An Inventory Application of Renewal Theory

By

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ABSTRACT

Much interest has been generated in the development of optimum control doctrine for inventory systems, due to the tremendous amount of money tied up in inventories of physical goods in both the private and public sectors of the economy. Many procedures have been devised which result in optimum control doctrine for very specific types of inventory systems. This is perhaps due to the difficulties involved in obtaining optimum solutions to cost equations of inventory models of a more general nature. Hence the importance, in the more general case, of easily solved approximate solution procedures for the development of optimum control doctrine is obvious.

This thesis is concerned with the development of such an approximation procedure in the form of an iterative solution method. This method is valid over a wide range of types of inventory systems which have a periodic inventory level check occuring at regular intervals in time. Where successful, the iterative procedure converges to a solution rapidly and can be solved by hand using standard Poisson distribution tables.

INTRODUCTION

Inventories of physical goods have become a major portion of the total investment of both the private and public sectors of the economy. Inventories are a 'cost-only' part of any supply organization not explicitly returning any profit as such. Thus the study of inventory control and management procedures, possibly resulting in a decrease in the cost of an inventory system, is easily justified. The reasons for holding inventories of physical goods can be classified by an analogy to the reasons for holding cash (or an inventory of money) as pointed out by Arrow (1963, p. 4). In general, three motives can be thought of: transaction, precaution, and speculation. The transaction motive is the cost of having to order and deliver items as they are demanded; the precaution motive is the cost of not being able to supply a good when it is demanded; and the speculation motive is the possibility of gain due to future changes in price or interest rate.

The transaction motive can best be shown by illustration. Suppose in the simplest deterministic case we define the annual demand rate by $\lambda \geq 0$, the quantity ordered each period by $Q > 0$, and the cost of placing an order by $A \geq 0$. Then if C is the cost per item (independent of quantity ordered) and I is the alternative (1) interest rate, and if we assume the safety stock (amount on hand when an order arrives) to be

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(1) The alternative interest rate is defined as the rate at which the money invested in inventory stock could earn revenue if it were invested differently (i.e. stocks, bonds, mutual funds, loans, etc.)

zero, the average annual cost expression can be written:

$$\kappa = \frac{\lambda \cdot A}{Q} + \frac{I \cdot C \cdot Q}{2} .$$

By differentiation we see

$$\frac{d\kappa}{dQ} = \frac{-\lambda \cdot A}{Q^2} + \frac{I \cdot C}{2} ,$$

or Q^*, the optimum amount to order, is given by

$$Q^* = \sqrt{\frac{2\lambda \cdot A}{I \cdot C}} .$$

since the second derivation of κ with respect to Q is positive everywhere. All other quantities being positive, $A > 0$ implies $Q^* > 0$ and $A \rightarrow 0$ implies $Q^* \rightarrow 0$. Thus, if the procurement cost is zero the optimum order quantity is zero, hence the importance of A (the procurement cost) is obvious.

The precautionary motive is the cost associated with being out-of-stock when a demand occurs. This cost may be in the form of a loss of goodwill and lost profit in the case where sales are lost, or the cost of backordering in the case where demands are backordered. In a steady state situation the transaction and precautionary motives are prevalent, but when prices or interest rates fluctuate sufficiently the speculative motive may also operate. This thesis is only concerned however, with steady state inventory situations and speculative inventory systems will not be considered.

The parameters which affect an inventory system model naturally form a good basis for a classification process of inventory systems. The first of these parameters is that one concerning the nature of the processes which generate demands on the system. The demand parameter may be deterministic as in the case of an inventory of parts to be used on an assembly line, or it may be stochastic as demands on a retail sales outlet. The second parameter concerns the nature of the supply processes to an inventory system and can be divided into three separate categories. First, the lead time (from placement of an order until delivery), if there is such, is either deterministic or stochastic. Second, the order quantities may not have any constraint, or may be allowed only in lot sizes. Finally, quantity discounts may or may not be available on orders placed for sufficient quantities of goods, and when available may apply to the entire order or only to the incremental quantities in the order which affect the discount rate. As may be seen in the literature, e.g., (Hadley, 1963, p. 40ff and p. 62ff) ordering in lot size quantities or quantity discounts does not affect the basic nature of the inventory model used; they simply involve continuous approximations or iterative procedures to solve. The final parameter which affects the inventory system control doctrine is the method of data collection. Data collection procedures are of two types, transactions reporting and periodic review. The transactions reporting method entails the recording of each sale and immediate(2) subtraction from stock records of the amount of goods sold. The periodic review method, which for many types of inventories is the only practical method, in which an accounting

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(2) Some large retail stores keep a record of items sold on paper or magnetic tape at their cash registers which is used to update the inventory records once each day. Given a sufficiently small demand rate (in comparison to amount of stock on hand) this may also be thought of as a transactions reporting system.

of the number of units of a particular item is performed periodically, will be the basis of the renewal theory application.

The costs of an inventory system are numerous; however, not all of them can be affected by or can themselves affect the choice of control doctrine and consequently need not be included in an analysis of an inventory system. We will only be concerned with those costs which have a direct relationship with the choice of control doctrine. Thus such costs as insurance or shrinkage which may not be affected by the stock level will not be considered; however, they may be a factor in the determination of I (the alternative interest rate). Those costs directly affected by or directly affecting the choice of control doctrine are: 1) Procurement costs, generally a fixed cost of placing an order, i.e., administrative and/or transportation costs; 2) Carrying costs or alternative costs, referred to as such since it is the cost of having money tied up in an inventory of goods that could otherwise be invested and accruing interest; 3) Stockout costs, such as the cost of notifying a customer of the out-of-stock condition and goodwill lost when the customer is inconvenienced as well as the cost of lost profit or backorder cost associated with the particular stockout policy in use; 4) Review cost, only considered in the case of a periodic review system since in the transactions reporting system the review cost does not affect the control doctrine.

The first chapter will review the basic transactions reporting inventory systems and basic periodic review system with deterministic supply and demand. The first part of the second chapter will present some results of renewal theory as applied to periodic review inventory systems, while the second part is concerned with developing an approximate solution to the stochastic periodic review model using two applications of renewal theory.

BASIC SYSTEMS

Transactions Reporting Models

Introduction: The difficulty of operating an inventory system is due primarily to uncertainty in the information used to make predictions and decisions Transactions reporting systems, even though giving an exact determination of stock levels at any instant in time thereby reducing this uncertainty, are possible only through the use of complicated procedures or expensive equipment. When the financial investment in an inventory is sufficiently large, however, a transactions reporting system is justified since expenditures on equipment can be offset by increased efficiency. Transactions reporting models are much easier to devise and results are more exact than is the case with periodic review systems. Additionally, basic models of transactions reporting systems provide background material leading to models of periodic review systems. Consequently, the first part of this chapter is devoted to basic transactions reporting models. In the second part of this chapter is presented a deterministic periodic review model as an introduction to the stochastic periodic review model which is presented in the second chapter.(3)

In a transactions reporting system the two questions which need to be answered in order to operate the system are: When to order and

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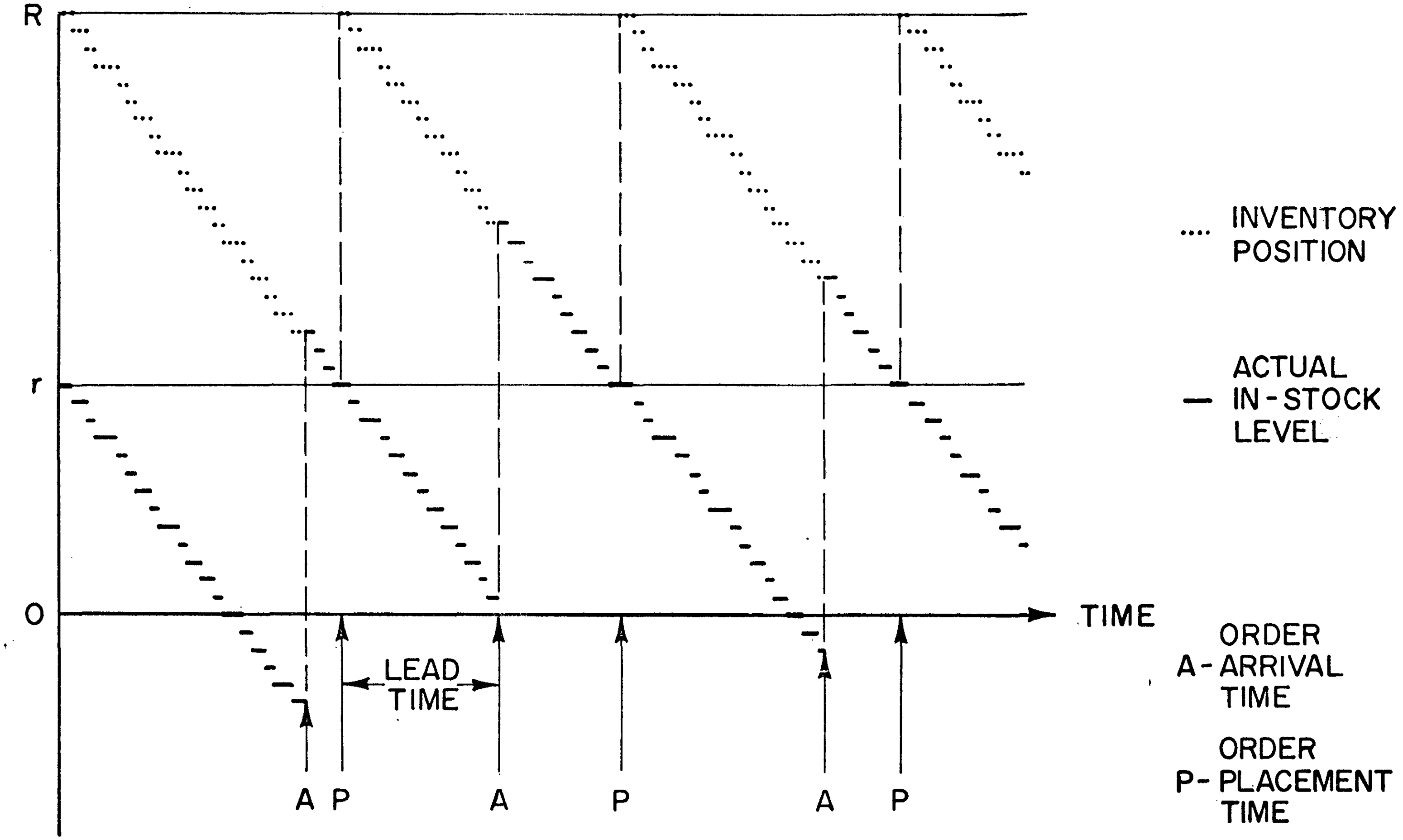
(3) Many of the basic models and some of the notation is from Hadley (1963).



