BURIED PENNY-SHAPED CRACKS

by

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ABSTRACT

Penny-shaped cracks are commonly used mathematical models, generally used in the field of fracture mechanics. One specific application is the modeling of microstructures, within elastic materials. From a purely mathematical perspective, a penny-shaped crack can be described as a flat, disk-shaped crack. In this work, we consider the buried penny-shaped crack problem, consisting of a single crack, buried below the surface of a half-space. Specifically, the flat surface of the crack is taken to be parallel to the boundary, and the radius of the crack is held constant. The primary point of interest in this problem is the depth dependence of the stress intensity factor, which characterizes the fracture conditions near the tip of the crack. Determining the stress intensity factor for this problem is reduced to solving a pair of dual integral equations, specifically looking at these equations evaluated at the upper bound of integration. These equations were amenable to numerical solution, where the distance between the crack and the boundary was allowed to become small. The values of these equations, at the upper bound of integration, both tend toward 0. Based on the numerical results, the stress intensity factors for this problem were dependent on the depth at which the penny-shaped crack is buried.

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CHAPTER 1

INTRODUCTION

In fracture mechanics, *penny-shaped cracks* are often used to model the microstructure of elastic materials. In fact, such models are suitable for capturing many of the essential physical features of fluid saturated rocks. Mathematically speaking, a penny-shaped crack is simply a flat, disk-shaped crack.

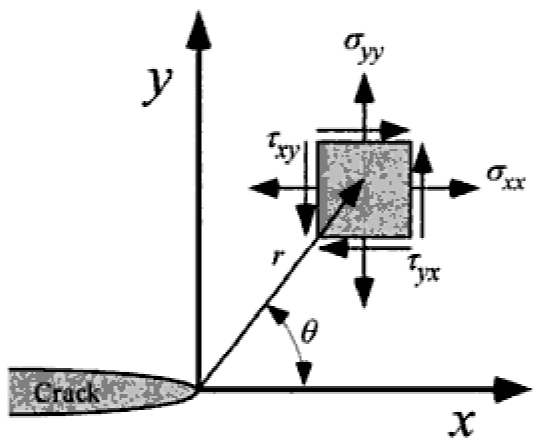
When considering such cracks, it is convenient to use a cylindrical polar coordinate system, defined by (r, θ, z), with the z-axis perpendicular to the disk. It is also convenient to define the coordinate system so that the origin coincides with the center of the disk. In general, we assume that the crack has a constant radius, a. We begin examining this type of crack in Section 3.2.

The simplest problem to consider involves a single crack within an infinite elastic solid, where the crack is being inflated, or opened, by an axisymmetric pressure, denoted $p(r)$. As the pressure is axisymmetric, there is no θ dependence. Typically, solving this type of problem leads to integral equations, which can be solved explicitly.

A more complicated problem involves placing the crack within an elastic half-space, where the crack is buried at a distance h below, but parallel to, the surface of the half-space. This is the *buried penny-shaped crack* problem. Similar to the infinite elastic solid case, solving problems of this type leads to a pair of coupled integral equations.

There are two main physical aspects that are of particular interest. The first point of interest is to consider what happens when we let $h \to 0$. This corresponds to placing the crack near the boundary of the half-space. The second is a theoretical construct known as the *stress intensity factor*.

To define the stress intensity factor, let us consider an element near the tip of a crack, within an elastic material, depicted in (Anderson, 2005, Figure 1.9) and Figure 1.1 below. Looking at the in-plane stresses on the element, each stress component is proportional to a single constant, say K_I. Typically, the subscript is used to denote the mode of loading; Mode I refers to an opening stress, Mode II refers to an in-plane shear, and Mode III refers to an out-of-plane shear. As an example, for a crack with a radius of a in an infinite plate, subjected to a remote tensile stress, σ, the stress intensity factor is $K_I = \sigma\sqrt{\pi a}$.



$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)$$

Figure 1.1: Stresses near the tip of a crack in an elastic material

This single constant completely characterizes the crack-tip conditions for a linear elastic material. Assuming that a material fails locally at some combination of stress and strain, then that fracture must occure at a specific stress intensity K_k, allowing for a measure of fracture toughness. Failure then occurs when the stress intensity factor reaches this specific stress intensity, when $K_I = K_k$. Thus, K_k could be considered a measure of the material resistance, while K_I, the stress intensity factor, is the driving force for fracture. Mathematically speaking, the stress intensity factor corresponds to the solution of the integral equations near the endpoints of integration.

CHAPTER 2

BASIC PROBLEMS FOR LAPLACE'S EQUATION

We begin by considering a simple problem, and build up to more complicated problems. The purpose of this chapter is to become familiar with the mathematical tools and procedures necessary to examine the crack problems to come.

2.1 A Simple Boundary-Value Problem

Consider solving the three dimensional Laplace's equation, $\nabla^2 u = 0$, in the half-space $z > 0$ with boundary condition

$$\frac{\partial u}{\partial z} = \begin{cases} 1, & x^2 + y^2 \leq a^2, \\ 0, & x^2 + y^2 > a^2, \end{cases}$$

on $z = 0$, and with the requirement that $u \to 0$ as $z \to \infty$.

Define a two-dimensional Fourier transform by

$$\mathcal{F}\{u\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y, z) e^{i(\xi x + \eta y)}\, dx\, dy = U(\xi, \eta, z). \tag{2.1}$$

Therefore, the corresponding inverse transform is

$$\mathcal{F}^{-1}\{U\} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(\xi, \eta, z) e^{-i(\xi x + \eta y)}\, d\xi\, d\eta = u(x, y, z). \tag{2.2}$$

Then,

$$\begin{aligned} \mathcal{F}\left\{\frac{\partial u}{\partial x}\right\} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i(\xi x + \eta y)}\, dx\, dy \\ &= \int_{-\infty}^{\infty} \left[u e^{i(\xi x + \eta y)}\right]_{-\infty}^{\infty} dy - i\xi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u e^{i(\xi x + \eta y)}\, dx\, dy \\ &= 0 - i\xi U \qquad \text{(since } u \to 0 \text{ at } \infty) \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{i(\xi x + \eta y)}\, dx\, dy \\ &= \int_{-\infty}^{\infty} \left[\frac{\partial u}{\partial x} e^{i(\xi x + \eta y)}\right]_{-\infty}^{\infty} dy - i\xi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i(\xi x + \eta y)}\, dx\, dy \\ &= 0 - i\xi \mathcal{F}\left\{\frac{\partial u}{\partial x}\right\} = -\xi^2 U. \qquad \text{(assuming } \tfrac{\partial u}{\partial x} \to 0 \text{ at } \infty) \end{aligned}$$

Similarly,

$$\mathcal{F}\left\{\frac{\partial u}{\partial y}\right\} = -i\eta U \quad \text{and} \quad \mathcal{F}\left\{\frac{\partial^2 u}{\partial y^2}\right\} = -\eta^2 U.$$

Next,

$$\begin{aligned}\mathcal{F}\left\{\frac{\partial u}{\partial z}\right\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial u}{\partial z} e^{i(\xi x+\eta y)}\,dx\,dy \\ &= \frac{\partial}{\partial z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u e^{i(\xi x+\eta y)}\,dx\,dy = \frac{\partial U}{\partial z}\end{aligned}$$

and

$$\mathcal{F}\left\{\frac{\partial^2 u}{\partial z^2}\right\} = \frac{\partial^2 U}{\partial z^2}.$$

Therefore, applying $\mathcal{F}$ to Laplace's equation,

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

gives

$$-\xi^2 U - \eta^2 U + \frac{\partial^2 U}{\partial z^2} = 0.$$

This is an ODE for U regarded as a function of z. Using primes to indicate z-derivatives, we now have

$$U'' = (\xi^2 + \eta^2)U.$$

The general solution of this equation is

$$U(\xi,\eta,z) = Ae^{\kappa z} + Be^{-\kappa z} \quad \text{with} \quad \kappa = \sqrt{\xi^2+\eta^2}. \tag{2.3}$$

The quantities A and B can depend on ξ and η but not on z.

We want to solve $\nabla^2 u = 0$ in the half-space $z > 0$. We require $u \to 0$ as $z \to \infty$, so $U \to 0$ as $z \to \infty$. This condition implies that

$$A = 0 \quad \text{and so} \quad U(\xi,\eta,z) = Be^{-\kappa z}. \tag{2.4}$$

To find B, we use the boundary condition on $z = 0$. We take this as

$$\frac{\partial u}{\partial z} = f(x,y) \quad \text{on} \quad z = 0, \quad \text{where} \quad f(x,y) = \begin{cases} 1, & x^2+y^2 \le a^2, \\ 0, & x^2+y^2 > a^2. \end{cases} \tag{2.5}$$

Applying $\mathcal{F}$ gives

$$\left.\frac{\partial U}{\partial z}\right|_{z=0} = \mathcal{F}\left\{f(x,y)\right\}.$$

Substituting for U from (2.4) gives

$$\left.\frac{\partial}{\partial z}(Be^{-\kappa z})\right|_{z=0} = -\kappa B = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{i(\xi x+\eta y)}\,dx\,dy. \tag{2.6}$$

On the right-hand side, we only need to integrate over a disk because of the definition of f, see (2.5). We use polar coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \xi = \kappa\cos\beta, \quad \eta = \kappa\sin\beta.$$

We have $dx\,dy = r\,dr\,d\theta$, $f = 1$ for $0 \le r \le a$ and

$$\xi x + \eta y = \kappa r\cos\theta\cos\beta + \kappa r\sin\theta\sin\beta = \kappa r\cos(\theta - \beta).$$

Thus, (2.6) becomes

$$-\kappa B = \int_0^a r\int_0^{2\pi} e^{i\kappa r\cos(\theta-\beta)}\,d\theta\,dr. \tag{2.7}$$

Consider the inner integral with respect to θ. The integrand is 2π-periodic, so we can set $\beta = 0$. The integral can be evaluated using Bessel functions, J_n. Starting from the generating function (Abramowitz & Stegun, 1972, Equation 9.1.41)

$$e^{(w/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(w),$$

put $t = ie^{i\theta}$ to obtain the Fourier series

$$\begin{aligned} e^{iw\cos\theta} &= \sum_{n=-\infty}^{\infty} i^n J_n(w)e^{in\theta} \qquad (2.8)\\ &= J_0(w) + 2\sum_{n=1}^{\infty} i^n J_n(w)\cos n\theta. \end{aligned}$$

Hence

$$i^n J_n(w) = \frac{1}{2\pi}\int_0^{2\pi} e^{iw\cos\theta}e^{-in\theta}\,d\theta. \tag{2.9}$$

Setting $n = 0$, (2.9) becomes

$$J_0(w) = \frac{1}{2\pi}\int_0^{2\pi} e^{iw\cos\theta}\,d\theta. \tag{2.10}$$

To evaluate B, we use (2.10) in (2.7):

$$
\begin{aligned}
-\kappa B &= 2\pi \int_0^a r J_0(\kappa r)\, dr \\
&= \frac{2\pi}{\kappa^2} \int_0^{\kappa a} x J_0(x)\, dx = \frac{2\pi}{\kappa^2} \left\{ \kappa a J_1(\kappa a) \right\},
\end{aligned}
$$

using the third line of (Duffy, 2008, Table 1.4.1),

$$
\frac{d}{dz} \left[z^n J_n(z) \right] = z^n J_{n-1}(z),
$$

with $n = 1$. Hence, $B = -2\pi a \kappa^{-2} J_1(\kappa a)$ and U is given by (2.4).

Inverting U gives

$$
\begin{aligned}
u(x, y, z) &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{\infty} -\frac{2\pi a}{\kappa^2} J_1(\kappa a) e^{-\kappa z} e^{-i\kappa r \cos(\theta - \beta)} \kappa\, d\kappa\, d\beta \\
&= -\frac{a}{2\pi} \int_0^{\infty} \frac{1}{\kappa} J_1(\kappa a) e^{-\kappa z} \int_0^{2\pi} e^{-i\kappa r \cos(\beta - \theta)}\, d\beta\, d\kappa
\end{aligned}
$$

Consider the inner integral with respect to β. The integrand is 2π-periodic, so we can set $\theta = 0$. As in the evaluation of B, the integral can be evaluated using the complex conjugate of (2.10), which gives

$$
\begin{aligned}
u(x, y, z) &= -\frac{a}{2\pi} \int_0^{\infty} \frac{1}{\kappa} J_1(\kappa a) e^{-\kappa z} \left(2\pi J_0(\kappa r) \right)\, d\kappa \\
&= -a \int_0^{\infty} \frac{1}{\kappa} J_0(\kappa r) J_1(\kappa a) e^{-\kappa z}\, d\kappa.
\end{aligned}
$$

2.2 A Mixed Boundary-Value Problem

Moving on to a more complicated problem, consider solving the three dimensional Laplace's equation, $\nabla^2 u = 0$, in the half-space $z > 0$ with mixed boundary conditions

$$
\frac{\partial u}{\partial z} = 1 \quad \text{for} \quad x^2 + y^2 \leq a^2, \tag{2.11}
$$

and

$$
u = 0 \quad \text{for} \quad x^2 + y^2 > a^2 \tag{2.12}
$$

on $z = 0$, together with the requirement that $u \to 0$ as $z \to \infty$.

