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INCORPORATION OF POWER-LAW FLUID EFFECTS IN THE PRESENCE OF LEAK-OFF IN THE PERKINS AND KERN MODEL OF VERTICAL FRACTURE PROPAGATION

by

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ABSTRACT

Non-Newtonian effects through power-law fluid in the presence of leak-off have been investigated, extending Nordgren's version of the Perkins and Kern model of vertical fracture propagation. The results show the substantial effect of the non-Newtonian fluid relative to the Newtonian cases for both fracture width and length. The present solutions are first compared with Nordgren's results for the Newtonian fluid. It gives good agreement between the two results. On the basis of the same method, effects of the non-Newtonian fluid on the fracture width and length are obtained.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont'd)

LIST OF FIGURES

LIST OF TABLES

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LITERATURE SURVEY

INTRODUCTION

Fracture propagation theories are based on the assumption that the rock deforms in a linear elastic manner when subjected to a liquid pressure during the hydraulic fracture treatment. The theory of elasticity provides a fracture-width equation which determines what shape the fracture will have. Viscous-flow theory establishes a relationship between the fluid pressure-gradient and the fracture-width. The computation of the geometry of hydraulic fractures consist of the mathematical combination of the fluid and fracture mechanics of the process.

Two different models of linearly expanding vertical fracture have been derived. In one model, introduced by Perkins and Kern(1), the fracture elasticity is concentrated in the vertical direction. Thus the fracture-width equation is utilized to describe the shape of vertical cross sections perpendicular to the direction of propagation. No elastic resistance of the formation in the horizontal direction is considered. In the other model, by Geertsma and De Klerk(2), the formation stiffness is considered only in the horizontal direction and neglected in the vertical direction. The sides of the fracture are assumed to be parallel, and the vertical cross sections are of rectangular shape (Fig. 1).

The classical procedures for designing hydraulic fractures belonging to the first model are those of Perkins and Kern and Nordgren[3]. The second model includes the methods of Geertsma and De Klerk and Daneshy[4]. The present paper concentrates on those methods based on the Perkins and Kern model of propagation, particularly Nordgren's method.

PERKINS AND KERN MODEL

Consider a hydraulic fracture extending linearly away from the wellbore with a fixed height. The fluid pressure decreases in the direction of fracture propagation, but in each vertical cross section the pressure is constant and slightly higher than the in-situ rock-stress component perpendicular to the fracture plane. The elastic deformations of the rock are concentrated in these cross sections. In the horizontal direction only frictional resistance of the fracturing fluid flowing in the narrow space between the fracture walls is taken into account. This is basically the model conceived by Perkins and Kern.

A relationship between fracture width and an arbitrary normal pressure distribution over the fracture wall was derived by England and Green[5]. This is a rather complex equation, which on the basis of constant pressure in the fracture, results in a more simplified version:

$$w(x) = \frac{(1-\nu) L \Delta P}{G} \left[1 - \left(\frac{x}{L}\right)^2 \right]^{\frac{1}{2}} \qquad (1)$$

where x is the coordinate in the direction of the length of the crack, and ΔP represents the difference between the fluid pressure and the rock-stress component perpendicular to the fracture plane. Fracture width, fluid pressure, and fracture height of the present model are interrelated according to this equation, if L is replaced by H and x by y.

$$w(y) = \frac{(1-\nu) H \Delta P}{G} \left[1 - \left(\frac{y}{H}\right)^2 \right]^{\frac{1}{2}}$$

These cross sections are thus elliptical in shape. Expression for the maximum width is obtained by setting y = 0, which results in

$$W(x,t) = \frac{(1-\nu) H \ \Delta P(x,t)}{G} \qquad (2)$$

and x is now in the direction of fracture propagation. It is also noted that both fracture width and fluid pressure vary with time of injection.

Fluid pressure may be related to the fracture width through viscous flow theories. Considering Newtonian fracturing fluid in laminar flow behavior, and noting that the vertical cross sections are of elliptical shape, the pressure gradient decreases with distance from the injection point according to Poisseuille's equation

$$\frac{\partial \Delta p(x,t)}{\partial x} = \frac{-64}{3\pi} \frac{\mu q(x,t)}{HW^3(x,t)} \tag{3}$$

Combination of Eqs. 2 and 3 gives

$$\frac{\partial W^4(x,t)}{\partial x} = \frac{-256(1-\upsilon)q(x,t)}{\pi G} \tag{4}$$

Perkins and Kern obtained a quick solution for this equation by assuming q(x,t) = Q (const.). Direct integration of Eq. 4 between x = 0 and x = L yields

$$W(o,t) = \left[\frac{256(1-\upsilon)\mu\ QL(t)}{\pi G}\right]^{1/4} \tag{5}$$

CARTER'S SOLUTION OF THE LEAK-OFF PROBLEM

During the fracture propagation, a certain volume of the fluid is lost to the formation and is not effective in extending the fracture. The basic solution for the effect of fluid losses was published by Carter [6]. He assumed that the flow of the fracturing fluid into the formation is linear and the direction perpendicular to the fracture faces.

The leak-off velocity at any point in the fracture is a function of the time this point has been exposed to flow according to

$$v(x,t) = \frac{C}{\sqrt{t-\tau(x)}} \tag{6}$$

This velocity function is the same for every point in the fracture face.

Carter combined this equation with a balance of volume in the fracture, and derived an analytical expression for the fracture length

$$L(t) = \frac{QW}{4\pi HC^2}\left[\frac{2\alpha}{\sqrt{\pi}} - 1 + e^{\alpha^2}\operatorname{erfc}\alpha\right] \tag{7}$$

where

$$\alpha = \frac{2C}{W}\sqrt{\pi t}$$

A common procedure for designing hydraulic fracturing treatment is to combine Carter's formula with Perkins and Kern's fracture width equation. A fracture length is assumed and the corresponding fracture width calculated from Eq. 5. A new fracture length is then calculated from Eq. 7. By means of iteration corresponding values for fracture width and length are thus obtained.

NORDGREN'S METHOD

Combination of Carter's formula and the fracture width theory of Perkins and Kern has been generally used to predict fracture dimensions. However, in deriving his fracture-length equation, Carter assumed a constant width. This assumption is not compatible with the theory of Perkins and Kern.

Nordgren considered this problem with a more rigorous mathematical approach. He tried to combine these two parts of the theory of fracture propagation by considering the local continuity equation

$$\frac{\partial q(x,t)}{\partial x} + \frac{A(x,t)}{\partial t} + q_1(x,t) = 0$$

Thus the effect of reduction in flow rate due to growth of the cross-sectional area with time as well as rate of leak-off was considered by Nordgren. Obviously this complicates the treatment of the problem. In the simplified fracture-width equation of Perkins and Kern this effect is completely ignored.

The area of the elliptical vertical cross sections is

$$A(x,t) = \frac{\pi}{4} HW(x,t)$$

therefore,

$$\frac{\partial q(x,t)}{\partial x} + \frac{\pi H}{4} \frac{\partial W(x,t)}{\partial t} + q_1(x,t) = 0 \tag{8}$$

Combination of this equation with Eqs. 4 and 6 gives

$$\frac{\pi G}{256(1-\nu)\mu} \frac{\partial^2 W^4(x,t)}{\partial x} = \frac{2HC}{\sqrt{t-\tau(x)}} + \frac{\pi H}{4} \frac{\partial W(x,t)}{\partial t}$$

This nonlinear version of the local continuity equation, subjected to the following boundary and initial conditions,

$$W(x,t) = 0, \qquad x > L(t)$$

$$\left[\frac{\partial W^4(x,t)}{\partial x}\right]_{x=0} = \frac{256\mu(1-\nu)Q}{\pi G}$$

$$W(x,0) = 0$$

constitutes the problem proposed by Nordgren.

DEVELOPMENT OF NORDGREN'S SOLUTION

DISCUSSION

One of the difficulties of Nordgren's problem is that Eq. 8 is to be solved for W(x,t) with the fracture length also to be determined as part of the solution. The function $\tau(x)$ - the time necessary for the fracture to extend to a distance x from the wellbore - is a function inverse of the fracture length. Thus

$$\tau\left[L(t)\right] = t$$

which implies that Eq. 8 actually comprises both fracture width and length as unknowns. Whatever method of numerical computation used, an iteration will be necessary to solve this equation. In the solution proposed by Nordgren, the fracture length increment is taken as constant. Thus the time increment of each step of computation cannot be arbitrarily assumed, but must take a definite value corresponding to the increment of fracture length. Nordgren, however, did not indicate how to avoid this and other difficulties of his numerical scheme.

An explanation of Nordgren's method is next presented in detail. An understanding of this method is necessary since it provides a basis for an alternative method of numerical computation, which was developed in the present paper to accommodate power-law fluid in the equations. Nordgren's finite-difference analogs, based on the block-centered grid configuration shown in Fig. 2 in presented. Analytical approximate solutions for fracture length and width are derived and compared with similar solutions obtained by Nordgren. Also a procedure for Nordgren's numerical computation is suggested.

FINITE-DIFFERENCE ANALOGS

Upon introduction of the dimensionless group

$$\bar{\tau} = \tau(x)/K$$

$$\bar{X} = x/I$$

$$\bar{L} = L(t)/I$$

$$\bar{t} = t/K$$

$$\bar{W} = W(x,t)/J$$

$$\bar{q} = q(x,t)/Q$$

where

$$I = \pi \left[\frac{(1-\nu)\mu Q^5}{256GC^8H^4}\right]^{1/3}$$

$$J = \left[\frac{16(1-\nu)\mu Q^2}{GC^2H^4}\right]^{1/3}$$

$$K = \pi^2 \left[\frac{(1-\nu)\mu\ Q^2}{32GC^5H} \right]^{2/3}$$

one may write Eqs. 4 and 8 as

$$\bar{q} = - \frac{\partial \bar{W}^4}{\partial \bar{X}} \tag{9}$$

and

$$\frac{\partial \bar{q}}{\partial \bar{x}} + \frac{\partial \bar{W}}{\partial \bar{t}} + \frac{1}{\sqrt{\bar{t}-\bar{\tau}}} = 0 \tag{10}$$

Finite-difference analogs, based on the block-centered grid configuration of Fig. 2, may now be developed by first integrating Eq. 10 with respect to time from $\bar{t}^m$ to $\bar{t}^{m+1}$ with use of the trapezoidal integration formula.

$$\frac{\Delta \bar{t}^m}{2} \frac{\partial}{\partial \bar{x}} \left(\bar{q}^{m+1} + \bar{q}^m \right) + 2 \left(\sqrt{\bar{t}^{m+1} - \bar{\tau}} - \sqrt{\bar{t}^m - \bar{\tau}} \right) + \bar{W}^{m+1} - \bar{W}^m = 0$$

Integration of this equation with respect to distance from $\bar{x}_{i-\frac{1}{2}}$ to $\bar{x}_{i+\frac{1}{2}}$ with use of the central value for the integral gives

$$\frac{\Delta \bar{t}^m}{2\Delta \bar{x}} \left(\bar{q}^{m+1}_{i+\frac{1}{2}} - \bar{q}^{m+1}_{i-\frac{1}{2}} + \bar{q}^m_{i+\frac{1}{2}} - \bar{q}^m_{i-\frac{1}{2}} \right) + 2f(\bar{\tau}_i) + \Delta \bar{W}^m_i = 0 \tag{11}$$

where

$$f(\bar{\tau}_i) = \sqrt{\bar{t}^{m+1} - \bar{\tau}_i} - \sqrt{\bar{t}^{m} - \bar{\tau}_i}$$

$$\Delta \bar{w}_i^m = \bar{w}_i^{m+1} - \bar{w}_i^m$$

and, as depicted in Fig. 2,

$$\Delta \bar{x} = \bar{x}_{i+\frac{1}{2}} - \bar{x}_{i-\frac{1}{2}} = \bar{x}_i - \bar{x}_{i-1}$$

$$\bar{x}_i = \bar{x}_{i-1} + \Delta \bar{x}$$

A centered-difference analog of Eq. 4 is

$$\bar{q}_{i+\frac{1}{2}}^{m+1} = \frac{\left(\bar{w}_i^{m+1}\right)^4 - \left(\bar{w}_{i+1}^{m+1}\right)^4}{\Delta \bar{x}} \tag{12}$$

