AN INNOVATIVE METHOD FOR TEACHING LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

by

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ABSTRACT

Differential equations lend themselves to many applications in applied mathematics, science and engineering. A unique method of teaching them is proposed; in particular, a unit on linear second order differential equations is presented. The format is a combination of popular teaching methods already in use in college classrooms. Among the methods employed are mastery and spiral learning, discovery learning, technology-assisted learning, cooperative learning, oral and written communication, and the traditional lecture method. It is reasoned that a combination of these methods may produce an increase in student understanding, due to variations in learning styles among students.

The introduction explains the various learning methods in use in college classrooms. In particular, it looks at the methods which will be combined into the textbook unit. It gives research support for the various methods and discusses their possible uses and limitations. An actual textbook unit is presented which combines these methods in a unique format. A sample problem set is included to show how the methods will be incorporated there, also. Future research, testing, and implementation are discussed.

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DEDICATION

To my wife, and best friend, Sheila, with all my love.

INTRODUCTION

0.1 Motivation

Differential equations lend themselves to many applications. The study of them is a key course in applied mathematics, science and engineering at most colleges and universities [23]. This thesis is written with the intention of helping students learn the fundamentals of differential equations in a new and exciting way. In particular, we focus on a unit that is normally one of the major topics covered in differential equations courses, linear second order differential equations. The choice of this unit is an arbitrary one, and we are more concerned with the method of text construction used than in the unit itself.

The basic motivation for the development of this text-writing method stems from

> a growing uneasiness among teachers who feel that their old teaching methods simply do not work as well as they used to. This uneasiness is confirmed by a growing body of research on learning theory that challenges traditional ways we think about teaching. (Meyers and Jones, 1993, p. 4) [39]

In recent years, new information and insights have challenged the traditional ways that we think about learning. We have found that: "Learning is not so much an additive process, with new learning simply piling upon top of existing knowledge, as it is an active, dynamic process in which the connections are constantly changing and the structure reformatted" (Cross, 1991, p. 9) [14]. Thus, we have the motivation for new techniques of teaching and learning which have been introduced into the classroom.

The main reason for investigating different methods of learning is found in the research that teaches us that people learn in different ways [10, 34]. Learning style consists of cognitive along with affective and physiological styles [33]. The changing demographics of college students include a substantial increase in adult students in the undergraduate population since the 1970's [2], as well as growing numbers of women and culturally diverse students that bring differing ideas and needs for their education [41].

As one example of different learning styles, Banks [3] found important differences in the preferred learning styles of students from African American and Mexican American cultures and their Anglo-American counterparts. He found that the former are generally more *field sensitive* [56] in their learning styles. One characteristic of field sensitive learners is that they like to work in groups to achieve a common goal. This might suggest that these students would do better in a cooperative-learning environment. He cautioned, however, to avoid stereotyping: "Teachers should recognize that students bring a variety of learning, cognitive, and motivational styles to the classroom, and that while certain characteristics are associated with specific ethnic and social class groups, these characteristics are distributed throughout the total student population" (Banks, p. 466) [3].

The important thing to remember is that people learn in different ways and that individuals can learn how to learn in a variety of ways [39, p. 10]. We do not necessarily need to know the exact learning style of each student. It is more important that we "act upon the assumption that in any group of people a diversity of [learning] styles will be present" (Guild and Garger, 1985, p. 89) [24]. Thus, it is with this in mind that we consider some different methods of learning, including mastery and spiral learning, discovery learning, technology-assisted learning, cooperative learning, and techniques of increasing communication in the classroom.

0.1.1 Mastery and Spiral Learning

Mastery and spiral learning are two slightly different methods with similar outcomes and general ideas. The first of these generally involves sequential learning of specific objectives that must be mastered, before proceeding to new objectives. Spiral learning is usually sequential as well, but does not have to be. Topics are revisited repeatedly in order that mastery might occur through repetition over a long period of time. The idea is that: "If mathematical concepts are practiced for a long time, they become familiar concepts, and familiar concepts are not troublesome" (Saxon, 1989a, p. xi) [62].

Mastery Learning

The main belief of mastery learning is that all students can learn when the conditions are appropriate for their learning [6]. The applications of mastery learning are generally based on Bloom's [6] ideas on mastery learning theory and the refinements of it by Block [5]. It is a learning model that is generally group-based and teacher-paced, where students learn in cooperation with their classmates.

The instruction focuses not on the content of the material, but on the mastering of the content. Specific performance objectives are set which the student must reach before proceeding to the next sequential unit of study. Teachers present the students with regular and specific feedback on their learning progress through the use of diagnostic formative tests. Students are evaluated through criterion referenced tests rather than norm referenced tests and are given alternative corrective activities when needed [66]. According to Bloom [6], mastery learning is most effective when the student has achieved the prerequisite learning for the subject under consideration.

Slavin states that "mastery learning students achieved at twice the level of nonmastery students in terms of percent correct on daily chapter tests". However, he goes on to say that they "spent more than twice as much time learning the same material" (Slavin, 1987, p. 8) [66]. The other major concern with mastery learning is that due to the fact that some students learn faster than others, one of two things must happen. Either the corrective instruction needed for the slower students to catch up must be given outside the regular class time, or faster students will have to spend considerable amounts of time waiting for their classmates to catch up [66, p. 6].

Spiral Learning

Closely related to the idea of mastery learning is the concept of spiralling. This method involves repeatedly touching on previous concepts while slowly incrementing new ones. This method once again lends itself to mastery through the continuous reinforcement of previous ideas. The general belief is that "there is only one way to acquire [academic] skills and abilities, and that is to practice them" (Bouton and Garth, 1983, p. 79) [8].

The idea of spiralling is not a new one, but one of the latest renovators of the method is John Saxon in his K-12 math and physics books [62, 64, 63]. Saxon's method involves the introduction of a new concept on an incremental level. The problem sets consist of only a few problems concerned with the new concept. The rest of the problems are review problems. Gradually, more abstract concepts are added to the previous concept after the concept has been practiced for several lessons. The method is controversial, but has produced significant increases in ACT scores and increased enrollment in upper level high school math classes, according to Saxon [63]. For example, Maine East High School in Des Plaines, Illinois found their enrollments in calculus go from two sections to ten sections, "including 48 students who will complete the full three-semester sequence of Calculus I, II, and III and a full semester of college differential equations by the end of May. The ACT scores in mathematics at Maine East have increased 19 percent" (Saxon, 1993, p. xvii) [63].

Critics of the Saxon method state that it does not conform to the NCTM Standards [43] published in 1989 [61]. Thus, several states have not allowed Saxon textbooks to be put on their textbook adoption list [61]. The Saxon method is controversial, to say the least, but it is this author's personal opinion that the spiral approach is a valuable learning tool, although not the only one.

0.1.2 Technology Assisted Learning

In 1989, the National Council of Teachers of Mathematics (NCTM) published their *Curriculum and Evaluation Standards for School Mathematics* [43], in which they called for the full use of technology in the classroom at all grade levels. We investigate the use of two technologies in more detail; the graphing calculator, and symbolic manipulators, which are two of the more popular technologies in use today.

Graphing Calculators

The use of graphing calculators in the classroom is a relatively inexpensive and convenient way of accessing a visual aide in problem solving. Students can quickly graph a multitude of functions, which would take a great deal of time to do by hand. They can also *zoom* in and out to investigate the short-term or long-term behavior of a function. The multitude of menus allow them to do much more, such as *trace* the graph of a function, *evaluate* the function at a point, or even look at the slope of the tangent line at a point on the graph. This is only a very short list of the many capabilities of these powerful little units [70]. Newer models now have the ability to do symbolic manipulations [44].

Graphing utilities can help make graphing a fast and effective problem-solving strategy. They allow students to graph many functions quickly and establish common properties of classes of functions, explore and discover mathematical concepts, and use graphs to solve problems. This was not usually possible in the past due to the time involved in producing accurate graphs [17].

Symbolic Manipulators

The power and popularity of symbolic manipulators appeal to many in mathematics education. Among these symbolic packages are Mathematica, Maple, Macsyma, and also Derive, which can be found on many of today's supercalculators [44]. These packages have the ability to simplify algebraic expressions, solve equations, differentiate and integrate functions. The idea behind their use is, once again, primarily to reduce the time used in algebraic manipulations. They allow students to explore general concepts and solutions in more detail, without spending time on the actual hand derivations. Many of them can also produce two-dimensional and even three-dimensional graphs which students could not otherwise perceive. Symbolic packages can take mathematical manipulations which would take days by hand and solve them in a few minutes, or even seconds [78]. Since "the process of uncovering mathematical ideas on one's own is exciting and stimulating" (Sours, 1991, p. 223) [69], and is, indeed, "*the very essence of the learning process*" (Sours, p. 223) [69], these symbolic manipulators can create a great opportunity for student discovery.

One of the main problems with using symbolic manipulators is the availability of laboratory access to computers. It is impractical today for schools to acquire enough desk-top computers to give mathematics classes regular access to them [75]. Waits and Demana [75] also claim that pocket computers with Computer Algebra Systems (CAS) are impractical because of their expense and complexity. They believed that CAS systems are the way to go in the future, but for now the use of simple graphing calculators is sufficient to make mathematics enjoyable [75]. On the other hand, Dick [19] suggests that these so-called "supercalculators" which contain symbolic manipulators, can be used in a recitation session very readily. He also says that skill labs should "center on the skills needed to use a supercalculator *intelligently*" (Dick, pp. 27-29) [19]. Very few students (and instructors, for that matter) have really analyzed the effects that numerical limitations can cause [19].

0.1.3 Cooperative Learning

Two of the more recent methods of group learning are cooperative learning and what is known as "workshopping".

Workshopping

The method of workshopping was first popularized by Uri Treisman [73] when he tried to understand and remedy the high failure rate of very talented African-Americans in beginning calculus courses at UC Berkeley. By comparing their habits with those of very successful Asian-American students, he concluded that most of the African-Americans worked almost exclusively on their own, while Asian-American students tended to move into "study gangs" to clarify their understanding and work toward completing assignments and studying for tests. Thus, Treisman developed a formal program of workshops for high risk students which focused on the skills required to be a successful student of mathematics, rather than on remedial skills. At

Berkeley, the workshop approach is now a formal part of the curriculum, and it has been tried by more than 30 universities with consistently positive results [76].

A similar well-known technique is cooperative learning. At the university level, one of the originators is Karl Smith at the University of Minnesota. He has worked with some of the educational leaders in the field to apply cooperative learning to engineering education [32, 68]. Cooperative learning may be applied to assignments given during class, or assigned as homework. Each student is given a role essential to the completion of the task, and students are to support each other's learning. They are accountable as a group for developing skills, as well as individually. The work is done when every student knows how to solve the problem, not just their individual part of the problem. Johnson and Johnson [30] have statistically combined the results of many studies into a mega-analysis and found that the average student working in cooperative class assignments scored two-thirds of a standard deviation above students working competitively or individually.

Cooperative learning methods are highly structured, when done correctly. It is important to use cooperative learning correctly. For example, simply putting students in groups to learn is not cooperative learning. Cooperation is not having students sit side-by-side to talk with each other as they do in their individual assignments. It is not assigning a project in which one student does all the work. Nor is cooperative learning assigning a task where the first students done help the slower students [31]. According to Johnson, Johnson, and Smith, in order to be cooperative, a group must have:

- *positive interdependence* - the members of the group must believe that one cannot succeed unless the other members of the group succeed (and vice versa).
- *face-to-face promotive interaction* - students help, assist, encourage, and support

each other's efforts to learn.