We start exactly as in Section 2.1, with two-dimensional Fourier transforms, arriving at (2.4):

$$U(\xi, \eta, z) = \mathcal{F}\{u\} = Be^{-\kappa z}. \tag{2.13}$$

We need to apply the boundary conditions on $z = 0$ to find B. We start by inverting the transform,

$$u(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B(\xi, \eta) e^{-\kappa z} e^{-i(\xi x + \eta y)} \, d\xi \, d\eta.$$

We use polar coordinates, just as before, which gives

$$u(r, \theta, z) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} B(\kappa, \beta) e^{-\kappa z} e^{-i\kappa r \cos(\theta - \beta)} \kappa \, d\kappa \, d\beta. \tag{2.14}$$

Therefore, the first boundary condition (2.11), valid for $r \leq a$, gives

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} B(\kappa, \beta)(-\kappa) e^{-\kappa(0)} e^{-i\kappa r \cos(\theta - \beta)} \kappa \, d\kappa \, d\beta = 1,$$

since B does not depend on z.

Similarly, the second boundary condition (2.12), valid for $r > a$, gives

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} B(\kappa, \beta) e^{-\kappa(0)} e^{-i\kappa r \cos(\theta - \beta)} \kappa \, d\kappa \, d\beta = 0.$$

So, we have the "dual integral equations",

$$\int_0^{2\pi} \int_0^{\infty} B(\kappa, \beta) \kappa^2 e^{-i\kappa r \cos(\theta - \beta)} \, d\kappa \, d\beta = -4\pi^2 \qquad \text{on } r \leq a, \tag{2.15}$$

$$\int_0^{2\pi} \int_0^{\infty} B(\kappa, \beta) \kappa e^{-i\kappa r \cos(\theta - \beta)} \, d\kappa \, d\beta = 0 \qquad \text{on } r > a. \tag{2.16}$$

These hold for $0 \leq \theta < 2\pi$.

The right-hand sides of these equations do not depend on θ, so we should expand the left-hand sides as Fourier series in θ and then retain only the term that is independent of θ. Taking the complex conjugate of (2.8) gives

$$e^{-iw\cos(\theta - \beta)} = \sum_{n=-\infty}^{\infty} (-i)^n J_n(w) e^{-in(\theta - \beta)}. \tag{2.17}$$

Substituting into (2.15) and (2.16) gives

$$\sum_{n=-\infty}^{\infty} (-i)^n e^{-in\theta} \int_0^\infty B_n(\kappa)\kappa^2 J_n(\kappa r)\, d\kappa = -4\pi^2 \qquad \text{on } r \le a, \tag{2.18}$$

$$\sum_{n=-\infty}^{\infty} (-i)^n e^{-in\theta} \int_0^\infty B_n(\kappa)\kappa J_n(\kappa r)\, d\kappa = 0 \qquad \text{on } r > a, \tag{2.19}$$

for $0 \le \theta < 2\pi$, where

$$B_n(\kappa) = \int_0^{2\pi} B(\kappa, \beta) e^{in\beta}\, d\beta.$$

As the right-hand sides of (2.18) and (2.19) do not depend on θ, we deduce that $B_n = 0$ for $n \neq 0$. Instead of using B_0, we simplify notation a little by defining

$$A(\kappa) = \frac{\kappa B_0}{4\pi^2} = \frac{\kappa}{4\pi^2} \int_0^{2\pi} B(\kappa, \beta)\, d\beta, \tag{2.20}$$

and then (2.18) and (2.19) give

$$\int_0^\infty \kappa A(\kappa) J_0(\kappa r)\, d\kappa = -1 \qquad \text{on } r \le a, \tag{2.21}$$

$$\int_0^\infty A(\kappa) J_0(\kappa r)\, d\kappa = 0 \qquad \text{on } r > a. \tag{2.22}$$

Going back to the formula for u and using (2.20) with $B_n = 0$ for $n \neq 0$, (2.14) simplifies to

$$u(r, z) = \int_0^\infty A(\kappa) J_0(\kappa r) e^{-\kappa z}\, d\kappa, \tag{2.23}$$

which depends only on r and z, not on θ.

Let us solve (2.21) and (2.22) for $A(\kappa)$. We begin by writing

$$A(\kappa) = \int_0^a g(t) \sin(\kappa t)\, dt, \tag{2.24}$$

where g is to be found. Then,

$$\int_0^\infty A(\kappa) J_0(\kappa r)\, d\kappa = \int_0^a g(t) \left[\int_0^\infty \sin(\kappa t) J_0(\kappa r)\, d\kappa \right] dt. \tag{2.25}$$

From (Duffy, 2008, Equation 1.4.13), we have

$$\int_0^\infty \sin(\kappa t) J_0(\kappa r)\, d\kappa = \begin{cases} 0, & t < r, \\ (t^2 - r^2)^{-1/2}, & t > r. \end{cases} \tag{2.26}$$

Then, we see that the inner integral in (2.25) vanishes for $r > a$ as $0 \le t \le a$. Therefore the expression of $A(\kappa)$ as (2.24) ensures that (2.22) is satisfied exactly, for any choice of g.

Next, we integrate by parts in (2.24), giving

$$A(\kappa) = \int_0^a g(t)\sin(\kappa t)\,dt = -g(a)\frac{\cos(\kappa a)}{\kappa} + \frac{1}{\kappa}\int_0^a g'(t)\cos(\kappa t)\,dt, \tag{2.27}$$

where we assume $g(0) = 0$ (see Appendix A).

Substituting into (2.21) gives

$$-g(a)\int_0^\infty \cos(\kappa a)J_0(\kappa r)\,d\kappa + \int_0^a g'(t)\left[\int_0^\infty \cos(\kappa t)J_0(\kappa r)\,d\kappa\right]dt = -1. \tag{2.28}$$

From (Duffy, 2008, Equation 1.4.14), we have

$$\int_0^\infty \cos(\kappa t)J_0(\kappa r)\,d\kappa = \begin{cases} (r^2 - t^2)^{-1/2}, & t < r, \\ 0, & t > r. \end{cases} \tag{2.29}$$

Then, we see that the first integral in (2.28) vanishes, and we are left only with the portion of the second integral corresponding to $t < r$,

$$\int_0^r \frac{g'(t)}{\sqrt{r^2 - t^2}}\,dt = -1,$$

which is an Abel-type integral equation.

Now, from (Duffy, 2008, Equation 1.2.13) and (Duffy, 2008, Equation 1.2.14), we have

$$g'(t) = \frac{2}{\pi}\frac{d}{dt}\left[\int_0^t \frac{-r}{\sqrt{t^2 - r^2}}\,dr\right] = \frac{2}{\pi}\frac{d}{dt}\left[-t\right].$$

Hence, $g(t) = -2t/\pi$, where the constant of integration is zero, since $g(0) = 0$, and then $A(\kappa)$ is given by (2.27).

Using (2.24) in (2.23) gives

$$\begin{aligned} u(r,z) &= \int_0^\infty \left[\int_0^a -\frac{2}{\pi}t\sin(\kappa t)\,dt\right] J_0(\kappa r)e^{-\kappa z}\,d\kappa \\ &= \frac{2}{\pi}\int_0^\infty \left[\frac{a}{\kappa}\cos(\kappa a) - \frac{1}{\kappa^2}\sin(\kappa a)\right] J_0(\kappa r)e^{-\kappa z}\,d\kappa. \end{aligned}$$

From (Abramowitz & Stegun, 1972, Equation 10.1.11), we have

$$j_1(\kappa a) = \frac{\sin(\kappa a)}{(\kappa a)^2} - \frac{\cos(\kappa a)}{\kappa a},$$

a spherical Bessel Function of the first kind.

Therefore,

$$u(r,z) = \frac{-2a^2}{\pi} \int_0^\infty j_1(\kappa a) J_0(\kappa r) e^{-\kappa z}\, d\kappa.$$

Finally, using (Abramowitz & Stegun, 1972, Equation 10.1.1),

$$j_1(\kappa a) = \sqrt{\frac{\pi}{2\kappa a}} J_{\frac{3}{2}}(\kappa a),$$

and we have

$$u(r,z) = -\sqrt{\frac{2a^3}{\pi}} \int_0^\infty \frac{1}{\sqrt{\kappa}} J_0(\kappa r) J_{\frac{3}{2}}(\kappa a) e^{-\kappa z}\, d\kappa.$$

Although this integral can be evaluated, we will not pursue this.

2.3 A Disk in a Half-Space

Consider solving the three dimensional Laplace's equation, $\nabla^2 u = 0$, in the half-space $z > 0$ with boundary condition

$$\frac{\partial u}{\partial z} = 0 \tag{2.30}$$

on $z = 0$, and with the requirement that $u \to 0$ as $z \to \infty$. Additionally, using polar coordinates,

$$\frac{\partial u}{\partial z} = 1 \quad \text{for} \quad r \leq a \tag{2.31}$$

on $z = h$. This problem is depicted in Figure 2.1, below.

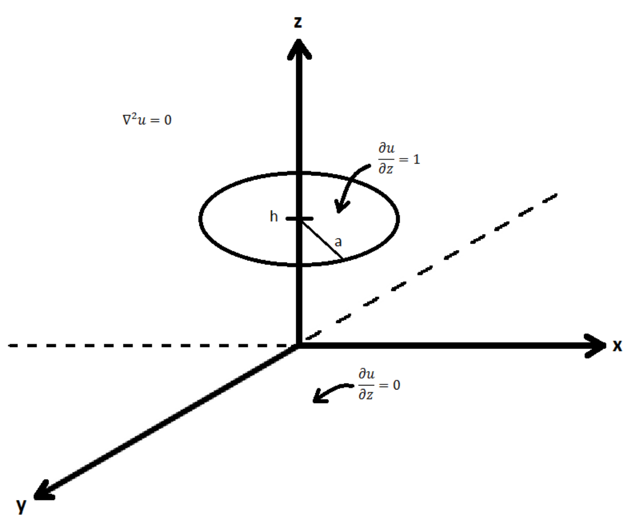


Figure 2.1: Disk in a Half-Space

Based on our results from Section 2.1 and Section 2.2, we take

$$u(r,z;\kappa) = J_0(\kappa r)\left\{Ae^{\kappa z} + Be^{-\kappa z}\right\}, \tag{2.32}$$

which solves $\nabla^2 u = 0$ for any κ. Later, we will integrate over κ, giving a superposition.

There are now two regions of the half-space to consider; Region 1, corresponding to $z > h$, and Region 2, corresponding to $0 < z < h$. We use $A = A_j$ and $B = B_j$ in Region j.

In Region 1, we require $u \to 0$ as $z \to \infty$. This condition implies that

$$A_1 = 0 \quad \text{and so} \quad u(r,z;\kappa) = B_1 J_0(\kappa r)e^{-\kappa z}, \tag{2.33}$$

for $z > h$, assuming that $\kappa > 0$.

In Region 2, we need to apply boundary condition (2.30) on $z = 0$, which gives

$$J_0(\kappa r)\left\{\kappa A_2 - \kappa B_2\right\} = 0,$$

on $z = 0$. Therefore,

$$A_2 = B_2 \quad \text{and so} \quad u(r,z;\kappa) = C_2 J_0(\kappa r)\cosh(\kappa z), \tag{2.34}$$

for $0 < z < h$.

Next, we require $\frac{\partial u}{\partial z}$ to be continuous across $z = h$, which gives

$$-\kappa B_1 J_0(\kappa r)e^{-\kappa h} = \kappa C_2 J_0(\kappa r)\sinh(\kappa h).$$

Therefore, we have a relation between B_1 and C_2,

$$-B_1 e^{-\kappa h} = C_2 \sinh(\kappa h). \tag{2.35}$$

Now, allowing B_1 and C_2 to depend on κ, and applying superposition, we have

$$u_1(r,z) = \int_0^\infty B(\kappa)J_0(\kappa r)e^{-\kappa z}\,d\kappa \qquad \text{for } z > h, \tag{2.36}$$

$$u_2(r,z) = \int_0^\infty C(\kappa)J_0(\kappa r)\cosh(\kappa z)\,d\kappa \qquad \text{for } 0 < z < h, \tag{2.37}$$

with relation (2.35), where $B = B_1$ and $C = C_2$.