The basic idea of the present method is to combined Eqs. 12 and 11 to obtain an expression involving the fracture widths at the node i and the two adjacent nodes, i-1 and i+1. However, the exponential functions of the widths complicates the treatment of the problem. Nordgren handled this problem by taking the following approximation:

$$\bar{w}_i^{m+1} \simeq \left(\bar{w}_i^m\right)^4 + 4\left(\bar{w}_i^m\right)^3 \Delta \bar{w}_i^m \tag{13}$$

Implied in this equation are the assumptions that $\Delta\bar{W}_i^m$ is sufficiently small and that

$$\bar{W}_i^m \neq 0$$

Substitution of Eq. 13 into Eq. 12 gives

$$\bar{q}_{i+\frac{1}{2}}^{m+1} = \frac{\left(\bar{W}_i^m\right)^4 + 4\left(\bar{W}_i^m\right)^3 \Delta\bar{W}_i^m - \left(\Delta\bar{W}_{i+1}^m\right)^4 - 4\left(\bar{W}_{i+1}^m\right)^3 \Delta\bar{W}_{i+1}^m}{\Delta\bar{x}} \tag{14}$$

which may be combined with Eq. 11 yielding

$$- \alpha_{i-1}\Delta\bar{W}_{i-1}^m + 2(\alpha_i+1)\Delta\bar{W}_i^m - \alpha_{i+1}\Delta W_{i+1} = \frac{\Delta\bar{t}^m}{\Delta\bar{x}} \left(\bar{W}_{i-1}^m\right)^4 - 2\left(\bar{W}_i^m\right)^4 + \left(\bar{W}_{i+1}^m\right)^4 - 4f(\bar{\tau}_i) \tag{15}$$

where

$$\alpha_i = 4\,\frac{\Delta\bar{t}^m}{\Delta\bar{x}^2}\left(\bar{W}_i^m\right)^3$$

Equation 15 is the finite-difference analog of the governing equation of Nordgren's problem for a typical node i extending from $\bar{x}_{i-\frac{1}{2}}$ to $\bar{x}_{i+\frac{1}{2}}$ (Fig. 2).

Consideration is now given to the boundary conditions of this problem. The fracture is supposed to be closed at

$\bar{x}_{\ell+\frac{1}{2}}$ at the time step m and at $\bar{x}_{\ell+1\frac{1}{2}}$ at the time step m+1. Thus at any step of computation the fracture length increment is constant and equal to $\bar{x}_{\ell+1\frac{1}{2}} - \bar{x}_{\ell-\frac{1}{2}}$. This implies the following boundary conditions:

$$\bar{W}^{m}_{\ell+1},\ \bar{q}^{m}_{\ell+\frac{1}{2}},\ \bar{q}^{m+1}_{\ell+1\frac{1}{2}} = 0 \qquad (16)$$

On the other hand, the flow rate at the fracture entrance is assumed to be constant during the fracturing treatment. In dimensionless form this may be stated by

$$\bar{q}^{m}_{\frac{1}{2}} = 1 \qquad (17)$$

Equation 11 is now written for the first node (i = 1) and for the last node at the time step m(i = ℓ). Upon setting i = 1 and taking Eq. 16 into account one obtains

$$\frac{\Delta\bar{t}^{m}}{2\Delta\bar{x}}\left(\bar{q}^{m+1}_{1+\frac{1}{2}} - 1 + \bar{q}^{m}_{1+\frac{1}{2}} - 1\right) + 2f(\bar{\tau}_{1}) + \Delta\bar{W}^{m}_{1} = 0 \qquad (18)$$

Similarly, Eq. 11 for i = ℓ combined with Eq. 16 becomes

$$\frac{\Delta\bar{t}^{m}}{2\Delta\bar{x}}\left(\bar{q}^{m+1}_{\ell+\frac{1}{2}} - \bar{q}^{m+1}_{\ell-\frac{1}{2}} - \bar{q}^{m}_{\ell-\frac{1}{2}}\right) - 2f(\bar{\tau}_{\ell}) + \Delta\bar{W}^{m}_{\ell} = 0 \qquad (19)$$

Substitution of Eq. 14 into Eq. 18 leads to

$$(\alpha_1+2)\Delta\bar{W}_1^m - \alpha_2(\Delta\bar{W}_2^m) = 2\,\frac{\Delta\bar{t}^m}{\Delta\bar{x}}\left[1 - \frac{\left(\bar{W}_1^m\right)^4 - \left(\bar{W}_2^m\right)^4}{\Delta\bar{x}}\right] - 4f(\bar{\tau}_1) \quad (20)$$

Equation 19, however, cannot be treated on the same basis of Eq. 18. Since $\bar{W}_{\ell+1}^m = 0$ (boundary condition), Eq. 14 is not valid for $i = \ell$. Thus Eq. 12 is written for $i = \ell$

$$\bar{q}_{\ell+\frac{1}{2}}^{m+1} = \frac{1}{\Delta\bar{x}}\left[4\left(\bar{W}_\ell^m\right)^3\Delta\bar{W}_\ell^m + \left(\bar{W}_\ell^m\right)^3 - \left(\bar{W}_{\ell+1}^{m+1}\right)^4\right] \quad (21)$$

Substituting Eq. 21 into Eq. 19 and using Eq. 14 to replace $\bar{q}_{\ell-\frac{1}{2}}^{m+1}$ and $\bar{q}_{\ell-\frac{1}{2}}^{m}$ gives

$$-\alpha_{\ell-1}\Delta\bar{W}_{\ell-1}^m + 2\left(\alpha_\ell+1\right)\Delta\bar{W}_\ell^m = \frac{\Delta\bar{t}^m}{\Delta\bar{x}^2}\left[2\left(\bar{W}_{\ell-1}^m\right)^4 - 3\left(\bar{W}_\ell^m\right)^4 + \left(\bar{W}_{\ell+1}^{m+1}\right)^4\right] - f(\bar{\tau}_\ell) \quad (22)$$

Writing Eq. 15 for $2 \leq i \leq \ell-1$, and taking Eqs. 20 and 22, respectively, for the particular cases when $i = 1$ and $i = \ell$, provides the following tridiagonal system:

$$\begin{bmatrix} \alpha_1+2 & -\alpha_2 & & & 0 \\ -\alpha_1 & 2\alpha_2+2 & -\alpha_3 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -\alpha_\ell \\ 0 & & & -\alpha_{\ell-1} & 2\alpha_\ell+2 \end{bmatrix} \times \begin{bmatrix} \bar{W}_1^m \\ \bar{W}_w^m \\ \vdots \\ \Delta\bar{W}_\ell^m \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_\ell \end{bmatrix}$$

where

$$D_i = \frac{\Delta\bar{t}^m}{\Delta\bar{x}^2}\left[\left(\bar{W}_{i-1}^m\right)^4 - 2\left(\bar{W}_i^m\right)^4 + \left(\bar{W}_{i+1}^m\right)^4\right], \quad 2 \le i \le \ell-1$$

$$D_1 = \frac{2\Delta\bar{t}}{\Delta\bar{x}}\left[1 - \frac{\left(\bar{W}_1^m\right)^4 - \left(\bar{W}_2^m\right)^4}{\Delta\bar{x}}\right]$$

$$D_\ell = \frac{\Delta\bar{t}}{\Delta\bar{x}^2}\left[2\left(\bar{W}_{\ell-1}^m\right)^4 - 3\left(\bar{W}_\ell^m\right)^4 + \left(\bar{W}_{\ell+1}^{m+1}\right)^4\right]$$

This system is to be solved for $\bar{W}_i^{m+1}$ $(i = 1, 2, \ldots\ell)$ with two additional unknowns: the time increment ($\Delta\bar{t}m$), and the fracture width at the center of the last node ($\bar{W}_{\ell+1}^{m+1}$). The time increment must take a definitive value corresponding to the fracture length increase during the period of time $\Delta\bar{t}^m$. Therefore, an iteration is required to solve the problem. A value of time increment is assumed and is to be compared in some way with the resulting fracture dimensions solved by the matrix problem. A relationship between time increment and fracture dimensions may be derived on the basis of the following balance of volume during fracture propagation:

$$\text{volume injected during time } \Delta\bar{t} = \text{sum of fracture volume increment} + \text{sum of leak-off volume increment}$$

In dimensionless form this can be written as

$$\Delta\bar{t}^m = \sum_{i=1}^{\ell+1} \Delta\bar{W}_i^{m+1} \cdot \Delta\bar{x} + 2 \sum_{i=1}^{\ell+1} f(\bar{\tau}_i) \cdot \Delta\bar{x} \tag{23}$$

Analytical solutions for $\bar{W}_{\ell+1}^{m+1}$ is also required because this variable is included in the right-hand side of the tri-diagonal system. Concentrating in the node $\ell+1$, and neglecting leak-off effects during the period of time $\Delta\bar{t}^m$ one may write a simplified version of the local continuity equation

$$\frac{\partial\bar{q}}{\partial\bar{x}} + \frac{\partial\bar{W}}{\partial\bar{t}} = 0$$

This can be approximated by the method described above, according to Eq. 11. Now, if i is set equal to $\ell + 1$, and taking the boundary conditions into account, Eq. 11 becomes

$$\frac{\Delta\bar{t}^m}{2}\left(\bar{q}_{\ell+1\frac{1}{2}}^{m+1} - \bar{q}_{\ell+\frac{1}{2}}^{m+1}\right) + \bar{W}_i^{m+1} = 0$$

From Eq. 16,

$$\bar{q}^{m+1}_{\ell+\frac{1}{2}} = 0$$

Therefore,

$$\bar{q}^{m+1}_{\ell+\frac{1}{2}} = \frac{2\Delta\bar{x}}{\Delta\bar{t}^{m}} \bar{W}^{m+1}_{\ell+1} \tag{24}$$

The forward-difference analog of Eq. 9 is

$$\bar{q}^{m+1}_{i} = \frac{\left(\bar{W}^{m+1}_{i}\right)^{4} - \left(\bar{W}^{m+1}_{i+\frac{1}{2}}\right)^{4}}{\Delta\bar{x}/2} \tag{25}$$

which gives an approximate expression for the flow rate at the center of the node, by simply replacing i by $\ell + 1$ and noting that $\bar{W}^{m+1}_{\ell+1\frac{1}{2}} = 0$ (boundary condition). Thus

$$\bar{q}^{m+1}_{\ell+1} = \frac{\left(\bar{W}^{m+1}_{\ell+1}\right)^{4}}{\Delta\bar{x}} \tag{26}$$

Combination of Eqs. 24 and 26 gives

$$\frac{\bar{q}^{m+1}_{\ell+1}}{\bar{q}^{m+1}_{\ell+\frac{1}{2}}} = \frac{\Delta\bar{t}^{m}\left(\bar{W}^{m+1}_{\ell+1}\right)^{3}}{\Delta\bar{x}} \tag{27}$$

No other relationship between $\bar{q}^{m+1}_{\ell+1}$ and $\bar{q}^{m+1}_{\ell+\frac{1}{2}}$ is available. In the present paper, the quotient between those two variables was assumed to be equal to ½, independently of the distance of fracture propagation. Although this approximation proved extremely useful in the present investigation, there is no justification for such assumption. It was found, however, that results obtained with other arbitrary quotients were approximately the same, specially for large volumes of treatment. Now Eq. 27 explicitly in the fracture width becomes

$$\bar{W}_{\ell+1}^{m+1} = \left[\frac{\Delta\bar{x}^2}{2\Delta\bar{t}^m}\right]^{1/3} \tag{28}$$

The problem is thus solved by first assuming a value for the time increment, then calculating the fracture width at the center of the last node from Eq. 28. The matrix system is next solved for $\bar{W}_i^m$, $1 \leq i \leq \ell$. This procedure is repeated until the value of the time increment is confirmed through Eq. 23.