TRANSACTIONS REPORTING SYSTEM

FIG. I

how much to order. They can be most easily answered in terms of stock levels. If we define 'inventory position' as the total amount in stock plus stock on order then r and R will represent the order-placement and immediately after-ordering inventory positions, respectively (see fig. 1). For a system in which the demand and supply process parameters are deterministic, r and R can be found exactly. However, when either demand or supply processes or both are stochastic then r and R can only be determined so that the expectation of profit or cost is optimized. 'Actual in-stock level' is negative when backorders occur.

Deterministic Situation: The simplest transactions reporting systems occur in cases where the demand and supply processes to the system are known with certainty and are assumed to be constant over time. Yet in such a simple model there are many physical situations which can be included. To avoid unnecessary complication we will make the following assumptions: 1) The cost of one unit of an item in inventory is unaffected by choice of control doctrine and constant over time; 2) An item does not become obsolete, i.e., can be kept in inventory for an indefinite period of time; 3) Each order is delivered in its entirety; 4) Advertising or any other promotional changes in selling price will not be considered. We will also treat the inventory position, demand, and supply as continuous variables even though they must by definition be integer values. This descrepancy can be accounted for and corrected using techniques presented by Hadley (1963, p. 40ff). Two additional assumptions will also be made: 5) The procurement lead time (ℓ) is assumed constant independent of demand rate (λ) and quantity ordered;

6) Lead time is small enough so that each order arrives before the next order is placed.

We can see intuitively that in the case of lost sales it is optimal either never to stockout or not to run the system at all. Suppose that the periods in stock during a year are thought of as occuring all together at the beginning of the year and the out-of-stock periods as occuring all together the remainder of the year. Then if it is plausible to operate the system at all, costs can be reduced (or profits increased) by stocking during the out-of-stock period at the end of the year.

Also noting that the carrying cost will be minimized by letting the on-hand inventory level be zero when an order arrives, we choose r to be the number of demands during the lead time period, i.e.,

$$r = \lambda \cdot \ell .$$

Since λ is a constant and no stockouts occur, profits derived from the sale of an item are constant, independent of control doctrine. Thus minimizing annual cost and maximizing annual profit will yield equivalent control doctrines. Let us define a cycle as the time between placement of orders, then if T is the length of a cycle the inventory carrying costs per cycle are given by:

$$I \cdot C \left[\int_0^T (R - \lambda t)\, dt \right] = ICT\left(R - \frac{\lambda \cdot T}{2}\right) .$$

Noting however that

$$T = \frac{R}{\lambda} ,$$

the inventory carrying costs per cycle become

$$IC\left(\frac{R^2}{\lambda} - \frac{R^2}{2\lambda}\right) = \frac{ICR^2}{2\lambda} .$$

The number of cycles per year is λ/R, thus the annual inventory carrying cost is $ICR/2$. The procurement cost is simply A (the procurement cost per cycle) multiplied by the number of cycles per year, or $A\lambda/R$. Thus the average annual variable cost expression is

$$\kappa = \frac{\lambda A}{R} + \frac{ICR}{2} .$$

Critical values of R which minimize κ occur when the derivative of κ with respect to R vanishes i.e.,

$$\frac{d\kappa}{dR} \equiv \frac{-\lambda A}{R^2} + \frac{I\cdot C}{2} = 0 \quad .$$

Solving for R we obtain

$$R = \sqrt{\frac{2\lambda A}{I\cdot C}} ,$$

which results in minimum average annual variable cost since the second derivative of κ with respect to R is positive everywhere.

<u>Quantity Discounts</u>: If we relax assumption (1), (see page 8) and allow the cost per unit (C) to vary with quantity ordered then there are two cases to consider: All units quantity discounts, in which the discount rate for a particular quantity increment is applied to the entire order; and incremental quantity discounts, in which the discount rate for a particular quantity increment is applied only to that portion of the order within that quantity increment. Iterative procedures using

the cost expressions already devised are fairly simple to implement (see Hadley, 1963, p. 62ff) and will yield good approximations.

Backorder Case: In the case where backorders are allowed, we must define two new inventory parameters, Q and s, as the amount ordered and amount on backorder when an order is received, respectively. Thus in terms of Q and s we have

$$R = Q - s + \lambda\ell$$

and

$$r = \lambda\ell - s.$$

If we let $t = 0$ be the time of arrival of an order, and $t = t_1$ the time of arrival of the particular demand which results in zero on-hand inventory, then the inventory carrying cost per cycle is given by

$$IC\int_0^{t_1} (Q - s - \lambda t)\,dt = \frac{IC}{2\lambda}(Q - s)^2$$

Thus since there is an average of λ/Q cycles per year the annual inventory carrying cost is

$$\frac{IC}{2Q}(Q - s)^2 \quad ;$$

We assume here that the backorder cost is of the form $\pi + \hat{\pi}t$ where t is the length of time that the backorder is on the books. The backorder cost consists then of π, a fixed cost, and $\hat{\pi}t$, a cost proportional to the length of time that the backorder is on the books. This is not the only possible choice for a backorder cost relation, but is fairly accurate in many cases, and any more complicated relation in t increases the difficulty of solution significantly. Letting $t = t_2$ represent the time that an order arrives and $t = 0$ the time that stock-out occurs, we can obtain the backorder cost per cycle as

$$\pi s + \hat{\pi} \int_{0}^{t_2} \lambda t \, dt = \pi s + \frac{1}{2} \frac{\hat{\pi} s^2}{\lambda}$$

using the fact that $\lambda t_2 = s$ and treating Q and s as continuous. Since the number of cycles per year is λ/Q, we obtain a relation for the average annual cost of backorders,

$$\frac{1}{Q}\left[\pi\lambda s + \frac{1}{2}(\hat{\pi} s^2)\right] ;$$

and thus for average annual variable cost,

$$\kappa = \frac{\lambda}{Q} A + \frac{IC}{2Q}(Q - s)^2 + \frac{1}{Q}\pi\lambda s + \frac{\hat{\pi} s^2}{2}$$

The optimum values of Q and s, Q^* and s^*, respectively, must satisfy

$$\frac{d\kappa}{dQ} = \frac{d\kappa}{ds} = 0 \quad .$$

Assuming $\hat{\pi} \neq 0$ and $\nu = IC/\pi$, we find

$$s^* = (\hat{\pi} + IC)^{-1}\left\{-\pi\lambda + \left[2\lambda A IC(1 - \nu) - \nu\pi^2\lambda^2\right]^{\frac{1}{2}}\right\}$$

$$Q^* = (1 + \nu)^{\frac{1}{2}}\left[\frac{2\lambda}{IC} A - \frac{\pi^2\lambda^2}{IC(\hat{\pi} + IC)}\right]^{\frac{1}{2}}$$

since the second derivatives of κ with respect to Q and s are both positive everywhere.