For $r > a$, we require $u_1 = u_2$ on $z = h$. This gives

$$\int_0^\infty B(\kappa)J_0(\kappa r)e^{-\kappa h}\,d\kappa = \int_0^\infty C(\kappa)J_0(\kappa r)\cosh(\kappa h)\,d\kappa, \quad r > a.$$

For $0 < r < a$, we require $\frac{\partial u_1}{\partial z} = 1$ on $z = h$. This gives

$$\int_0^\infty -\kappa B(\kappa) J_0(\kappa r) e^{-\kappa h}\, d\kappa = 1, \quad 0 < r < a.$$

So, we have

$$\int_0^\infty \kappa B(\kappa) e^{-\kappa h} J_0(\kappa r)\, d\kappa = -1 \quad \text{on } 0 < r < a, \tag{2.38}$$

and

$$\int_0^\infty \left[B(\kappa)e^{-\kappa h} - C(\kappa)\cosh(\kappa h)\right] J_0(\kappa r)\, d\kappa = 0 \quad \text{on } r > a, \tag{2.39}$$

with $-B(\kappa)e^{-\kappa h} = C(\kappa)\sinh(\kappa h)$.

As $h \to \infty$, we might expect to get back to the problem in Section 2.2, so we try to write these two equations as dual integral equations, similar to (2.21) and (2.22).

So, substituting the relation between B and C into the square brackets in (2.39) gives

$$B(\kappa)e^{-\kappa h} - C(\kappa)\cosh(\kappa h) = B(\kappa)e^{-\kappa h}\left[1 + \coth(\kappa h)\right].$$

Since $1 + \coth(\kappa h) \to 2$ as $\kappa h \to \infty$, we set

$$2D(\kappa) = B(\kappa)e^{-\kappa h}\left[1 + \coth(\kappa h)\right].$$

Rearranging, this gives

$$B(\kappa)e^{-\kappa h} = 2D(\kappa)(1 + \coth(\kappa h))^{-1} = D(\kappa)\left[1 - e^{-2\kappa h}\right].$$

Therefore, (2.38) and (2.39) become

$$\int_0^\infty \kappa D(\kappa)\left[1 - e^{-2\kappa h}\right] J_0(\kappa r)\, d\kappa = -1 \qquad \text{on } 0 < r < a, \tag{2.40}$$

$$\int_0^\infty D(\kappa) J_0(\kappa r)\, d\kappa = 0 \qquad \text{on } r > a. \tag{2.41}$$

In order to solve these dual integral equations, we first split up the integral in (2.40), giving

$$\int_0^\infty \kappa D(\kappa) J_0(\kappa r)\, d\kappa = \int_0^\infty \kappa D(\kappa) e^{-2\kappa h} J_0(\kappa r)\, d\kappa - 1, \quad 0 < r < a.$$

Now, renaming the right hand side, we have

$$\int_0^\infty \kappa D(\kappa) J_0(\kappa r)\, d\kappa = f(r) \qquad \text{on } 0 < r < a, \tag{2.42}$$

$$\int_0^\infty D(\kappa) J_0(\kappa r)\, d\kappa = 0 \qquad \text{on } r > a, \tag{2.43}$$

where

$$f(r) = \int_0^\infty \kappa D(\kappa) e^{-2\kappa h} J_0(\kappa r)\, d\kappa - 1, \quad 0 < r < a. \tag{2.44}$$

We now begin to solve (2.42) and (2.43) for $D(\kappa)$ by writing

$$D(\kappa) = \int_0^a g(t) \sin(\kappa t)\, dt, \tag{2.45}$$

where g is to be found. As in Section 2.2, the expression of $D(\kappa)$ as (2.45) ensures that (2.43) is satisfied exactly, for any choice of g.

After integrating by parts in (2.45), assuming $g(0) = 0$, and substituting into (2.42), we get, much like in Section 2.2,

$$\int_0^r \frac{g'(t)}{\sqrt{r^2 - t^2}}\, dt = f(r),$$

another Abel-type integral equation.

Then, from (Duffy, 2008, Equation 1.2.13) and (Duffy, 2008, Equation 1.2.14), we have

$$g'(t) = \frac{2}{\pi} \frac{d}{dt} \left[\int_0^t \frac{r f(r)}{\sqrt{t^2 - r^2}}\, dr \right].$$

Integrating g gives

$$g(t) = \frac{2}{\pi} \int_0^t \frac{r f(r)}{\sqrt{t^2 - r^2}}\, dr, \tag{2.46}$$

where the constant of integration must be zero since $g(0) = 0$.

Next, we define

$$f_1(r) = \int_0^\infty \kappa D(\kappa) J_0(\kappa r) e^{-2\kappa h}\, d\kappa, \tag{2.47}$$

so that

$$f(r) = f_1(r) - 1.$$

Similarly, we define

$$g_1(t) = \frac{2}{\pi} \int_0^t \frac{r f_1(r)}{\sqrt{t^2 - r^2}}\, dr, \tag{2.48}$$

so that

$$g(t) = g_1(t) - \frac{2}{\pi} \int_0^t \frac{r}{\sqrt{t^2 - r^2}}\, dr = g_1(t) - \frac{2}{\pi} t. \tag{2.49}$$

Substituting (2.47) into (2.48) gives

$$\begin{aligned} g_1(t) &= \frac{2}{\pi} \int_0^t \frac{r}{\sqrt{t^2 - r^2}} \int_0^\infty \kappa D(\kappa) J_0(\kappa r) e^{-2\kappa h}\, d\kappa\, dr \\ &= \frac{2}{\pi} \int_0^\infty \kappa D(\kappa) e^{-2\kappa h} \int_0^t \frac{r J_0(\kappa r)}{\sqrt{t^2 - r^2}}\, dr\, d\kappa. \end{aligned}$$

Since we have already assumed $\kappa > 0$, we can use (Duffy, 2008, Equation 1.4.9),

$$\int_0^t \frac{r J_0(\kappa r)}{\sqrt{t^2 - r^2}}\, dr = \frac{\sin(\kappa t)}{\kappa},$$

to simplify g_1. So, we have

$$g_1(t) = \frac{2}{\pi} \int_0^\infty D(\kappa) \sin(\kappa t) e^{-2\kappa h}\, d\kappa. \tag{2.50}$$

Now, using (2.45), g_1 becomes

$$\begin{aligned} g_1(t) &= \frac{2}{\pi} \int_0^\infty \left[\int_0^a g(s) \sin(\kappa s)\, ds \right] \sin(\kappa t) e^{-2\kappa h}\, d\kappa \\ &= \frac{2}{\pi} \int_0^a g(s) \int_0^\infty \sin(\kappa s) \sin(\kappa t) e^{-2\kappa h}\, d\kappa\, ds \\ &= \frac{2}{\pi} \int_0^a g(s) \int_0^\infty \left[\frac{1}{2} \cos(\kappa(s-t)) - \frac{1}{2} \cos(\kappa(s+t)) \right] e^{-2\kappa h}\, d\kappa\, ds. \end{aligned}$$

Evaluating the Laplace transform gives

$$\begin{aligned} g_1(t) &= \frac{2}{\pi} \int_0^a g(s) \left[\frac{h}{4h^2 + (s-t)^2} - \frac{h}{4h^2 + (s+t)^2} \right] ds \\ &= \int_0^a g(s) \frac{8hst}{\pi(16h^4 + 8h^2(s^2 + t^2) + (s^2 - t^2)^2)}\, ds. \end{aligned} \tag{2.51}$$

Now, substituting into (2.49), we have

$$g(t) = \int_0^a g(s) \frac{8hst}{\pi(16h^4 + 8h^2(s^2 + t^2) + (s^2 - t^2)^2)}\, ds - \frac{2}{\pi} t, \quad 0 < t < a, \tag{2.52}$$

which is a Fredholm integral equation of the second kind for g.

To simplify this integral equation, we notice that the right hand side of (2.52) is an odd function of t. Therefore, we can extend $g(t)$ as an odd function to the interval $-a < t < a$. Also, putting $t = 0$ into (2.52) gives $g(0) = 0$, which is consistent with this extension.

Next, from (2.51), we have

$$g_1(t) = \frac{2h}{\pi}\left\{G_-(t) - G_+(t)\right\},$$

where

$$G_\pm = \int_0^a \frac{g(s)}{4h^2 + (t \pm s)^2}\, dy.$$

Now, using the substitution $s \to -s$ in G_+ gives

$$G_+ = -\int_0^{-a} \frac{g(-s)}{4h^2 + (t-s)^2}\, ds = -\int_{-a}^0 \frac{g(s)}{4h^2 + (t-s)^2}\, ds,$$

since $g(s)$ is an odd function. Therefore,

$$g_1(t) = \frac{2h}{\pi}\left\{\int_0^a \frac{g(s)}{4h^2 + (t-s)^2}\, ds + \int_{-a}^0 \frac{g(s)}{4h^2 + (t-s)^2}\, ds\right\},$$

and we obtain

$$g(t) = \frac{2h}{\pi}\int_{-a}^a \frac{g(s)}{4h^2 + (t-s)^2}\, ds - \frac{2}{\pi}t, \quad -a < t < a.$$

To make this equation more amenable to solving numerically, we apply a change of variables,

$$t = ax, \quad s = ay, \quad 2h = ad.$$

Therefore, we have

$$\phi(x) = \frac{d}{\pi}\int_{-1}^1 \frac{\phi(y)}{d^2 + (x-y)^2}\, dy - \frac{2}{\pi}x, \quad -1 < x < 1, \tag{2.53}$$

where we define $a\phi(x) = g(ax) = g(t)$, so that our new integral equation does not depend on the constant a.

This Fredholm integral equation of the second kind is known as *Love's integral equation*. There has been a great deal of work done on this equation, but the exact solution is not known. It has a continuous kernel, which indicates that it can be solved numerically using a standard Nyström method.

The Nyström method begins with a standard discretization of an integral equation using a quadrature rule,

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i), \tag{2.54}$$

where each w_i is the weight applied to the function evaluated at the quadrature point x_i. Applying this quadrature rule to the Fredholm integral equation of the second kind,

$$f(x) = u(x) - \int_a^b K(x,y)u(y)\,dy,$$

gives

$$f(x) \approx u(x) - \sum_{i=1}^{n} w_i K(x,y_i)u(y_i).$$

To numerically solve an equation of this type, we set up a linear system of n equations, with j fixed for each equation, $j = 1,\ldots,n$. So, each equation is of the form

$$f(x_j) \approx u(x_j) - \sum_{i=1}^{n} w_i K(x_j,y_i)u(y_i) \quad j = 1,\ldots,n.$$

Applying this method to (2.53) gives

$$-\frac{2}{\pi}x_j \approx \phi_j - \frac{d}{\pi}\sum_{i=1}^{n} w_i \frac{\phi_i}{d^2 + (x_j - y_i)^2} \quad j = 1,\ldots,n,$$

where w_i are the weights for the particular rule being applied, and $x_i = y_i$ are the quadrature points. Also, we write $\phi_j = \phi(x_j)$, and for the terms in the summation, we write $\phi_i = \phi(y_i)$ with x_j fixed. We can apply each quadrature rule as we would when integrating a function of only y, with respect to y. Thus, setting

$$K_{ji} = K(x_j,y_i) = \frac{-d}{\pi[d^2 + (x_j - y_i)^2]},$$

we have the system

$$\begin{bmatrix} 1 + w_1K_{11} & w_2K_{12} & \cdots & w_nK_{1n} \\ w_1K_{21} & 1 + w_2K_{22} & \cdots & w_nK_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_1K_{n1} & w_2K_{n2} & \cdots & 1 + w_nK_{nn} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix} = \begin{bmatrix} -\frac{2}{\pi}x_1 \\ -\frac{2}{\pi}x_2 \\ \vdots \\ -\frac{2}{\pi}x_n \end{bmatrix}.$$

We need to choose a specific quadrature rule to determine the weights and quadrature points. Since we are interested in the value of the unknown function at the endpoints, we use the repeated trapezium rule. The quadrature points are distributed evenly along the interval $[a, b]$, so that we have N equally spaced subintervals. Using the notation from (2.54), we let $n = N + 1$, and the weights for this rule are given by

$$w_1 = \frac{(b-a)}{2N} = w_n \quad \text{and} \quad w_i = \frac{(b-a)}{N} \quad \text{for } i = 2, \ldots, N.$$

Therefore, the weights for our numerical approach to *Love's equation* are given by

$$w_1 = \frac{1}{N} = w_n \quad \text{and} \quad w_i = \frac{2}{N} \quad \text{for } i = 2, \ldots, N.$$

We now use MatLab to solve for the vector $(\phi_1, \ldots, \phi_n)$. Depicted in Figure 2.2, we have plots of the solution $\phi(x)$, as ϕ_i vs. x_i, $i = 1, \ldots, n$, for various values of d. We see that as d becomes small, the oscillations in $\phi(x)$ increase considerably in amplitude.