The fracture length is computed by

$$\bar{L}^{m+1} = (\ell+1)\ \Delta\bar{x}$$

An expression for the fracture width at the wellbore is obtained by writing Eq. 25 with i equal to ½ and noting that $\bar{q}_{1/2}^{m+1} = \dot{1}$. This results in

$$\bar{w}_{\frac{1}{2}}^{m+1} = \left[\bar{w}_{1}^{m+1} + \frac{\Delta\bar{x}}{2}\right]^{\frac{1}{4}} \tag{29}$$

This numerical method of solution also requires analytical approximations for both fracture width at the wellbore and fracture length. As was implied in the preceding discussion, the fracture dimensions at the time step m are supposed to be known. Fortunately, such approximations are relatively easy to be derived, on the basis of the same method utilized for deriving the equation for fracture width at the node $\ell + 1$.

The fracture is initially closed. After the period of time $\Delta\bar{t}^0$ it will propagate through a distance equal to $\Delta\bar{x}$. Thus the problem now is reduced to the computation of only one node (i = 1). The boundary conditions are

$$\bar{w}_{1+\frac{1}{2}}^{1} = 0$$

$$\bar{q}_{\frac{1}{2}}^{1} = 1$$

Therefore Eq. 25 becomes

$$\bar{q}_{1}^{1} = \frac{\left(\bar{w}_{1}^{1}\right)^{4}}{\Delta\bar{x}/2} \tag{30}$$

Taking the approximation

$$\bar{q}_1^1 = \frac{\bar{q}_{1/2}^1}{2} = 1/2$$

one can write Eq. 30 explicit in $\Delta\bar{x}$.

$$\Delta\bar{x} = 4\left(\bar{w}_1^1\right)^4 \tag{31}$$

If leak-off is neglected, Eq. 23 now can be written as

$$\Delta\bar{t}^0 = \bar{w}_1^1 \; \Delta\bar{x} \tag{32}$$

Combination of Eqs. 31 and 32 leads to

$$\Delta\bar{x} = 4^{1/5}(\Delta\bar{t}^0)^{4/5} = 1.3195(\Delta\bar{t}^0)^{4/5} \tag{33}$$

which is to be compared with Nordgren's fracture-length equation for small time of injection:

$$\bar{L} = 1.32 \times \bar{t}^{4/5}$$

Solution for fracture width at the wellbore are derived by first combining Eqs. 32 and 33 to obtain an expression for $\bar{w}_1^1$ as a function of $\Delta\bar{t}^0$:

$$\bar{W}_1^1 = \left[\frac{\Delta \bar{t}^0}{4} \right]^{1/5} \tag{34}$$

Now substitution of Eqs. 33 and 34 into Eq. 29 gives

$$\bar{W}_{\frac{1}{2}}^1 = .9974 (\Delta \bar{t}^0)^{1/5}$$

which is also in perfect agreement with Nordgren's fracture-width equation for small time of injection:

$$\bar{W} = 1.00 \times \bar{t}^{1/5}$$

ALTERNATIVE NUMERICAL COMPUTATION

An alternative method of numerical computation using the same equations and finite replacements for the derivatives of Nordgren's problem have been derived to accommodate the power-law fluid effect. Now, instead of solving a matrix problem, the fracture width at each node is calculated in an explicit manner. The finite-difference analogs of the continuity equation (Eq. 10) and that of the pressure drop equation (Eq. 9) are used separately rather than combined in a linear system. An immediate consequence of this approach is that the problem is solved for $\bar{W}_i^{m+1}$, unlike the other method which is solved for $\Delta\bar{W}_i^m$. Thus the approximation according to Eq. 13 - essential approach in that method - is unnecessary in the present numerical computation.

The basic idea of the present method is first to calculate the fracture width at the center of the last node from Eq. 12. This requires analytical solutions for both the fracture width at the center of the node $\ell + 1$, and for the flow rate at the point $\ell + \frac{1}{2}$. Analytical solutions for those variables have been discussed before. The flow rate at the point $\ell - \frac{1}{2}$ is next calculated explicitly from Eq. 19. The procedure is repeated with alternate use of Eqs. 12 and 19 until the fracture width in the first node is calculated.

Finally, the value of $\Delta\bar{t}^m$ is confronted with the fracture dimensions according to Eq. 23.

The method is detailed in the following algorithm. Although the equations are defined in the preceding section, they are repeated here for completeness.

1. Select a value for $\Delta\bar{t}^0$. Calculate $\Delta\bar{x}$ from Eq. 33:

$$\Delta\bar{x} = 1.3195(\Delta\bar{t}^0)^{4/5}$$

2. Calculate $\bar{W}_1^1$ from Eq. 32:

$$\bar{W}_1^1 = \frac{\Delta t^o}{\Delta\bar{x}}$$

3. Set i = 2 and m = 1.
4. Assume $\Delta\bar{t}^m$.
5. Calculate $\bar{W}_i^{m+1}$ from Eq. 28:

$$\bar{W}_i^{m+1} = \left[\frac{\Delta\bar{x}^2}{2\Delta\bar{t}^m}\right]^{1/3}$$

6. Calculate $\bar{q}_{i-\frac{1}{2}}^{m+1}$ from Eq. 24:

$$\bar{q}_{i-\frac{1}{2}}^{m+1} = \frac{2\Delta\bar{x}}{\Delta\bar{t}^m}\,\bar{W}_i^{m+1}$$

7. Calculate $\bar{w}_{i-1}^{m+1}$ from Eq. 12:

$$\bar{w}_{i-1}^{m+1} = \left[\left(\bar{w}_{i-\frac{1}{2}}^{m+1} \right)^4 + \bar{q}_{i-\frac{1}{2}}^{m+1} \cdot \Delta\bar{x} \right]^{\frac{1}{4}}$$

8. Calculate $\bar{q}_{i-1\frac{1}{2}}^{m+1}$ from Eq. 11:

$$\bar{q}_{i-1\frac{1}{2}}^{m+1} = \bar{q}_{i-\frac{1}{2}}^{m+1} + \bar{q}_{i-1\frac{1}{2}}^{m} - \bar{q}_{i-\frac{1}{2}}^{m} + \frac{2\Delta\bar{x}}{\Delta\bar{t}} \left[2f\left(\bar{\tau}_{i-1}\right) + \Delta\bar{w}_{i-1}^{m} \right]$$

9. Set i = i - 1. Repeate steps 5 to 8 until $\bar{w}_1^{m+1}$ is calculated.

10. Calculate $\Delta\bar{t}^m$ from Eq. 23:

$$\Delta\bar{t}^m = \Delta\bar{x} \left[\sum_{i=1}^{\ell+1} \Delta\bar{w}_i^m + 2 \sum_{i=1}^{\ell+1} f(\bar{\tau}_i) \right]$$

11. Repeat steps 3 to 10 until $\bar{t}^m$ calculated is approximated to the value assumed in step 4.

12. Set i = i + 1 and m = m + 1. Repeat the procedure from step 4.

Obviously, $\Delta\bar{t}^m$ represents the time necessary for the fracture to be increased of $\Delta\bar{x}$ at the time step m. The increment of fracture length has a fixed value. Thus the time of injection corresponding to $\bar{L}^{m+1}$ is simply the cumulative time increment. Finally, the fracture width at the wellbore is calculated from Eq. 29:

$$\bar{w}_{\frac{1}{2}}^{m+1} = \left[\left(\bar{w}_1^{m+1}\right)^4 + \frac{\Delta\bar{x}}{2}\right]^{\frac{1}{4}} .$$

POWER-LAW FLUID EFFECTS

FORMULATION OF THE PROBLEM

The present problem is the same proposed by Nordgren, with the difference that it may accommodate non-Newtonian flow in the fracture. The Perkins and Kern model of vertical fracture propagation is utilized, as well as the classical solution for the leak-off velocity due to Carter. Thus the problem is formulated on the basis of the following assumptions:

1. The fracture propagates with a fixed height and in a straight line away from the wellbore.
2. The formation is homogeneous and isotropic as regards those of its properties that influence the fracture-propagation process.
3. The formation deforms in a linear elastic manner in the vertical direction.
4. In the direction of fracture propagation there is no elastic deformations of the rock.
5. The fracturing fluid pressure is constant in any cross section perpendicular to the direction of fracture propagation.
6. These cross sections are of elliptical shape, and the width varies in the vertical direction according to England

and Green's formula.

7. The surface fluid flow rate is constant.

8. The flow rate in the fracture varies with rate of leak-off as well as rate of fracture volume increase during propagation.

9. The fracturing fluid is incompressible and behaves like a power-law model.

GOVERNING EQUATIONS

Solution of the present model has been obtained in the basis of the same method used by Nordgren. The local continuity equation is coupled with an expression borrowed from fluid mechanics which relates fluid flow rate and pressure drop gradient for power-law fluids flowing in an elliptical conduit. These equations are next casted in dimensionless form, and the derivation of the dimensionless group presented in details. Finally, it is shown how the algorithm of the alternative method for numerical computation was used to solve the present equations.

In deriving a simplified solution for vertical fracture dimensions, Perkins and Kern presented Eq. 3 relating pressure drop gradient of power-law fluids flowing in laminar regime in an elliptical conduit of nearly zero eccentricity.

$$\frac{dP}{dx} = \frac{-16}{3\pi}\ 2^{n+1}K'\left(\frac{2n'+1}{n'}\right)^{n'}\ \frac{1}{H^{n'}}\ \frac{Q^{n'}}{W^{2n'+1}} \tag{35}$$

Upon taking the derivatives of Eq. 2

$$\frac{\partial W(x,t)}{\partial x} = \frac{(1-\nu)H}{G}\ \frac{\partial \Delta p(x,t)}{\partial x}$$

and combining this expression with Eq. 35, one obtains Eq. 36, explicit in the local fluid flow rate.

$$q(x,t) = \gamma H^{n'-1}\left[\frac{\partial\left(W(x,t)\right)^{2n'+2}}{\partial x}\right]^{1/n'} \tag{36}$$

where

$$\gamma = \frac{3\pi G(n'/2n'+1)^{n'}}{16\cdot 2^{n'+1}(1-\nu)(2n'+2)K'}$$

Thus the governing equations of the present model are set up. As in Nordgren's problem, Eq. 8 is used to interrelate flow rate, rate of fracture growth, and rate of leak-off. Equation 36 -- which is replacing Eq. 4 used in the Newtonian case -- interrelates elastic properties of the formation, fracture width, and flow rate.