Stochastic Situation: We now examine a transactions reporting system in which the demands on the system cannot be predicted with certainty. The model discussed here is termed a lot size reorder point model in which an order is placed each time for an amount Q known as the lot size and it is assumed that we do not overshoot the reorder point r, i.e., the system must be reviewed after each demand. Thus those inventory systems which use a computerized "transactions reporting" system, which reviews the level of inventory once each day, can only use this model as an approximation. Optimum values of Q and r will be determined by formulating an expected annual variable cost expression and, again using the methods of calculus, finding critical values of Q and r, namely Q^* and r^*, which minimize the cost.

The expected annual variable cost can be written

$$\begin{aligned}\kappa(Q,r) &= E\{\text{annual procurement cost}\} + E\{\text{annual carrying cost}\} \\ &\quad + E\{\text{annual stockout cost}\}.\end{aligned}$$

We will make the same assumptions as we did in the deterministic case (see p. 8), except amend assumption (5) to include a Poisson distributed demand process; and assumption (6) to include the situation in which more than one order may be outstanding, thus more closely approximating a real world situation, although we will assume the probability of this event to be very small. We must now determine expressions for the three expected annual costs in the annual variable cost expression. However, before these quantities can be determined, a specific demand distribution must be specified.

Description of Demands: The nature of the process generating demands on the system is not known with certainty in the stochastic situation; however, by making certain assumptions about the process we can describe it probabilisticly. Choosing a specific probabilistic description of the demand process will enable us to discover certain properties of the demand process which will be useful in describing the inventory system, such as the expected number of demands in a particular time interval.

The first assumption we will make is that the process generating the demands is not changing with time, and that demands occuring in a specific interval of time are independent of demands occuring during all other intervals of time. This assumption may seem very restrictive; however, if the rate of change of the demand process is sufficiently small so as to be assumed zero over the time period of interest, then an ensemble average rather than a time average concept can be used in calculating estimators of expected values and probabilities can be thought of, as the number of systems in the ensemble approaches infinity, as the fraction of the systems in the ensemble which yield the appropriate value of the given random variable.

The number of demands on an inventory system is related to the time between arrivals of a demanding agent and the number of demands made by the agent. As a second assumption, we will define the probability of more than one demand occuring in a time interval of length h, divided by h, as approaching zero as h approaches zero. This assumption requires that only one unit be demanded by each demanding agent. Suppose we define G(t) as the distribution of the time between demands on the system. In general, G(t) could depend upon the particular calendar time, the times of occurance of all previous demands, and perhaps many other things.

We will restrict ourselves, first by considering only those G(t)'s which depend entirely on the time of the last previous demand. Then G(t) must be unity at t = 0 since the demand under consideration must occur after the last previous demand and assuming that another demand will occur at some future time, G(t) must approach zero as $t \to \infty$. A form of G(t) which can be of interest and which is the one chosen for consideration here is

$$G(t) = e^{-\lambda t}$$

Then G(t) is the cumulative distribution function of an exponentially distributed random variable which we will refer to as the "demand interarrival time".

We can compute the probability that during time (t + dt) after a specific demand, another demand occurs. If we let the event A be the occurrance of no demands up to time t following the specific demand and the event B be the occurrance of a demand in time interval (t,t + dt) following the specific demand.

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)} ,$$

$$= \frac{1 \cdot \lambda e^{-\lambda t} dt}{e^{-\lambda t}} = \lambda dt .$$

We can also find the probability ($P_o(t)$) that no demands occur in a time interval of length t from the time when we begin observing the system, a random point in time. Since we have made the assumption that demands occuring in an interval of time are independent of demands occuring during all other periods of time, we can calculate the probability of no demands occuring in a time period of length $(t + dt)$. Given λdt as the probability that a demand occurs in the time interval $(t,t + dt)$, the probability that no demand occurs in time interval $(t,t + dt)$ is $1 - \lambda dt$. Thus

$$P_o(t + dt) = (1 - \lambda dt)P_o(t)$$

or

$$\frac{dP_o(t)}{P_o(t)} = -\lambda dt \ .$$

Integrating, and using the fact that $P_o(0) = 1$ we obtain

$$P_o(t) = e^{-\lambda t}$$

This gives us the result that the probability that the time to the next demand, given that the observation begins at a random point in time, is $e^{-\lambda t}$ independent of the time that observation began. It can be shown that this particular probability distribution function for demand interarrival time leads to the result that the demand process is Poisson distributed with mean rate λ.

<u>Analysis of Cost Equation:</u> The inventory position can only be in states $\{r + Q, r + Q - 1, r + Q - 2, \cdots , r + 2, r + 1\}$ since the system cannot exist in state r for a finite amount of time (an order is placed immediately after the inventory position reaches r). Suppose we define the stochastic process $\{x(t), t \geq 0\}$ with state space $\{0, 1, 2,\cdots,Q-1\}$ as the process of the inventory position, where $x(t)$ being in state j corresponds to the inventory position being $r + j + 1$. If we define T_1 as a time point when the process (probabilisticly) first restarts itself,

i.e., the continuation of the process beyond T_1 is a probabilistic replica of the whole process starting at zero, such a process is known as a "regenerative" process. The probability density of T_1 is given by

$$P\{T_1 = t\} = P\{Q \text{ demands in time } t\} = \frac{(\lambda t)^Q e^{-\lambda t}}{Q!}$$

which exists on the interval $[0,\infty)$. The demand interarrival times are independent and identically distributed random variables thus the expected time of arrival of the Q^{th} demand after a reorder is the sum of the expected interarrival times of the first Q demands following a reorder and is given by:

$$E[T_1] = Q\int_0^\infty te^{-\lambda t}\,dt = Q/\lambda < \infty$$

Thus, by a result of the Key Renewal Theorem, quoted in Ross (1970, p. 95) P_j, the limiting probability of being in state j is given by:

$$P_j = \frac{E[\text{time in state } j \text{ per cycle}]}{E[\text{time of one cycle}]}$$

where one cycle is defined as the time between reorders. The

$$E[\text{time in state } j \text{ per cycle}] = E[\text{demand interarrival time}]$$
$$= \int_0^\infty te^{-\lambda t}\,dt = \frac{1}{\lambda}$$

and

$$E[\text{time of one cycle}] = E[\text{time for } Q \text{ demands to occur}]$$
$$= E[T_1] = Q/\lambda\ .$$

Hence, $P_j = \frac{1/\lambda}{Q/\lambda} = \frac{1}{Q}$ independent of j.

Thus the state probabilities $\rho(r + j)$ of being in state $r + j$ can be written

$$\rho(r + j) = \begin{cases} 1/Q & j = 1,2,\ldots,Q \\ 0 & \text{elsewhere} \end{cases}$$

If ℓ is the lead time then the probability that x units are on hand at a time t, given that $r + j$ units were on hand at time $t - \ell$ is

$$p(r + j - x;\lambda\ell),$$

where $p(y;wz)$ is the Poisson probability of having y demands in time z when the expected demand rate is w. But the probability of having $r + j$ units on hand at time $t - \ell$ is simply $1/Q$. Thus we can find $\psi_1(x)$ the probability that x units are on hand at time t:

$$\psi_1(x) = (1/Q) \sum_{j=1}^{Q} p(r + j - x;\ \lambda\ell)\ .$$

Plus some terms which occur since we have allowed more than one order on the books at a time; however, we have assumed these terms to be very small and will omit them. Similarly we can calculate $\psi_2(z)$ the probability that z backorders are on the books at time t, from which can be calculated $B(Q,r)$ the expected number of backorders at any time t:

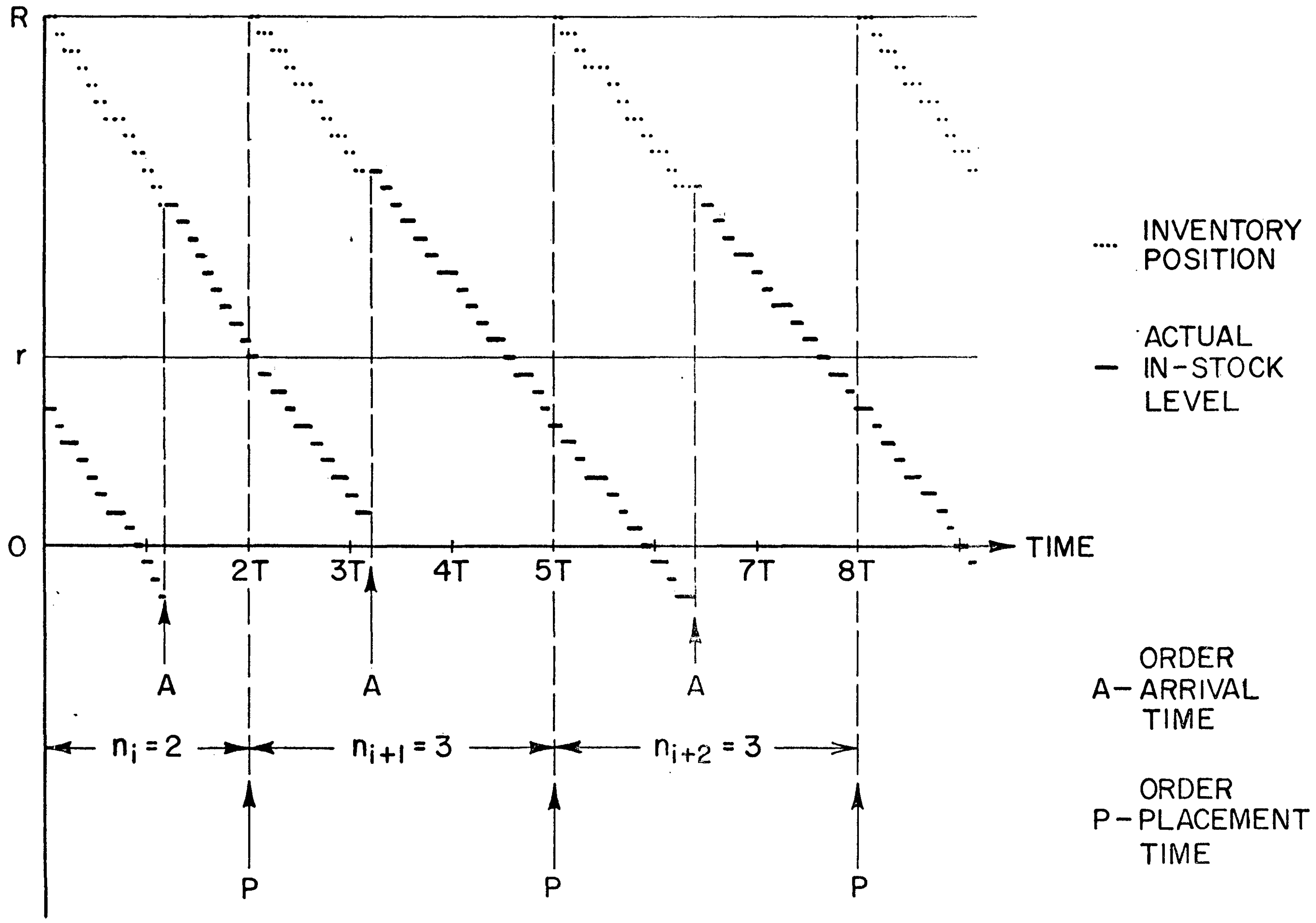
$$B(Q,r) = \sum_{z=0}^{\infty} z\cdot\psi_2(z)\ .$$

Also, we can calculate $D(Q,r)$ the expected on hand inventory at any time t:

$$D(Q,r) = \sum_{x=0}^{r+Q} x\cdot\psi_1(x)\ ,$$

and $E(Q,r)$ the expected number of backorders which are incurred per year:

$$E(Q,r) = \sum_{z=0}^{\infty} z\cdot\psi_2(z)\ .$$



PERIODIC REVIEW SYSTEM

FIG. 2

We can now write the expected annual variable cost expression

$$\kappa(Q,r) = (\lambda/Q)\cdot A + IC\cdot D(Q,r) + \pi\cdot E(Q,r) + \hat{\pi}\cdot B(Q,r).$$

It is difficult in general to find optimum values of Q^* and r^* from this exact expression for $\kappa(Q,r)$ since r and Q both appear in the limit of summation in the expected value expression $D(Q,r)$. Iterative procedures of direct search methods have been employed with much success, however, in obtaining relative minima. As pointed out by Hadley (1963, p. 188) although it is difficult to theoretically rule out the existance of local minima different from the absolute minimum, it appears from direct calculation that these local minima do not exist.

Periodic Review Model

Introduction: In a moderately large inventory situation it is often too expensive to employ a digital transactions reporting system and physically impossible to record each sale for the purpose of keeping a continuous accounting of inventory levels. An occasional check on inventory level must be made, however, in order that a procurement of replenishment stock can be ordered whenever the inventory level drops sufficiently. If these occasional checks on inventory are made with regularity (i.e., the time (T) between reviews of the system stock level is a constant) a periodic review inventory model can be used to determine optimum control doctrine for the system (see fig. 2).

Again, as with transactions reporting models, certain simplifying assumptions must be made. They are: 1) The cost of one unit of an item in inventory is unaffected by choice of control doctrine and constant over time; 2) An item does not become obsolete; 3) Each order is delivered in its entirety; 4) Advertising, promotional, or any other changes in selling price will not be considered; 5) The procurement lead time (ℓ) is assumed constant independent of demand rate and quantity ordered; 6) Lead time is small enough so that each order arrives before the next order is placed; 7) λ, the demand rate, is also assumed constant. Again we will treat all variables as continuous; the integrality of certain variables will be taken account of in the final procedures.

Let us define a "period" as the length of time (T) between reviews of the system. A determination of T must involve more than just the costs involved with the inventory system, in fact, physical limitations may be more important than cost limitations on T. In addition, a cost analysis determination of an optimum value for T would be difficult analytically in the procedures outlined here. For this reason we will assume that T is a predetermined constant for the system being studied.

Let us assume that orders are placed for procurement of replenishment stocks just after completion of a review which discloses the stock level to be below a particular number of units (r). An order is placed in suffucient quantity to bring the inventory position exactly to a level R. The inventory level when an order is placed is S, and a cycle is defined as the period of time between two successive procurement orders, thus cycle length is an integral multiple of T.

Deterministic Situation: In our first analysis we will assume that the lead time (ℓ) and demand processes (λ) are deterministic. These assumptions are not generally possible in a real situation; however, they allow us to examine the general structure of a periodic review model without the additional complication of uncertainty of data. These assumptions will be eliminated in the final model. We can write a general cost equation for a particular cycle

κ = review cost + procurement cost + carrying cost + stockout cost.

It is important to notice that the cost equation for a periodic review system includes the additional cost of the reviews during the cycle which was not a part of the transactions reporting cost equation. If we were to simply optimize the cost per cycle, the review cost would not be a constant; it would depend on the number of periods in a cycle. This procedure would not necessarily optimize the cost of operating the system however, since it is possible to minimize the cost per cycle by ordering after each review. A more correct method of optimizing the cost of operating the system is to optimize the cost per unit time (or rate of cost). The length of a cycle is a constant once values of R and r have been chosen, thus we can obtain a cost per unit time equation by simply dividing the cost per cycle by the length of a cycle. Since the cost of a review is constant and the length of a period is constant, the review cost per unit time is constant independent of control doctrine and need not be considered in the cost equation.

If we let n be the number of periods in a cycle then

$$n = \frac{R - S}{\lambda \cdot T} \qquad (1)$$

and

$$nT = \frac{R - S}{\lambda} \doteq \text{ length of a cycle.}$$

If we assume the penalty of a stockout to be sufficiently high then since lead time and demand processes are deterministic, we let S be the number of demands during the lead time. Thus stockout is zero.(4) The procurement cost (A) is assumed constant for each order thus A/nT is the procurement cost per unit time. Since we assumed the demand processes to be deterministic, the average unit years of stock held per cycle is

$$\frac{R + S}{2}\, nT,$$

where T is measured in years. Thus the carrying cost per cycle can be written

$$\tfrac{1}{2}(R + S)nTIC;$$

and the rate at which carrying cost is incurred is given by

$$\tfrac{1}{2}(R + S)IC.$$

Thus the cost per unit time ($\dot{k}$) expression can be written

$$\dot{k} = A/nT + \tfrac{1}{2}(R + S)IC = \frac{A\,\lambda}{R-S} + \frac{R + S}{2}\, IC \quad .$$

To optimize cost, we take the partial derivative with respect to R and set it equal to zero

$$\frac{\partial \dot{k}}{\partial R} \equiv \tfrac{1}{2}IC - \frac{A\lambda}{(R-S)^2} = 0 \;;$$

solving for R we obtain

$$R = \sqrt{2} \cdot (A\lambda/IC)^{\frac{1}{2}} + S$$

The above calculations were made allowing the variables to be

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(4) By decreasing S by 1 unit we increase stockout cost by π and decrease carrying cost by ICnT. Thus if π is not greater than ICnT in the final solution, incurring some stockouts might be more optimum and stockout costs should be included in the model.

continuous; however, we must have integer values for R, S, and n. Using equation (1) we can find a value for n

$$n = (R - S)/\lambda T = \sqrt{\frac{2A}{\lambda CIT^2}}$$

If we choose the two integers closest to the value found for n and, using these, find values for $\dot{\kappa}$. The integer (n) which when substituted into the equation for $\dot{\kappa}$ results in the smallest cost is the optimum n value. Then r can be chosen as S + 1.

RENEWAL APPLICATION

Introduction

In this chapter we will be concerned with a periodic review inventory system which has stochastic demand processes and procurement lead times. An application of renewal theory will be used to determine many useful quantities pertaining to the system being studied without explicitly solving for values of R and r. Then an iterative procedure is developed which will yeild approximate values for R and r. Finally, several example systems are proposed and the results obtained are discussed.