- *individual accountability* - students must hold each other personally and individually accountable for doing their share of the work.
- *appropriate social skills* - individuals must have and use leadership, decision-making, trust-building, communication, and conflict-management skills appropriately.
- *group processes* - the group must analyze how effectively group members are working together [31, pp. 1:18-1:20].

There is considerable evidence that cooperative learning also promotes higher-order thinking [29, p. 57] [65, p. 251]

0.1.4 Discovery Learning and Active Learning

Webster's Dictionary [77] defines the word *discover* as, "to obtain sight or knowledge of for the first time." The root word *cover* "presupposes exploration, investigation, or chance encounter and always implies the previous existence of what becomes known." This, then, is the essence of discovery learning: learning through experience and investigation. Recent mathematics education reform papers [42, 43, 58, 71] call for students to have more flexible problem-solving skills, and show a growing consensus that students should learn through inquiry and through the construction of their own mathematics [16, 27, 43, 52]. It is helpful to consider Piaget's concept of mental structures [53, p. 119], which says that "children do not receive knowledge passively but rather discover and construct knowledge through activities" [39, p. 13]. As Mary Field Belenky (1986) and her coauthors suggest in *Women's Ways of Knowing*, our task as educators is to be "midwife-teachers" who help students give "birth to their own ideas, in making their own tacit knowledge explicit and elaborating it" [4, pp. 217-218].

The teacher or facilitator determines the degree of linearity or non- linearity in the students' inquiries. The amount of guidance can be high or low. For example, a student's reasoning can be guided by little more than a series of yes/no questions as in Plato's *Protagoras and Meno* [54]; or the student can be engaged in inquiry with no active support, as in Bruner's *The Act of Discovery* [11]. The limits on the amount and direction of discovery are controlled by the mentor and the external environment.

Closely related to the idea of discovery learning is the phrase *active learning* which stands in contrast to traditional classroom styles where teachers do most of the work and students remain passive [39]. Active learning involves two basic assumptions:

- that learning is by nature an active endeavor;
- that different people learn in different ways.

Two corollaries seem to follow from these: first, that students learn best by doing, and second, that teachers who rely exclusively on any one teaching approach often fail to get through to significant numbers of students. Active learning provides opportunities for students to *talk and listen, read, write*, and *reflect* using problem-solving exercises, informal small groups, simulations, case studies, role playing, and other activities - all of which require students to *apply* what they are learning [39, p. xi].

0.1.5 Communication in Mathematics

The National Council of Teachers of Mathematics (NCTM) in its publication "Curriculum and Evaluation Standards for School Mathematics" in 1989 [43], states the following:

> In grades 9-12, the mathematics curriculum should include the continued development of language and symbolism to communicate mathematical ideas so that all students can:
>
> - reflect upon and clarify their thinking about mathematical ideas and relationships;
> - formulate mathematical definitions and express generalizations discovered through investigations;
> - express mathematical ideas orally and in writing;
> - read written presentations of mathematics with understanding;
> - ask clarifying and extending questions related to mathematics they have read or heard about;
> - appreciate the economy, power, and elegance of mathematical notation and its role in the development of mathematical ideas.

Writing in Mathematics

Writing in mathematics can force students to think about concepts and relationships. It provides an opportunity for them "to organize, interpret, and explain, to construct, symbolize, and communicate, to plan, infer, and reflect" (Countryman, 1992, p. 12) [13]. Writing in mathematics can take the form of essays, short answers to questions, listing steps in a problem-solving method, and others. Countryman [13, pp. 13-20] suggests that teachers introduce writing into their classrooms by means of learning logs - personal accounts of work done in class; freewrites, which often get at the heart of students' attitudes and feelings; finishing sentences; commenting on assignments; finding definitions; and writing comparisons of different procedures. Journal writing is another vehicle for student self-expression, in which students keep journals that the teacher comments in every week.

Some of the goals of journal writing can be found in Countryman's purposes of journals:

- to increase confidence;

- to increase participation;
- to decentralize authority;
- to encourage independence;
- to replace quizzes and tests as a means of assessment;
- to monitor progress;
- to enhance communication between teacher and student;
- to record growth [13, pp. 42-43].

She goes on to state that students often have trouble with word problems because of a larger inability to understand math language, but that writing about their learning processes helps reveal how much they do or do not understand [13, pp. 45-57].

Oral Communication

The value of oral communication in mathematics cannot be overlooked, also. "The reason we need to provide time and activities for students to talk and listen to each other is that talking and listening discipline them to be clearer about their thinking" [39, p. 22]. Lochhead and Whimbey [36] advocate a process of problem solving called Thinking Aloud Pair Problem Solving (TAPPS). Students are grouped in pairs, and one student "talks aloud" while the other listens and tries to clarify what is being said. It is successful because students must be aware of their thought processes as they solve problems.

Student-to-teacher communication is also important. As an example, Wulff, et al., [79] did a three-year study of large lecture hall classes. They found that students surveyed felt that one of the biggest impediments to their learning was "the lack of instructor-student interaction, with opportunities for questions and discussion" (Wulff, 1987, p. 29) [79].

Questions can be a good method of starting oral communication and feedback. It is important that "wait time" be used in order to allow students time to reflect. One study found that the typical teacher pauses only 0.9 second between asking a question and supplying the answer [67]. As the noted educator John Dewey once observed: "All reflection involves, at some point, stopping external observations and reactions so that an idea may mature" [18, p. 210]. Questions and the proper "wait time" can allow reflection and elicit responses from students who normally would not respond.

0.1.6 Lecture

The traditional teaching method is the teacher-centered lecture format. As William E. Cashin [12] points out, lectures can:

- *provide information* that is new, based on original research, and generally not found in textbooks and other printed sources;
- *highlight similarities and differences* between key concepts;
- *help communicate* the enthusiasm of teachers for their subjects;
- *model* how a particular discipline deals with questions of evidence, critical analysis, problem solving, and the like;
- *dramatize* important concepts and *share* personal insights;
- *organize* subject matter in a way that is best suited to a particular class and course objectives [12].

College teachers in many disciplines contend that students need background information, concepts, and methods *before* they can learn much on their own and become effective participants in classroom discussions [39, pp. 13-14].

Some of the documented weaknesses of the lecture method can be found in the following:

What Teachers Would Rather Not Know

- While teachers are lecturing, students are not attending to what is being said 40 percent of the time [55, p. 11].
- In the first ten minutes of a lecture, students retain 70 percent of the information; in the last ten minutes, 20 percent [38, p. 72].
- Students lose their initial interest, and attention levels continue to drop, as a lecture proceeds [74, pp. 90-91].
- Four months after taking an introductory psychological course, students knew only 8 percent more than a control group who had never taken the course [59, pp. 151-152].

As Eleanor Duckworth, quoting Dawkins, states in *The Having of Wonderful Ideas* [21], we are usually so busy "covering the material" that we miss the chance to "uncover it" with our students. We need to give the students more time to dig into the subject matter and make their own sense of it. Chances are they will be more likely to retain and use the information, if we do [39, p. 14].

0.2 Implementation

It is with these methods in mind that we discuss their integration into the study of linear second order differential equations. We implement the methods of spiralling, discovery, communication, technology- assistance, and lecture into the text in two locations, the lesson itself and the problem set.

0.2.1 The Lesson

We intertwine the lecture method with questions that involve both spiralling and discovery, which helps create student-teacher communication and feedback. A graphing calculator section is used as a convenient access to visual explorations and discovery. Also, since "the CAS (Computer Algebra System) is an excellent *discovery* tool" (Sours, 1991, p. 223), we use the popular symbolic programming language Mathematica to do explorations and discoveries of new problem-solving methods, as well as investigating the problems, themselves.

0.2.2 The Problem Set

The underlying foundation for the construction of the problem sets is the spiral method. In each problem set, there are two to three problems pertaining to the material from that particular lesson. The rest of the problems, except for the group/challenge problem, are review problems covering previous lessons, with a greater number of problems from the more recent material, in general. Each topic is reviewed for at least four to five lessons, then interspersed at increasing intervals. It is conjectured that the constant review over a longer period of time will lead to long-term retention and mastery of the concepts. Once the concepts are practiced for a period of time, more abstract use of them is emphasized.

Within the problem set is a group/challenge problem. This provides an opportunity for students to work cooperatively in groups on a discovery-type problem. It is conjectured that this will lead to better communication between students, promote active learning, and create a positive atmosphere in which students will share their ideas and learn from each other. It is suggested that heterogeneous groups of four are formed, and specific group behavior guidelines established early. It is recommended that a method such as that employed by Johnson, et al. [31], or one similar to the "workshopping" technique first popularized by Treisman [73] be used.

Students are then asked to write individually about what they learned in this lesson, as well as to record their thoughts and feelings about the lesson, problem set, and group work. This provides feedback to the teacher about how the student feels about his/her educational experience, which can help the teacher make needed corrections to improve learning conditions.

TEXTBOOK UNIT INTRODUCTION AND MOTIVATION

Linear equations of second order are of crucial importance in the study of differential equations as they are vital to the study of mathematical models of some important physical processes. In the investigation of fluid mechanics, heat conduction, wave motion and electromagnetic phenomena, it is necessary to be able to solve second order linear differential equations. Specifically, in this study, we will investigate mechanical and electrical systems, and immediate applications of differential equations to real world phenomena.

The student should read each lesson thoroughly and work through the examples carefully with pencil in hand. The *Questions* sections should be answered to the best of the student's ability. The questions labeled *Spiral* are designed to reinforce earlier concepts. If they cannot be answered confidently, a review of previous material may be in order. The questions labeled *Discovery* are included to help the student transfer his knowledge to other concepts. Some of them are open-ended questions, that are designed to promote active thinking. There are some *graphing calculator* exercises to perform that allow the student to visualize and investigate functions. At the end of the lesson are some *Mathematica* examples that can help introduce the student to the power of computer algebra systems.

The study of differential equations requires that one master certain concepts. As one rarely masters a concept on the first encounter, problem sets are written with the spiral concept. There is also a group/challenge problem that requires students to work cooperatively, and to apply their knowledge to solving problems. Spiralling is the main focus of the problem sets, however. That is, there are only a few problems of the actual type the lesson introduces. The rest of the problem set is primarily made up of review problems. The specific type of problem introduced in that lesson will occur again in future problem sets. The idea is that if students practice a concept, they will not only have learned it, they will have mastered it. Long-term retention will replace short-term memory. This will allow students to apply that knowledge more easily to other (possibly more abstract) concepts. The student should focus on DOING EVERY PROBLEM IN EVERY PROBLEM SET. Even the world's greatest performers and athletes did not get to be great without practicing daily.