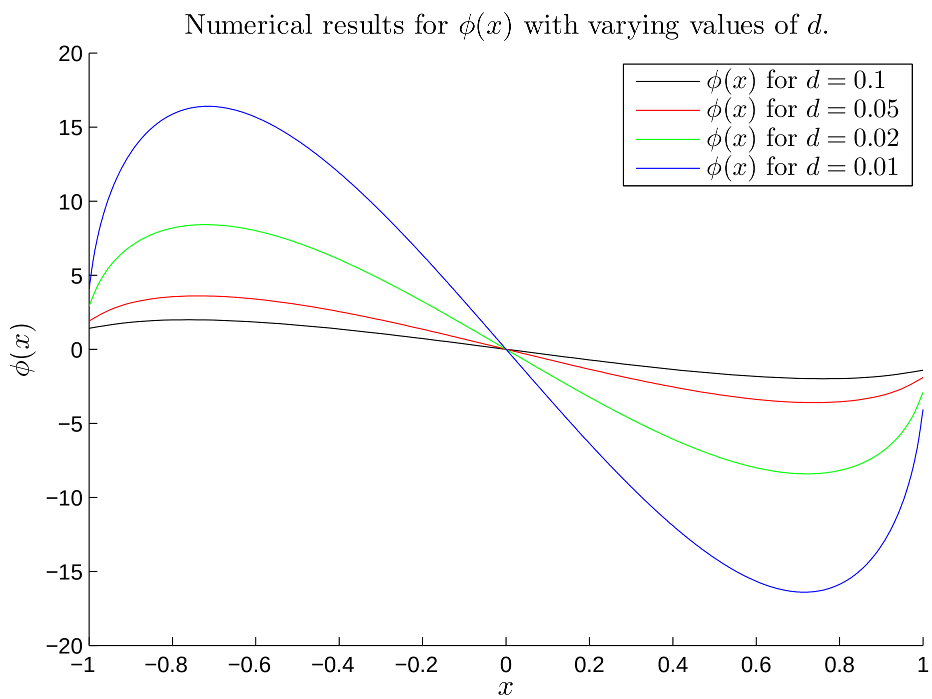


Figure 2.2: Numerical solution to *Love's equation*

Of particular interest is $\phi(1)$, as this relates to the stress intensity factor for the crack problems to come. We now use MatLab in a similar fashion, but examine $\phi(1)$ for varying d, with special interest in small values of d. Depicted in Figure 2.3, we have a plot of $\phi(1)$ as a function of d. Again, when d is small, slight changes in d create large variances in ϕ.

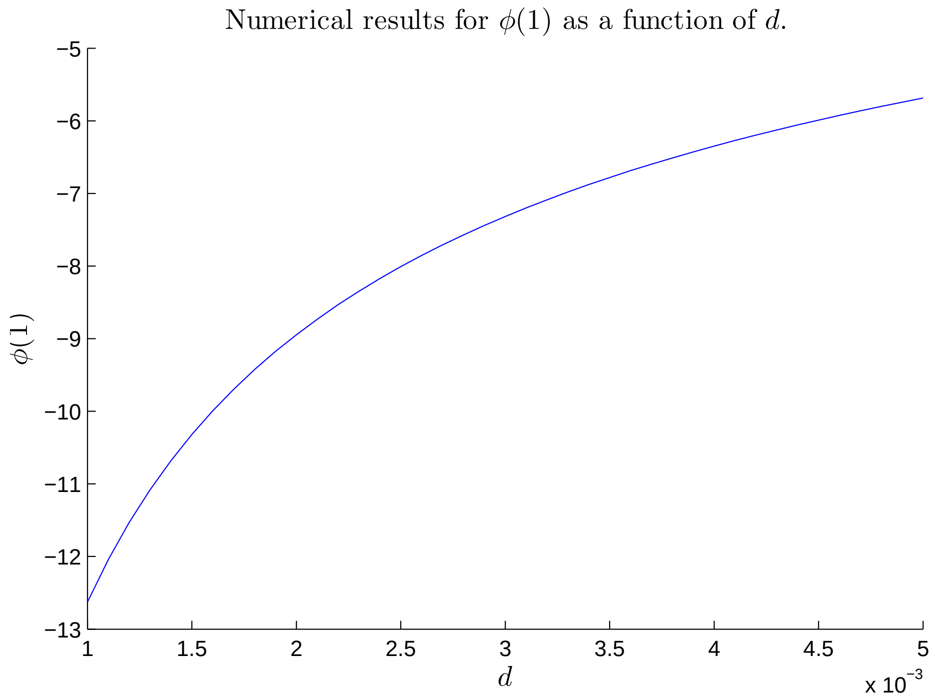


Figure 2.3: $\phi(1)$ for varying d

For the repeated trapezium rule used here, the error can be written as

$$E_N(f) = \int_a^b f(x)\,dx - \frac{(b-a)}{N}\left[\frac{f(a)+f(b)}{2} + \sum_{i=2}^{N} f(x_i)\right].$$

As shown in Atkinson (1989), there exists some number α between a and b such that

$$E_N(f) = -\frac{(b-a)^3}{12N^2} f''(\alpha).$$

Thus, for constant or linear $f(x)$, this error is zero, and the repeated trapezium rule is exact. Additionally, this rule often converges very quickly when integrating a periodic function over one full period, since numerous cancellations tend to occur. Asymptotically, the error for $N \to \infty$ is given by

$$E_N(f) = -\frac{(b-a)^2}{12N^2}\left[f'(b) - f'(a)\right] + O\left(N^{-3}\right) = O\left(N^{-2}\right).$$

With our specific function, the value of d impacts convergence. For large values of d, convergence occurs quickly, and N may be small. If d is small, then N must be large in order for the solution to converge. For each of the above plots, the value of N was taken to be large enough to ensure convergence for the chosen value of d.

CHAPTER 3

CRACK PROBLEMS

In this chapter, we consider the physical properties of stress and strain, and we begin our crack problem analysis.

3.1 Axisymmetric elasticity

The basic unknown in the theory of linear elasticity is the displacement vector, $\mathbf{u}(r,\theta,z) = (u_r, u_\theta, u_z)$. The strains, from (Graff, 1991, p. 600), are listed below:

$$\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r}, \quad e_{rz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right), \\
e_{zz} &= \frac{\partial u_z}{\partial z}, \quad e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{\partial u_\theta}{\partial r}\right), \\
e_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \quad e_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right).
\end{aligned}$$

In order to obtain axisymmetric solutions, we require that $u_\theta \equiv 0$ and that all θ derivatives are also 0, giving

$$e_{\theta\theta} = \frac{u_r}{r} \quad \text{and} \quad e_{r\theta} = e_{\theta z} = 0.$$

Next, the stresses are given by

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},$$

where λ and μ are the Lamé moduli, δ_{ij} is the Kronecker delta, and

$$e_{kk} = \Delta = \frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{\partial u_z}{\partial z}.$$

Therefore, we have

$$\begin{aligned}
\tau_{rr} &= \lambda\Delta + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{rz} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right), \\
\tau_{zz} &= \lambda\Delta + 2\mu\frac{\partial u_z}{\partial z}, \quad \tau_{r\theta} = 0, \\
\tau_{\theta\theta} &= \lambda\Delta + 2\mu\frac{u_r}{r}, \quad \tau_{\theta z} = 0.
\end{aligned}$$

In the absence of body forces, the axisymmetric equilibrium equations are

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0, \tag{3.1}$$

$$\frac{1}{r}\frac{\partial (r\tau_{rz})}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} = 0. \tag{3.2}$$

We look for solutions of the form

$$u_z(r,z) = J_0(\kappa r)Z_0(z), \quad u_r(r,z) = J_1(\kappa r)Z_1(z), \tag{3.3}$$

where J_n is a Bessel function, Z_0 and Z_1 are to be found, and κ is a constant. We have the following facts regarding Bessel functions:

$$\frac{d}{dr}(J_0(\kappa r)) = -\kappa J_1(\kappa r), \quad \frac{d}{dr}(rJ_1(\kappa r)) = \kappa r J_0(\kappa r),$$
$$r\frac{d}{dr}\left(r\frac{d}{dr}J_0(\kappa r)\right) = -(\kappa r)^2 J_0(\kappa r), \quad r\frac{d}{dr}\left(r\frac{d}{dr}J_1(\kappa r)\right) = [1-(\kappa r)^2]J_1(\kappa r).$$

Next, we have

$$\Delta = (\kappa Z_1 + Z_0')J_0(\kappa r),$$

and so, we have:

$$\tau_{rr} = \lambda(\kappa Z_1 + Z_0')J_0(\kappa r) + 2\mu Z_1 \frac{d}{dr}(J_1(\kappa r)), \quad \tau_{zz} = \{\lambda\kappa Z_1 + (\lambda+2\mu)Z_0'\}J_0(\kappa r),$$
$$\tau_{\theta\theta} = \lambda(\kappa Z_1 + Z_0')J_0(\kappa r) + \frac{2\mu}{r}Z_1 J_1(\kappa r), \quad \tau_{rz} = \mu(Z_1' - \kappa Z_0)J_1(\kappa r).$$

Now, the terms in the first equilibrium equation, (3.1), become:

$$\frac{\partial \tau_{rr}}{\partial r} = \{-\lambda\kappa^2 Z_1 - \lambda\kappa Z_0'\}J_1(\kappa r) + 2\mu Z_1 \frac{\partial}{\partial r}\left(\frac{\partial J_1(\kappa r)}{\partial r}\right),$$
$$\frac{\partial \tau_{rz}}{\partial z} = \{\mu Z_1'' - \mu\kappa Z_0'\}J_1(\kappa r),$$
$$\frac{\tau_{rr} - \tau_{\theta\theta}}{r} = \frac{2\mu}{r}Z_1\frac{\partial J_1(\kappa r)}{\partial r} - \frac{2\mu}{r^2}Z_1 J_1(\kappa r).$$

Plugging these terms into (3.1) yields

$$\{\mu Z_1'' - \lambda\kappa^2 Z_1 - (\lambda+\mu)\kappa Z_0'\}J_1(\kappa r) - \frac{2\mu}{r^2}Z_1 J_1(\kappa r)$$
$$+ 2\mu Z_1 \frac{\partial}{\partial r}\left(\frac{\partial J_1(\kappa r)}{\partial r}\right) + \frac{2\mu}{r}Z_1\frac{\partial J_1(\kappa r)}{\partial r} = 0.$$

Considering the last two terms, we find that

$$2\mu Z_1 \left\{ \frac{\partial}{\partial r}\left(\frac{\partial J_1(\kappa r)}{\partial r}\right) + \frac{1}{r}\frac{\partial J_1(\kappa r)}{\partial r} \right\} = \frac{2\mu}{r^2} Z_1 \left\{ r\frac{\partial}{\partial r}\left(r\frac{\partial J_1(\kappa r)}{\partial r}\right)\right\}$$
$$= \frac{2\mu}{r^2} Z_1 J_1(\kappa r) - 2\mu\kappa^2 Z_1 J_1(\kappa r).$$

Additionally, the terms in the second equilibrium equation, (3.2), become:

$$\frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r} = \{\mu\kappa Z_1' - \mu\kappa^2 Z_0\} J_0(\kappa r), \quad \frac{\partial \tau_{zz}}{\partial z} = \{\lambda\kappa Z_1' + (\lambda + 2\mu) Z_0''\} J_0(\kappa r).$$

Therefore, the equilibrium equations, (3.1) and (3.2), become

$$\{\mu Z_1'' - (\lambda + 2\mu)\kappa^2 Z_1 - (\lambda+\mu)\kappa Z_0'\} J_1(\kappa r) = 0,$$
$$\{(\lambda+\mu)\kappa Z_1' + (\lambda + 2\mu) Z_0'' - \mu\kappa^2 Z_0\} J_0(\kappa r) = 0,$$

respectively. Both expressions within the braces must vanish in order to satisfy these equations. Eliminating the Z_1 terms from the first expression gives

$$Z_0^{iv} - 2\kappa^2 Z_0'' + \kappa^4 Z_0 = 0,$$

with the general solution

$$Z_0(z) = Ae^{\kappa z} + Be^{-\kappa z} + Cze^{\kappa z} + Dze^{-\kappa z}, \tag{3.4}$$

where A, B, C, and D are all arbitrary constants. Then, we have

$$Z_1(z) = -Ae^{\kappa z} + Be^{-\kappa z} - C(3 - 4\nu + \kappa z)\frac{e^{\kappa z}}{\kappa} - D(3 - 4\nu - \kappa z)\frac{e^{-\kappa z}}{\kappa}, \tag{3.5}$$

where $\nu = \frac{1}{2}\lambda/(\lambda+\mu)$ is Poisson's ratio. Now, the stresses are given by

$$\tau_{rz} = -2\mu J_1(\kappa r)\left\{\kappa(Ae^{\kappa z} + Be^{-\kappa z}) + C[2(1-\nu) + \kappa z]e^{\kappa z} - D[2(1-\nu) - \kappa z]e^{-\kappa z}\right\}, \tag{3.6}$$

$$\tau_{zz} = 2\mu J_0(\kappa r)\left\{\kappa(Ae^{\kappa z} - Be^{-\kappa z}) + C[1 - 2\nu + \kappa z]e^{\kappa z} + D[1 - 2\nu - \kappa z]e^{-\kappa z}\right\}. \tag{3.7}$$

All of the above expressions are valid for any $\kappa > 0$. Therefore, we can apply superposition, by integrating with respect to κ, which yields expressions in the form of Hankel transforms.