DIMENSIONLESS VARIABLES

Similarly to the solution for the Newtonian case, the following dimensionless group is introduced:

$$\bar{\tau} = \tau(x)/N$$

$$\bar{x} = x/M$$

$$\bar{L} = L(x)/M$$

$$\bar{t} = t/N$$

$$\bar{q} = q(x,t)/Q$$

$$\bar{W} = W(x,t)/R$$

From Eq. 36 and 11 one obtains the following expressions

$$\bar{q} = - \left[\gamma \frac{H^{n'-1}}{Q^{n'}} \frac{R^{2n'+2}}{M} \right]^{1/n'} \cdot \left[\frac{\partial \bar{W}^{2n'+2}}{\partial \bar{x}} \right]^{1/n'}$$

$$\frac{Q}{M} \frac{\partial \bar{q}}{\partial \bar{x}} + \frac{\pi HR}{4N} \frac{\partial \bar{W}}{\partial \bar{t}} + \frac{2HC}{\sqrt{N}} \frac{1}{\sqrt{\bar{t}-\bar{\tau}}} = 0$$

which provide

$$\frac{\gamma H^{n'-1} R^{2n'+2}}{Q^{n'} M} = 1 \tag{37}$$

and

$$\frac{Q}{M} = \frac{\pi HR}{4N} = \frac{2HC}{\sqrt{N}} \tag{38}$$

Equation 37 gives

$$M = \frac{\gamma H^{n'-'} \; R^{2n'+2}}{Q^{n'}} \tag{39}$$

and Eq. 38,

$$M = \frac{4NQ}{\pi HR}$$

$$N = \frac{\pi^2 \; R^2}{64 \; C^2}$$

Therefore,

$$M = \frac{\pi QR}{16 \; HC^2} \tag{40}$$

Combination of Eqs. 39 and 40 leads to

$$R = \left[Z \; \frac{Q^{n'+1}}{H^{n'} \; C^2} \right]^{1/(2n'+1)} \tag{41.1}$$

whereas combination of the other equations gives

$$N = \frac{\pi^2}{64} \left[Z \; \frac{Q^{n'+1}}{H^{n'} \; C^{2n'+3}} \right]^{1/(2n'+1)} \tag{41.2}$$

$$M = \frac{\pi}{16} \left[Z \; \frac{Q^{3n'+2}}{H^{3n'+1} \; C^{4n'+4}} \right]^{2/(2n'+1)} \tag{41.3}$$

where

$$Z = \frac{2^{n'+1}(1-\nu)(2n'+2)K'}{3\,G(n'/2n'+1)^{n'}}$$

Now Eqs. 36 and 8 in dimensionless form become, respectively:

$$\bar{q} = -\left[\frac{\partial \bar{W}^{2n'+2}}{\partial x}\right]^{1/n'} \tag{42}$$

$$\frac{\partial \bar{q}}{\partial \bar{x}} + \frac{1}{\sqrt{\bar{t}-\bar{\tau}}} + \frac{\partial \bar{W}}{\partial \bar{t}} = 0 \tag{43}$$

A finite-difference analog of Eq. 43 were derived in the discussion of the Newtonian fluid case (Eq. 11). The derivative included in Eq. 42 can be replaced by either a centered- or by a foward-difference quotient, yielding the following approximations:

$$\bar{q}_{i+\frac{1}{2}}^{m+1} = \left[\frac{\left(\bar{W}_{i}^{m+1}\right)^{2n'+2} - \left(\bar{W}_{i+1}^{m+1}\right)^{2n'+2}}{\Delta\bar{x}}\right]^{1/n'} \tag{44}$$

$$\bar{q}_{i+\frac{1}{2}}^{m+1} = \left[\frac{\left(\bar{W}_{i+\frac{1}{2}}^{m+1}\right)^{2n'+2} - \left(\bar{W}_{i+1}^{m+1}\right)^{2n'+2}}{\Delta\bar{x}/2}\right]^{1/n'} \tag{45}$$

Concentrating in the grid configuration of Fig. 2, the fracture is again supposed to be close at the point $\ell + \frac{1}{2}$ at the step m. It propagates a distance $\Delta\bar{x}$ and closes at $\ell + 1\frac{1}{2}$

at the time step m + 1. If leak-off is neglected, the same consideration for the case of Newtonian fluid as regards the continuity equation may be applied in the present problem. Thus Eq. 24 is used here with no modifications.

An expression for the flow rate at the center of the node $\ell + 1$ is provided by Eq. 45, written for $i = \ell + \frac{1}{2}$

$$\bar{q}^{m+1}_{\ell+1} = \left[\frac{\left(\bar{w}^{m+1}_{\ell+1}\right)^{2n'+2} - \left(\bar{w}^{m+1}_{\ell+1\frac{1}{2}}\right)^{2n'+2}}{\Delta\bar{x}/2}\right]^{1/n'}$$

Boundary conditions impose that

$$\bar{w}^{m+1}_{\ell+1\frac{1}{2}} = 0$$

Therefore

$$\bar{q}^{m+1}_{\ell+1} = \left[\frac{2\left(\bar{w}^{m+1}_{\ell+1}\right)^{2n'+2}}{\Delta\bar{x}}\right]^{1/n'} \tag{46}$$

Combination of Eq. 24 with Eq. 46 with the use of the approximation

$$\bar{q}^{m+1}_{\ell+1} \simeq \frac{1}{2}\,\bar{q}^{m+1}_{\ell+\frac{1}{2}}$$

gives

$$\bar{w}^{m+1}_{\ell+1} = \left[\frac{\Delta\bar{x}^{n'+1}}{2(\Delta t^m)^{n'}}\right]^{1/(n'+2)} \tag{47}$$

ANALYTICAL SOLUTIONS

Approximate analytical solutions for small time of injection has also been derived on the basis of the same considerations for the Newtonian case. Now the time of injection is small, leak-off is neglected, and the problem is to derive equations for both fracture width and length considering only one node (Fig. 2). If this is the case, Eq. 23 gives

$$\Delta\bar{t}^0 = \bar{w}^1_1 \; \Delta\bar{x} \tag{48}$$

If i is set equal to ½, one obtains (from Eq. 45)

$$\bar{q}^1_1 = \left[\frac{\left(\bar{w}^1_1\right)^{2n'+2} - \left(\bar{w}^1_{1\frac{1}{2}}\right)^{2n'+2}}{\Delta x/2}\right]^{1/n'}$$

Boundary conditions lead to

$$\bar{w}^1_{1\frac{1}{2}} = 0$$

therefore

$$\bar{q}_1^1 = \left[\frac{\left(\bar{w}_1^1\right)^{2n'+2}}{\Delta\bar{x}/2}\right]^{1/n'} \simeq \frac{1}{2} \tag{49}$$

Combination of Eqs. 48 and 49 gives

$$\Delta\bar{x} = \left[2(\Delta t^o)^2\right]^{\frac{n'+1}{2n'+3}} \tag{50}$$

Equation 45 gives a relationship between the fracture width at the center of the node and at the wellbore:

$$\bar{q}_{1_{1/2}}^1 = \left[\frac{\left(\bar{w}_{1_{1/2}}^1\right)^{2n'+2} - \left(\bar{w}_1^1\right)^{2n'+2}}{\Delta x/2}\right]^{1/n'}$$

For the case of constant flow rate at the fracture entrance, $\bar{q}_{1_{1/2}}^m = 1$. Therefore

$$\bar{w}_{1_{1/2}}^1 = \left[\left(\bar{w}_1^1\right)^{2n'+2} + \frac{\Delta\bar{x}}{2}\right]^{1/(2n'+2)} \tag{51}$$

Substitution of Eqs. 49 and 50 into Eq. 51 provides

$$\bar{w}_{1_{1/2}}^1 = \frac{(1+2^{n'})^{1/(2n'+2)}}{2^{\frac{n'+1}{2n'+3}}} (\Delta\bar{t}^0)^{1/(2n'+3)}$$

which may be approximated to

$$\bar{w}_{1_{1/2}}^1 = (\Delta\bar{t}^0)^{1/(2n'+3)}$$

with an error of less than 8% for $.2 \leq n' \leq 1$.

ALGORITHM

The equations of the present model have been solved with the algorithm for the Newtonian fluid case. Although the present method is indicated for the power-law fluid problem, it can also be used for Newtonian fracturing fluids if n' is set equal to 1, and the fluid viscosity taken as numerically equal to k'.

Results of the Newtonian fluid problem using the present computation were plotted in Figs. 3.1 and 3.2. Fracture dimensions have been obtained for the power-law fluid case through different values of n', and plotted in Figs. 4.1 and 4.2.

1. Select a value for $\Delta\bar{t}^o$. Calculate $\Delta\bar{x}$ from Eq. 50.
2. Calculate $\bar{W}_1^1$ from Eq. 48.
3. Set i = 2, and m = 1.
4. Assume $\Delta\bar{t}^m$.
5. Calculate $\bar{W}_i^{m+1}$ from Eq. 47.
6. Calculate $\bar{q}_{i-\frac{1}{2}}^{m+1}$ from Eq. 24.
7. Calculate $\bar{W}_{i-1}^{m+1}$ from Eq. 44.
8. Calculate $\bar{q}_{i-1\frac{1}{2}}^{m+1}$ from Eq. 11.
9. Set i = i-1. Repeat steps 5 to 8 until W_1^{m+1} is calculated.

10. Calculate $\Delta\bar{t}^m$ from Eq. 23.

11. Repeat steps 3 to 10 until $\Delta\bar{t}^m$ calculated is approximated to the value assumed in step 4.

12. Set i = i + 1, and m = m+1. Repeat the procedure from step 1.

APPLICATION OF THE PRESENT METHOD

Fracture length and maximum fracture width at the wellbore are evaluated by first calculating the parameters R, N, and M according to Eqs. 41.1, 41.2, and 41.3, respectively. The function Z should be written as

$$Z = \frac{.004629\ K'(1-\nu)(2n'+2)}{(30n'/2n'+1)^{n'}\ G}$$

if the following practical units are used:

Q..............................cu ft/min

C..............................$ft/min^{1/2}$

H..............................ft

G..............................psi

k'.............................$lb.sec^{n'}/sq\ ft$

R..............................ft

N..............................min

M..............................ft

The dimensionless time is calculated from

$$\bar{t} = \frac{t}{N}$$

The corresponding values of dimensionless maximum fracture width and dimensionless length are determined from Figs. 4.1 and 4.2. The maximum fracture width is then calculated from

$$W = \bar{W} \times R$$

and the fracture length from

$$L = \bar{L} \times M$$

Figures 4.1 and 4.2 are restricted to values of dimensionless time between .01 and 1.5. This range is suitable for most practical application. However, an estimation of the dimensionless fracture length and width for times shorter than .01 can be obtained from Eqs. 50 and 48. For dimensionless time in excess of 1.5, the dimensionless geometry can be estimated by extrapolating the straight line portions of Figs. 4.1 and 4.2.

EXAMPLE PROBLEM

Two methods for calculating fracture dimensions allowing for power-law fluid behavior are available. One is a simplified formulation suggested by Perkins and Kern[2] who derived analytically a fracture width equation assuming no leak-off, and constant flow rate at any point in the fracture. The other method, based on the horizontal plane strain model of vertical propagation, is a numerical scheme proposed by Daneshy. In this method, both leak-off and variation of local flow rate are taken into account. However, the method is not available for general use.

An approximate estimation of the dimensions of fracture induced by power-law fluid with use of equations for Newtonian behavior was suggested by Geertsma and Haafkens[7]. The viscosity is replaced by an effective value based in the flow rate and fracture width at the wellbore, and derived from a comparison of the equations for pressure drop for Newtonian and power-law fluids. Geertsma, however, did not indicate how his classical numerical procedure should be altered to include this viscosity approximation.

For these reasons, comparison of existing theories with the method proposed here was only possible for those cases for which solutions are available. Fracture dimensions were calculated from the set of parameters given in Table 2, and the results plotted in Figs. 6.1 and 6.2, along with solutions published in reference 7.

Also, in order to evaluate the effect of power-law fluid behavior on fracture dimensions, a hypothetical fracture treatment was calculated assuming the set of parameters shown in Table 1, for different values of consistency index, flow behavior index and treatment volume. The results are presented graphically in Figs. 6.1 and 6.2.

SUMMARY AND CONCLUSIONS

A method has been developed for estimating the dimensions of hydraulic fractures induced by power-law fluids. The method is an extension of Nordgren's version of the Perkins and Kern model of vertical fracture propagation. An investigation of Nordgren's numerical solution is presented in detail. An alternative method of numerical computation is developed to accommodate the power-law fluid in the equations. The present solution is first compared with Nordgren's results for the Newtonian fluid case. Good agreement was found between the two results (Figs. 3.1 and 3.2). On the basis of the same method, effects of the non-Newtonian fluid on both the fracture width and length are obtained (Figs. 4.1 and 4.2).

Effects of the flow behavior index and consistency index on fracture dimensions have been investigated on the basis of the present method, assuming the set of basic parameters specified in Table 1. The fracture length increases with the flow behavior index and decreases with the consistency index. A converse tendency was observed as regards the variations of fracture width (Figs. 5 and 6).