Chapter 1

DERIVATION OF THE CHARACTERISTIC EQUATION

Recall from calculus that $\frac{d}{dx}\, e^u = e^u\, \frac{du}{dx}$. Then, if (Spiral)

$$y = e^{6x}, \text{we have } y' = e^{6x} * 6 = 6e^{6x}, \text{ and } y'' = 36e^{6x}.$$

In fact, in general,

$$y = e^{\lambda x},\ y' = \lambda e^{\lambda x}, \text{ and } y'' = \lambda^2 e^{\lambda x}.$$

Each time we differentiate we obtain $e^{\lambda x}$ multiplied by a constant. Can you think of another function for which this is true? (Discovery)

Now, consider the equation

$$ay'' + by' + cy = 0, \tag{1.1}$$

where $a \neq 0$, b, and c are real constants. Equation (1.1) is called a **homogeneous linear second order differential equation** with constant coefficients. It is **homogeneous** because the right hand side of the equation is zero for all x. If the right hand side is not zero, we call the equation **nonhomogeneous**.It is called **linear** since y and all its derivatives occur to the first power,and y does not occur as a product with any of its derivatives. Nor do products of its derivatives occur with each other. Also, no transcendental functions of y and/or its derivatives may occur. For example, the homogeneous differential equation

$$x^2y'' + yy' + 3xy = 0$$

is **nonlinear** because of the middle term's product. Looking at equation (1.1), we can see that a solution must have the property that a constant times its second derivative plus a constant times its first derivative plus a constant times itself must sum to zero.

Questions

What types of functions might we try for this?

What difficulties arise from guessing $y = \sin x$?

What about the polynomial $y = x^3$?

Suppose we try a solution of the form $y = e^{\lambda x}$. Substituting $y = e^{\lambda x}$ and it derivatives into (1.1), we obtain

$$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0 \tag{1.2}$$

or

$$e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0. \tag{1.3}$$

How do we get from (1.2) to (1.3)?

Graphing Calculator

Investigate the graph of $e^{\lambda x}$ on your graphing calculator for $\lambda = -1, -1/2, 0,$ and 1, respectively. Does it ever cross the x-axis?

This is not a proof that the functions never cross the axis, but it is very suggestive.

If $e^{\lambda x}$ is never zero, we can divide by it to obtain

$$a\lambda^2 + b\lambda + c = 0. \tag{1.4}$$

Questions

What is this type of equation called?

How do we solve it? (Spiral)

We have shown that $y = e^{\lambda x}$ is a solution to (1.1) if λ satisfies equation (1.4). Equation (1.4) is called the **characteristic equation** or **auxiliary equation** associated with the homogeneous equation (1.1).

Mathematica

You can use the Mathematica commands

```
y = Exp[lambda*x]
yp = D[y,x]
ypp=D[yp,x]
```

to input the function $e^{\lambda x}$ and differentiate it twice. Try finding *yp* and *ypp* for the functions

y=Sin[lambda*x] and y=lamda*x^3.

Can you find constants a, b, and c so that these equations will solve equation (1.4)?

Chapter 2

CHARACTERISTIC EQUATION (DISTINCT REAL ROOTS)

Previously, we derived the characteristic equation (Spiral)

$$a\lambda^2 + b\lambda + c = 0 \qquad (1.4)$$

from the homogeneous linear second order equation

$$ay'' + by' + cy = 0. \qquad (1.1)$$

Then, $y = e^{\lambda x}$ is a solution to (1.1) if λ satisfies equation (1.4).

What kind of equation is (1.4) and how do we solve it? (Spiral & Discovery)

We find the roots of equation (1.4) are

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Thus, $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$ are solutions to equation (1.1).

Questions

How would you verify, using two different methods, that $y_1 = e^{-3x}$ and $y_2 = e^x$ are both solutions to the differential equation

$$y'' + 2y' - 3y = 0? \qquad (2.1)$$

Are there other exponential functions that solve the differential equation (2.1), say $y = e^{12x}$? How do you know?

Are $y_1 = 2e^{-3x}$ and $y_2 = 4e^x$ both solutions to (2.1)?

Are $y_1 = -3e^{-3x}$ and $y_2 = -e^x$ solutions?

Does $y = e^{-3x} + e^x$ also solve (2.1)?

What about $y = 2e^{-3x} + 4e^x$?

From your answers to the above questions can you state a general rule for the form of the solutions to equation (2.1)? (Discovery)

It is in fact true that if y_1 and y_2 are solutions to (1.1), then so is $cy_1 + cy_2$.

Theorem 1 *Principle of Superposition*

If y_1 and y_2 are two solutions of the differential equation,

$$y'' + p(x)y' + q(x)y = 0,$$

then the linear combination

$$y(x) = c_1y_1 + c_2y_2$$

is also a solution for any values of the constants c_1 and c_2.

Proof The proof follows directly from the fact that differentiation is a *linear* operation. In other words, since the derivative of a sum is the sum of the derivatives, we have

$$y' = c_1y_1' + c_2y_2' \text{ and } y'' = c_1y_1'' + c_2y_2''.$$

Then

$$\begin{aligned} y'' + py' + qy &= (c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2) \\ &= (c_1y_1'' + c_2y_2'') + p(c_1y_1' + c_2y_2') + q(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) \\ &= (c_1)(0) + (c_2)(0) = 0, \end{aligned}$$

since y_1 and y_2 are solutions. Thus $y = c_1y_1 + c_2y_2$ is also a solution to (1.1). ∎

The principle of superposition does not hold for nonlinear differential equations. It also does not hold for nonhomogeneous linear equations.

(Note: If $c_1 = c_2 = 0$, we have what is called the *trivial solution* $y(x) \equiv 0$, which always satisfies the homogeneous linear differential equation.)

The principle of superposition also generalizes to *homogeneous* n^{th} order *linear* differential equations of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0. \tag{2.2}$$

The proof is omitted since it is essentially the same as for the second order case.

Theorem 1 *Principle of Superposition*

If y_1, y_2, $\cdots$, y_n are n solutions of the homogeneous linear equation (2.2) on some interval I, then the linear combination

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

is also a solution of (2.2) on I for any values of the constants c_1, c_2, $\cdots$, and c_n.

Question

If we know that $y_1 = \cos 2x$, $y_2 = e^{\pi x}$, and $y_3 = x^2$ are solutions to a third order linear homogeneous equation, what is another solution? (Discovery)

Returning to our solutions of the characteristic equation, we obtained the roots

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Question

From the sign of the **discriminant,** $b^2 - 4ac$, what can you tell about the roots of equation (1.4)? In particular, what are the three cases for the roots of the quadratic equation? (Spiral)

Distinct Real Roots ($b^2 - 4ac > 0$)

If the discriminant $b^2 - 4ac$ is greater than zero, then the characteristic equation has two distinct real roots λ_1 and λ_2. (Spiral)

It follows that $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are solutions to (1.1). From the Principle of Superposition, any linear combination of these solutions is also a solution. A general solution of a differential equation is the set of all solutions. We discuss this in more detail in

Chapter 3. For now, we say that a **general solution** of (1.1) is

$$y_g(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}, \tag{2.3}$$

where c_1 and c_2 are arbitrary constants.

Example 2.1 *Find a general solution to*

$$y'' + 5y' - 6y = 0 \tag{2.4}$$

Solution *The characteristic equation associated with (2.4) is*

$$\lambda^2 + 5\lambda - 6y = 0 \Longrightarrow (\lambda - 1)(\lambda + 6) = 0.$$

Thus the possible values of λ *are* $\lambda_1 = 1$ *and* $\lambda_2 = -6$. *Then we call* $\{e^{1x}, e^{-6x}\}$ *a fundamental solution set for equation (2.4), and a general solution is*

$$y_g(x) = c_1 e^{x} + c_2 e^{-6x}.$$

Example 2.2 *Solve the initial value problem (IVP)*

$$y'' + 2y' - y = 0; \qquad y(0) = 0, \qquad y'(0) = -1. \tag{2.5}$$

Solution *The characteristic equation is*

$$\lambda^2 + 2\lambda - 1 = 0.$$

Applying the quadratic formula, we find that

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)} \Longrightarrow \lambda = \frac{-2 \pm \sqrt{8}}{2} \Longrightarrow \lambda_1 = -1 + \sqrt{2}, \lambda_2 = -1 - \sqrt{2}$$

Consequently, a general solution to the differential equation in (2.5) is

$$y_g(x) = c_1 e^{(-1+\sqrt{2})x} + c_2 e^{(-1-\sqrt{2})x}, \tag{2.6}$$

where c_1 *and* c_2 *are arbitrary constants. To find the particular solution that satisfies the initial conditions given in (2.5), we differentiate (2.6) to find* y'*, then apply the initial conditions of (2.5).*

$$y'(x) = c_1(-1 + \sqrt{2})e^{(-1+\sqrt{2})x} + c_2(-1 - \sqrt{2})e^{(-1-\sqrt{2})x}. \tag{2.7}$$

Using (2.6),

$$y(0) = 0 \Rightarrow 0 = c_1 e^0 + c_2 e^0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1.$$

Using (2.7),

$$y'(0) = -1 \Rightarrow -1 = c_1(-1+\sqrt{2})e^0 + c_2(-1-\sqrt{2})e^0 \Rightarrow -1 = c_1(-1+\sqrt{2}) + c_2(-1-\sqrt{2}).$$

Then, substituting $-c_1$ *in for* c_2 *in the latter equation, combining like terms and solving produces*

$$c_1 = \frac{-\sqrt{2}}{4} \qquad \text{and} \qquad c_2 = \frac{\sqrt{2}}{4}.$$

Thus, the desired solution is

$$y_{IVP}(x) = \frac{-\sqrt{2}}{4}e^{(-1+\sqrt{2})x} + \frac{\sqrt{2}}{4}e^{(-1-\sqrt{2})x}. \quad (2.6')$$

Graphing Calculator

Graph the general solution of Example 2 using values of: 1,1; 2,-1; -1,2; and -2,-2 for c_1 and c_2, respectively on the same graph. Describe the graphs. What happens as x gets larger?

Now graph the IVP solution on the previous graph. How are the initial conditions related to the graph that you observe?

Mathematica

You can use the Mathematica command

```
Plot[-Sqrt[2]/4*Exp[(-1+Sqrt[2]) x] + Sqrt[2]/4*Exp[(-1-Sqrt[2]) x],
        {x, -2, 8}]
```

to plot the IVP solution in the domain $-2 \le x \le 8$.

More elegantly, we can use a series of commands to plot the general solution for various values of c_1 and c_2, as follows:

```
yivp[c1_,c2_]=c1*(-Sqrt[2]/4)*Exp[(-1+Sqrt[2]) x] +
        c2*Sqrt[2]/4*Exp[( -1-Sqrt[2]) x]

sols=Table[yivp[c1,c2],{c1,-2,1},{c2,-1,1}];

Plot[Evaluate[sols],{x,-1,4}, PlotRange -> {-5,5}]
```

A little explanation is in order; the first line is defining the general solution as a function in terms of our two constants c_1 and c_2. The second line is creating a table of functions with c_1 being allowed to take integer values between -2 and 1. Similarly, c_2 is taking on the values -1, 0, and 1. The semicolon keeps the output from being shown, since it is a fairly long table of values. The third line evaluates the functions that we created at the various points in the domain, then calls the Plot function to plot the points.

Try changing the values that c_1 and c_2 are allowed to take. Then try varying your domain and range values to see how the graphs look.

Chapter 3

FUNDAMENTAL SOLUTIONS, LINEAR INDEPENDENCE, AND THE WRONSKIAN

In Example 2.2 of the previous lesson, we solved an initial value problem (IVP) which included a homogeneous linear second order differential equation and two initial conditions.

