition in the symmetric case since the regions are defined by inequalities of similar structure. Further investigation into this possibility is necessary.

However, this result does seem to indicate an even greater possibility of spatial chaos in the symmetric case than in the general case.

This analysis presents stability conditions and the regions formed by these conditions. The spatial entropy analysis indicates the significant possibility of chaotic behavior. More remains to be done to make this analysis applicable to creating neural networks and using this information to streamline existing models. Currently, it is understood that random initial weights avoid problems of convergence to local minima as opposed to global minima. But more research into the effects of the weight-related coefficient μ_η needs to be done. The role of the bias term has not been explored, as it could manage the ξ_η term and hence the μ_{eta} term. We see a further dependence in the symmetric case where region S9 is a region of high entropy or a region of non-stability depending on the value of μ_η and the value of ξ_η . A similar analysis could be done in terms of μ_η and α_η as well. More experimental research into this dependence is certainly indicated to enhance the intuition regarding these results.

We also leave for future research the explicit analysis of those systems of higher dimension. Most neural networks are of much higher dimension than the one-dimensional case. In general, those methods are in development. Explicit region definition and entropy calculation for those models of higher dimension would give better understanding of the roles of the coupling coefficients and the bias terms. The technique will be best tested when applied directly to a Hopfield net, a CNN, or a Backpropagation formula.

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APPENDIX A

In this appendix, we present the full calculations for the eigenvalues of the 32 regions defined in the general one-dimensional case. We list each possible triple transfer pair and the resulting doubles needed to construct the **transfer matrix** to be used in calculating the entropy of the region. The regions are defined in Chapter 4.

R1 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, \pm 1, \pm 1) \rightarrow (\pm 1, \pm 1, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(1, 1), (-1, 1), (-1, -1), (1, -1)$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The largest eigenvalue: 2

R2 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, \pm 1, \pm 1) \rightarrow (\pm 1, \pm 1, 0)$ has eligible doubles in order of rows and columns of matrix :

$(1, 1), (-1, 1), (-1, -1), (1, -1)$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The largest eigenvalue: 2

R3 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, \pm 1, \pm 1) \rightarrow (\pm 1, 0, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(1, 1), (-1, 1), (1, 0), (-1, -1), (1, -1), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R4 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, \pm 1, \pm 1) \rightarrow (\pm 1, 0, 0)$ has eligible doubles in order of rows and columns of matrix :

$(1, 1), (-1, 1), (1, 0), (-1, -1), (1, -1), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R5 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, \pm 1, 0) \rightarrow (0, \pm 1, 0)$ has eligible doubles in order of rows and columns of matrix :

$(1, 0), (0, 1), (-1, 0), (0, -1)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R6 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, \pm 1, 0) \rightarrow (0, \pm 1, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(1, 0), (0, 1), (-1, 0), (0, -1)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R7 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, \pm 1, 0) \rightarrow (0, 0, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(1, 0), (0, 0), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R8 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, \pm 1, 0) \rightarrow (0, 0, 0)$ has eligible doubles in order of rows and columns of matrix :

$(1, 0), (0, 1), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R9 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, \pm 1, 0) \rightarrow (0, \pm 1, 0)$ has eligible doubles in order of rows and columns of matrix :

$(1, 0), (0, 1), (-1, 0), (0, -1)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R10 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, \pm 1, 0) \rightarrow (0, \pm 1, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(1, 0), (0, 1), (-1, 0), (0, -1)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R11 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, \pm 1, 0) \rightarrow (0, 0, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(1, 0), (0, 1), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R12 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, \pm 1, 0) \rightarrow (0, 0, 0)$ has eligible doubles in order of rows and columns of matrix :

$(1, 0), (0, 1), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R13 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, \pm 1, \pm 1) \rightarrow (\pm 1, \pm 1, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(1, 1), (-1, 1), (-1, -1), (1, -1)$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The largest eigenvalue: 2

R14 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, \pm 1, \pm 1) \rightarrow (\pm 1, \pm 1, 0)$ has eligible doubles in order of rows and columns of matrix :

$(1, 1), (-1, 1), (-1, -1), (1, -1)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R15 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, \pm 1, \pm 1) \rightarrow (\pm 1, 0, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(1, 1), (1, -1), (1, 0), (-1, -1), (-1, 1), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R16 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, \pm 1, \pm 1) \rightarrow (\pm 1, 0, 0)$ has eligible doubles in order of rows and columns of matrix :

$(1, 1), (-1, 1), (1, 0), (-1, -1), (1, -1), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R17 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, 0, \pm 1) \rightarrow (\pm 1, \pm 1, \pm 1)$

has eligible doubles in order of rows and columns of matrix :

$(0, 1), (1, 1), (-1, 1), (0, -1), (-1, -1), (1, -1)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R18 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, 0, \pm 1) \rightarrow (\pm 1, \pm 1, 0)$ has

eligible doubles in order of rows and columns of matrix :

$(0, 1), (1, 1), (-1, 1), (0, -1), (-1, -1), (1, -1)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R19 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, 0, \pm 1) \rightarrow (\pm 1, 0, \pm 1)$ has

eligible doubles in order of rows and columns of matrix :

$(0, 1), (1, 0), (0, -1), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R20 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, 0, \pm 1) \rightarrow (\pm 1, 0, 0)$ has eligible doubles in order of rows and columns of matrix :

$(0, 1), (1, 0), (0, -1), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R21 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, 0, \pm 1) \rightarrow (\pm 1, \pm 1, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(0, 1), (1, 1), (-1, 1), (0, -1), (-1, -1), (1, -1)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R22 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, 0, \pm 1) \rightarrow (\pm 1, \pm 1, 0)$ has eligible doubles in order of rows and columns of matrix :