3.2 A Pressurized Penny-Shaped Crack

Next, we consider a pressurized penny-shaped crack in an infinite solid, a classical problem. To begin, we put the crack on the plane $z = 0$, with the center at the origin and with radius a. Through symmetry, we only need to consider $z > 0$, with $\tau_{rz} = 0$ on $z = 0$ for $r \geq 0$. To ensure that $u \to 0$ as $z \to \infty$, we retain only terms proportional to $e^{-\kappa z}$, assuming $\kappa > 0$, so that $A = C = 0$. Therefore, we have

$$\tau_{rz} = -2\mu J_1(\kappa r)e^{-\kappa z}\{\kappa B - D[2(1-\nu) - \kappa z]\}.$$

For this to vanish on $z = 0$, we take $\kappa B = 2D(1-\nu)$.

Using this in (3.7), we have

$$\tau_{zz} = -\frac{\mu\kappa B}{1-\nu} J_0(\kappa r)$$

on $z = 0$. Also on $z = 0$, we have

$$u_z = BJ_0(\kappa r).$$

As with equation (2.36), we allow B to depend on κ and apply superposition, by integrating over κ, giving

$$u_z(r,0) = \int_0^\infty B(\kappa)J_0(\kappa r)\,d\kappa \quad \text{and} \quad \tau_{zz}(r,0) = -\frac{\mu}{1-\nu}\int_0^\infty \kappa B(\kappa)J_0(\kappa r)\,d\kappa.$$

As in Section 2.1 and Section 2.3, we have the boundary condition

$$u_z(r,0) = 0 \quad \text{for } r > a.$$

Additionally, we have the boundary condition

$$\tau_{zz}(r,0) = -\frac{\mu p(r)}{1-\nu} \quad \text{for } r < a,$$

where p is a given dimensionless pressure. These conditions yield a pair of dual integral equations;

$$\int_0^\infty \kappa B(\kappa)J_0(\kappa r)\,d\kappa = p(r), \qquad r < a, \tag{3.8}$$

$$\int_0^\infty B(\kappa)J_0(\kappa r)\,d\kappa = 0, \qquad r > a. \tag{3.9}$$

These are the same equations as (2.42) and (2.43).

From (2.45) and (2.46), the solution of these equations is

$$B(\kappa) = \frac{2}{\pi} \int_0^a \sin(\kappa t) \int_0^t \frac{r p(r)}{\sqrt{t^2 - r^2}} \, dr \, dt. \tag{3.10}$$

Now, if $p(r) = p_0$, a constant, then, as in Section 2.2, we obtain

$$B(\kappa) = \frac{2a^2}{\pi} p_0 \left[\frac{1}{(\kappa a)^2} \sin(\kappa a) - \frac{1}{\kappa a} \cos(\kappa a) \right] = \frac{2a^2}{\pi} p_0 j_1(\kappa a), \tag{3.11}$$

where j_n is a spherical Bessel function of the first kind.

CHAPTER 4

A BURIED PENNY-SHAPED CRACK

In this chapter, we consider a pressurized penny-shaped crack in an elastic half-space. The plane of the crack is parallel to the boundary of the half-space, and we choose coordinates so that the crack is on the plane $z = 0$, with radius a. The half-space boundary is $z = -h$, with $h > 0$.

As in Section 2.3, there are two regions of the half-space; Region 1, corresponding to $z > 0$, and Region 2, corresponding to $-h < z < 0$. We use subscripts on A, B, C, and D, as well as on u, to denote the region.

In Region 1, we require $u \to 0$ as $z \to \infty$, so, from equations (3.3), (3.4), and (3.5), we retain only terms proportional to $e^{-\kappa z}$, assuming $\kappa > 0$, so that $A_1 = C_1 = 0$. Thus, for $z > 0$, we have

$$u_{1z}(r,z) = J_0(\kappa r)e^{-\kappa z}\left\{B_1 + D_1 z\right\}, \quad u_{1r}(r,z) = J_1(\kappa r)e^{-\kappa z}\left\{B_1 - D_1(3 - 4\nu - \kappa z)/\kappa\right\}.$$

In Region 2, we must keep all terms from the previous equations. The surface $z = -h$ is traction free, so (3.6) and (3.7) give

$$\kappa(A_2 e^{-\kappa h} + B_2 e^{\kappa h}) + C_2[2(1-\nu) - \kappa h]e^{-\kappa h} - D_2[2(1-\nu) + \kappa h]e^{\kappa h} = 0, \tag{4.1}$$
$$\kappa(A_2 e^{-\kappa h} - B_2 e^{\kappa h}) + C_2[1 - 2\nu - \kappa h]e^{-\kappa h} + D_2[1 - 2\nu + \kappa h]e^{\kappa h} = 0. \tag{4.2}$$

Now, continuity of tractions across $z = 0$ gives

$$\kappa(B_1 - A_2 - B_2) = 2(C_2 + D_1 - D_2)(1-\nu), \tag{4.3}$$
$$-\kappa(B_1 + A_2 - B_2) = (C_2 - D_1 + D_2)(1 - 2\nu). \tag{4.4}$$

Let $[u_z(r)]$ and $[u_r(r)]$ be the discontinuities in u_z and u_r, respectively, across $z = 0$,

$$[u_z(r)] = u_{1z}(r,0) - u_{2z}(r,0), \quad [u_r(r)] = u_{1r}(r,0) - u_{2r}(r,0).$$

For these displacement discontinuities, we obtain

$$[u_z(r)] = \mathcal{A} J_0(\kappa r) \quad \text{and} \quad [u_r(r)] = \mathcal{B} J_1(\kappa r),$$

where

$$\mathcal{A} = B_1 - A_2 - B_2, \tag{4.5}$$

$$\mathcal{B} = B_1 + A_2 - B_2 + \frac{3-4\nu}{\kappa}(C_2 - D_1 + D_2) = -\frac{2(1-\nu)}{1-2\nu}(B_1 + A_2 - B_2), \tag{4.6}$$

using (4.4).

We now have six unknowns (A_2, B_1, B_2, C_2, D_1, and D_2) with four relations between them, so we can express each unknown in terms of $\mathcal{A}$ and $\mathcal{B}$. From (4.5) and (4.6), we obtain

$$A_2 = -\frac{\mathcal{A}}{2} - \frac{1-2\nu}{4(1-\nu)}\mathcal{B} \quad \text{and} \quad B_1 - B_2 = \frac{\mathcal{A}}{2} - \frac{1-2\nu}{4(1-\nu)}\mathcal{B},$$

and from (4.3) and (4.4), we obtain

$$C_2 = \frac{\kappa(\mathcal{A}+\mathcal{B})}{4(1-\nu)} \quad \text{and} \quad D_1 - D_2 = \frac{\kappa(\mathcal{A}-\mathcal{B})}{4(1-\nu)}.$$

Now, adding (4.1) and (4.2) gives

$$\begin{aligned} D_2 e^{2\kappa h} &= 2\kappa A_2 + C_2(3 - 4\nu - 2\kappa h) \\ &= \frac{\kappa[\mathcal{B} - \mathcal{A} - 2\kappa h(\mathcal{A}+\mathcal{B})]}{4(1-\nu)}, \end{aligned}$$

so that

$$D_1 = \frac{\kappa[(\mathcal{A}-\mathcal{B})(1-e^{-2\kappa h}) - 2\kappa h(\mathcal{A}+\mathcal{B})e^{-2\kappa h}]}{4(1-\nu)}.$$

Finally, from (4.1), we obtain

$$\begin{aligned} B_2 e^{2\kappa h} &= -A_2 - \frac{C_2}{\kappa}[2(1-\nu) - \kappa h] + \frac{D_2}{\kappa}[2(1-\nu) + \kappa h]e^{2\kappa h} \\ &= \frac{(1-2\nu)\mathcal{B} - 2(1-\nu)\mathcal{A} - 2\kappa h[\mathcal{A} + (\mathcal{A}+\mathcal{B})(1-2\nu+\kappa h)]}{4(1-\nu)}, \end{aligned}$$

so that

$$B_1 = \frac{[2(1-\nu)\mathcal{A} - (1-2\nu)\mathcal{B}](1-e^{-2\kappa h}) - 2\kappa h[\mathcal{A} + (\mathcal{A}+\mathcal{B})(1-2\nu+\kappa h)]e^{-2\kappa h}}{4(1-\nu)}.$$

We are interested in τ_{rz} and τ_{zz} on $z = 0$. For τ_{rz}, we have

$$
\begin{aligned}
\tau_{rz} =& -2\mu J_1(\kappa r)\left\{\kappa B_1 - D_1[2(1-\nu) - \kappa z]\right\} e^{-\kappa z}, \quad \text{z} \ > \ 0, \\
\tau_{rz} =& -2\mu J_1(\kappa r)\left\{\kappa A_2 + C_2[2(1-\nu) + \kappa z]\right\} e^{\kappa z} \\
& -2\mu J_1(\kappa r)\left\{\kappa B_2 - D_2[2(1-\nu) - \kappa z]\right\} e^{-\kappa z}, \quad -h < z < 0.
\end{aligned}
$$

Since the stresses must be continuous across $z = 0$ for $r > a$, we see that τ_{rz} is continuous across $z = 0$ for all r. Therefore, for the tractions on $z = 0$, we can take the average of these two τ_{rz} equations, giving

$$
\tau_{rz}(r, 0) = -\frac{\mu\kappa T_r}{2(1-\nu)} J_1(\kappa r) \tag{4.7}
$$

where

$$
\begin{aligned}
T_r &= 2(1-\nu)\left\{B_1 + A_2 + B_2 + \frac{2(1-\nu)}{\kappa}(-D_1 + C_2 - D_2)\right\} \\
&= \mathcal{B}(1 - e^{-2\kappa h}) - 2\kappa h[\mathcal{A} + (\mathcal{A} + \mathcal{B})(-1 + \kappa h)]e^{-2\kappa h}.
\end{aligned} \tag{4.8}
$$

Now, for τ_{zz}, we have

$$
\begin{aligned}
\tau_{zz} =& -2\mu J_0(\kappa r)\left\{\kappa B_1 - D_1[1 - 2\nu - \kappa z]\right\} e^{-\kappa z}, \quad \text{z} \ > \ 0, \\
\tau_{zz} =& -2\mu J_0(\kappa r)\left\{-\kappa A_2 - C_2[1 - 2\nu + \kappa z]\right\} e^{\kappa z} \\
& -2\mu J_0(\kappa r)\left\{\kappa B_2 - D_2[1 - 2\nu - \kappa z]\right\} e^{-\kappa z}, \quad -h < z < 0.
\end{aligned}
$$

Again, continuity of the stresses across $z = 0$ for $r > a$ indicates that τ_{zz} is continuous across $z = 0$ for all r. As before, we can take the average of these two τ_{zz} equations for the tractions on $z = 0$, giving

$$
\tau_{zz}(r, 0) = -\frac{\mu\kappa T_z}{2(1-\nu)} J_0(\kappa r) \tag{4.9}
$$

where

$$
\begin{aligned}
T_z &= 2(1-\nu)\left\{B_1 - A_2 + B_2 + \frac{1-2\nu}{\kappa}(-D_1 - C_2 - D_2)\right\} \\
&= \mathcal{A}(1 - e^{-2\kappa h}) - 2\kappa h[\mathcal{A} + (\mathcal{A} + \mathcal{B})\kappa h]e^{-2\kappa h}.
\end{aligned} \tag{4.10}
$$

Notice that, as $h \to \infty$, $T_z \to \mathcal{A}$ and $T_r \to \mathcal{B}$.

4.1 Integral Equations for $\mathcal{A}$ and $\mathcal{B}$

As in Section 3.2, suppose that we have the boundary conditions

$$\tau_{zz}(r,0) = -\frac{\mu p(r)}{1-\nu} \quad \text{and} \quad \tau_{rz}(r,0) = -\frac{\mu q(r)}{1-\nu} \quad \text{for } r < a, \tag{4.11}$$

where p and q are given dimensionless pressures.