Questions

Do you think $y(x) = \frac{-\sqrt{2}}{4}e^{(-1+\sqrt{2})x} + \frac{\sqrt{2}}{4}e^{(-1-\sqrt{2})x}$ is the only solution to the initial value problem in (2.5)? Why or why not? (Discovery)

The answer is found in the following theorem which is stated without proof since the proof is fairly advanced. See, for example, Hale(1980) [26].

Theorem 3.1 (Existence and Uniqueness) *Suppose $p(x)$, $q(x)$, and $g(x)$ are continuous on an interval (a, b) that contains the point x_0. Then there is exactly one solution $y(x)$ on the whole interval (a, b) to the initial value problem*

$$y'' + p(x)y' + q(x)y = g(x), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0. \tag{3.1}$$

(Note: g(x) does not have to be zero, which implies that this theorem applies equally well to nonhomogeneous equations.)

The theorem says that a solution to the IVP exists, that there is only one solution, and that the solution is at least a twice differentiable function throughout the interval (a, b). For the IVP in (2.5), $p(x) = 2$, $q(x) = 1$, and $g(x) = 0$. These are continuous functions on $(-\infty, \infty)$, which contains $x_0 = 0$. Thus, the solution given in (2.6') is unique.

In the more general case, we showed $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are each solutions to the differential equation. So, from the Principle of Superposition, $c_1 y_1 + c_2 y_2$ is a solution, also.

The next question is whether the constants c_1 and c_2 can be chosen so that the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$ are satisfied. Substituting, we have

$$y_0 = c_1 y_1(x_0) + c_2 y_2(x_0)$$

$$y_0' = c_1 y_1'(x_0) + c_2 y_2'(x_0)$$

Using basic algebra, we get

$$c_1 = \frac{y_0 y_2'(x_0) - y_0' y_2(x_0)}{y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)}$$

$$c_2 = \frac{-y_0 y_1'(x_0) + y_0' y_1(x_0)}{y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)}$$

When will these constants be undefined? (Spiral)

The denominator in both constants is

$$y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)$$

which is called the Wronskian[1]. If this Wronskian is non-zero, both c_1 and c_2 can be

[1]The Wronskian was named after Josef Hoëné de Wronski (1778-1853) who was born in Poland.

found.

Question

Can you think of a more convenient way of writing the Wronskian, using linear algebra concepts? (Discovery)

In terms of the determinant the Wronskian is

$$W\left[y_1, y_2\right](x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}$$

One reason for writing the Wronskian in the determinant form is that it will generalize to higher order linear equations.

Question

How do we evaluate the determinant?

What is a good rule for remembering the signs? (Spiral)

If $W \neq 0$, we have shown the existence of constants c_1 and c_2 such that $y = c_1 y_1(x) + c_2 y_2(x)$ will satisfy the homogeneous equation (3.1). Thus, we have established the following theorem.

For many years Wronski's work was considered worthless. However, a closer examination in recent times shows that, although some is wrong, and he had an incredibly high opinion of himself and his ideas, there is also some mathematical insight of great brilliance hidden within the papers [51].

Theorem 3.2 (Constants Existence and Uniqueness) *If y_1 and y_2 are two solutions of the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \tag{3.2}$$

and if the Wronskian

$$W[y_1, y_2] = y_1y_2' - y_1'y_2 \neq 0$$

at x_0 where the initial conditions

$$y(x_0) = y_0, \qquad y'(x_0) = y_0' \tag{3.3}$$

are given, then there exist unique constants c_1 and c_2 such that $y = c_1y_1(x) + c_2y_2(x)$ satisfies the differential equation (3.2) and the initial conditions (3.3).

We now justify the term "general solution" that we introduced in Chapter 2, by establishing the following theorem.

Theorem 3.3 (General Solution) *If y_1 and y_2 are solutions of the differential equation (3.2)*

$$y'' + p(x)y' + q(x)y = 0,$$

and if there exists a point x_0 where $W[y_1, y_2] \neq 0$, then

$$y = c_1 y_1(x) + c_2 y_2(x)$$

includes every solution of (3.2), where the c_1 and c_2 are arbitrary coefficients. Since y includes every solution, it is called the general solution *to the differential equation.*

Proof

Let $\xi(x)$ be any solution of equation (3.2). We must show there are constants c_1 and c_2 such that $\xi(x) = c_1 y_1(x) + c_2 y_2(x)$.

Consider the system of equations

$$c_1 y_1(x_0) + c_2 y_2(x_0) = \xi(x_0) \tag{3.4}$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = \xi'(x_0), \tag{3.5}$$

where x_0 is a point where the Wronskian

$$y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)$$

is not zero. Now, solve for c_1 by multiplying equation (3.4) by $y_2'(x_0)$ and multiplying equation (3.5) by $y_2(x_0)$, and subtracting to obtain

$$c_1\left[y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)\right] = \xi(x_0)y_2'(x_0) - \xi'(x_0)y_2(x_0).$$

Since the quantity in the brackets is the Wronskian and not equal to zero, we can divide by it to obtain

$$c_1 = \frac{\xi(x_0)y_2'(x_0) - \xi'(x_0)y_2(x_0)}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)}.$$

Following similar procedures, we find that

$$c_2 = \frac{\xi'(x_0)y_1(x_0) - \xi(x_0)y_1'(x_0)}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)}.$$

Since the denominators are not equal to zero, c_1 and c_2 are determined. Thus, if we use these choices for c_1 and c_2, the functions $c_1y_1 + c_2y_2$ and ξ are solutions to (3.2) that satisfy the initial conditions. Then, by the uniqueness part of Theorem 3.1, $c_1y_1 + c_2y_2 = \xi$. Since ξ was an *arbitrary* solution of (3.2), it follows that *every* solution of this equation is included in this **general solution**. ■

We say that $\{y_1, y_2\}$ is a **fundamental solution set**. Closely related to the Wronskian is the idea of linear independence. Two functions y_1 and y_2 are said to be **linearly dependent** on an interval (a, b) if there exist two constants c_1 and c_2, not both zero, such that

$$c_1f(x) + c_2g(x) = 0,$$

for all x in (a, b). If we set $c_1 \neq 0$, we have

$$f(x) = \frac{-c_2}{c_1}g(x)$$

which implies y_1 *is a constant multiple of* y_2 in the case of two functions. The functions y_1 and y_2 are said to be **linearly independent** on an interval if they are not **linearly dependent**. In other words,

$$c_1y_1(x) + c_2y_2(x) = 0 \text{ if and only if } c_1 = c_2 = 0.$$

Now, we relate linear independence and dependence to the Wronskian using the following theorem.

Theorem 3.4 (Wronskian and Independence) *If y_1 and y_2 are differentiable functions on an interval (a, b) and if*

$$W[y_1, y_2](x_0) \neq 0,$$

for some point x_0 in (a, b), then y_1 and y_2 are linearly independent on (a, b). The contrapositive says, if y_1 and y_2 are linearly dependent on (a, b), then $W[y_1, y_2](x_0) = 0$ for every x in (a, b).

Proof

We prove the first part of the statement and the contrapositive gives us the second part. First, we assume that a linear combination $c_1y_1(x) + c_2y_2(x) \equiv 0$ on (a, b). Then, evaluating the function and its derivative at x_0, we have

$$c_1y_1(x_0) + c_2y_2(x_0) = 0,$$

$$c_1y_1'(x_0) + c_2y_2'(x_0) = 0.$$

The determinant of coefficients of this set of equations is $W[y_1, y_2](x_0)$, which was given to be non-zero. Since the only solution to the set of equations is $c_1 = c_2 = 0$, y_1 and y_2 are linearly independent.

Example 3.1 *Verify that* $\{\cos 3x, \sin 3x\}$ *is a fundamental solution set of* $y''+9y=0$ *on* $(-\infty, \infty)$ *and find a general solution.* *(Spiral)*

Solution

1. *Show that* $y_1(x) = \cos 3x, y_2(x) = \sin 3x$ *satisfy* $y'' + 9y = 0$.

$$\begin{aligned} y_1(x) &= \cos 3x, & y_2(x) &= \sin 3x, \\ y_1'(x) &= -3\sin 3x, & y_2'(x) &= 3\cos 3x, \\ y_1''(x) &= -9\cos 3x, & y_2''(x) &= -9\sin 3x. \end{aligned}$$

Then,

$$y_1'' + 9y = -9\cos 3x + 9\cos 3x = 0,$$

and

$$y_2'' + 9y = -9\sin 3x + 9\sin 3x = 0.$$

2. *Check for linear independence using the Wronskian*

$$\begin{aligned} W\left[y_1, y_2\right](x) &= \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \\ &= y_1 y_2' - y_1' y_2 \\ &= (\cos 3x)(3\cos 3x) - (-3\sin 3x)(\sin 3x) \end{aligned}$$

$$\begin{aligned} &= 3\cos^2 3x + 3\sin^2 3x \\ &= 3(\cos^2 3x + \sin^2 3x) \\ &= 3(1) \\ &= 3. \end{aligned}$$

Since $3 \neq 0$, we have linearly independent solutions and a general solution is

$$y_g(x) = c_1 \cos 3x + c_2 \sin 3x.$$

Example 3.2 *Determine whether the following pairs of functions y_1 and y_2 are linearly dependent on $(-\infty, \infty)$:*

1. $y_1(x) = e^{3x}, \quad y_2(x) = -2x.$

2. $y_1(x) = \sin 2x, \quad y_2(x) = \cos x \sin x.$

3. $y_1(x) = x, \quad y_2(x) = |x|.$

Solution

(1) By observation, it appears that neither function $y_1(x) = e^{3x}$ nor $y_2(x) = -2x$ is a constant multiple of the other. Indeed, if there exist a c_1 and a c_2, both not zero, such that $c_1y_1 + c_2y_2 = 0$, for all x in $(-\infty, \infty)$, then $c_1e^{3x} + c_2(-2x) = 0$. Setting $x = 0$, we have

$$c_1e^0 + c_2(0) = 0 \Rightarrow c_1 = 0.$$

Then, setting $x = 1$, *we have*

$$0 \cdot e^{3 \cdot 1} + c_2(-2 \cdot 1) = 0 \Rightarrow c_2 = 0.$$

So $c_1 = c_2 = 0$, *which says that* y_1 *and* y_2 *are linearly independent.*

Alternate Solution

We can also use the Wronskian to check for linear independence.

$$\begin{aligned} W\left[e^{3x}, -2x\right](x) &= \begin{vmatrix} e^{3x} & -2x \\ 3e^{3x} & -2 \end{vmatrix} \\ &= -2e^{3x} - (-6e^{3x}) \\ &= 4e^{3x} \\ &\neq 0 \text{ for all } x. \end{aligned}$$

Since the Wronskian is never zero, we see that e^{3x} *and* $-2x$ *are linearly independent on* $(-\infty, \infty)$.

(2) Since $y_1(x) = \sin 2x = 2 \sin x \cos x$ *(trigonometric identity), we see that* $y_1(x) = 2y_2(x)$, *or equivalently,* $1y_1(x) + (-2)y_2(x) = 0$. *Thus, there exists a* $c_1 = 1$, *and a* $c_2 = -2$, *such that* $c_1y_1 + c_2y_2 = 0$. *Then,* y_1 *and* y_2 *are linearly dependent on* $(-\infty, \infty)$.