Renewal Theory As Applied to Stochastic Periodic Review Systems

Results Concerning N: Since the inventory position is reset to the point R at the beginning of each cycle then the cycle lengths are non-negative identically distributed lattice random variables with period T which we will denote by τ_i. If we again make the assumption that demands during each period are independent from demands occuring during all other periods then the τ_i's are independent. The length of a cycle is not independent of all other cycles if sales are lost when the system is out-of-stock since the previous cycles' activities determine the length of time out-of-stock (at least partially) during the next cycle. Thus we must assume that all demands occuring when the system is out-of-stock are backordered.

If we let S_n be the time up to end of the n^{th} cycle then

$$S_n = \sum_{i=0}^{n} \tau_i$$

and define

$$N(t) = \sup \{n: \quad S_n \leq t\}$$

the process $\{N(t), t \geq 0\}$ is a renewal process.(5) We can determine many properties of the inventory system without formulation of a model by using the results of renewal theory. Many of the results in renewal theory are applicable to periodic review inventory systems as we will show.

If we let n_i be the number of periods between the starting point of the i^{th} cycle and the starting point of the $(i + 1)^{st}$ cycle, then the n_i's are positive independent identically distributed random variables. If we have k periods in the i^{th} cycle then there are n $(m \geq 1)$ less than R-r demands in k-1 periods and at least m demands in the k^{th} period. If we again assume that the demand is Poisson distributed with mean rate λ then the demand during each period is Poisson distributed with the same expected rate. Using the Poisson assumption of independent increments we thus obtain the probability mass function of n

$$p\{n = k\} = \sum_{m=1}^{R-r} \sum_{j=m}^{\infty} p\{j; \lambda t\} \, p\{R - r - m; \lambda(k - 1)T\}$$

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(5) A process $\{N(t), t \geq 0\}$ is defined as a renewal process if $N(t) = \sup \{n: S_n \leq t\}$ where S_n is the sum of n non-negative independent, identically distributed random variables X_i; where $E[X_i]$ (not necessarily finite) exists.

where p(j; λt) is the Poisson probability of j demands in a time period of length t and p{A} is the probability of the event A.

If we define m(t) as the expected value of N(t) then we can develop an integral equation for m(t). By conditioning on the length of the first cycle we obtain

$$m(t) = \int_0^\infty E[N(t) \mid \tau_1 = x] \, dF(x)$$

where F is the common distribution of the τ_i's. We can find the conditioned expectation of N(t):

$$E\left[N(t) \mid \tau_1 = x\right] = \begin{cases} 0 & x > t \\ 1 + m(t - x) & x \leq t \end{cases} ;$$

since if the first cycle ends at a time $x \leq t$ the process starts all over again at time x. Thus the expected number of renewals in 0,t is just l plus the expected number of renewals in time (t - x). We then have

$$m(t) = \int_0^t [1 + m(t - x)] \, dF(x)$$

or

$$m(t) = F(t) + \int_0^t m(t - x) \, dF(x)$$

assuming that F(0) = 0. According to Feller (1971, p. 358ff), this renewal equation can be solved, depending on the functional form of F. If F(t) is directly Riemann integrable and arithmatic(6) with span K then the solution is given by

$$m(t + nK) \rightarrow K\mu^{-1} \sum_{j=1}^{\infty} F(t + jK), \qquad n \rightarrow \infty$$

where μ is defined as

$$\mu = \int_0^{\infty} x dF(x).$$

We know that

F(x) = p{u $\leq$ R-r demands in the first m-l periods and R-r-u demands in the m^{th} period},

where

$$m = \sup\{q : qT < x\}.$$

Thus

$$F(x) = \sum_{u=0}^{R-r-1} p\{u \leq R\text{-}r \text{ demands in the first } m\text{-}l \text{ periods}\}\; p\{R\text{-}r\text{-}u \text{ demands in the } m^{th} \text{ period}\}$$

$$\sum_{u=0}^{R-r-1} \left[\frac{(\lambda(m-1)T)^u \; e^{-\lambda(m-1)T}}{u!}\right] \cdot \left[\frac{(\lambda T)^{(R-r-u)} \; e^{-\lambda T}}{(R-r-u)!}\right]$$

The distribution of F(x) is arithmetic with span T and is Riemann integrable since it is bounded and peicewise continuous; however, it would seem at least cumbersome to compute the solution for m(t+n•k) as given above.

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(6) A distribution concentrated on multiples of a number K, where K is called the span of the distribution.

Central Limit Theorem Result: Suppose we define $T^{(r)}$, as the number of periods up to the r^{th} cycle. We can write

$$T^{(r)} = \sum_{i=0}^{r-1} n_i \quad ,$$

where n_i denotes the number of periods in the i^{th} cycle, and if we let

$$\mu = E[n]$$

and

$$\sigma^2 = \mathrm{Var}[n]$$

then since $T^{(r)}$ is the sum of r independent and identically distributed random variables (not necessarily with finite variance - see Feller, 1971, p. 260) the Central Limit Theorem asserts that

$$\lim_{r\to\infty} p\left\{\frac{T^{(r)} - r\mu}{\sigma\sqrt{r}} < a\right\} = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\, dx$$

The right hand side of the equation is a cumulative normal distribution function with mean zero and variance 1. If

$$T^{(r)} \leq k \quad ,$$

then

$$\frac{T^{(r)} - r\mu}{\sigma\sqrt{r}} \leq \frac{k - r\mu}{\sigma\sqrt{r}}$$

Also we know that if the number of periods up to the r^{th} cycle is less than k then the number of cycles in k periods must be more than r; so if $T^{(r)} \leq k$ then $N(kt) \geq r$. Also, if the number of cycles in k periods is greater than r then the number of periods up to the r^{th} cycle is less than k. Consequently,

$$p\{T^{(r)} \leq k\} = p\{N(kt) \geq r\} .$$

But we have that

$$\lim_{\substack{r\to\infty \\ k\to\infty}} p\{N(kt) \geq r\} = \lim_{\substack{r\to\infty \\ k\to\infty}} p\{T^{(r)} \leq k\} ,$$

where

$$\lim_{\substack{r\to\infty \\ k\to\infty}} \frac{k - r\mu}{\sigma \sqrt{r}} = a .$$

Thus we have shown that the complementary cumulative distribution of the number of cycles in a time kt asymptotically approaches a normal distribution with mean zero and variance 1 for sufficiently large values of r and k.

Approximate Solution Procedure

Preliminary Discussion: If a control doctrine for the system is not needed, but simply one of the above properties, namely the expected member of cycles up to time t (m(t)), or some parameter of the complementary cumulative distribution of the number of cycles in a time kt, then explicit values of R and r are not needed. However, if an explicit control doctrine is desired then a general cost equation is needed. As with the deterministic model the cost equation consists of four parts: 1) Review cost; 2) Procurement cost; 3) Carrying cost; 4) Stock-out cost. An exact cost expression can be developed (Hadley, 1963, p.275):

$$\kappa(R,r,T) = \frac{J}{T} + \frac{A + \sum_{n=0}^{\infty} \sum_{j=1}^{R-r} p\{R-r-j;\ n\lambda T\}\ H(r+j,T)}{T \sum_{n=1}^{\infty} \sum_{j=1}^{R-r} np\{R-r-j;(n-1)\lambda T\}p\{j;\lambda T\}}$$

where $p(y;wz)$ is the Poisson probability of having y demands in time z when the expected demand rate is w; and $H(r+j,T)$ is the conditional expected cost contribution from period $(n+1)$ given that the inventory position at the beginning of the period is $r + j$. However, solution for R^* and r^* (values of R and r which result in minimum cost) is difficult manually and might take as long as a minute on a large, high-speed, digital computer. The utility of an approximate solution method is therefore obvious if it can be solved manually or by computer fairly easily and quickly. R and r are not the only parameters which describe a periodic review inventory system. Alternatively, the procedure which we will develop uses N (the expected number of periods per cycle) and S (the stock on hand when an order is placed) as the parameters of the system. Later in the development, we will use N and S to find optimizing values for R and r so that the control doctrine can be implemented.

Additional Renewal Theory: We will denote by $\{Y_n, n=1,2,\ldots\}$ a renewal reward process where Y_n is the reward earned during the n^{th} renewal interval of length X_n. If we suppose that the pairs $\{(X_n,Y_n), n=1,2,\ldots\}$ are independent and identically distributed and we let

$$Y(t) = \sum_{n=1}^{N(t)} Y_n \quad ,$$

then Y(t) is the total reward earned up to time t. By a well known renewal theorem (Ross, 1970, p. 52) we know that if E $|Y|$ and E[X] are finite, then

$$E\left[\frac{Y(t)}{t}\right] \rightarrow \frac{EY}{EX} \qquad \text{as } t \rightarrow \infty \quad .$$

Since each cycle in a periodic review inventory system is in fact a renewal interval, the cost per cycle is a renewal reward process. Thus, the expected cost per unit time is given by the expected cost per cycle divided by the expected cycle length.