$(0, 1), (1, 1), (-1, 1), (0, -1), (-1, -1), (1, -1)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R23 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, 0, \pm 1) \rightarrow (\pm 1, 0, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(0, 1), (1, 0), (0, -1), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R24 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, 0, \pm 1) \rightarrow (\pm 1, 0, 0)$ has eligible doubles in order of rows and columns of matrix :

$(0, 1), (1, 0), (0, -1), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R25 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, 0, 0) \rightarrow (0, \pm 1, 0)$ has eligible doubles in order of rows and columns of matrix :

$(1, 0), (0, 0), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R26 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, 0, 0) \rightarrow (0, \pm 1, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(0, 1), (0, 0), (-1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R27 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, 0, 0) \rightarrow (0, 0, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(0, 0)$

$$\begin{pmatrix} 0 \end{pmatrix}$$

The largest eigenvalue: 0

R28 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(\pm 1, 0, 0) \rightarrow (0, 0, 0)$ has eligible doubles in order of rows and columns of matrix :

$(0, 0)$

$$\begin{pmatrix} 1 \end{pmatrix}$$

The largest eigenvalue: 1

R29 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, 0, 0) \rightarrow (0, \pm 1, 0)$ has eligible doubles in order of rows and columns of matrix :

$(0, 0), (0, 1), (0, -1)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R30 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, 0, 0) \rightarrow (0, \pm 1, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(0, 0), (0, 1), (0, -1)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

R31 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, 0, 0) \rightarrow (0, 0, \pm 1)$ has eligible doubles in order of rows and columns of matrix :

$(0, 0)$

$$\begin{pmatrix} 1 \end{pmatrix}$$

The largest eigenvalue: 1

R32 $T_1(x_{\eta-1}^{(1)}, x_{\eta}^{(1)}, x_{\eta+1}^{(1)}) \rightarrow T_2(x_{\eta-1}^{(2)}, x_{\eta}^{(2)}, x_{\eta+1}^{(2)})$ gives $(0, 0, 0) \rightarrow (0, 0, 0)$ has eligible doubles in order of rows and columns of matrix :

$(0, 0)$

$$\begin{pmatrix} 1 \end{pmatrix}$$

The largest eigenvalue: 1

APPENDIX B

In this appendix, we present the doubles indicated by the transfer possibilities of the triples in the symmetric one-dimensional case. The **transfer matrices** are enumerated and eigenvalues computed. A summary of the regional definitions and resulting entropies is presented in Chapter 4.

The sequence of admissible doubles used to construct the transfer matrices is as follows:

Region S1 $\{(1,1),(1,-1),(1,0),(0,1),(0,0),(-1,-1),(-1,1),(-1,0),(0,-1)\}$

Region S2 $\{(1,1),(1,-1),(1,0),(0,1),(0,0),(-1,-1),(-1,1),(-1,0),(0,-1)\}$

Region S3 $\{(1,1),(1,-1),(1,0),(0,1),(-1,-1),(-1,1),(-1,0),(0,-1)\}$

Region S6 $\{(1,1),(1,-1),(1,0),(0,1),(0,0),(-1,-1),(-1,1),(-1,0),(0,-1)\}$

Region S5 $\{(1,1),(1,-1),(1,0),(0,1),(-1,1)\}$

Region S4 $\{(-1,-1),(-1,1),(-1,0),(0,1),(0,-1),(0,-1)\}$

Region S7 $\{(1,1),(1,-1),(-1,1)\}$

Region S8 $\{(1,1),(1,-1),(1,0),(0,1),(0,0),(-1,-1),(-1,1),(-1,0),(0,-1)\}$

Region S9 $\{(1,1),(1,-1),(1,0),(0,1),(-1,-1),(-1,1),(-1,0),(0,-1)\}$

Region S10 $\{(1,1),(1,-1),(1,0),(0,1),(-1,-1),(-1,1),(-1,0),(0,-1)\}$

Region S11 $\{(1,-1),(-1,-1),(-1,1),(-1,0),(0,-1)\}$

Region S12 $\{(1,1),(1,-1),(-1,1)\}$

Region S13 $\{(1,-1),(-1,-1),(-1,1),(-1,0),(0,-1)\}$

Region S14 $\{(1,-1),(1,0),(0,1),(-1,-1),(-1,1),(-1,0),(0,-1)\}$

Region S15 $\{(1,-1),(-1,-1),(-1,1),(-1,0),(0,-1)\}$

Region S16 $\{(1,1),(1,-1),(1,0),(0,1),(-1,-1),(-1,1),(-1,0),(0,-1)\}$

Region S17 $\{(1,1),(1,-1),(1,0),(0,1),(-1,1)\}$

The transfer matrices and largest eigenvalues are as follows:

Region S1

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The largest eigenvalue: 2.89329

Region S2

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue:2

Region S3

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 1

Region S6

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 1.61803

Region S5

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue:0

Region S4

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The largest eigenvalue: 1.32472

Region S7

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

The largest eigenvalue: 1

Region S8

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 1

Region S9

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The largest eigenvalue: 2

Region S10 Although S10 is not a region of stability, A possible transfer matrix is included. It doesn't seem to change the outcome.

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

Region S11

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 1

Region S12

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 1

Region S13

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0

Region S14

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The largest eigenvalue: 1

Region S15

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 1

Region S16

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The largest eigenvalue: 1

Region S17

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue: 0