(3) Here we have two functions that are identical on the subinterval $(0, \infty)$. *However, neither is a constant multiple of the other on the whole interval* $(-\infty, \infty)$. *For example, on* $(0, \infty)$, *we know that* $|x| = x$, *which gives us* $|x| + (-1)x = 0$. *In other*

words, we have

$$(1)y_1(x) + (-1)y_2(x) = 0,$$

but on $(-\infty, 0)$*, we know that* $|x| = -x$*, which gives us*

$$(1)y_1(x) + (1)y_2(x) = 0.$$

We see that c_2 *must be* -1 *on one interval, and* $+1$ *on the other interval. This would violate the fact that it must be fixed on the whole interval. It is easy to see that no non-zero constants* c_1 *and* c_2 *can be found such that*

$$c_1y_1(x) + c_2y_2(x) = 0$$

on the whole interval $(-\infty, \infty)$*. Thus,* y_1 *and* y_2 *are linearly independent on* $(-\infty, \infty)$*.*

Graphing Calculator

Graph each pair of functions in Example 2 on the same graph. What does it mean *graphically* for two functions to be dependent?

Example 3.3 *Show that* $y_1(x) = x^{-1}$ *and* $y_2(x) = x^3$ *are solutions to*

$$x^2y'' - xy' - 3y = 0. \tag{3.6}$$

on the interval $(0, \infty)$ *and give a general solution.*

Solution *Since* $y_1 = x^{-1}$*, we have*

$$y_1' = -x^{-2} \qquad \text{and} \qquad y_1'' = 2x^{-3}.$$

Then, substituting into the left-hand side of equation (3.6), we have:

$$x^2(2x^{-3}) - x(-x^{-2}) - 3(x^{-1})$$

which simplifies to 0.

Also, since $y_2 = x^3$*, we obtain*

$$x^2(6x) - x(3x^2) - 3(x^3),$$

which simplifies to 0. *Thus* y_1 *and* y_2 *are both solutions to (3.6). Furthermore,* x^{-1} *and* x^3 *are linearly independent on* $(0, \infty)$ *since neither is a constant multiple of the other.*

What is another way of showing that the two functions are linearly independent?

Do this to confirm the above.

Then, we have that x^{-1}, x^3 *is a fundamental solution set on* $(0, \infty)$*, and a general solution to (3.6) is*

$$y(x) = c_1 x^{-1} + c_2 x^3.$$

Mathematica

Mathematica can greatly reduce the amount of work in finding a Wronskian. For example, to find the Wronskian of the solutions in Example 2(b), we can use the commands:

```
rowone = {x^(-1), x^3}
rowtwo = D[rowone, x]
matrix = {rowone, rowtwo}
MatrixForm[matrix]
```

```
wronskian = Det[matrix]
Expand[wronskian, Trig -> True]
```

More elegantly, we can use a self-defined function

```
Clear[wronskian]
wronskian[functions_]:=
     Module[{number, deriv, matrix},
     number = Length[functions];
     deriv[1] = functions;
     deriv[k_]:=deriv[k] = D[deriv[k-1],x];
     matrix = Table[deriv[i], {i, 1, number}];
     Expand[Det[matrix], Trig -> True]
     ]
```

A little explanation is in order. The **Module** command allows us to set up local variables within the block of code. The **Length** command gives us the number of functions in the list. The variable **deriv** is an array to store the derivatives as they are calculated. The variable **matrix** creates a table where the first through the n^{th} derivatives are stored. The **Det** command finds the determinant of the matrix, and the **Expand** command simplifies the expression with the **Trig - > True** option allowing you to simplify the trigonometric expressions.

Now, we try the functions in Example 2(b), using

```
w1 = wronskian[Sin[2*x], Cos[x]*Sin[x]]
```

Try finding the Wronskian (call it w2) of the four functions
Sin[x], Sin[2 x], Sin[3 x], Sin[4 x].

What do you find? Are the functions independent? You can check to see if the resulting function is the zero function in two ways:
First, we can evaluate it at some points using

w2 /. x -> Pi
w2 /. x -> Pi/2

We can also graph it using

Plot[w2, {x, -5, 5}]

What do you observe?

Chapter 4

CHARACTERISTIC EQUATION (COMPLEX ROOTS)

In Chapter 1 we assumed a solution of the form $y = ce^{\lambda x}$ for the homogeneous linear second order differential equation

$$ay'' + by' + cy = 0 \qquad (1.1)$$

which produced the characteristic equation

$$a\lambda^2 + b\lambda + c = 0. \qquad (1.4)$$

To review, what are the three cases for the types of roots to (1.4)? (Spiral)

Questions

What is $b^2 - 4ac$ called?

From Lesson 2, what kind of roots did the characteristic equation have when $b^2 - 4ac > 0$?

What was the form of the general solution? (Spiral)

We now look at the second case.

If $b^2 - 4ac$ is negative, then what do we know about the roots of the quadratic (characteristic) equation? Are they real or complex?

How many are there?

Do they have any other special properties?

How do we solve the characteristic equation?

What do we get when we solve the characteristic equation? (Spiral)

If $b^2 - 4ac < 0$, then the characteristic equation has two complex conjugate roots

$$\lambda_1 = \alpha + i\beta \qquad \text{and} \qquad \lambda_2 = \alpha - i\beta \qquad\qquad (i = \sqrt{-1})$$

where α and β are the real numbers

$$\alpha = \frac{-b}{2a} \qquad \text{and} \qquad \beta = \frac{4ac - b^2}{2a}.$$

Questions

Why is it $4ac - b^2$ instead of $b^2 - 4ac$?

Using the form of the general solution (2.3) discovered in Lesson 2, what do you think we will have for solutions to equation (1.1) here?

Using the complex roots, we have

$$y_1 = e^{(\alpha + i\beta)x} \qquad \text{and} \qquad y_2 = e^{(\alpha - i\beta)x} \tag{4.1}$$

or, could we write this?

$$y_1 = e^{\alpha x} e^{i\beta x} \qquad \text{and} \qquad y_2 = e^{\alpha x} e^{-i\beta x} \tag{4.2}$$

What does $e^{i\beta x}$ mean? The answer is found in Euler's formula. [1]

[1] Leonhard Euler (1707 - 1783) was the most prolific writer of mathematics of all time. His complete works contain 886 books and papers. Among the notations credited to him are that of $f(x)$ for a function, e for the base of natural logarithms, i for the square root of -1, π for pi, and Σ for summation. Euler made giant contributions to modern analytic geometry and trigonometry, as well as contributing to geometry, calculus and number theory. He had five children that lived past infancy, and thirteen children total. His memory and powers of concentration are legendry. He produced almost half his works after the age of 58, despite having become totally blind [47].

Euler's formula

Do you recall from calculus how to form the Taylor series expansion for a function about a point $x = c$? If $c = 0$, we have the Maclaurin series.

Can you form the Maclaurin series for e^z, $\cos z$, and $\sin z$? (Spiral & Discovery)

You should find that

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots. \tag{4.3}$$

From complex analysis, it can be shown that the Taylor expansion holds for complex numbers as well [37]. If $z = i\theta$, the Maclaurin series for $e^{i\theta}$ is

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \cdots + \frac{(i\theta)^n}{n!} + \cdots.$$

Questions

If we look at even-powered terms of i, what is the value of i^2?

What is i^4? What about i^6?

Can you state a general pattern for the even powers of i?

What about the odd powers? (Spiral & Discovery)

What do we get when we simplify our expression for $e^{i\theta}$?

After combining like terms and factoring out i, you should have found

$$e^{i\theta} = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)$$

Recall that a complex number has the form $a + ib$, where a is called the *real part* of the number and b is called the *imaginary part.* (Spiral)

In this case, they are the Maclaurin series you should have obtained for $\cos\theta$ and

$\sin\theta$, respectively. Thus, we can simplify our expression to

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{4.4}$$

which is known as *Euler's formula.*

Recall from trigonometry that since $\cos\theta$ is even and $\sin\theta$ is odd that (Spiral)

$$\cos(-\theta) = \cos\theta \qquad \text{and} \qquad \sin(-\theta) = -\sin\theta.$$

Replacing θ by $-\theta$ in equation (4.4), we have

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{4.5}$$

Recalling our original intention from (4.2) of finding out what $e^{\pm i\beta x}$ means, we now let $\theta = \beta x$. What do we obtain for $e^{i\beta x}$ and $e^{-i\beta x}$ when we substitute into equations (4.4) and (4.5)?

Now, we substitute our expressions for $e^{\pm i\beta x}$ into the equations in (4.2) and obtain

$$y_1 = e^{\alpha x}(\cos\beta x + i\sin\beta x) \qquad \text{and} \qquad y_2 = e^{\alpha x}(\cos\beta x - i\sin\beta x). \tag{4.6}$$

Equivalently, we have

$$y_1 = e^{\alpha x}\cos\beta x + ie^{\alpha x}\sin\beta x \qquad \text{and} \qquad y_2 = e^{\alpha x}\cos\beta x - ie^{\alpha x}\sin\beta x. \tag{4.7}$$

Unfortunately, y_1 and y_2 are complex-valued solutions, whereas we would generally like to have real-valued functions.

Are there real-valued functions present in y_1 and y_2? (Discovery)

Observe that $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are two such elementary real-valued functions.

Questions

Would you guess that $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are each solutions to (1.1)?

What is your reasoning? (Discovery)

It turns out that it is the case that the real and imaginary parts of our complex-valued solutions y_1 and y_2 are also solutions to the differential equation (1.1). We state this in the form of a theorem.

Theorem 4.1 (Real Solutions from Complex Solutions) *If $y(x) = u(x) + iv(x)$ is a solution to the differential equation*

$$ay'' + by' + cy = 0 \qquad (1.1)$$

then the real part $u(x)$ and the imaginary part $v(x)$ of $y(x)$ are each real-valued solutions to (1.1).

Proof

By assumption, $ay'' + by' + cy = 0$, and since $y = u + iv$, we have

$$a(u+iv)'' + b(u+iv)' + c(u+iv) = 0.$$

Using the linearity of differentiation, and the properties of derivatives of complex functions [37], we have

$$a(u'' + iv'') + b(u' + iv') + c(u + iv) = 0,$$

$$(au'' + bu' + cu) + i(av'' + bv' + cv) = 0.$$

But, when is a complex number equal to zero? (Spiral)

This says that we have both

$$au'' + bu' + cu = 0 \qquad \text{and} \qquad av'' + bv' + cv = 0.$$

Thus, $u(x)$ and $v(x)$ are both real-valued solutions of (1.1). ■

Applying Theorem 4.1 to

$$e^{(\alpha+i\beta)x} = e^{\alpha x}\cos\beta x + ie^{\alpha x}\sin\beta x,$$

we obtain a general result for complex roots.

Complex Conjugate Roots

If the characteristic equation has complex conjugate roots $\alpha \pm i\beta$, then two (linearly independent) solutions to (1.1) are

$$e^{\alpha x}\cos\beta x \qquad \text{and} \qquad e^{\alpha x}\sin\beta x,$$

and a general solution is

$$y(x) = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x, \tag{4.8}$$

where c_1 and c_2 are arbitrary constants.

Questions

If $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are solutions, what is their Wronskian?