As in previous sections, we allow $\mathcal{A}$ and $\mathcal{B}$ to depend on κ. We rewrite the functions T_r and T_z, given by (4.8) and (4.10), as

$$T_z(\kappa) = \mathcal{A}(\kappa) - S_z(\kappa)e^{-2\kappa h} \quad \text{and} \quad T_r(\kappa) = \mathcal{B} - S_r(\kappa)e^{-2\kappa h},$$

where

$$S_z(\kappa) = [1 + 2\kappa h + 2(\kappa h)^2]\mathcal{A}(\kappa) + 2(\kappa h)^2\mathcal{B}(\kappa),$$
$$S_r(\kappa) = [1 - 2\kappa h + 2(\kappa h)^2]\mathcal{B}(\kappa) + 2(\kappa h)^2\mathcal{A}(\kappa).$$

Now, consider a buried crack with $p(r) = p_0$, a constant, and $q \equiv 0$. To find more general expressions, we integrate over κ, as in previous sections. Using the first of (4.11) and (4.9), together with $[u_z(r)] = 0$ for $r > a$ gives

$$\int_0^\infty \kappa T_z(\kappa) J_0(\kappa r)\, d\kappa = 2p_0, \qquad r < a, \tag{4.12}$$
$$\int_0^\infty \mathcal{A}(\kappa) J_0(\kappa r)\, d\kappa = 0, \qquad r > a. \tag{4.13}$$

We rewrite (4.12) as

$$\int_0^\infty \kappa \mathcal{A}(\kappa) J_0(\kappa r)\, d\kappa = 2p_0 + g_0(r), \quad r < a, \tag{4.14}$$

where

$$g_0(r) = \int_0^\infty \kappa S_z(\kappa) J_0(\kappa r) e^{-2\kappa h}\, d\kappa.$$

Treating the right-hand side of (4.14) as known, and using (3.10), we can 'solve' (4.14) and (4.13) for $\mathcal{A}$, giving

$$\mathcal{A}(\kappa) = \frac{2}{\pi}\int_0^a \sin(\kappa t) \int_0^t \frac{r(2p_0 + g_0(r))}{\sqrt{t^2 - r^2}}\, dr\, dt.$$

We can now write this as

$$\mathcal{A}(\kappa) = \mathcal{A}_\infty(\kappa) + \mathcal{C}_0(\kappa), \tag{4.15}$$

where

$$\mathcal{A}_\infty(\kappa) = \frac{4}{\pi}\int_0^a \sin(\kappa t)\int_0^t \frac{rp_0}{\sqrt{t^2-r^2}}\,dr\,dt$$
$$= \frac{4a^2}{\pi}p_0 j_1(\kappa a) \tag{4.16}$$

is the solution when $h=\infty$, and $\mathcal{C}_0$, which depends on $\mathcal{A}$ and $\mathcal{B}$, is given by

$$\mathcal{C}_0(\kappa) = \frac{2}{\pi}\int_0^a \sin(\kappa t)\int_0^t \frac{rg_0(r)}{\sqrt{t^2-r^2}}\,dr\,dt$$
$$= \frac{2}{\pi}\int_0^\infty S_z(\eta)e^{-2\eta h}\int_0^a \sin(\kappa t)\int_0^t \frac{\eta r J_0(\eta r)}{\sqrt{t^2-r^2}}\,dr\,dt\,d\eta$$
$$= \frac{2}{\pi}\int_0^\infty S_z(\eta)e^{-2\eta h}\int_0^a \sin(\kappa t)\sin(\eta t)\,dt\,d\eta. \tag{4.17}$$

Evidently, (4.15) gives an integral equation relating $\mathcal{A}$ and $\mathcal{B}$. We now look for a second such equation.

Using the second of (4.11) and (4.7), together with $[u_r(r)]=0$ for $r>a$ gives

$$\int_0^\infty \kappa T_r(\kappa)J_1(\kappa r)\,d\kappa = 0, \qquad r<a, \tag{4.18}$$
$$\int_0^\infty \mathcal{B}(\kappa)J_1(\kappa r)\,d\kappa = 0, \qquad r>a. \tag{4.19}$$

We rewrite (4.18) as

$$\int_0^\infty \kappa\mathcal{B}(\kappa)J_1(\kappa r)\,d\kappa = g_1(r), \quad r<a, \tag{4.20}$$

where

$$g_1(r) = \int_0^\infty \kappa S_r(\kappa)J_1(\kappa r)e^{-2\kappa h}\,d\kappa.$$

From (Sneddon, 1951, p. 65-70), the solution of the dual integral equations

$$\int_0^\infty y\tilde{F}(y)J_n(xy)\,dy = \tilde{G}(x), \qquad x<1,$$
$$\int_0^\infty \tilde{F}(y)J_n(xy)\,dy = 0, \qquad x>1,$$

is given by

$$\tilde{F}(y) = \sqrt{\frac{2y}{\pi}}\int_0^1 \eta^{3/2}J_{n+1/2}(\eta y)\int_0^1 \frac{\zeta^{n+1}\tilde{G}(\eta\zeta)}{\sqrt{1-\zeta^2}}\,d\zeta\,d\eta.$$

In these formulas, we put $x = r/a$, $y = \kappa a$, $\eta = t/a$, $\zeta = r/t$, $F(\kappa) = a\tilde{F}(\kappa a)$, and $G(r) = a^{-1}\tilde{G}(r/a)$. The dual integral equations become

$$\int_0^\infty \kappa F(\kappa) J_n(\kappa r)\, d\kappa = G(r), \qquad r < a,$$
$$\int_0^\infty F(\kappa) J_n(\kappa r)\, d\kappa = 0, \qquad r > a,$$

with their solution given by

$$F(\kappa) = \frac{2\kappa}{\pi} \int_0^a t^{1-n} j_n(\kappa t) \int_0^t \frac{r^{n+1} G(r)}{\sqrt{t^2 - r^2}}\, dr\, dt,$$

where $j_n(x) = (\pi/[2x])^{1/2} J_{n+1/2}(x)$ is a spherical Bessel function.

Treating the right-hand side of (4.20) as known, and using Sneddon's solution, we can 'solve' (4.20) and (4.19) for $\mathcal{B}$, giving

$$\mathcal{B}(\kappa) = \frac{2\kappa}{\pi} \int_0^a j_1(\kappa t) \int_0^t \frac{r^2 g_1(r)}{\sqrt{t^2 - r^2}}\, dr\, dt.$$

We now have our second integral equation relating $\mathcal{A}$ and $\mathcal{B}$. It is

$$\mathcal{B}(\kappa) = \mathcal{C}_1(\kappa), \tag{4.21}$$

where $\mathcal{C}_1$, which depends on $\mathcal{A}$ and $\mathcal{B}$, is given by

$$\begin{aligned} \mathcal{C}_1(\kappa) &= \frac{2\kappa}{\pi} \int_0^a j_1(\kappa t) \int_0^t \frac{r^2 g_1(r)}{\sqrt{t^2 - r^2}}\, dr\, dt \\ &= \frac{2\kappa}{\pi} \int_0^\infty \eta S_r(\eta) e^{-2\eta h} \int_0^a j_1(\kappa t) \int_0^t \frac{r^2 J_1(\eta r)}{\sqrt{t^2 - r^2}}\, dr\, dt\, d\eta. \end{aligned}$$

Using the following forumla, from (Gradshteyn & Ryzhik, 1980, Equation 6.567),

$$\int_0^a \frac{r^{n+1} J_n(\eta r)}{\sqrt{a^2 - r^2}}\, dr = a^{n+1} j_n(\eta a),$$

we now have

$$\mathcal{C}_1(\kappa) = \frac{2\kappa}{\pi} \int_0^\infty \eta S_r(\eta) e^{-2\eta h} \int_0^a t^2 j_1(\kappa t) j_1(\eta t)\, dt\, d\eta. \tag{4.22}$$

4.2 Simplified Integral Equations

In an effort to obtain simpler equations, we introduce new unknown functions ψ_0 and ψ_1. Motivated by our expressions for $\mathcal{A}_\infty(\kappa)$ and $\mathcal{B}(\kappa)$, we put

$$\mathcal{A}(\kappa) = \frac{4p_0}{\pi} \kappa \int_0^a t\psi_0(t) j_0(\kappa t)\, dt = \frac{4p_0}{\pi} \int_0^a \psi_0(t) \sin(\kappa t)\, dt$$

and

$$\mathcal{B}(\kappa) = \frac{4p_0}{\pi}\kappa \int_0^a t^2 \psi_1(t) j_1(\kappa t)\, dt.$$

Then, using (4.16) and (4.17), (4.15) becomes

$$\begin{aligned}\int_0^a \psi_0(t)\sin(\kappa t)\, dt &= a^2 j_1(\kappa a) + \frac{1}{2p_0}\int_0^\infty S_z(\eta) e^{-2\eta h}\int_0^a \sin(\kappa t)\sin(\eta t)\, dt\, d\eta \\ &= \int_0^a t\sin(\kappa t)\, dt + \int_0^a \frac{1}{2p_0}\sin(\kappa t)\int_0^\infty S_z(\eta)\sin(\eta t) e^{-2\eta h}\, d\eta\, dt.\end{aligned}$$

Therefore,

$$\psi_0(t) = t + \frac{1}{2p_0}\int_0^\infty S_z(\kappa)\sin(\kappa t) e^{-2\kappa h}\, d\kappa.$$

Now, substituting for $\mathcal{A}$ and $\mathcal{B}$ in S_z, we obtain

$$\psi_0(t) = t + \int_0^a \{K_{00}(t,s;h)\psi_0(s) + K_{01}(t,s;h)\psi_1(s)\}\, ds, \quad 0 < t < a, \tag{4.23}$$

where

$$\begin{aligned}K_{00}(t,s;h) &= \frac{2}{\pi}\int_0^\infty [1 + 2\kappa h + 2(\kappa h)^2]\sin(\kappa t)\sin(\kappa s) e^{-2\kappa h}\, d\kappa, \\ K_{01}(t,s;h) &= \frac{4s^2}{\pi}h^2 \int_0^\infty \kappa^3 \sin(\kappa t) j_1(\kappa s) e^{-2\kappa h}\, d\kappa.\end{aligned}$$

Similarly, using (4.22), (4.21) gives

$$\begin{aligned}\int_0^a t^2\psi_1(t) j_1(\kappa t)\, dt &= \frac{1}{2p_0}\int_0^\infty \eta S_r(\eta) e^{-2\eta h}\int_0^a t^2 j_1(\kappa t) j_1(\eta t)\, dt\, d\eta \\ &= \int_0^a \frac{1}{2p_0} t^2 j_1(\kappa t)\int_0^\infty \eta S_r(\eta) j_1(\eta t) e^{-2\eta h}\, d\eta\, dt.\end{aligned}$$

Therefore

$$t\psi_1(t) = \frac{t}{2p_0}\int_0^\infty \kappa S_r(\kappa) j_1(\kappa t) e^{-2\kappa h}\, d\kappa.$$

Now, substituting for $\mathcal{A}$ and $\mathcal{B}$ in S_r, we obtain

$$t^2\psi_1(t) = \int_0^a \{K_{10}(t,s;h)\psi_0(s) + K_{11}(t,s;h)\psi_1(s)\}\, ds, \quad 0 < t < a, \tag{4.24}$$

where

$$K_{10}(t,s;h) = \frac{4t^2}{\pi} h^2 \int_0^\infty \kappa^3 \sin(\kappa s) j_1(\kappa t) e^{-2\kappa h}\, d\kappa = K_{01}(s,t;h),$$
$$K_{11}(t,s;h) = \frac{2}{\pi}(ts)^2 \int_0^\infty \kappa^2 [1 - 2\kappa h + 2(\kappa h)^2] j_1(\kappa t) j_1(\kappa s) e^{-2\kappa h}\, d\kappa.$$

We now define

$$L_{ij}(T,S) = \pi h K_{ij}(2hT, 2hS; h), \quad \text{i, j} = 0,\ 1,$$

so that, letting $x = 2\kappa h$, we have

$$L_{00}(T,S) = \int_0^\infty (1 + x + x^2/2) \sin(xT) \sin(xS) e^{-x}\, dx, \tag{4.25}$$
$$L_{01}(T,S) = hS^2 \int_0^\infty x^3 \sin(xT) j_1(xS) e^{-x}\, dx = L_{10}(S,T) \tag{4.26}$$
$$L_{11}(T,S) = 4h^2(TS)^2 \int_0^\infty x^2 (1 - x + x^2/2) j_1(xT) j_1(xS) e^{-x}\, dx. \tag{4.27}$$

Considering $L_{00}(T,S)$, defined by (4.25), we can write,

$$L_{00}(T,S) = M_{00}(T-S) - M_{00}(T+S), \tag{4.28}$$

where

$$M_{00}(R) = \frac{1}{2} \int_0^\infty (1 + x + x^2/2) \cos(xR) e^{-x}\, dx.$$

Then, we have the Laplace integrals

$$\frac{1}{2}\int_0^\infty \cos(xR) e^{-px}\, dx = \frac{p}{2(p^2+R^2)},$$
$$\frac{1}{2}\int_0^\infty x\cos(xR) e^{-px}\, dx = \frac{p^2-R^2}{2(p^2+R^2)^2},$$
$$\frac{1}{4}\int_0^\infty x^2\cos(xR) e^{-px}\, dx = \frac{p(p^2-3R^2)}{(p^2+R^2)^3},$$

which, with $p = 1$, gives us

$$M_{00}(R) = \frac{3-R^2}{2(1+R^2)^3}. \tag{4.29}$$

Now, considering $L_{01}(T,S)$, defined by (4.26), and using $(d/dS) j_0(xS) = -x j_1(xS)$ and $j_0(x) = x^{-1}\sin(x)$, we have

$$
\begin{aligned}
L_{01}(T,S) &= hS^2 \frac{\partial}{\partial S} \int_0^\infty x^2 \sin(xT) j_0(xS) e^{-x}\, dx \\
&= -hS^2 \frac{\partial}{\partial S} \left\{ S^{-1} \int_0^\infty x \sin(xT) \sin(xS) e^{-x}\, dx \right\} \\
&= -\frac{hS^2}{2} \frac{\partial}{\partial S} \left\{ S^{-1} [L_1(T-S) - L_1(T+S)] \right\},
\end{aligned}
$$

where

$$
L_1(R) = \int_0^\infty x \cos(xR) e^{-x}\, dx = \frac{1-R^2}{(1+R^2)^2}.
$$

Then, we have

$$
L_{01}(T,S) = \frac{h}{2} L_1(T-S) + \frac{hS}{2} L_1'(T-S) - \frac{h}{2} L_1(T+S) + \frac{hS}{2} L_1'(T+S),
$$

where

$$
L_1'(R) = \frac{2R(R^2-3)}{(1+R^2)^3}.
$$

We can now write

$$
L_{01}(T,S) = M_{01}(T,S) - M_{01}(T,-S), \tag{4.30}
$$

where

$$
\begin{aligned}
M_{01}(T,S) &= \frac{h}{2} L_1(T-S) + \frac{hS}{2} L_1'(T-S) \\
&= \frac{h[1 - W^4 - 2SW(3-W^2)]}{2(1+W^2)^3},
\end{aligned} \tag{4.31}
$$

with $W = T - S$.