The present method was also used to calculate fracture dimensions for another set of parameters specified in Table 2, in order to provide a basis for comparison with other methods of predicting fracture dimensions. Relative to the results of Daneshy and Geertsma and De Klerk, the present solutions are larger for small volumes of treatment but shorter for larger volumes (Figs. 7.1 and 7.2).

TABLE 1 BASIC PARAMETERS FOR FRACTURE DIMENSION CALCULATIONS (Results Plotted in Figs. 5 and 6)

Pump rate	- 15 bbl/min
Poisson's ratio	- .15
Shear modulus	- 2.6 10^6
Fluid-loss coefficient	- .001 ft/min$^{1/2}$
Fracture height	- 150 ft
Consistency index	- .00416 lb.sec$^{n'}$/sq ft
Flow behavior index	- .5

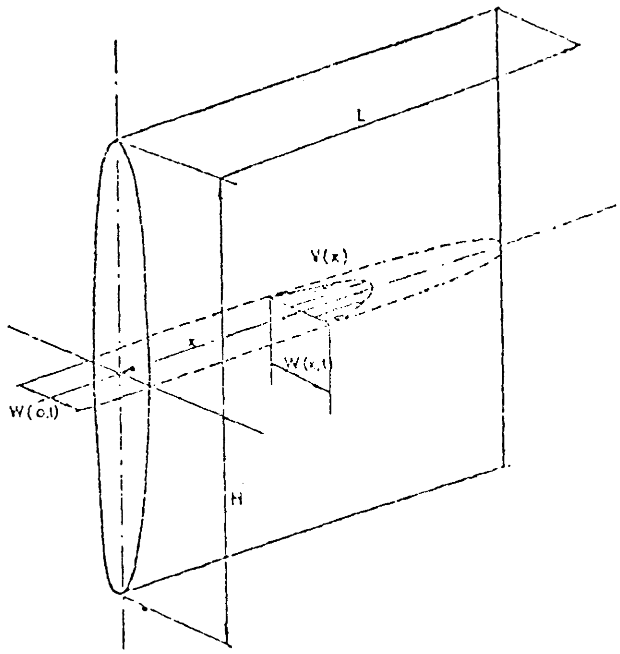
TABLE 2 BASIC PARAMETERS FOR FRACTURE DIMENSION CALCULATIONS (Results Plotted in Fig. 7)

Pump rate	- 10 bbl/min
Poisson's ratio	- .15
Shear modulus	- 2.6 10^6 psi
Fluid loss coefficient	- .0015 ft/min$^{1/2}$
Fracture height	- 100 ft
Consistency index	- .0044 lb.sec$^{n'}$/sq ft
Flow behavior index	- .63

FIGURE 1

SCHEMATIC REPRESENTATION OF A LINEARLY PROPAGATING FRACTURE [7]

PERKINS AND KERN MODEL



GEERTSMA AND DE KLERK MODEL

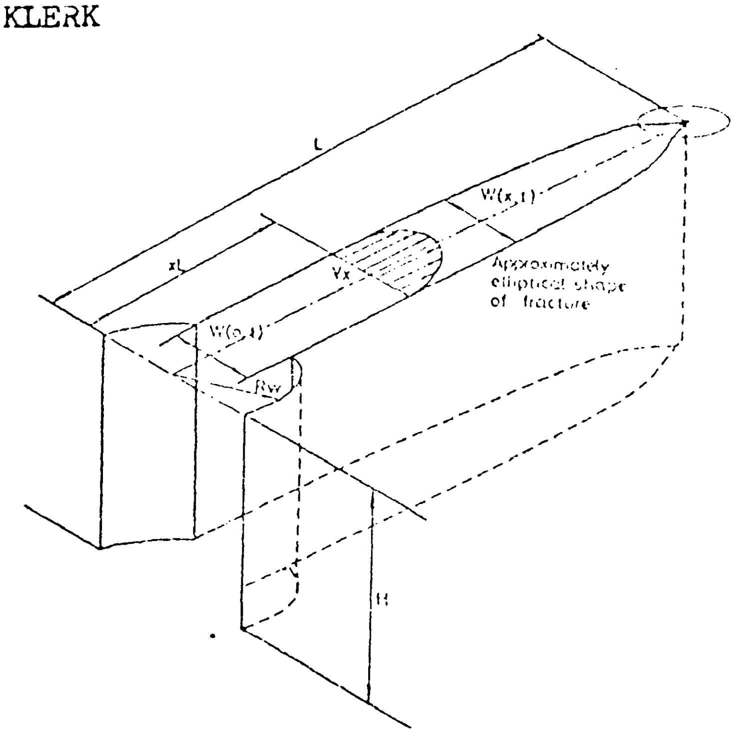
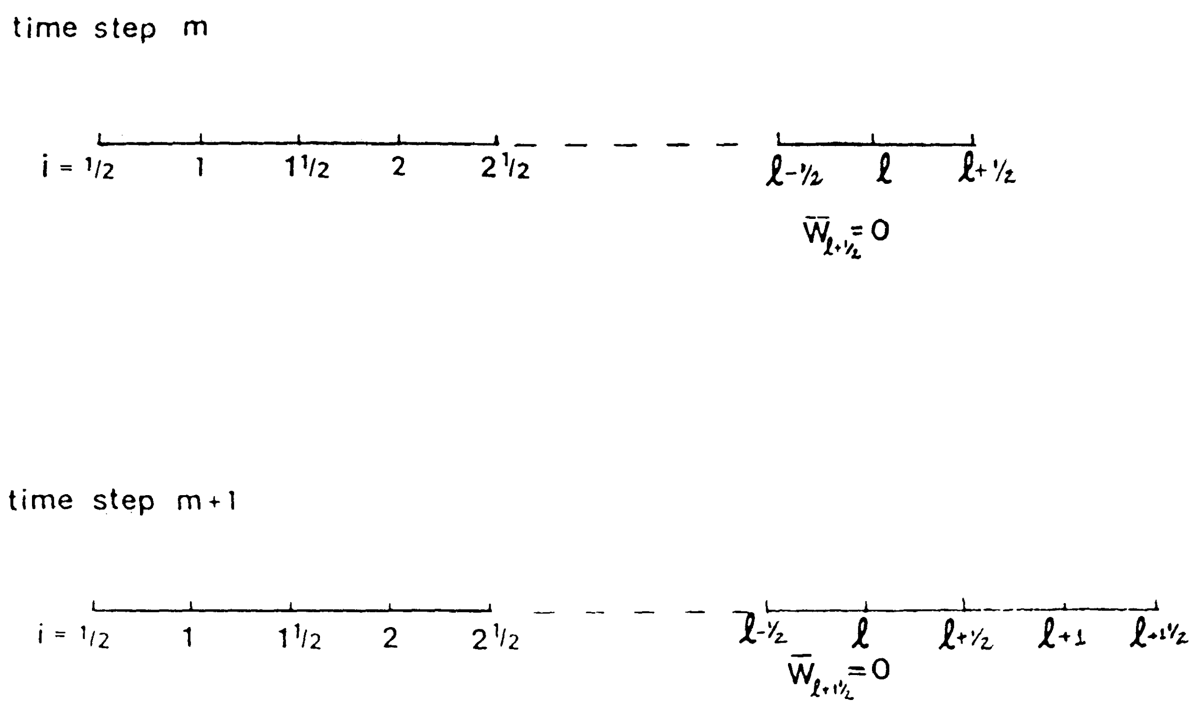


FIGURE 2

BLOCK-CENTERED GRID CONFIGURATION



$\bar{x}_{1/2} = 0$ (fracture entrance)

$\Delta \bar{x} = (x_i - x_{i-1})/2$

FIGURE 3.1

DIMENSIONLESS FRACTURE LENGTH (NEWTONIAN FRACTURING FLUID)

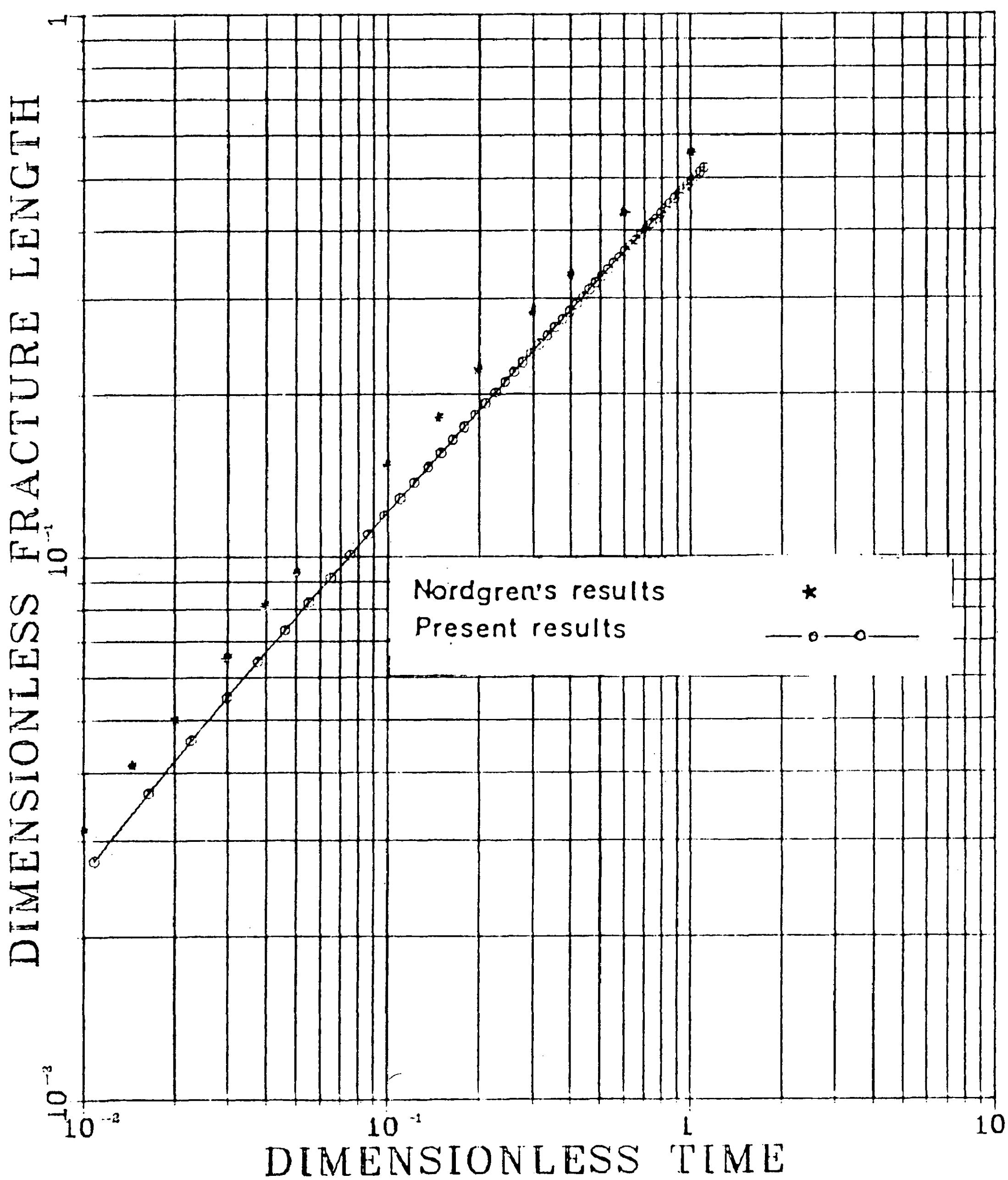


FIGURE 3.2

DIMENSIONLESS FRACTURE WIDTH (NEWTONIAN FRACTURING FLUID)