How do you prove that the two solutions are linearly independent? (Spiral)

Example 4.1 *Find a general solution to*

$$y'' + 2y' + 4y = 0. \tag{4.9}$$

Solution *The characteristic equation is*

$$\lambda^2 + 2\lambda + 4 = 0,$$

which has roots

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)} = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm 2\sqrt{-3}}{2} = -1 \pm i\sqrt{3}.$$

So, $\lambda_1 = -1 + i\sqrt{3}$ *and* $\lambda_2 = -1 - i\sqrt{3}$*, which gives*

$$\alpha = -1 \qquad \textit{and} \qquad \beta = \sqrt{3}.$$

Then a general solution for (4.9) is

$$y(x) = c_1 e^{-x} \cos(\sqrt{3}x) + c_2 e^{-x} \sin(\sqrt{3}x).$$

Graphing Calculator

Try graphing the general solution to Example 1 using values of {-4,3} and {2,-5} for c_1 and c_2, respectively. Start with a domain interval of $(-0.5, 5.0)$, and a range interval of $(-3, 2)$.

Questions

What do the graphs look like?

How does the appearance of the graphs relate to the basic exponential and trigonometric functions in the general solution?

Now try zooming in on the graphs in the domain of $(4, 5)$.

What do your range values need to change to in order to get a good graph of the functions?

What is happening to the functions as x gets larger?

Try graphing the general solution for other values of of c_1 and c_2.

How do the signs of c_1 and c_2 affect the resulting graphs?

How do the magnitudes of c_1 and c_2 affect the resulting graphs?

Example 4.2 *Solve the initial value problem*

$$36x''(t) + 12x'(t) + 37x(t) = 0; \qquad x(0) = 70, \qquad x'(0) = 10. \tag{4.10}$$

(Notice that t is now the independent variable and x is the dependent variable.) The equation has the general form

$$mx''(t) + bx'(t) + kx(t) = 0; \qquad x(0) = x_0, \qquad x'(0) = v_0.$$

This equation is found in the study of a vibrating spring with damping. The constant m *is the mass of the system,* b *is the damping constant,* k *is called the spring constant,* x_0 *is the object's initial displacement from its equilibrium position,* v_0 *is the initial velocity, and* $x(t)$ *is the displacement of the object from equilibrium at time* t*. Thus, for example, we have a mass of* $36kg$*, a damping coefficient of* $12kg/sec$*, a spring constant of* $37kg/sec^2$*, an initial displacement of* $70cm$*, and an initial velocity of* $10cm/sec$*.*

Solution *Equation (4.10) yields the characteristic equation*

$$36\lambda^2 + 12\lambda + 37 = 0,$$

which has roots

$$\lambda = \frac{-12 \pm \sqrt{12^2 - 4(36)(37)}}{2(36)} = \frac{-12 \pm 12\sqrt{1-37}}{72} = -1/6 \pm 1i$$

Hence with $\alpha = -1/6$ *and* $\beta = 1$*, the displacement is*

$$x(t) = c_1 e^{(-1/6)t} \cos(1t) + c_2 e^{(-1/6)t} \sin(1t).$$

In other words,

$$x(t) = c_1 e^{-t/6} \cos(t) + c_2 e^{-t/6} \sin(t). \tag{4.11}$$

is the general solution to (4.10).

Substituting our initial conditions into (4.11), we have,

$$\begin{aligned}70 &= c_1 e^0 \cos 0 + c_2 e^0 \sin 0\\ 70 &= c_1(1)(1) + c_2(1)(0)\\ c_1 &= 70.\end{aligned}$$

Also, since

$$x(t) = c_1 e^{-t/6} \cos(t) + c_2 e^{-t/6} \sin(t),$$

we have

$$\begin{aligned}x'(t) &= c_1 e^{-t/6}(-\sin t) + c_1 \cos t(\frac{-1}{6} e^{-t/6}) + c_2 e^{-t/6} \cos t - c_2 \sin t(\frac{-1}{6} e^{-t/6}),\\ &= -c_1 e^{-t/6} \sin t + -16 c_1 e^{-t/6} \cos t + c_2 e^{-t/6} \cos t - -16 c_2 e^{-t/6} \sin t.\end{aligned}$$

Then, using $x'(0) = 10$ *yields*

$$\begin{aligned}10 &= -c_1 e^0 \sin 0 + \frac{-1}{6} c_1 e^0 \cos 0 + c_2 e^0 \cos 0 - \frac{-1}{6} c_2 e^0 \sin 0.\\ 10 &= 0 - \frac{1}{6} c_1 + c_2 - 0\\ 10 &= \frac{-1}{6} c_1 + c_2.\end{aligned}$$

Since $c_1 = 70$, *we have* $10 = \frac{-1}{6}(70) + c_2$, *which gives* $c_2 = 65/3$.

With these values for c_1 *and* c_2, *the specific solution to the IVP is*

$$x(t) = 70 e^{-t/6} \cos t + \frac{65}{3} e^{-t/6} \sin t.$$

Mathematica

Mathematica can help us verify that a function is a solution to our differential equation. For example, to check Example 1, we can define the general solution y as a function of c_1 and c_2 as follows:

```
Clear[y]
y[{c1_,c2_}]=c1*Exp[-x]*Cos[Sqrt[3]*x] + c2*Exp[-x]*Sin[Sqrt[3]*x]
newy=%
```

Note: We define the expression *newy* to use when we differentiate later. We cannot differentiate the original function with respect to x, as we would normally think of doing, because it is a function of two new variables c_1 and c_2, instead of x. Now we define a list of ordered pairs of values we want to assign to c_1 and c_2 as follows:

```
cvals={{0,1},{1,0},{1,-1},{2,1},{1,-2}};
```

Then the **Map** function can be used to apply the function y to our list of c-values:

```
functs=Map[y,cvals]
```

Then Mathematica can plot the functions on some interval, say [-1,5]:

```
Plot[Evaluate[functs],{x,-1,5}]
```

Notice how Mathematica sets the range for you, automatically. You can use the option **PlotRange** to change it as follows:

Plot[Evaluate[functs],{x,-1,5}, PlotRange -> {-0.2,0.2}]

Try plotting the functions we obtained in the interval $(2,5)$, using the **PlotRange** option, then plot them without **PlotRange**.
What difference do you see, if any, from Mathematica's choice of range?
Which do you like better?

Now we show that our function y is a general solution of equation (4.9), using the following series of commands:

resultone=D[newy,{x,2}]+2*D[newy,x]+4*newy
Simplify[resultone]

We get a result of zero which shows that our solution is, in fact, a solution of the differential equation (4.9). Notice the new method introduced to find the second derivative. In fact, to find the n^{th} derivative of function *newy*, we could use the function **D[newy,{x,n}]**
Try showing that
y = c1*Exp[x] + c2*x*Exp[x] + c3*x^2*Exp[x]
is a solution of the third order homogeneous differential equation

$$y''' - 3y'' + 3y' - y = 0.$$

Also try plotting y for various values of c_1 and c_2.

SAMPLE PROBLEM SET - CHAPTER 4

1. Find a general solution to

$$y'' + 4y' + 29y = 0.$$

2. Solve the initial value problem

$$5y'' - 15y' + 10y = 0; \qquad y(0) = 1, \quad y'(0) = 0.$$

3. Solve the initial value problem

$$\frac{d^2y}{dx^2} + 16y = 0,$$

subject to the initial conditions: $y = 0$ and $\dfrac{dy}{dx} = -1$ when $x = 0$.
Also, sketch the graph of the solution.

4. Verify that $\{e^x, e^{2x}\}$ is a fundamental solution set to $2y'' - 6y' + 4y = 0$ on $(-\infty, \infty)$ and find a general solution. Then, show that e^x and e^{2x} are linearly independent on the interval.

5. Show that $\{e^x, xe^x\}$ is a fundamental solution set to $y'' - 2y' + y = 0$ on $(-\infty, \infty)$. Then, show that e^x and xe^x are linearly independent on the interval.

6. Find a general solution to

$$32u'' + 16u' + u = 0.$$

7. Find a solution to the IVP

$$y'' - 2y' - 2y = 0; \qquad y(0) = 0, \quad y'(0) = 2.$$

8. Derive the characteristic equation (1.4).

9. Derive a form of the characteristic equation if we guess a solution of the form

$$y = e^{\lambda^2 x}.$$

10. Give the order of the following differential equation and tell if it is linear and homogeneous, respectively.

$$\frac{d^4y}{dx^4} + x^3\frac{d^3y}{dx^3} + x\frac{dy}{dx} = xe^x.$$

11. Solve the first order differential equation

$$dx + x^2dy + y^2dx + dy = 0.$$

12. Solve the linear first order equation

$$y' + (\frac{4}{x})y = x^4.$$

13. A body weighing 1 lb. falls from rest from a tall building. Assume the air resistance $R = \frac{v}{40}$ lb., where v is the velocity. Find the velocity of the body at time t.

14. Evaluate

$$\int \frac{x\,dx}{x^2 + 4x + 3}.$$

15. Evaluate

$$\int x^2 \ln x\,dx.$$

16. Group/Challenge Problem

(a) Using the techniques presented in Lessons 2 and 4, try to solve the *fourth* order homogeneous equation

$$y^{(4)} + 2y'' - 3y = 0.$$

(b) Try solving the *fifth* order equation

$$u^{(5)} - u^{(4)} - 4u''' - 2u'' + 8u' + 8u = 0.$$

(c) Write a full group description of all the methods you tried and explain why they did or did not work for parts (a) and (b).

(d) Write an individual description of how well the group worked together and why. Describe what you learned in this lesson and group problem and constructively critique the text and teacher presentation of the lesson, along with your own performance.

Chapter 5

CHARACTERISTIC EQUATION (REPEATED ROOTS)

In previous lessons, we showed that the homogeneous linear second order differential equation

$$ay'' + by' + cy = 0 \qquad (1.1)$$

leads to the characteristic equation

$$a\lambda^2 + b\lambda + c = 0. \qquad (1.4)$$

Questions

What were the corresponding solutions for the cases when λ_1 and λ_2 are real and unequal?

What were they when λ_1 and λ_2 are complex conjugates? (Spiral)

Now consider the case when the discriminant is zero.

When $b^2 - 4ac = 0$, what kind of roots do we get from the quadratic formula?

Why is that?

What are the roots? (Spiral)

Then, since both roots are the same, they both yield the same solution for the differential equation, namely

$$y_1(x) = y_2(x) = e^{-bx/2a}$$

Are these two solutions linearly independent?

We need a second *linearly independent* solution in order to write the general solution.

What theorem(s) tell us this?

A second linearly independent solution can be found by a method originated by D'Alembert[1] in the eighteenth century.

Rediscovering D'Alembert's Method

Previously, we found that if $y_1(x)$ is a solution of equation (1.1), so is $cy_1(x)$ for any constant c. What theorem gave us that? (Spiral)

D'Alembert suggested replacing c by some function $v(x)$ and then trying to determine $v(x)$ so that the *product* $v(x)y_1(x)$ is a solution of equation (1.1). We know that one solution of our equation is $y_1(x) = e^{-bx/2a}$. Thus, we assume that

$$y = v(x)y_1(x) = v(x)e^{-bx/2a}$$

is also a solution and substitute it and its derivatives into (1.1).