Since $L_{01}(T,S) = L_{10}(S,T)$, we can write

$$
L_{10}(T,S) = M_{10}(T,S) - M_{10}(T,-S), \tag{4.32}
$$

where

$$
M_{10}(T,S) = \frac{h[1 - W^4 + 2TW(3-W^2)]}{2(1+W^2)^3}, \tag{4.33}
$$

with $W = T - S$.

Finally, considering $L_{11}(T,S)$, defined by (4.27), using $(d/dR) j_0(xR) = -x j_1(xR)$ and $j_0(x) = x^{-1}\sin(x)$, and writing $\Phi(x) = (1 - x + x^2/2)e^{-x}$, we have

$$
\begin{aligned}
L_{11}(T,S) &= 4h^2(TS)^2\frac{\partial^2}{\partial T\partial S}\int_0^\infty \Phi(x)j_0(xT)j_0(xS)\,dx \\
&= 4h^2(TS)^2\frac{\partial^2}{\partial T\partial S}\left\{(TS)^{-1}\int_0^\infty \Phi(x)x^{-2}\sin(xT)\sin(xS)\,dx\right\} \\
&= 2h^2(TS)^2\frac{\partial^2}{\partial T\partial S}\left\{(TS)^{-1}[L_2(T-S)-L_2(T+S)]\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
L_2(R) &= \int_0^\infty \Phi(x)x^{-2}\{\cos(xR)-1\}\,dx \\
&= -\int_0^R\int_0^\infty \Phi(x)x^{-1}\sin(xt)\,dx\,dt \\
&= -\int_0^R\int_0^\infty (x^{-1}-1+x/2)\sin(xt)e^{-x}\,dx\,dt.
\end{aligned}
$$

Then, we have the Laplace integrals

$$
\begin{aligned}
\int_0^\infty x^{-1}\sin(xt)e^{-x}\,dx &= \tan^{-1}(t), \\
-\int_0^\infty \sin(xt)e^{-px}\,dx &= -\frac{t}{p^2+t^2}, \\
\frac{1}{2}\int_0^\infty x\sin(xt)e^{-px}\,dx &= \frac{pt}{(p^2+t^2)^2},
\end{aligned}
$$

which, with $p=1$, gives us

$$
\begin{aligned}
L_2(R) &= -\int_0^R\left\{\tan^{-1}(t)-\frac{t}{1+t^2}+\frac{t}{(1+t^2)^2}\right\}dt \\
&= \log(1+R^2)-R\tan^{-1}(R)-\frac{R^2}{2(1+R^2)}.
\end{aligned}
$$

Now, we can write

$$
L_{11}(T,S) = M_{11}(T,S) - M_{11}(T,-S), \tag{4.34}
$$

where

$$
M_{11}(T,S) = 2h^2L_2(T-S) - 2h^2(T-S)L_2'(T-S) - 2h^2TSL_2''(T-S)
$$

The derivatives of L_2 are given by

$$L_2'(R) = \frac{R^3}{(1+R^2)^2} - \tan^{-1}(R) \quad \text{and} \quad L_2''(R) = \frac{-2R^4 + R^2 - 1}{(1+R^2)^3}$$

so that

$$M_{11}(T,S) = 2h^2 \log(1+W^2) - \frac{h^2W^2(1+3W^2)}{(1+W^2)^2} + \frac{2h^2TS(2W^4 - W^2 + 1)}{(1+W^2)^3} \tag{4.35}$$

with $W = T - S$.

Back to our ψ functions, we examine integral equation (4.23). We find that the right-hand side is an odd function of t, and that $\psi_0(0) = 0$. This allows us to extend $\psi_0(t)$ as a continuous, odd function of t, defined for $-a < t < a$. Similarly, examining (4.24) indicates that we can extend $\psi_1(t)$ as an even function of t. Exchanging our K_{ij} kernels in favor of L_{ij}, we now have

$$\psi_0(t) = t + \frac{1}{\pi h}\int_0^a \left\{ L_{00}\left(\frac{t}{2h}, \frac{s}{2h}\right)\psi_0(s) + L_{01}\left(\frac{t}{2h}, \frac{s}{2h}\right)\psi_1(s)\right\} ds,$$
$$t^2\psi_1(t) = \frac{1}{\pi h}\int_0^a \left\{ L_{10}\left(\frac{t}{2h}, \frac{s}{2h}\right)\psi_0(s) + L_{11}\left(\frac{t}{2h}, \frac{s}{2h}\right)\psi_1(s)\right\} ds,$$

for $-a < t < a$. Now, using (4.28), (4.30), (4.32), and (4.34), we obtain

$$\begin{aligned}\psi_0(t) = t + \frac{1}{\pi h}\int_0^a \Bigg\{ &\left[M_{00}\left(\frac{t-s}{2h}\right) - M_{00}\left(\frac{t+s}{2h}\right)\right]\psi_0(s) \\ &+ \left[M_{01}\left(\frac{t}{2h}, \frac{s}{2h}\right) - M_{01}\left(\frac{t}{2h}, -\frac{s}{2h}\right)\right]\psi_1(s)\Bigg\} ds, \\ t^2\psi_1(t) = \frac{1}{\pi h}\int_0^a \Bigg\{ &\left[M_{10}\left(\frac{t}{2h}, \frac{s}{2h}\right) - M_{10}\left(\frac{t}{2h}, -\frac{s}{2h}\right)\right]\psi_0(s) \\ &+ \left[M_{11}\left(\frac{t}{2h}, \frac{s}{2h}\right) - M_{11}\left(\frac{t}{2h}, -\frac{s}{2h}\right)\right]\psi_1(s)\Bigg\} ds,\end{aligned}$$

for $-a < t < a$. We can write this simply as

$$\psi_0(t) = t + \frac{1}{\pi h}\int_{-a}^a \left\{ M_{00}\left(\frac{t-s}{2h}\right)\psi_0(s) + M_{01}\left(\frac{t}{2h}, \frac{s}{2h}\right)\psi_1(s)\right\} ds,$$
$$t^2\psi_1(t) = \frac{1}{\pi h}\int_{-a}^a \left\{ M_{10}\left(\frac{t}{2h}, \frac{s}{2h}\right)\psi_0(s) + M_{11}\left(\frac{t}{2h}, \frac{s}{2h}\right)\psi_1(s)\right\} ds,$$

for $-a < t < a$. Explicitly, using (4.29), (4.31), (4.33), and (4.35), we have

$$
\begin{aligned}
M_{00}\left(\frac{t-s}{2h}\right) &= \frac{8h^4[12h^2-(t-s)^2]}{[4h^2+(t-s)^2]^3}, \\
M_{01}\left(\frac{t}{2h},\frac{s}{2h}\right) &= \frac{2h^3\{16h^4-(t-s)^4-2s(t-s)[12h^2-(t-s)^2]\}}{[4h^2+(t-s)^2]^3}, \\
M_{10}\left(\frac{t}{2h},\frac{s}{2h}\right) &= \frac{2h^3\{16h^4-(t-s)^4+2t(t-s)[12h^2-(t-s)^2]\}}{[4h^2+(t-s)^2]^3}, \\
M_{11}\left(\frac{t}{2h},\frac{s}{2h}\right) &= 2h^2\log\left(\frac{4h^2+(t-s)^2}{4h^2}\right)-\frac{h^2(t-s)^2[4h^2+3(t-s)^2]}{[4h^2+(t-s)^2]^2} \\
&\quad+\frac{4tsh^2[(t-s)^4-2h^2(t-s)^2+8h^4]}{[4h^2+(t-s)^2]^3}.
\end{aligned}
$$

Let us now introduce some dimensionless quantities: $t = ax$, $s = ay$, $\psi_0(ax) = af_0(x)$, $\psi_1(ax) = f_1(x)$, and $\varepsilon = 2h/a$. The integral equations become

$$f_0(x) = x + \int_{-1}^{1}\{k_{00}(x,y;\varepsilon)f_0(y) + k_{01}(x,y;\varepsilon)f_1(y)\}\,dy, \tag{4.36}$$

$$x^2 f_1(x) = \int_{-1}^{1}\{k_{10}(x,y;\varepsilon)f_0(y) + k_{11}(x,y;\varepsilon)f_1(y)\}\,dy, \tag{4.37}$$

for $-1 < x < 1$, where

$$
\begin{aligned}
k_{00}(x,y;\varepsilon) &= \frac{\varepsilon^3[3\varepsilon^2-(x-y)^2]}{\pi[\varepsilon^2+(x-y)^2]^3}, \\
k_{01}(x,y;\varepsilon) &= \frac{\varepsilon^2\{\varepsilon^4-(x-y)^4-2y(x-y)[3\varepsilon^2-(x-y)^2]\}}{2\pi[\varepsilon^2+(x-y)^2]^3}, \\
k_{10}(x,y;\varepsilon) &= \frac{\varepsilon^2\{\varepsilon^4-(x-y)^4+2x(x-y)[3\varepsilon^2-(x-y)^2]\}}{2\pi[\varepsilon^2+(x-y)^2]^3}, \\
k_{11}(x,y;\varepsilon) &= \frac{\varepsilon}{\pi}\log\left(\frac{\varepsilon^2+(x-y)^2}{\varepsilon^2}\right)-\frac{\varepsilon(x-y)^2[\varepsilon^2+3(x-y)^2]}{2\pi[\varepsilon^2+(x-y)^2]^2} \\
&\quad+\frac{xy\varepsilon[2(x-y)^4-\varepsilon^2(x-y)^2+\varepsilon^4]}{\pi[\varepsilon^2+(x-y)^2]^3}.
\end{aligned}
$$

Equations (4.36) and (4.37) are now our basic integral equations. As in Section 2.3, we are interested in solving these equations numerically. We begin by appling the same Nyström method that we used in Section 2.3 to each of our integral equations, giving

$$x_j \approx f_{0,j} - \sum_{i=1}^{n} w_i\{k_{00,ji}f_{0,i} + k_{01,ji}f_{1,i}\} \quad j = 1, \ldots, n,$$
$$0 \approx x_j^2 f_{1,j} - \sum_{i=1}^{n} w_i\{k_{10,ji}f_{0,i} + k_{11,ji}f_{1,i}\} \quad j = 1, \ldots, n,$$

where w_i are the weights, and $x_i = y_i$ are the quadrature points. We write $f_{u,j} = f_u(x_j)$, and for the terms in the summation, we write $f_{v,i} = f_v(y_i)$, $k_{uv,ji} = k_{uv}(x_j, y_i; \varepsilon)$, with x_j fixed.