Another result which we will make use of is Wald's Equation. Simply, Wald's Equation states that if we have X_1, X_2, ... independent and identically distributed random variables with finite expectations and if N is a stopping time for X_1, X_2, ... such that EN < ∞ then

$$E\left[\sum_{i=1}^{N} X_i\right] = EX \cdot EN$$

For a proof, see (Ross, 1970, p. 38), where the stopping time (N) is an integer valued positive random variable which may be dependent on X_1, X_2, ... , X_N but which must be independent of X_{N+1}, X_{N+2}, ...

Development of the Rate-of-Cost Expressions: We will now develop expressions for the four costs involved in a periodic review system. Again making the same assumptions which we made for the deterministic model (see p. 20) with the exceptions that assumption (5) is amended to include stochastic procurement lead times with expected value ℓ, and assumption (7) is amended to include a stochastic (Poisson distributed) demand process with expected rate λ. As with the deterministic model we will treat all variables as continuous.

Suppose we let N represent the expected value of the number of periods in the i^{th} cycle (n_i). If J is the cost of each review, then the expected cost of reviews during the cycle is

$$E[n_i J] = J\, E[n_i] = JN$$

The procurement cost per cycle is simply the constant cost of a procurement (A), since exactly one order is placed during each cycle.

Suppose we let P_{ji} represent the number of demands during the j^{th} period in the i^{th} cycle and λ_{ji} represent the demand rate. Then

$$E\left[P_{ji}\right] = T\, E\left[\lambda_{ji}\right] = T\lambda$$

since the expected demand rate is a constant independent of time. The P_{ji}'s are independent and identically distributed random variables, so utilizing Wald's Equation we can find the expected number of demands per cycle,

$$E\left[\sum_{j=1}^{n_i} P_{ji}\right] = E\left[n_i\right]\cdot E\left[P_{ji}\right] = NT\lambda.$$

The carrying cost per cycle can be divided into two parts: The carrying cost incurred due to the units which are demanded during the cycle, and the cost incurred due to those units which, having been held the entire cycle, remain at the end of the cycle. We will first examine the cost due to the units which are demanded during the cycle. Since we have assumed the variables to be continuous, the expected rate of carrying cost is

$$ICNT\lambda - \frac{E\left[\int_0^{n_i T} t\lambda(t)\ dt\right] IC}{NT}$$

where the first term represents the expected carrying cost due to the total number of units demanded and the second term is the expected cost reduction due to the demands on the system divided by the expected length of the period resulting in the reduced expected rate-of-cost expression. Assuming λ to be constant over time and conditioning on the value of n_i (the number of periods in the i^{th} cycle),

$$E\left[\int_0^{n_i T} t\lambda(t)\ dt\right] = \int_0^{\infty} E\left[\int_0^{n_i T} t\lambda\ dt \mid n_i = n\right] dQ(n)$$

$$= \int_0^{\infty} \left[\int_0^{nT} t\lambda\ dt\right] dQ(n)$$

$$= \int_0^{\infty} \left[\lambda(t^2/2)\right]_0^{nT} dQ(n)$$

$$= \int_0^{\infty} \tfrac{1}{2} n^2 T^2 \lambda\ dQ(n)$$

$$= \tfrac{1}{2}\lambda T^2 \int_0^{\infty} n^2\ dQ(n)$$

$$= \tfrac{1}{2}\lambda T^2\ E[n^2]$$

where Q(n) is the distribution of the number of periods per cycle. Recalling that N is the expected number of periods per cycle, and if the variance of the number of periods per cycle is small,(7) then $E[n^2] - N^2 \approx 0$. Hence we can approximate the expected rate at which carrying cost (of the units not demanded) is incurred by

$$ICNT\lambda - \frac{IC\lambda T^2N^2}{2NT} = \frac{ICNT\lambda}{2}$$

Since the expected number of units not demanded is S, the expected rate at which carrying costs are incurred is

$$IC\left[\frac{NT\lambda}{2} + S\right]. \tag{1}$$

Again, letting π represent the backorder penalty cost associated with each backordered demand, the expected cost of backorders per cycle is π multiplied by the expected number of backorders per cycle. Frequently the penalty of a backorder is large enough so that the optimum expected number of backorders incurred per cycle is small. Consequently, we make the assumption that the probability of stocking out <u>before an order is placed</u> is small enough to be neglected (recall that this does not include the lead time). The expected number of backorders in a

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(7) The variance of the number of periods per cycle is directly related to the distribution of the demand process. It would seem that the assumption of small variance of the number of periods per cycle would indicate a fairly 'smooth' demand process, i.e., wide variations in the average demand rate over time intervals of a length close to a cycle length, would be fairly rare occurrences.

cycle will be j if there are exactly S + j demands during the lead time. Thus the expected number of backorders can be found by conditioning on the length of the lead time

$$\int_0^\infty \pi \sum_{j=S+1}^{\infty} j \cdot p\{j;\ \lambda x\}\ dG(x) \qquad S \geq 0 \qquad (2)$$

where G(x) is the lead time distribution and S is the expected number of units in stock when an order is placed.

Computational Procedure and Discussion

Introduction: We have developed expressions for the four costs involved in a periodic review system. However, it would not be easy to solve the resulting cost equation manually and difficult if not impossible to obtain an explicit solution for r* and R*. A numerical procedure could be developed and programmed on a large computer; however, depending upon the individual problem, this could be very expensive. Alternatively, an iterative solution procedure will be developed which can be solved manually without great difficulty for an approximate solution.

Iterative Solution Method: Let us suppose, for the purpose of obtaining initial approximations, that the probability of stocking out is small enough to be neglected. Since the costs per cycle of reviews and procurements are renewal reward processes, we can find the expected rate at which these costs are incurred by dividing them by the expected length of a cycle, see (Ross, 1970, p. 19) for more information.

Using the results of the previous section, the expected rate-of-cost expression can then be written

$$\tilde{\kappa}_1 = \frac{A}{N \cdot T} + \frac{J}{T} + IC\left[\frac{NT\lambda}{2} + S\right],$$

where $\tilde{\kappa}_1$ represents the first approximation of the rate-of-cost. To find the first approximation ($\tilde{N}_1$) of N (the expected number of periods per cycle) which results in a minimum value for $\tilde{\kappa}_1$ we take the partial derivative of $\tilde{\kappa}_1$ with respect to N

$$\frac{\partial \tilde{\kappa}_1}{\partial N} = \frac{-A}{N^2 T} + \frac{ICT\lambda}{2}$$

Setting this result equal to zero we obtain

$$\tilde{N}_1 = \left[\frac{2A}{T^2 IC\lambda}\right]^{\frac{1}{2}}$$

In order to find an approximate optimum value for S ($S \geq 0$, the expected number of units in stock when an order is placed), we would like to take the partial derivative of $\tilde{\kappa}_1$ with respect to S. However, S is the lower limit of the index of summation in the expression for backorder cost (see p. 36). We will therefore instead use the following procedure which is approximately analogous to differentiating $\tilde{\kappa}_1$ with respect to S. Suppose we let $\hat{S}$ represent the value of S which results in a minimum cost per cycle. An approximate value for $\hat{S}$ can be obtained by setting the expression for the change in cost per cycle (when $\hat{S}$ is changed by one unit) to zero and solving for $\hat{S}$. Since the value of N is already fixed, the only costs affected by the change in $\hat{S}$ are the carrying cost of those units not demanded during the cycle, and the backorder cost (expressions (1) and (2) on p. 35).

Let us choose to change the value of $\hat{S}$ by increasing $\hat{S}$ by one unit, the resultant change in carrying cost rate is IC, so the change in cycle carrying cost is ICNT. We recall that the expected cost of backordering is

$$\pi\int_0^\infty \sum_{j=S+1}^{\infty} j\cdot p\{j;\ \lambda x\}\ dG(x) \qquad\qquad S \geq 0$$

Depending on the functional nature of G(x), this integral may be very difficult to solve. However, since we assumed the variance of the lead time to be small, the entire probability mass will be concentrated about the mean. Recalling that ℓ is the expected lead time, we can obtain an approximation to the expected backorder cost:

$$\pi\sum_{j=S+1}^{\infty} j\cdot p\{j;\ \lambda\ell\} \quad ,$$

or equivalently,

$$\pi\sum_{j=1}^{\infty} j\cdot p\{S + j;\ \lambda\ell\} \qquad\qquad (3)$$

Thus, the approximate change in expected backorder cost resulting from adding one unit to $\hat{S}$ is

$$\pi\left[\sum_{j=1}^{\infty} j\cdot p\{S + j;\ \lambda\ell\} - \sum_{j=1}^{\infty} j\cdot p\{S + j + 1;\ \lambda\ell\}\right] =$$

$$= \pi\left[\sum_{j=1}^{\infty} j\cdot p\{S + j;\ \lambda\ell\} - \sum_{j=2}^{\infty} (j-1)\cdot p\{S + j;\ \lambda\ell\}\right]$$

$$= \pi\left[p\{S + 1;\ \lambda\ell\} + \sum_{j=2}^{\infty} p\{S + j;\ \lambda\ell\}\right]$$

$$= \pi \sum_{i=\hat{S}+1}^{\infty} p\{i;\ \lambda\ell\}$$

$$= \pi\left[1 - \sum_{j=1}^{\hat{S}} p\{j;\ \lambda\ell\}\right]$$

Hence we obtain the resultant approximate expected change in total cycle cost:

$$\tilde{N}_1\mathrm{TIC} - \pi\left[1 - \sum_{j=1}^{\hat{S}} p\{j;\ \lambda\ell\}\right] .$$