If

$$y'(x) = v(x)(\frac{-b}{2a})e^{-bx/2a} + e^{-bx/2a}v'(x),$$

then what is $y''(x)$? (Don't forget to use the product rule.) (Spiral & Discovery)

Substitute y' and y'' into equation (1.1), and you should obtain

$$\{a[v''(x) - \frac{b}{a}v'(x) + \frac{b^2}{4a^2}v(x)] + b[v'(x) - \frac{b}{2a}v(x)] + cv(x)\}e^{-bx/2a} = 0$$

[1]Jean d'Alembert (1717 - 1783) was a pioneer in the field of differential equations and applied them to equilibrium and motion of fluids. He also did important work in developing the foundations of analysis. He turned down a number of offers in his life, including an offer from Frederick II to go to Prussia as President of the Berlin Academy, as well as an offer from Catherine II to go to Russia and tutor her son [46].

Again, since $e^{-bx/2a}$ is never zero, we can eliminate it, and rearranging the remaining terms, we find that

$$av''(x) + (-b+b)v'(x) + (\frac{b^2}{4a} - \frac{b^2}{2a} + c)v(x) = 0$$

Then, simplifying the expression, and getting a common denominator, we have a factor of $\frac{-b^2+4ac}{4a}$ on the $v(x)$ term. But, since $b^2 - 4ac = 0$, what does $-b^2 + 4ac$ equal?

Therefore, we have

$$av''(x) = 0 \qquad (a \neq 0)$$

and so

$$v''(x) = 0.$$

How can we find an expression for $v(x)$?

Integrate both sides of the equation with respect to x.

Integrate again, and you should find

$$v(x) = c_1 x + c_2.$$

Thus,

$$y = v(x)e^{-bx/2a}$$

becomes

$$y = (c_1 x + c_2)e^{-bx/2a}$$

or, equivalently,

$$y = c_1 x e^{-bx/2a} + c_2 e^{-bx/2a}. \tag{5.1}$$

We see that y is then a linear combination of the two solutions

$$y_1(x) = e^{-bx/2a} \qquad \text{and} \qquad y_2(x) = xe^{-bx/2a}.$$

How do we check the independence of these two solutions?

The Wronskian of these two solutions is

$$W\left[y_1, y_2\right](x) = \begin{vmatrix} e^{-bx/2a} & xe^{-bx/2a} \\ -\frac{b}{2a}e^{-bx/2a} & (1-\frac{bx}{2a})e^{-bx/2a} \end{vmatrix}$$

Upon simplifying, you should obtain $e^{-bx/2a}$ for the Wronskian.

Is it ever zero? Can you prove it?

Thus, $\{y_1, y_2\} = \{e^{-bx/2a}, xe^{-bx/2a}\}$ is a fundamental set of solutions. Also, (5.1) is the general solution of equation (1.1) when the roots of the characteristic equation are equal. Notice that one solution corresponds to the repeated root, while a second solution is obtained by multiplying the first solution by x.

We now summarize the results of Lessons 2, 4, and 5. (Spiral)

If we have a homogeneous linear second order differential equation with constant coefficients,

$$ay'' + by' + cy = 0 \qquad (1.1)$$

with corresponding characteristic equation

$$a\lambda^2 + b\lambda + c = 0, \qquad (1.4)$$

then, we have three cases for the form of the general solution, which we summarize in a table:

Type of Roots	General Solution
real, unequal ($\lambda_1 \neq \lambda_2$)	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
complex, conjugates ($\alpha \pm i\beta$)	$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$
real, repeated ($\lambda_1 = \lambda_2$)	$y = c_1 e^{\lambda_1 x} + x c_2 e^{\lambda_2 x}$

Example 5.1 *Find a general solution to*

$$y'' + 4y' + 4y = 0. \tag{5.2}$$

Solution *The characteristic equation is*

$$\lambda^2 + 4\lambda + 4 = 0,$$

which has roots $\lambda_1 = \lambda_2 = -2$. *Then, two linearly independent solutions are*

$$e^{-2x} \qquad \textit{and} \qquad xe^{-2x},$$

which yields the following general solution to (5.2);

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}.$$

Check that it is a solution to (5.2).

Graphing Calculator

Graph the solution $y(x)$ of Example 5.1 for the following values of c_1 and c_2:

$$\{1,3\}, \{1,2\}, \{1,1\}, \text{and}\{1,4\}.$$

Questions

What point do the graphs have in common? Why do they share this point?

If you keep c_1 fixed at 1, discuss the shape of the curve near $y(0)$ for the various values of c_2.

State a conjecture about how c_2 affects the graph when $c_1 = 1$.

Now let c_1 and c_2 vary. What happens to the graph as $x \to \infty$?

From the form of the general solution explain why the function behaves that way, using the words "decays" and "dominates" in your discussion.

Example 5.2 *Find the solution to the initial value problem (IVP)*

$$y'' - 6y' + 9y = 0; \qquad y(0) = 2, \qquad y'(0) = 1. \tag{5.3}$$

Solution *The characteristic equation is*

$$\lambda^2 - 6\lambda + 9 = 0,$$

which has roots $\lambda_1 = \lambda_2 = 3$. *Thus, a fundamental set of solutions is* $\{e^{3x}, xe^{3x}\}$, *and a general solution is*

$$y(x) = c_1 e^{3x} + c_2 x e^{3x}.$$

Then, we have

$$y'(x) = 3c_1 e^{3x} + [c_2 x(3e^{3x}) + e^{3x}(c_2)].$$

Using our initial conditions $y(0) = 2$ *and* $y'(0) = 1$, *we have*

$$2 = c_1 e^0 + c_2(0)e^0,$$

which yields

$$c_1 = 2,$$

and

$$1 = 3c_1e^0 + (c_2(0)3e^0 + e^0c_2)$$

gives

$$1 = 3c_1 + c_2.$$

Substituting in $c_1 = 2$, *we have*

$$c_2 = 1 - 3(2) = -5.$$

Thus, a solution for the IVP in (5.3) is

$$y(x) = 2e^{3x} - 5xe^{3x}.$$

What happens to this function as $x \to \infty$?

Mathematica

Mathematica has a powerful function called **DSolve** which can solve n^{th} order linear homogeneous differential equations with constant coefficients for $n < 5$. We can solve Example 1 using the commands:

```
Clear[y]
DSolve[{y''[x]+4*y'[x]+4*y[x]==0},y[x],x]
```

Notice that the output contains two sets of braces around it. This is a list containing a list with two elements in it. We check that it is indeed a solution to equation (5.1), using the following commands:

```
ygen = %[[1,1,2]]
resultone=D[ygen,{x,2}] + 4*D[ygen,x] + 4*ygen
Simplify[resultone]
```

The **%[[1,1,2]]** command allows us to pick out the second element of the first element of the first element of **y**, which we then call **ygen**.

We can also solve initial value problems using **DSolve** with the initial conditions inserted within the braces. For example, we can use the following commands to solve Example 2:

```
Clear[y]
DSolve[{y"[x]-6 y'[x]+9 y[x]==0, y[0]==2, y'[0]==1},y[x],x]
```

Then we check:

```
yivp = %[[1,1,2]]
resultone=D[yivp,{x,2}] - 6*D[yivp,x] + 9*yivp
Simplify[resultone]
```

Try solving the initial value problem

$$2y''[x] + 5y'[x] + 5y[x] == 0; \qquad y(0) == 0, \qquad y'(0) == 1/2.$$

Notice you obtain a complex exponential. Why is that? How would we simplify it? Mathematica can simplify complex exponentials using the function **ComplexExpand**, as follows:

y = yivp//ComplexExpand

which is equivalent to:

y = ComplexExpand[yivp[[1,1,2]]]

Try plotting this function using the commands learned in previous lessons. What do you see? Try plotting the solution of Example 2. How do the two graphs compare?

Chapter 6

NONHOMOGENEOUS EQUATIONS; METHOD OF UNDETERMINED COEFFICIENTS

In this lesson, we look at solving nonhomogeneous linear equations with constant coefficents; that is, equations of the form

$$ay'' + by' + cy = f(x). \tag{6.1}$$

What was the general form of a linear second order homogeneous equations with constant coefficients?
How do you know the equation is linear?
What is the difference between homogeneous and nonhomogeneous equations? (Spiral)

These equations occur often in applications where an external force is applied to a system. For example, an object attached to the end of a vertical spring can be modeled by the equation

$$my'' = f(t) - \gamma y' - ky,$$

where y is the position of the object at time t, and $f(t)$ is some external force applied to the system. The constant m is the mass of the object.
Mass is measured in what units?
The coefficient γ is the damping coefficient, and k is the spring constant, the stiffness of the spring.

Questions

How can we manipulate this equation into a form similar to (6.1)?

What will that equation be?

We let y_p be a solution of equation (6.1).

What does it mean for y_p to be a solution of the differential equation?

We have

$$ay_p'' + by_p' + cy_p = f(x). \tag{6.2}$$

Now, we let y_c be a solution of the corresponding homogeneous equation

$$ay'' + by' + cy = 0, \qquad (1.1)$$

which says

$$ay_c'' + by_c' + cy_c = 0. \tag{6.3}$$

Adding (6.2) and (6.3), we obtain

$$ay_p'' + ay_c'' + by_p' + by_c' + cy_p + cy_c = f(x).$$

Why did we add the two equations? (Discovery)

Rewriting produces

$$a(y_p'' + y_c'') + b(y_p' + y_c') + c(y_p + y_c) = f(x),$$

which is equivalent to

$$a(y_p + y_c)'' + b(y_p + y_c)' + c(y_p + y_c) = f(x).$$

The previous equation shows that

$$y = y_p + y_c \tag{6.4}$$

is a solution to the nonhomogeneous linear second order equation

$$ay'' + by' + cy = f(x). \qquad (6.1)$$

Explain why.

We also saw in previous lessons that y_c has the form

$$y_c = c_1 y_1(x) + c_2 y_2(x).$$

What theorem tells us that?

Then, we have that a general solution to equation (6.1) is

$$y = y_p + y_c,$$

or, equivalently,

$$y = y_p + [c_1 y_1(x) + c_2 y_2(x)].$$

We call y_c (sometimes denoted by y_h) the *complementary solution* (or homogeneous solution).

Now we investigate how to find y_p, which is called a particular solution. [1]

In order to investigate y_p, we look at several examples with trial solutions.

Example 6.1 *Given the differential equation*

$$y'' + 4y' + 4y = 8x^2 + 2, \tag{6.5}$$

verify that

$$y_p(x) = 2x^2 - 4x + \frac{7}{2}$$

is a particular solution to (6.5), then find the general solution.

Solution

How do you verify that y_p is a particular solution? Do this.

If y_c is a solution to the homogeneous equation, how do we find it? *(Spiral)*

The homogeneous equation is

$$y'' + 4y' + 4y = 0.$$

What is the characteristic equation?

We solve it and find the roots are $\lambda_1 = \lambda_2 = -2$. Hence, the general solution of the homogeneous equation, or the complementary solution, *is*

$$y_c(x) = c_1 e^{-2x} + c_2 x e^{-2x}.$$

[1] This is the name used frequently, but it is a poor choice of terms since the term particular solution may also refer to a solution satisfying prescribed initial conditions. Fortunately, the meaning is usually clear from the context.