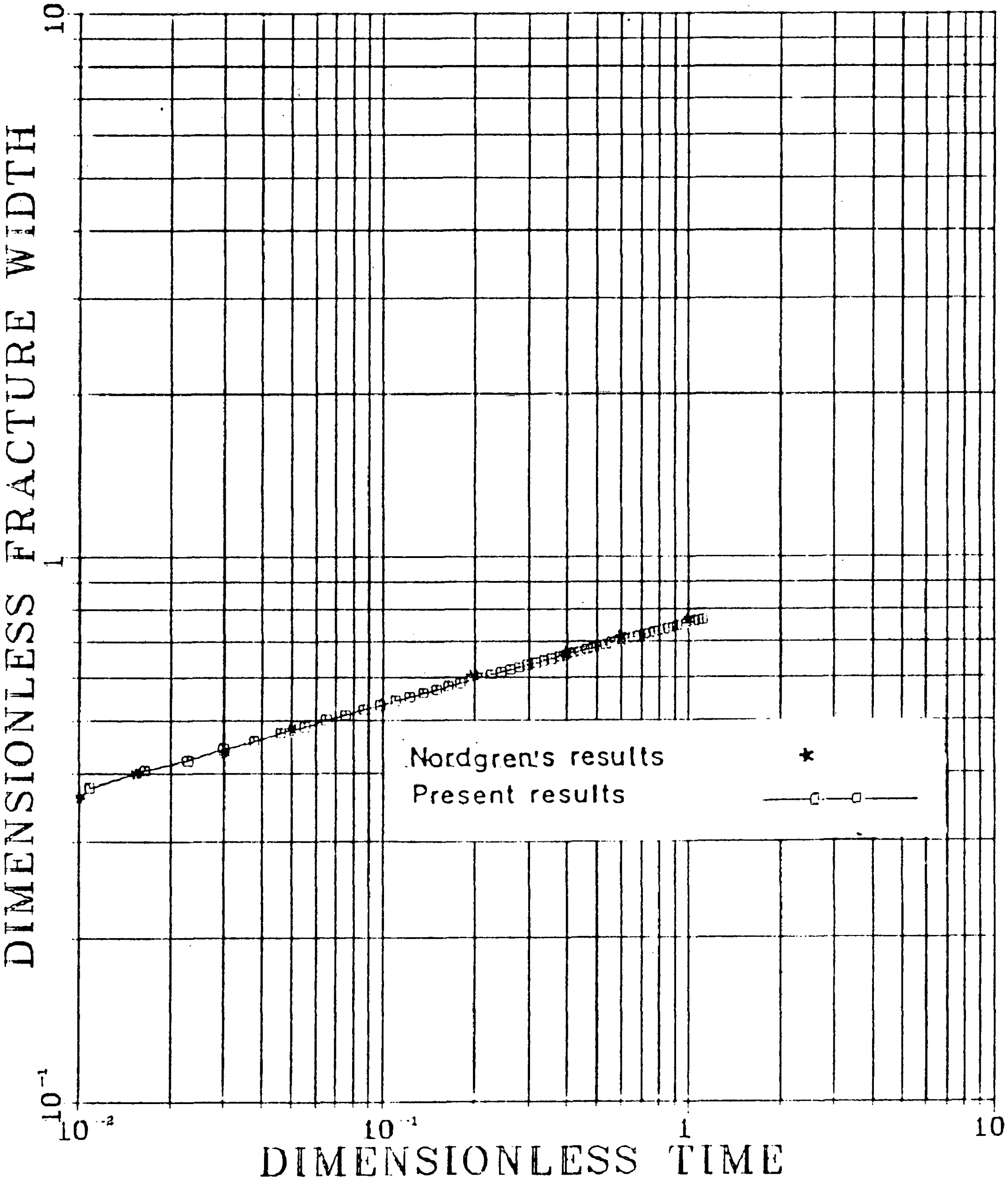


FIGURE 4.1

DIMENSIONLESS FRACTURE LENGTH
(POWER-LAW FRACTURING FLUID)

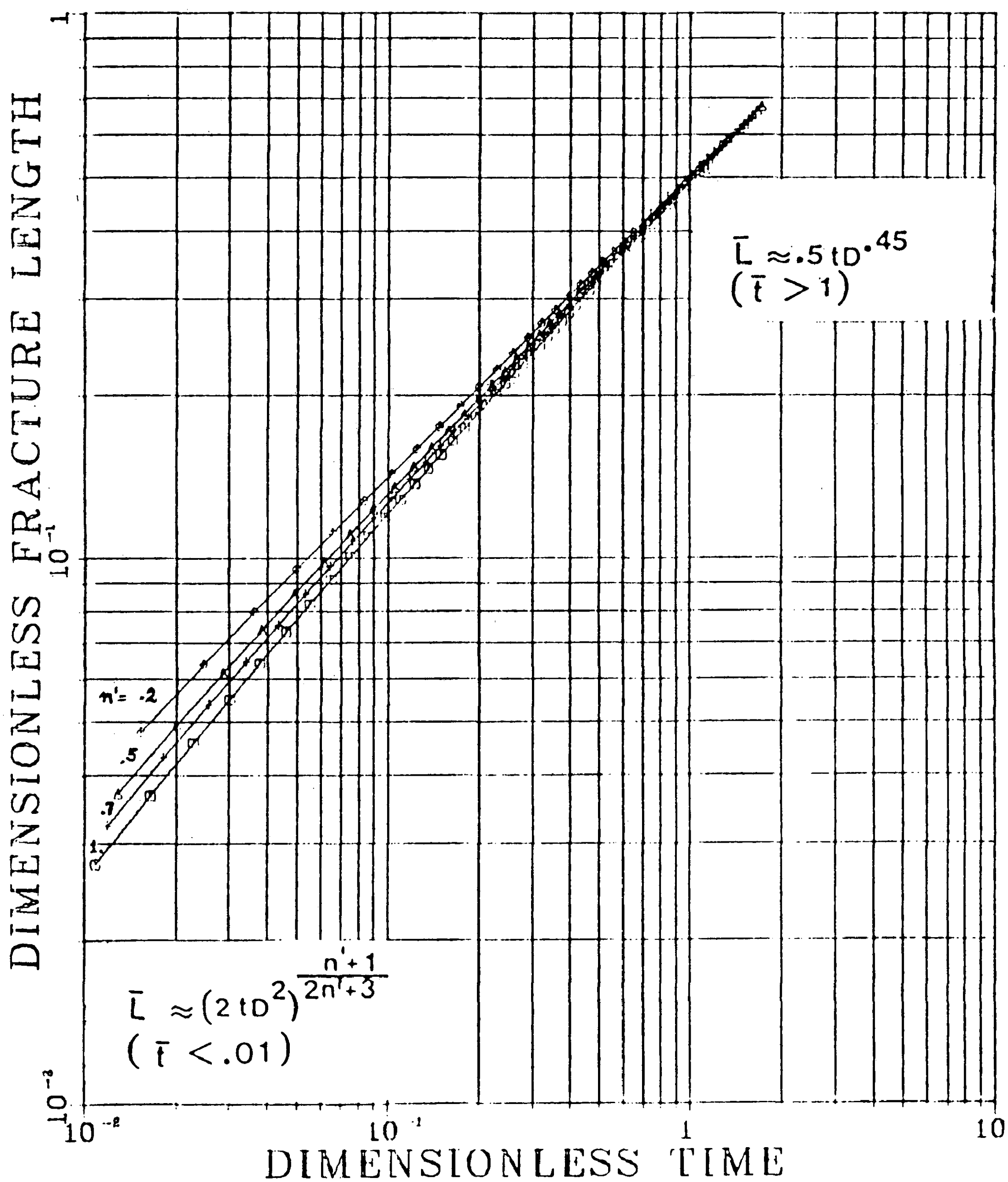


FIGURE 4.2

DIMENSIONLESS FRACTURE WIDTH (POWER-LAW FRACTURING FLUID)