Setting this expression equal to zero we obtain

$$\sum_{i=1}^{\hat{S}} p\{i;\ \lambda\ell\} = (\pi - \tilde{N}_1\mathrm{TIC})/\pi \qquad (4)$$

Similarly, if we change S by subtracting one unit, we obtain, for the approximate expected change in total cycle cost, the expression

$$\sum_{i=1}^{\hat{S}-1} p\{i;\ \lambda\ell\} = (\pi - \tilde{N}_1 TIC)/\pi \qquad (5)$$

Strict equality in expressions (4) or (5) will be unlikely to happen; however, we wish to find the value of $\hat{S}$ which results in the least difference between the sum of probabilities on the left hand sides and the expressions on the right hand side, i.e., we would like a value of $\hat{S}$ such that

$$\sum_{i=1}^{\hat{S}} p\{i;\ \lambda\ell\} \geq (\pi - \tilde{N}_1 TIC)/\pi$$

and

$$\sum_{i=1}^{\hat{S}-1} p\{i;\ \lambda\ell\} \leq (\pi - \tilde{N}_1 TIC)/\pi$$

Two possible expressions to use for $\tilde{S}_1$ (the first integer approximation of $\hat{S}$ which is the cost minimizing value of S) are first:

$$\tilde{S}_1 = \inf \left\{ S: \sum_{i=1}^{S} p\{i;\ \lambda\ell\} \geq (\pi - \tilde{N}_1 TIC)/\pi \right\} ;$$

or second,

$$\tilde{S}_1 = \sup \left\{ S: \sum_{i=1}^{S-1} p\{i;\ \lambda\ell\} \leq (\pi - \tilde{N}_1 TIC)/\pi \right\} .$$

We will choose the former due to its advantages in the iteration procedures. Since we have assumed that the demand processes are

Poisson distributed, the left hand side of the inequality inside the braces is the cumulative Poisson distribution.

The approximate expected cost of backorders per cycle ($\tilde{B}_1$) can now be computed from expression (3) on p. 38,

$$\tilde{B}_1 = \pi \sum_{j=\tilde{S}_1+1}^{\infty} j\cdot p\{j;\ \lambda\ell\}$$

or equivalently,

$$\tilde{B}_1 = \pi\left[\lambda\ell - \sum_{j=1}^{\tilde{S}_1} j\cdot p\{j;\ \lambda\ell\}\right]$$

This approximate expected backorder cost per cycle is, for each fixed value of $\tilde{S}_1$, independent of N. Thus we can obtain a second approximate value of $\tilde{N}$ (call it $\tilde{N}_2$), by now not assuming the probability of stockout to be small as opposed for previous formulation $\tilde{N}$, on p. 37,

$$\tilde{N}_2 = \sqrt{\frac{2(A + \tilde{B}_1)}{T^2IC\lambda}}$$

Following the above computations, second approximations of $\hat{S}$ and $\hat{B}$ are obtained

$$\tilde{S}_2 = \inf\left\{S:\ \sum_{i=1}^{S} p\{i;\ \lambda\ell\} \geq (\pi - \tilde{N}_2TIC)/\pi\right\}$$

$$\tilde{B}_2 = \pi \sum_{j=\tilde{S}_2+1}^{\infty} j\cdot p\{j;\ \lambda\ell\}$$

Thus we can develop an iterative procedure after first approximations of $\hat{N}$, $\hat{S}$, and $\hat{B}$ have been completed where

$$\tilde{N}_i = \sqrt{\frac{2(A + \tilde{B}_{i-1})}{T^2 IC\lambda}}$$

$$\tilde{S}_i = \inf \left\{ S: \sum_{k=1}^{S} p\{k;\ \lambda\ell\} \geq (\pi - \tilde{N}_i TIC)/\pi \right\} ,$$

$$\tilde{B}_i = \pi \sum_{j=\tilde{S}_i+1}^{\infty} j \cdot p\{j;\ \lambda\ell\}$$

The procedure is to compute successive approximate values of $\hat{N}$, $\hat{S}$, and $\hat{B}$, until $\tilde{S}_i = \tilde{S}_{i+1}$ at which time the procedure is terminated with

$$\tilde{S} = \tilde{S}_i$$

and

$$\tilde{N} = \tilde{N}_i$$

In this iteration procedure, convergence will always occur. We illustrate with two cases ($\gamma = \tilde{N}_2 TIC$):

Case I, $\tilde{N}_1 TIC \geq \pi$: In this case, $\tilde{S}_1$ will be 1, thus $\tilde{B}_1$ will be greater than zero then γ will be greater than π in which case $\tilde{S}_2$ will be 1 and the procedure will terminate.

Case II, $\tilde{N}_1 TIC \leq \pi$: In this case, the worst that can possibly happen is that N_{i+1} will be enough larger than $\tilde{N}_i$ in each iteration so that

$\tilde{S}_{i+1}$ is less than $\tilde{S}_i$. Eventually, this situation will lead to $\tilde{S}_i=0$ and again we have convergence.

Determination of R and r: R is the number of demands during the cycle plus the remaining stock unsold at the end of the cycle. Thus we obtain an approximate optimizing value of R (call it $\tilde{R}_I$) the integer closest to the value of $\tilde{R}$, where

$$\tilde{R} = \tilde{N}T\lambda + \tilde{S}$$

The operating doctrine is defined as: If a review discloses the inventory position to be below the level r, then an order is placed which brings the inventory position up to R. Thus the inventory position must reach the value r at some time during the period at the end of which an order is placed. If we denote the times during the i^{th} cycle at which the inventory position reaches r and at which an order is placed by t_{ri} and t_{oi}, respectively, then

$$0 \leq (t_{oi} - t_{ri}) < T$$

or equivalently,

$$t_{oi} - t_{ri} = a_i T \qquad\qquad 0 \leq a_i < 1 ,$$

where a_i, whose expected value we will denote by α, is the proportion of a period that remains until an order is placed when the inventory position reaches r. We will indicate how to use α to find an expression for r in terms of N and S.

Suppose we let λ_{ri} represent the demand rate during the time interval $\left[t_{ri}, t_{oi}\right]$, the expected number of demands in the time interval between the inventory position reaching r and the placement of an order is given by

$$E\left[a_i T\lambda_{ri}\right] = E\left[a_i\right]\cdot T\cdot\lambda_{ri} = \alpha T\lambda \quad .$$

Recalling that S is the expected number of units remaining in stock when an order is placed, we see that

$$S = r - \alpha T\lambda$$

or, equivalently,

$$r = S + \alpha T\lambda \qquad (6)$$

An exact determination of α is difficult, and if accomplished would probably involve an unwieldy expression involving the distribution of the n_i's. Instead, in order to preserve the simplicity of this approximate solution procedure we will present a heuristic argument to show that α should be $\frac{1}{2}$. We will therefore assume that t_{oi} and t_{ri} are both uniformly distributed with expectations of t_o and t_r, respectively, and let N_I represent the greatest integer in N.

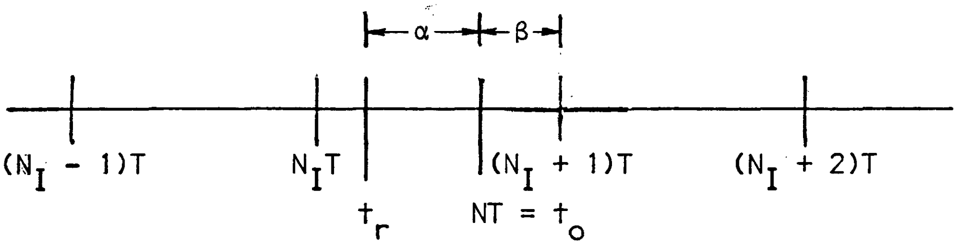

Fig. 3

We will examine the two cases; first, where t_{ri} and t_{oi} will both occur within $\frac{1}{2}T$ units of their respective expectations; second, where t_{ri} and t_{oi} will both occur within 1.5T units of their respective expectations. We could continue to look at cases where t_{ri} and t_{oi} occur within further expanded limits about their respective means; however, since we have assumed the variance of N to be small, we will not do so.