Now we can set up a linear system, where the unknown vector is composed of values from both f_0 and f_1. Therefore, we have

$$\begin{bmatrix} 1 - w_1k_{00,11} & -w_1k_{01,11} & \cdots & -w_nk_{00,1n} & -w_nk_{01,1n} \\ -w_1k_{10,11} & x_1^2 - w_1k_{11,11} & \cdots & -w_nk_{10,1n} & -w_nk_{11,1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_1k_{00,n1} & -w_1k_{01,n1} & \cdots & 1 - w_nk_{00,nn} & -w_nk_{01,nn} \\ -w_1k_{10,n1} & -w_1k_{11,n1} & \cdots & -w_nk_{10,nn} & x_n^2 - w_nk_{11,nn} \end{bmatrix} \begin{bmatrix} f_{0,1} \\ f_{1,1} \\ \vdots \\ f_{0,n} \\ f_{1,n} \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ x_n \\ 0 \end{bmatrix},$$

where $n = N + 1$, with weights given by

$$w_1 = \frac{1}{N} = w_n \quad \text{and} \quad w_i = \frac{2}{N} \quad \text{for } i = 2, N.$$

To simplifiy the linear system, we define $F_l = f_{0,i}$ for odd values of l, and we define $F_l = f_{1,i}$ for even values of l. So, we have

$$F_l = \begin{cases} f_{0,i}, & l = 2i - 1, \\ f_{1,i}, & l = 2i. \end{cases}$$

Similarly, we define $G_m = x_j$ for odd values of m, and we define $G_m = 0$ for even values of m. So, we also have

$$G_m = \begin{cases} x_j, & m = 2j - 1, \\ 0, & m = 2j. \end{cases}$$

Now, we define

$$K_{ml} = \begin{cases} -k_{00,ji}, & m = 2j - 1, l = 2i - 1, \\ -k_{01,ji}, & m = 2j - 1, l = 2i, \\ -k_{10,ji}, & m = 2j, l = 2i - 1, \\ -k_{11,ji}, & m = 2j, l = 2i, \end{cases}$$

and

$$I_m = \begin{cases} 1, & m = 2j-1, \\ x_j^2, & m = 2j. \end{cases}$$

Therefore, rewriting our system using m and l, we have

$$\begin{bmatrix} I_1 + w_1 K_{11} & \cdots & w_l K_{1l} & \cdots & w_\nu K_{1\nu} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ w_1 K_{m1} & \cdots & I_m + w_l K_{mm} & \cdots & w_\nu K_{m\nu} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ w_1 K_{\nu 1} & \cdots & w_l K_{\nu l} & \cdots & I_\nu + w_\nu K_{\nu\nu} \end{bmatrix} \begin{bmatrix} F_1 \\ \vdots \\ F_l \\ \vdots \\ F_\nu \end{bmatrix} = \begin{bmatrix} G_1 \\ \vdots \\ G_m \\ \vdots \\ G_\nu \end{bmatrix},$$

where $\nu = 2(N+1)$, with weights given by

$$w_1 = w_2 = \frac{1}{N} = w_{\nu-1} = w_\nu \quad \text{and} \quad w_l = \frac{2}{N} \quad \text{for } l = 3, \ldots, 2N.$$

We return to using MatLab to solve for the vector $(F_1, \ldots, F_\nu)$, and subsequently for the vectors $(f_{0,1}, \ldots, f_{0,n})$ and $(f_{1,1}, \ldots, f_{1,n})$. Depicted in Figure 4.1, we have plots of the solutions $f_0(x)$ and $f_1(x)$, as $f_{0,i}$, $f_{1,i}$ vs. x_i, $i = 1, \ldots, n$, for multiple values of ε.

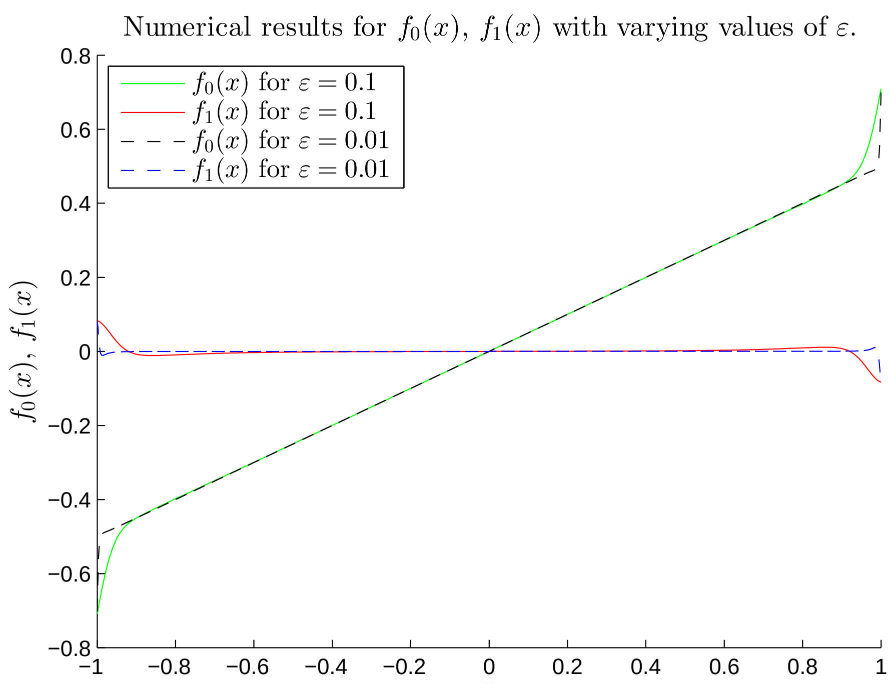


Figure 4.1: Numerical solution to dual integral equations

Of particular interest are the values $f_0(1)$ and $f_1(1)$. These determine the stress intensity factors. We now use MatLab in a similar fashion, but examine $f_0(1)$ and

$f_1(1)$ for varying ε, with special interest in small values of ε. Depicted in Figure 4.2, we have a plot of $f_0(1)$ and $f_1(1)$ as functions of ε.

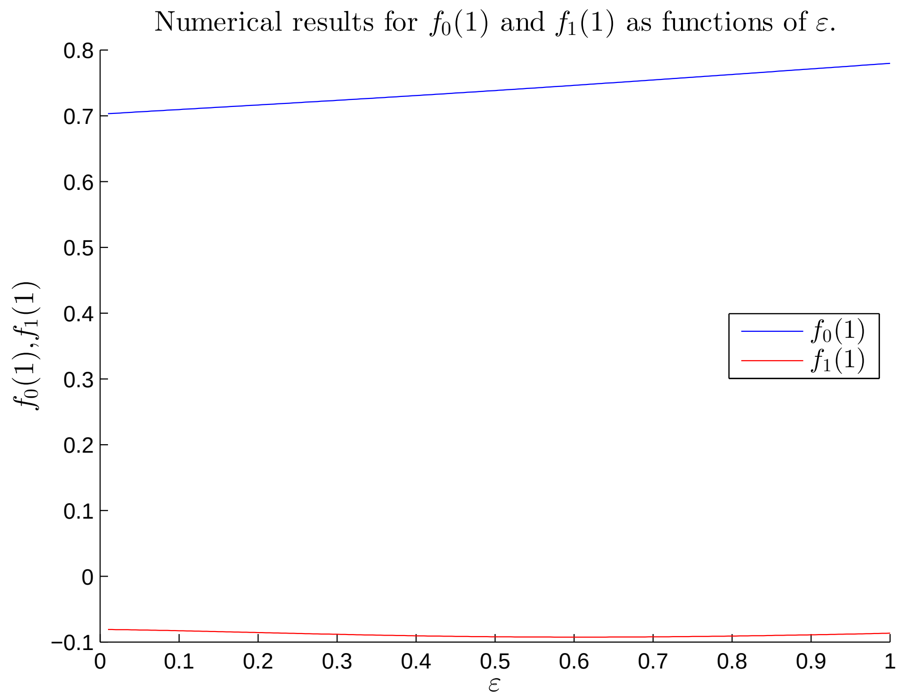


Figure 4.2: $f_0(1)$ and $f_1(1)$ for varying ε

For the range of ε covered in the plot for $f_0(1)$ and $f_1(1)$, the plot depicts a noticeable variance in $f_0(1)$ and a slight variance in $f_1(1)$. To get a better idea of how these values are varying, we examine a few specific values, depicted in Table 4.1.

Table 4.1: Values of $f_0(1)$ and $f_1(1)$ for various ε

ε	$f_0(1)$	$f_1(1)$
1.000	0.776200	-0.085027
0.100	0.707674	-0.080895
0.010	0.701542	-0.079042
0.005	0.701197	-0.078983

Based on these values, as ε decreases, $f_0(1)$ also decreases, tending toward 0, while $f_1(1)$ increases, also tending toward 0. Since $f_0(1)$ and $f_1(1)$ give the stress intensity factors, this implies that the stress intensity factors are dependent upon the distance between the crack and the half-space. Therefore, the fracture conditions at the edge of the crack are dependent upon the depth of the buried penny-shaped crack.

CHAPTER 5

CONCLUSION, RESULTS, AND EXTENSIONS

While our analysis has indicated that there is some depth dependence in the *stress intensity factors*, the exact nature of this dependence, for exceptionally small ε, is still unclear. The values of $f_0(1)$ and $f_1(1)$ appear to converge for small values of ε, but closer examination of smaller values could indicate a more complicated dependence. The numerical method used for this analysis requires an increasingly large number of quadrature points as ε becomes small, which requires increasing amounts of computational resources. Restrictions on computational resources led to the limitations in our current analysis, and indicate possible extensions.

There are numerous alternative numerical methods that could be applied to reduce resource consumption, allowing for the examination of the stress intensity factors at smaller ε values. Such alternatives could be applied by choosing a different quadrature rule, and thus choosing different weights within the linear system. The repeated trapezium rule was chosen for the sake of simplicity, offering a good first approach to the numerical methods. There are other methods that have proven to be more accurate, with fewer quadrature points, when applied to *Love's Integral equation.* Such methods could lead to a better choice for finding the stress intensity factors in the *buried penny-shaped crack* problem that we have examined here.

Additionally, there could be alterations made to the linear system to make it more amenable to numerical solution. Reworking the system of equations, or simplifying the matrix, could allow for more efficient computations. Alternatively, using a factorization method could also improve the efficiency of the numerical method. Creating a more efficient algorithm, or applying the current numerical method with more resources, would allow for smaller values of ε to be used. Therefore, these possible extensions would allow for further confirmation of the depth dependence of the stress intensity factors.

A final avenue for extension involves determining the exact stress intensity factors from the values found. The values of $f_0(1)$ and $f_1(1)$ are simply related to the stress intensity factors, and can be used to determine them exactly. The specifcs of this relation have not been explored here, and could be outlined further given a deeper understanding of the depth dependence in this crack problem.

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APPENDIX

In Section 2.2, if we choose not to assume $g(0) = 0$, then (2.27) becomes

$$A(\kappa) = \int_0^a g(t)\sin(\kappa t)\,dt = \frac{g(0)}{\kappa} - g(a)\frac{\cos(\kappa a)}{\kappa} + \frac{1}{\kappa}\int_0^a g'(t)\cos(\kappa t)\,dt. \tag{A.1}$$

Substituting into (2.21) gives

$$\int_0^\infty \left[g(0) - g(a)\cos(\kappa a) + \int_0^a g'(t)\cos(\kappa t)\,dt\right] J_0(\kappa r)\,d\kappa = -1.$$

Now, using (Abramowitz & Stegun, 1972, Equation 11.4.17),

$$\int_0^\infty J_0(t)\,dt = r\int_0^\infty J_0(\kappa r)\,d\kappa = 1,$$

we have

$$\frac{g(0)}{r} + \int_0^r \frac{g'(t)}{\sqrt{r^2 - t^2}}\,dt = -1. \tag{A.2}$$

Now, consider

$$I \equiv \frac{d}{dr}\int_0^r g(t)\frac{t}{\sqrt{r^2 - t^2}}\,dt.$$

To evaluate I, we first integrate by parts, letting $u = g(t)$ and $dv = t(r^2 - t^2)^{-1/2}\,dt$.

This gives

$$I = \frac{d}{dr}\left\{rg(0) + \int_0^r g'(t)\sqrt{(r^2 - t^2)}\,dt\right\}.$$

Now, we can differentiate, giving

$$I = g(0) + \int_0^r \frac{g'(t)r}{\sqrt{(r^2 - t^2)}}\,dt,$$

which is exactly r multiplied by the left-hand side of (A.2). Therefore, $I = -r$.

Integrating I gives

$$\int_0^r \frac{g(t)t}{\sqrt{r^2 - t^2}}\,dt = -\frac{1}{2}r^2 + A,$$

where $A = 0$, since the left-hand side is 0 when $r = 0$.

Therefore, we have an Abel-type integral equation for $tg(t)$. Using (Duffy, 2008, Equation 1.2.13) and (Duffy, 2008, Equation 1.2.14) gives

$$tg(t) = -\frac{1}{\pi}\frac{d}{dt}\left[\int_0^t \frac{r^3}{\sqrt{t^2 - r^2}}\,dr\right] = -\frac{1}{\pi}\frac{d}{dt}\left[\frac{2}{3}t^3\right].$$

Hence, $g(t) = -2t/\pi$, and so $g(0)$ does equal 0, and the solution is the same as in Section 2.2.