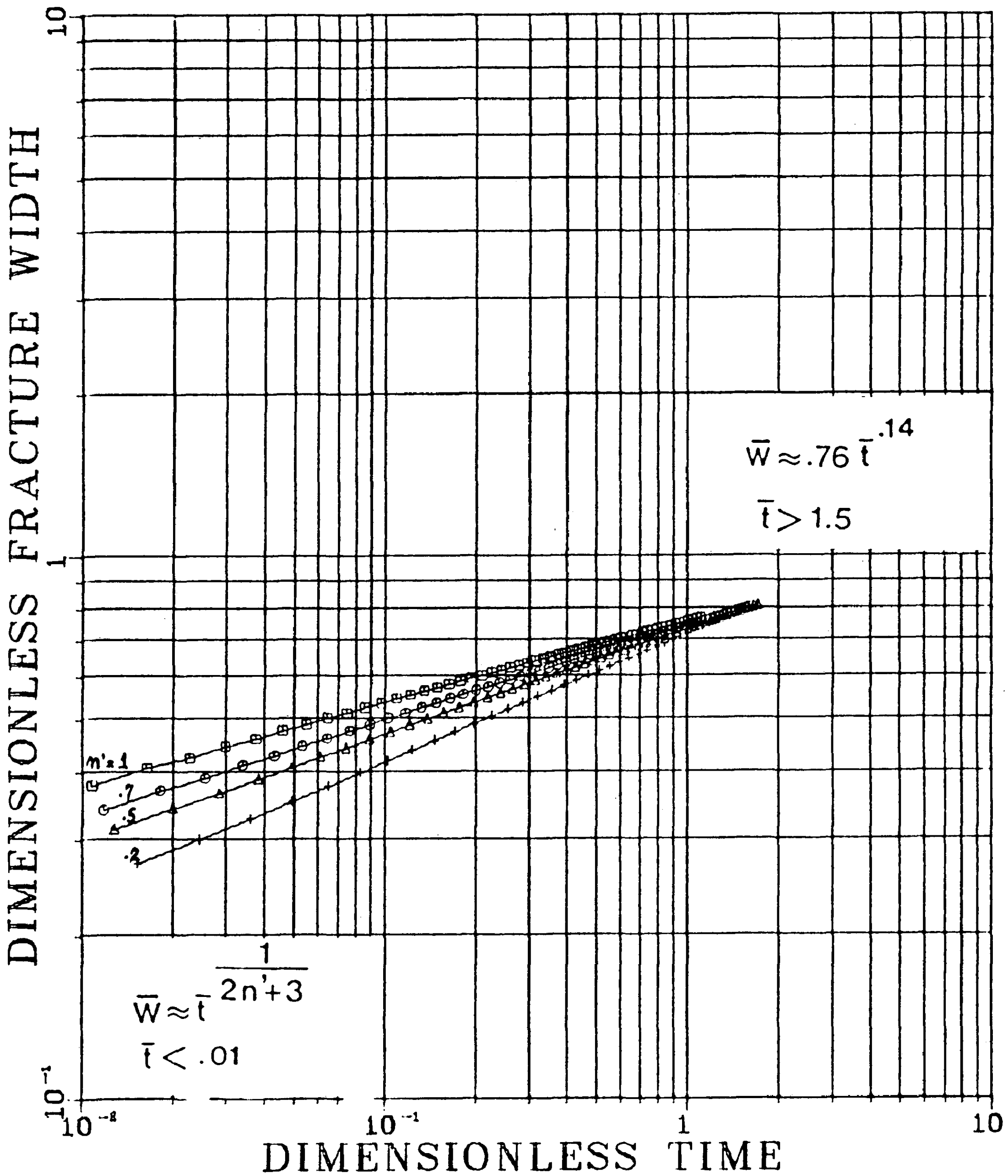


FIGURE 5.1

VARIATION OF FRACTURE LENGTH WITH FLOW BEHAVIOR INDEX

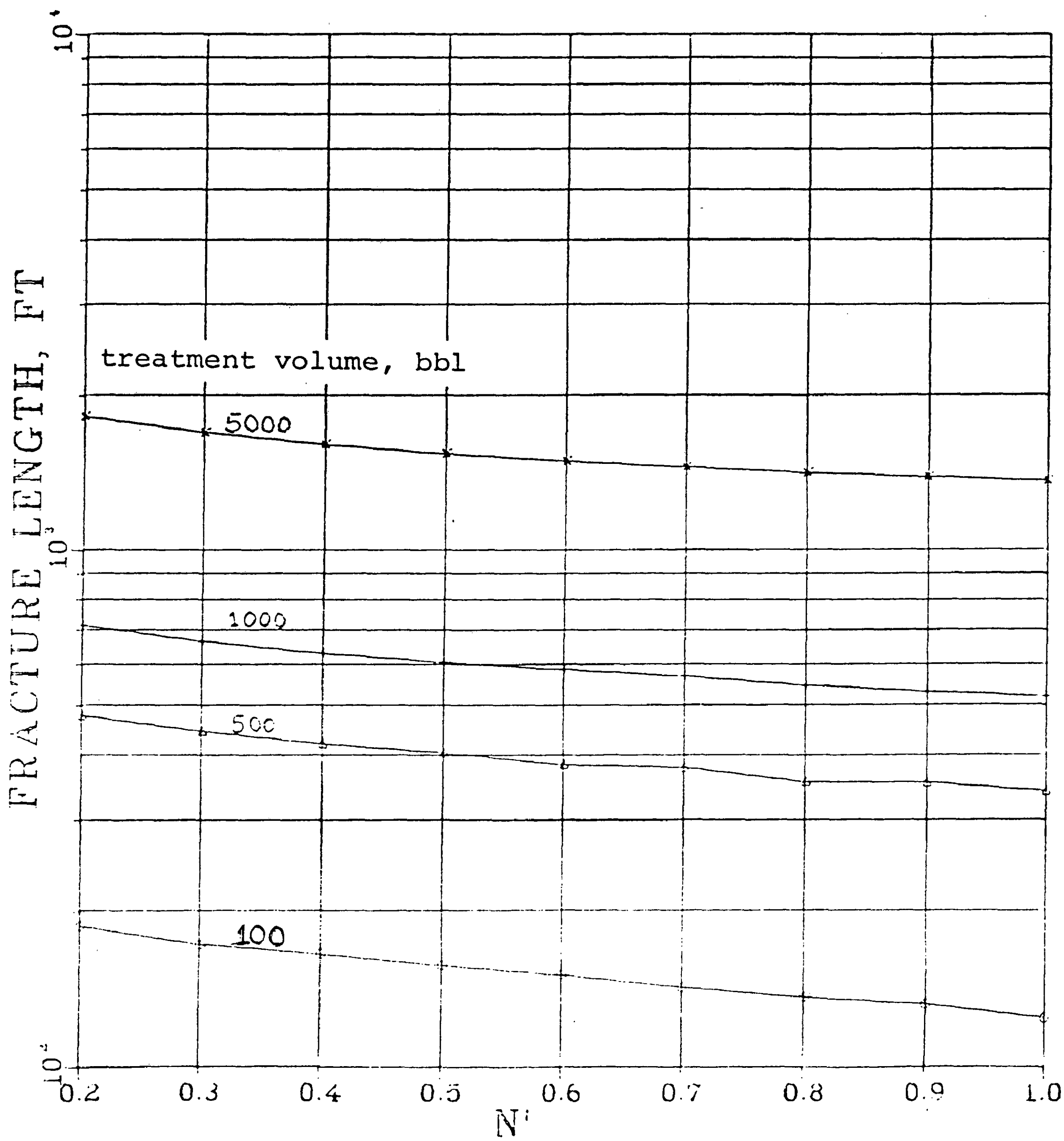


FIGURE 5.2

VARIATION OF FRACTURE WIDTH WITH FLOW BEHAVIOR INDEX

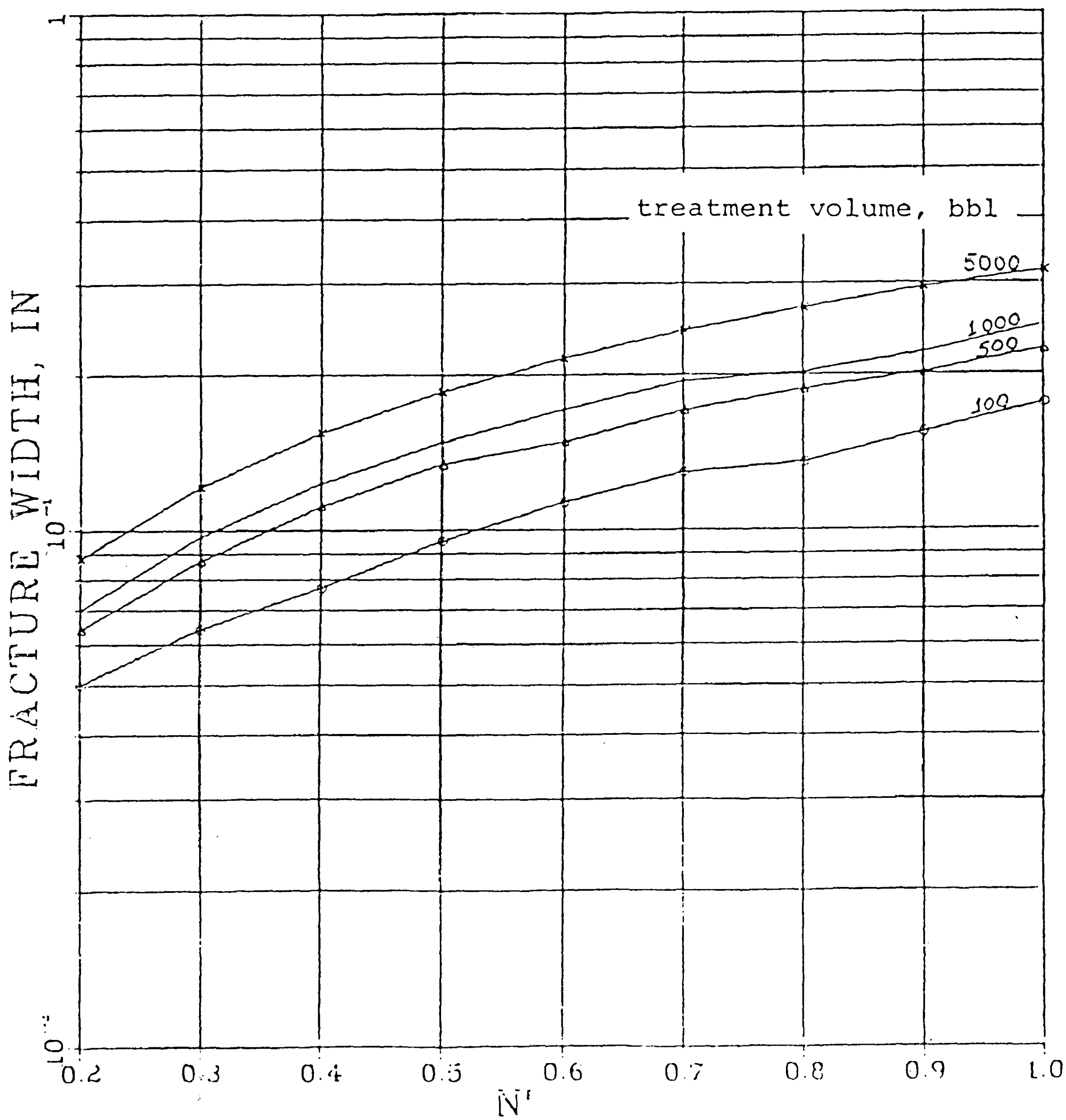


FIGURE 6.1

VARIATION OF FRACTURE LENGTH WITH CONSISTENCY INDEX

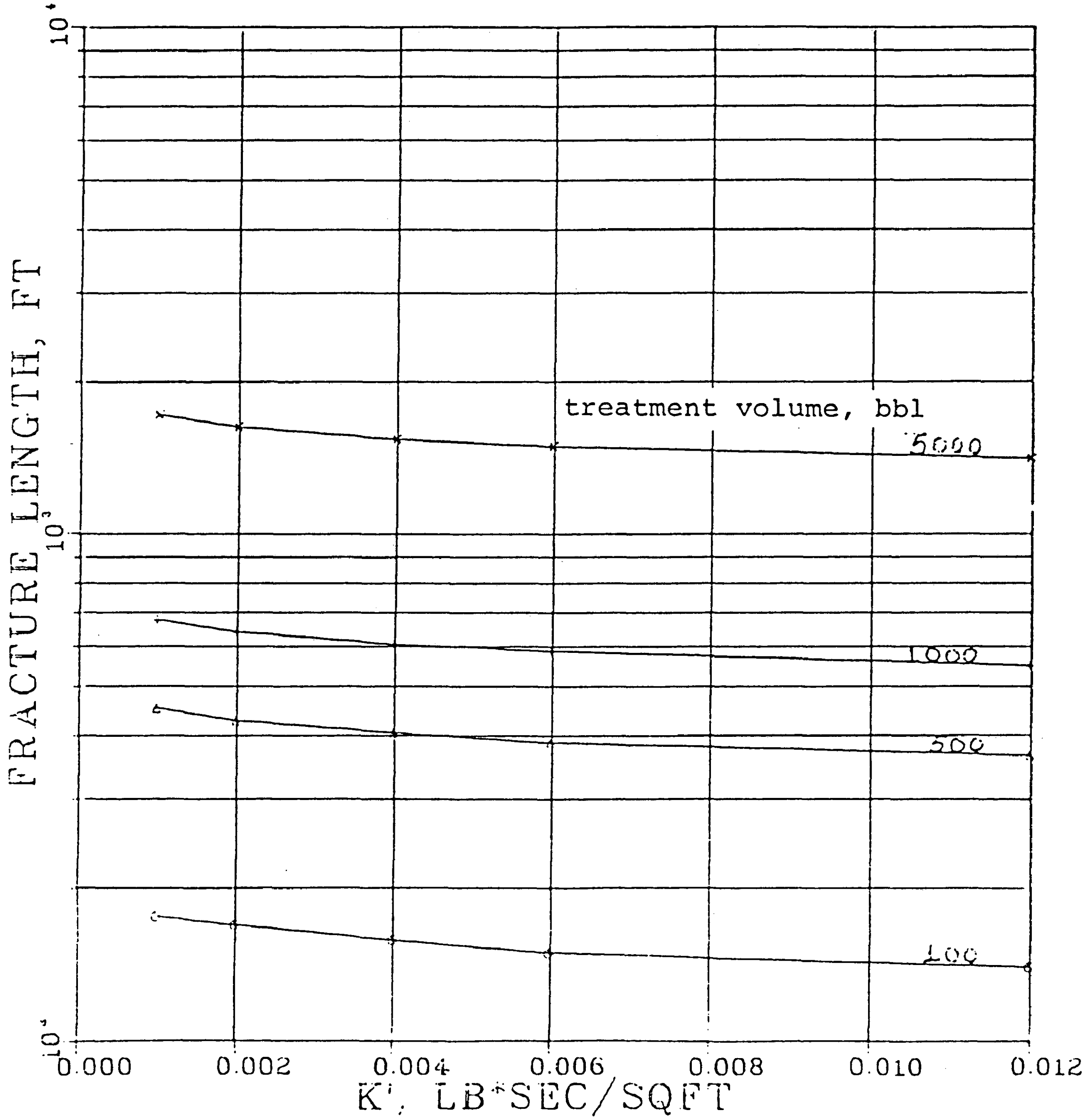


FIGURE 6.2

VARIATION OF FRACTURE WIDTH
WITH CONSISTENCY INDEX

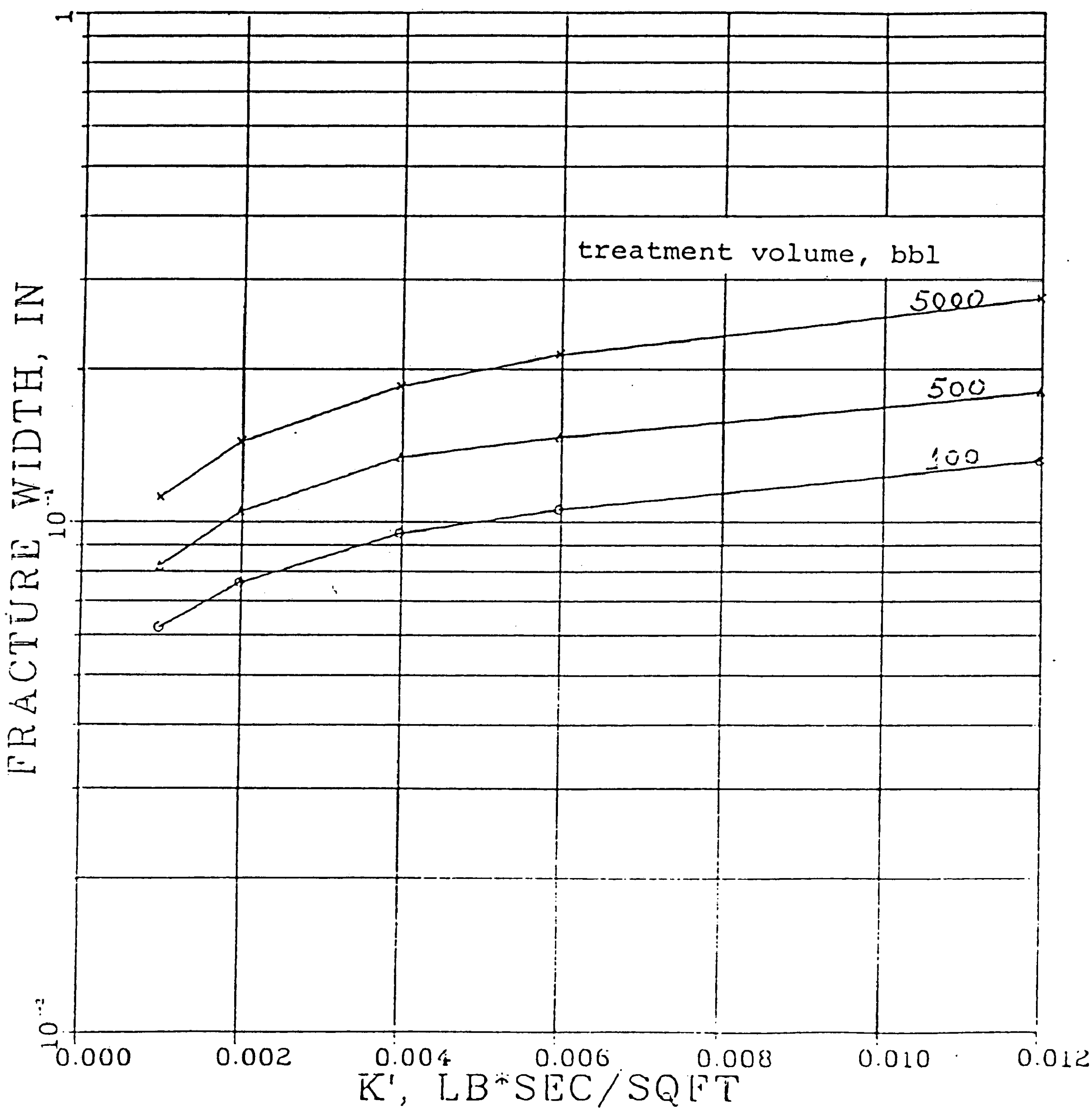


FIGURE 7.1

FRACTURE LENGTH VS TREATMENT VOLUME

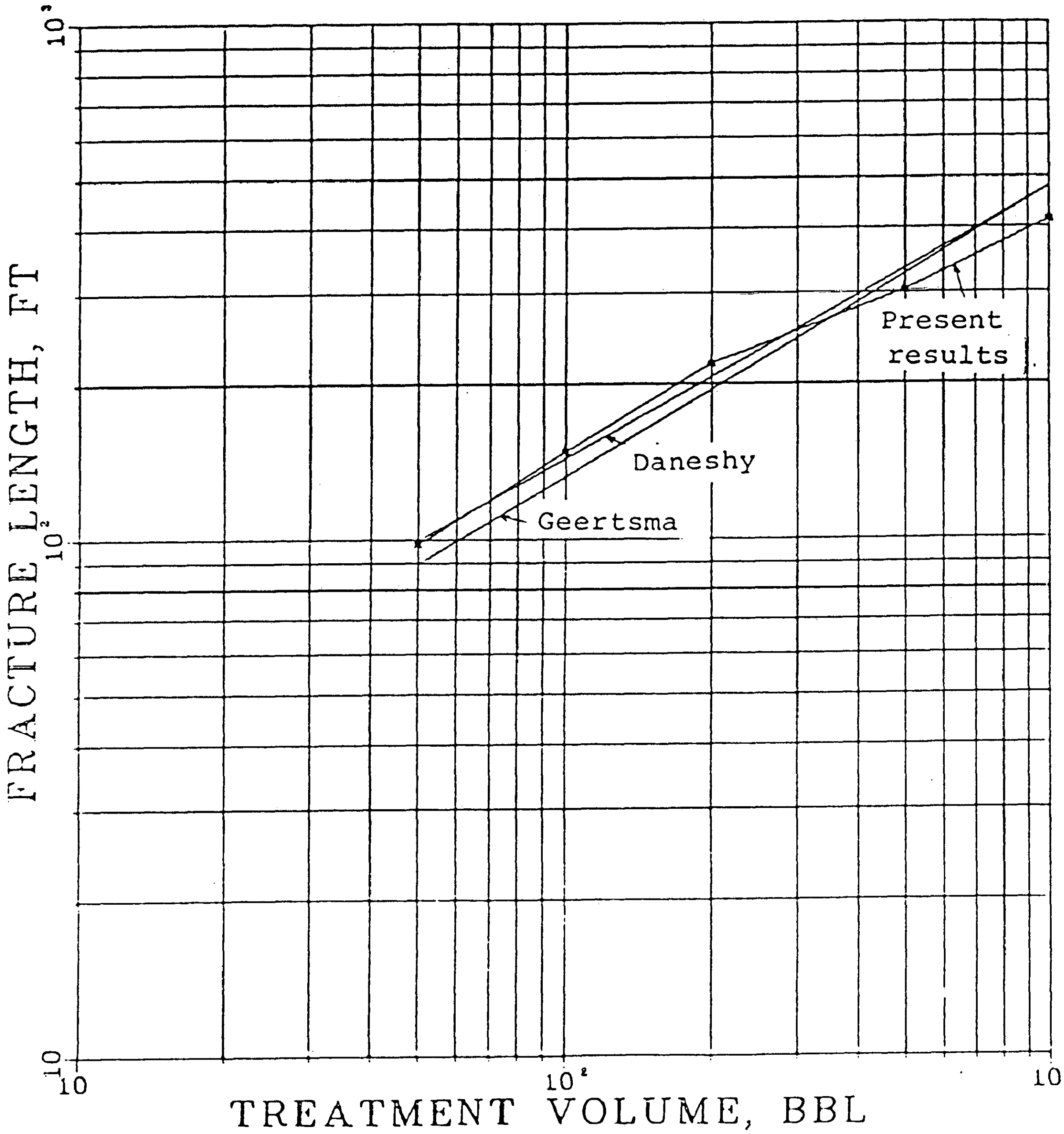
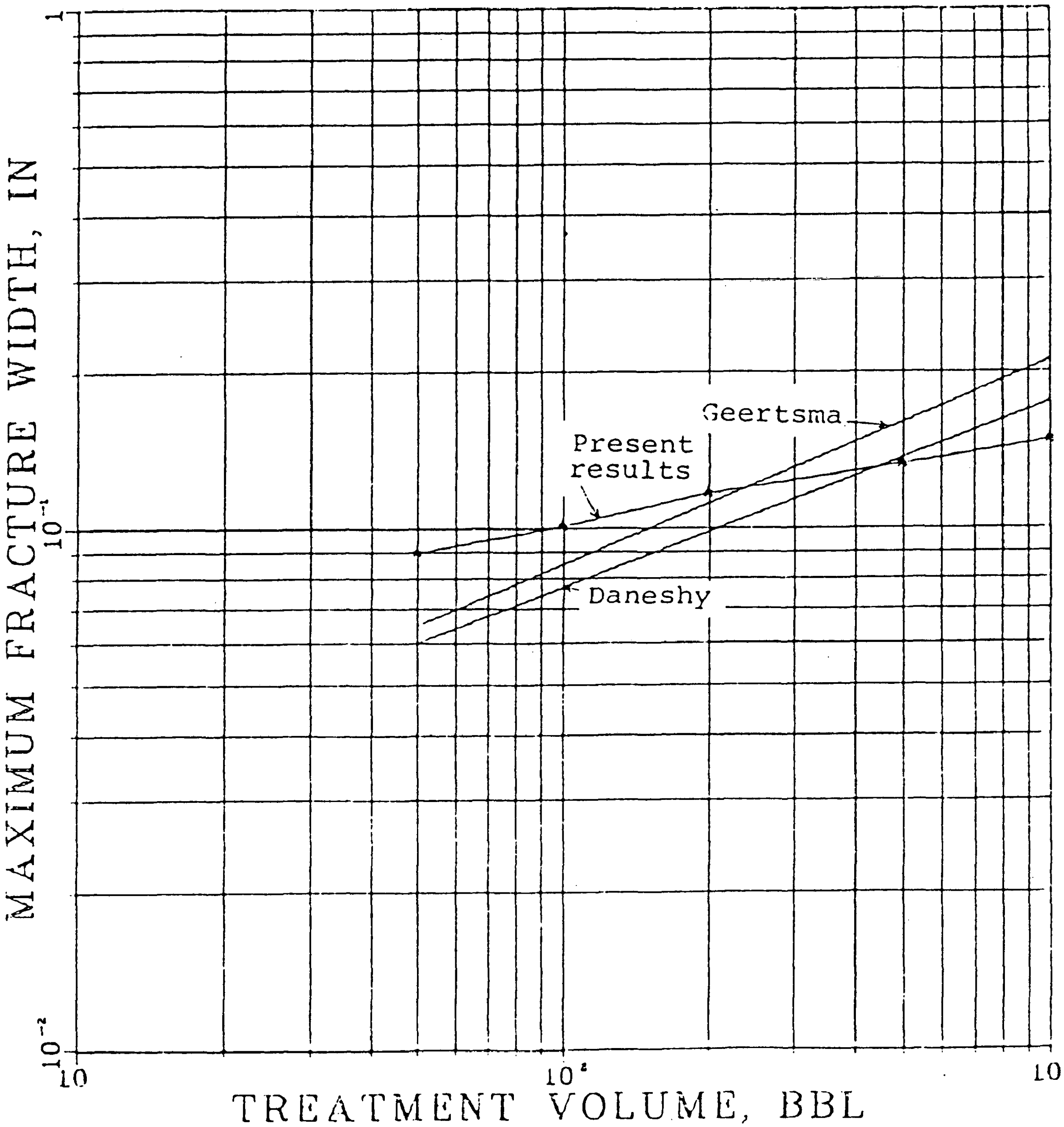


FIGURE 7.2

MAXIMUM FRACTURE WIDTH VS TREATMENT VOLUME



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APPENDIX - NOMENCLATURE

Greek letters

Δ	differential
μ	absolute viscosity
ν	Poisson's ratio
π	constant
τ	time necessary for the fracture to extend from the wellbore to a certain point

Roman letters

A	fracture vertical cross sections
C	over-all fluid leak-off coefficient
G	shear modulus
H	fracture height
k'	consistency index
l	number of nodes at time step m (grid configuration - Fig. 2)
L	fracture length
n'	flow behavior index
P	fluid pressure
q	flow rate in the fracture
q_l	rate of leak-off per unit fracture length
Q	surface flow rate

Roman letters (Cont'd)

t	injection time
v	leak-off velocity
w	fracture width at a given point in a vertical cross section
W	maximum fracture width in a vertical cross section
x	coordinate in the horizontal direction

Subscript

i	i^{th} node in the fracture grid configuration (Fig. 2)

Superscript

m	time step of computation
-	dimensionless variable