Case I: Suppose we first assume that t_{ri} and t_{oi} will both occur within $\frac{1}{2}T$ units of their respective expectations, then an order will be placed for each cycle in the $N_I{}^{th}$, $(N_I + 1)^{st}$, or $(N_I + 2)^{nd}$ period of the cycle. Recalling that N_I is the greatest integer in N, and defining β as the difference between the greatest integer in N plus 1 and N, we can find N in terms of N_I, α, and β. Due to the restrictions imposed on t_{ri} and t_{oi}, we see $0 \leq \alpha + \beta < 1.5$. If $1.5 > \alpha + \beta > .5$, then the probability that t_{ri} will occur in the $(N_I + 1)^{st}$ period is $1.5 - (\alpha + \beta)$ and the probability that t_{ri} will occur in the $N_I{}^{th}$ period is $\alpha + \beta - .5$ (see Fig. 3). If $\alpha + \beta \leq .5$, then the probability that t_{ri} will occur in the $(N_I + 1)^{st}$ period is $.5 + \alpha + \beta$ and the probability that t_{ri} will occur in the $(N_I + 2)^{nd}$ period is $.5 - \alpha - \beta$. In each situation the probability of t_{ri} occuring in any interval not mentioned is zero. Thus, for N we have (conditioning on t_{ri})

$$N = \begin{cases} (N_I + 1)\cdot(.5 + 1 - (\alpha + \beta)) + N_I(\alpha + \beta - .5) \\ \hfill 1.5 > \alpha + \beta > .5 \\ (N_I + 1)\cdot(.5 + \alpha + \beta) + (N_I + 2)\cdot(.5 - \alpha - \beta) \\ \hfill \alpha + \beta \leq .5 \quad . \end{cases}$$

In either case we obtain

$$N = N_I + 1.5 - \alpha - \beta$$

but since β is defined by

$$\beta = 1 + N_I - N$$

we have $\alpha = \frac{1}{2}$.

Case II: Now let us assume that t_{oi} and t_{ri} will both occur within 1.5T units of their respective expectations. Then an order will be placed in the $N_I{}^{th}$, $(N_I + 1)^{st}$, $(N_I + 2)^{nd}$, $(N_I + 3)^{rd}$, $(N_I - 1)^{st}$, or $(N_I - 2)^{nd}$ periods of the cycle. Similarly, we can find N in terms of N_I, α, and β as follows:

$$3N = \begin{cases} (\alpha + \beta - 1.5)\cdot(N_I - 2) + (N_I - 1) + N_I + \\ \qquad (1 - (\alpha + \beta - 1.5))\cdot(N_I + 1) & 2.5 > \alpha + \beta \geq 1.5 \\ (\alpha + \beta - .5)\cdot(N_I - 1) + (N_I + 1) + \\ \qquad (1.5 - \alpha - \beta)\cdot(N_I + 2) & 1.5 > \alpha + \beta \geq .5 \\ (.5 + \alpha + \beta)N_I + (N_I + 1) + (N_I + 2) + \\ \qquad (.5 - \alpha - \beta)\cdot(N_I + 3) & .5 > \alpha + \beta \geq 0 \end{cases}$$

Again using the fact that

$$\beta = 1 + N_I - N$$

we see that in all three situations

$$3N = 3N_I - 3\alpha - 3\beta + 4.5$$

or again,

$$N = N_I - \alpha - \beta + 1.5$$

Thus we see that $\alpha = .5$ is again the only solution. This argument can be extended by choosing even larger ranges of values for t_{oi} and t_{ri}; however, as was stated before, since variance of N was assumed to be small, continuation of the argument is not necessary. Hence we obtain the solution for $\tilde{r}$ the expected value of r (from equation (6) on p. 44)

$$\tilde{r} = \tilde{S} + \alpha T\lambda$$

or

$$\tilde{r} = \tilde{S} + .5T\lambda$$

Finally, since r must be an integer, we choose the closest integer to $\tilde{r}$, ($\tilde{r}_I$) as our final approximation of r.

Examples: Three examples have been chosen to illustrate the iterative proceedure:

Example 1: Suppose we have an item in an inventory system which costs \$ 1.00 (i.e., $C = 1$) and the inventory level is checked every four days (i.e., $T = .01$). It takes 12 days to recieve an order of this item on the average (i.e., $\ell = .03$). Transportation and administrative costs are \$ 60.00 for each order placed (i.e., $A = 60$). This item is demanded at an average rate of 900 per year (i.e., $\lambda = 900$) and studies indicate that the backorder penalty cost including lost profit is \$ 1.00 for each unit demanded when the system is out-of-stock (i.e., $\pi = 1$). If we assume the alternative interest rate to be 10 % per annum the optimum values of N and S are calculated as follows:

$$\tilde{N}_1 = \sqrt{\frac{2A}{T^2 IC\lambda}} = 115.47$$

$$\tilde{S}_1 = \inf \left\{ S: \sum_{k=1}^{S-1} p\{k;\ \lambda\ell\} \geq (\pi - \tilde{N}_1 TIC)/\pi \right\} = 33$$

$$\tilde{B}_1 = \pi \sum_{j=\tilde{S}_1+1}^{\infty} j \cdot p\{j;\ \lambda\ell\} = .35$$

$$\tilde{N}_2 = 115.81$$

$$\tilde{S}_2 = \tilde{S}_1 = 33$$

and

$$\tilde{B}_2 = \tilde{B}_1 = .35$$

Using the cost equation of page 37 and calculating the rate of cost per annum, we find N = 115.81 and S = 33 to be optimal values.

Example 2: Suppose an inventory system stocks an item which costs \$ 100.00 (i.e., $C = 100$) per unit. The item is fairly small so that transportation and administrative costs associated with a procuremant are fixed at \$ 900.00 (i.e., $A = 900$). An inventory level check is made every 36 days (i.e., $T = .1$), and orders are placed whenever the inventory level is sufficiently small. It takes an average of 73 days to receive an order (i.e., $\ell = .2$) and the lost profit and goodwill are estimated to be \$ 28.00 for each item backordered (i.e., $\pi = 28$). The item is only demanded at an average rate of one per week (so $\lambda = 52$). If we assume an alternative interest rate of 8 % (i.e., $I = .08$), the optimal values of N and S are calculated as follows:

$$\tilde{N}_1 = 21.21$$

$\tilde{S}_1 = 9$

$\tilde{B}_1 = 50.20$

$\tilde{N}_1 = 21.80$

$\tilde{S}_2 = \tilde{S}_1 = 9$

$\tilde{B}_2 = \tilde{B}_1 = 50.20$

Again using the cost equation on page 37, we find the optimal values of N and S to agree with the approximate iterative solution method values.

Example 3: Suppose in the previous example ($C = 100$, $\pi = 28$) we change the time between reviews to 15 days (i.e., $T = .041$) and increase the expected lead time to 146 days (i.e., $\ell = .4$), then if procurement cost is $ 500.00 (i.e., $A = 500$) and the alternative interest rate is 10 % (i.e., $I = .1$), the approximate optimal values of N and S are found to be

$\tilde{N} = 37.07$

$\tilde{S} = 20$

However, using the cost equation on page 37 and many different values for N and S, we find optimal values for N and S are

$N = 38$

and

$S = 20$

Although the difference in N is fairly significant, the difference in the values of the annual cost was less than 3 ¢ out of a total cost rate greater than $ 900.00.

CONCLUSIONS

The calculation of optimum control doctrine for inventory systems can be difficult. In all cases tested, the iterative approximation procedure as described herein appears to converge to optimum or nearly optimum values for N (the expected number of periods per cycle) and S (the expected inventory level at time of order placement) from which a control doctrine can be devised. It is, however, difficult to describe the exact conditions under which convergence to optimal values will occur except to delineate the restrictions imposed by the assumptions (discussed below) made in the determination of the iterative procedure.

We have assumed that no stockouts occur prior to the order placement time. Although this assumption may not be valid, if the penalty cost associated with a backorder is sufficiently large, the probability that the on hand inventory level drops to zero (thus causing backorders) before an order is placed, will be small. Hence, under these conditions, the approximate solution procedure will apparently converge to values which are sufficiently close to optimum as to result in a negligible difference in expected total cost rate.

The variance of n_i (the number of periods in the i^{th} cycle) should be small in comparison to its expected value, if the value of the expectation is sufficiently large. It is difficult to determine how large is sufficiently large, but for approximate values of N, $E[n_i]$, smaller than 20 direct calculation shows that the iterative procedure results in fairly large discrepancies in the expected total cost rate.

Our final assumption of a small variance in lead time should be valid in many cases since transportation of goods is often done on carriers which follow fairly regular schedules. A determination of the error caused by increased variance in lead time would be difficult and would probably depend to a great extent on the particular distribution of the lead time. It would seem though that larger expected values of lead time would yield approximations less sensitive to differing values of the variance.

We have made many other assumptions in the development of the approximate iterative procedure. However, these are standard assumptions (e.g., no variation in selling price over the period of interest, constant demand rate, etc.) and a discussion of their resultant restrictions on the types of inventory systems for which validity exists can be found in the literature (e.g., Hadley, 1963). FORTRAN coding for the procedure was quite easily developed for the examples; however, a manual solution is not difficult to perform using a standard Poisson distribution table for backorder costs.

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