Thus, the general solution of (6.5) is

$$y = \underbrace{c_1e^{-2x} + c_2xe^{-2x}}_{y_c} + \underbrace{2x^2 - 4x + \frac{7}{2}}_{y_p}.$$

Now, assume that we did not know what y_p *was, but assume it had a form of*

$$y_p = Ax^2 + Bx + C.$$

Then, what are y_p' *and* y_p''*?*

If we substitute these into (6.5), we find

$$2A + 4(2Ax + B) + 4(Ax^2 + Bx + C) = 8x^2 + 2.$$

Then, distributing and equating like powers of x*, what do we find for* A*,* B*, and* C*?* *(Discovery)*

Try a solution of the form $y_p = Ax^2 + B$*. What do you find?*

Where does the problem occur?

Now, we look at an $f(x)$ of a slightly different form.

Example 6.2 *Find a particular solution of*

$$y'' - 3y' - 4y = 3e^{2x}. \tag{6.6}$$

Solution

We want to find a function y_p *such that the sum of its second derivative minus three times its first derivative minus four times the function itself adds up to* $3e^{2x}$*.*

What type of function must y be to satisfy these requirements? (Hint: Think back to the derivation of the characteristic equation in Chapter 1.)

Thus, we make an "intelligent guess" that $y_p(x)$ is of the form

$$y_p(x) = Ae^{2x},$$

where A is some yet-to-be-determined constant.

Then, what are y_p' and y_p''?

Substituting for y, y', and y'' in equation (6.6) and collecting terms gives

$$(4A - 6A - 4A)e^{2x} = 3e^{2x}.$$

Thus, we must have $-6A = 3$. Why?

Then, $A = -1/2$. Hence, a particular solution is

$$y_p(x) = \frac{-1}{2}e^{2x}.$$

Is this of the form we "guessed" it would be? Explain.

Verify that this really is a particular solution.

Example 6.3 *Find a particular solution of*

$$y'' + 3y' + 4y = 3x + 2. \tag{6.7}$$

Solution

Here, $f(x) = 3x + 2$.

What kind of expression is $3x + 2$? Use the terms "polynomial" and "degree" in your *description.*

How does it compare with $f(x)$ in Example 6.1? What was y_p in that example?

Thus, for this example, we try a particular solution of the form

$$y_p = Ax + B.$$

Then, what do y_p' and y_p'' equal?

Substituting into (6.7), we have

$$0 + 3(A) + 4(Ax + B) = 3x + 2.$$

Simplify the left hand side and combine like powers of x. What do you get?

Two polynomials are equal if and only if the coefficients of like terms are equal. Thus, we set

$$4A = 3 \qquad \textit{and} \qquad 3A + 4B = 2.$$

Solve these two equations.

You should get

$$A = 3/4 \qquad \textit{and} \qquad B = -1/16.$$

Thus, a particular solution of (6.7) is

$$y_p(x) = \frac{3}{4}x - \frac{1}{16}.$$

Questions

How does the particular solution compare with our guess?

How does it compare with the right hand side of (6.7)?

From Examples 1 and 3, try making a conjecture about the form of particular solutions *to try when given $f(x)$ is a polynomial of degree n.*

Example 6.4 *Find a particular solution of*

$$3y'' + y' - 2y = 2\cos x. \tag{6.8}$$

Questions

What might you try for $y_p(x)$ in this example?

Try $y_p(x) = A\cos x$. What do you find on the left hand side?

Will it be easy to solve for A? Why not?

What kind of form do we need on the right hand side in order to set the coefficients of like terms equal to each other?

Solution

Now try

$$y_p = A\cos x + B\sin x.$$

What is the derivative of $\cos x$? Of $\sin x$? *(Spiral)*

What do you get for y_p'?

What about y_p''?

Substituting into (6.8) yields

$$3(-A\cos x - B\sin x) + (-A\sin x + B\cos x) - 2(A\cos x + B\sin x) = 2\cos x$$

and, combining like terms, we have

$$(-5A + B)\cos x + (-A - 5B)\sin x = 2\cos x.$$

Then, what must $-5A + B$ equal? What must $-A - 5B$ equal?

Using elimination, or substitution, we find

$$A = \frac{-5}{13} \text{ and } B = \frac{1}{13}.$$

Hence, our particular solution is

$$y_p(x) = \frac{-5}{13}\cos x + \frac{1}{13}\sin x.$$

How does this particular solution compare with $f(x)$ from equation (6.8)? What's different? Why?

To summarize the three cases we've looked at so far:

1. If $f(x)$ is a polynomial of degree n, we guess a particular solution of the form $A_1x^n + A_2x^{n-1} + \cdots + A_nx + A_{n-1}$.
2. If $f(x)$ is an exponential of the form $Ke^{\alpha x}$, where K and α are known, then we guess a particular solution of the form $Ae^{\alpha x}$.
3. If $f(x)$ is a function of the form $A_1 \sin kx$, $A_2 \cos kx$, or a linear combination of both, we guess a particular solution of the form $A \sin kx + B \cos kx$.

Now, that it looks straightforward, we throw in a new wrinkle:

Example 6.5 *Find a particular solution of*

$$y'' - 4y = 2e^{2x}. \tag{6.9}$$

Solution

Try $y_p = Ae^{2x}$. *Then,* $y_p' = 2Ae^{2x}$ *and* $y_p'' = 4Ae^{2x}$. *Substitute these values into (6.9), and simplify. What do you get on the left hand side?*

Does $0 = 2e^{2x}$*? Why not?* *(Discovery)*

No matter what A *is,* Ae^{2x} *cannot satisfy the given nonhomogeneous equation.*

Where does the problem occur? *(Discovery)*

We look at the corresponding homogeneous equation

$$y'' - 4y = 0,$$

with characteristic equation

$$\lambda^2 - 4 = 0.$$

This equation has roots $\lambda_1 = -2$ *and* $\lambda_2 = 2$*, which gives the complementary solution*

$$y_c = c_1 e^{-2x} + c_2 e^{2x}.$$

Compare the complementary solution with our guess for y_p.

What do you find?

Our initial guess $y_p = Ae^{2x}$ *actually satisfies the* homogeneous *equation (with* $c_1 = 0$*). Therefore, we should begin with a trial function whose derivatives involve* e^{2x} *and something else that can cancel upon substitution into the differential equation and leave the* e^{2x} *term that we need. What type of function will do that?*

We will guess

$$y_p(x) = Axe^{2x}.$$

Then, what do you get for y_p' and y_p''? (Don't forget you must use the product rule.) Substitution into (6.9) yields

$$(4Ae^{2x} + 4Axe^{2x}) - 4(Axe^{2x}) = 2e^{2x}.$$

Thus, the terms involving x cancel, and we get

$$4A = 2 \qquad \Rightarrow \qquad A = 1/2.$$

Then, our particular solution is

$$y_p(x) = 1/2xe^{2x}.$$

How does this particular solution compare with $f(x)$?

What if you would have guessed $y_p(x) = (Ax + B)e^{2x}$?

Will you find a solution?

Try it. Does it work? Why? *(Discovery)*

Example 6.6 *Find a particular solution of the differential equation*

$$y'' - 3y' - 4y = 3xe^{2x}. \tag{6.10}$$

Solution

Recalling that $f(x)$ is the right hand side of the equation, what is $f'(x)$? What is $f''(x)$?

Are $f(x)$, $f'(x)$, and $f''(x)$ linearly independent?

Notice that both $f'(x)$ and $f''(x)$ can be written in the form $(Ax + B)e^{2x}$. Thus, we

try a solution of the form $y_p = (Ax + B)e^{2x}$. *Then, what do we get for* y_p' *and* y_p''*?*
Now, substituting into equation (6.10), we obtain

$$(2Ae^{2x}+4Axe^{2x}+4Be^{2x}+2Ae^{2x})-3(Ae^{2x}+2Axe^{2x}+2Be^{2x})-4(Axe^{2x}+Be^{2x}) = 3xe^{2x}.$$

Collecting like terms, we have

$$-6Axe^{2x} + (A - 6B)e^{2x} = 3xe^{2x}.$$

What do you get when you equate the coefficients?
Then $A = -1/2$ *and* $B = -1/12$ *gives the particular solution of equation (6.10) is*

$$y_p(x) = e^{2x}[\frac{-1}{2}x + (\frac{-1}{12})].$$

How does the answer compare with $f(x)$*?*

A General Recipe for Solving Linear Second Order Nonhomogeneous Equations with Constant Coefficients

$$ay'' + by' + cy = f(x)$$

1. Solve the corresponding homogeneous equation

$$ay'' + by' + cy = 0 \qquad (1.1)$$

   by earlier methods and call this solution $y_c(x)$. Then

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

   is the complementary solution .

2. Write $f(x)$ and all its derivatives that are linearly independent. Multiply each by an arbitrary constant. Use this expression for the particular solution $y_p(x)$.

3. Compare $y_p(x)$ and $y_c(x)$.

   (a) If there are NO terms in common, substitute y_p into the original equation and find the constants that solve the equation.

   (b) If y_p and y_c do have terms in common, multiply all terms in y_p by the lowest power of x that will eliminate common terms.

4. The general solution will be

$$y(x) = y_c(x) + y_p(x).$$

Example 6.7 *Find the general solution of the differential equation*

$$y'' + 3y' + 2y = e^x \tag{6.11}$$

Solution

Step 1: *We solve the corresponding homogeneous equation*

$$y'' + 3y' + 2y = 0$$

using the characteristic equation $\lambda^2 + 3\lambda + 2 = 0$*, which factors into*

$$(\lambda + 2)(\lambda + 1) = 0.$$

Thus, we have $\lambda_1 = -2$ *and* $\lambda_2 = -1$*. So, the complementary solution is*

$$y_c(x) = c_1 e^{-2x} + c_2 e^{-x}.$$

Question

Is the ordering of the λ*'s important in the equation?*

Why or why not?

It is a common practice to write the smallest λ *first (and associate* c_1 *with it).*

Step 2: *We write the derivatives of* $f(x)$*, which are*

$$f(x) \;=\; e^x$$

$$\begin{aligned} f'(x) &= e^x \\ f''(x) &= e^x \\ &\vdots \\ f^{(n)} &= e^x \end{aligned}$$

There are no other linearly independent derivatives occurring.

How do you know that? Can you prove it?

Thus, our "intelligent guess" is

$$y_p(x) = Ae^x.$$

Step 3: *We compare*

$$Ae^x \qquad \textit{with} \qquad c_1e^{-2x} + c_2e^{-x}$$

and find they have no terms in common. Why not?

What are the derivatives of $y_p = Ae^x$?

Then, we substitute y_p and its derivatives into equation (6.11), and we obtain

$$Ae^x + 3(Ae^x) + 2(Ae^x) = e^x.$$

What do you get when you combine like terms? So, we have $A = \frac{1}{6}$. Then, a particular solution is

$$y_p(x) = \frac{1}{6}e^x.$$

Does this agree with our original guess?

Step 4: *The general solution is*

$$y(x) = \underbrace{c_1e^{-2x} + c_2e^{-x}}_{y_c(x)} + \underbrace{\frac{1}{6}e^x}_{y_p(x)}.$$

Example 6.8 *Solve the initial value problem*

$$y'' - y = 2e^x + x; \qquad y(0) = 0, \qquad y'(0) = 0 \tag{6.12}$$

Step 1: *Solve the homogeneous equation using the characteristic equation*

$$\lambda^2 - 1 = 0. \qquad \textit{(Why isn't it } \lambda^2 - \lambda \textit{ ?)}$$

Solving for λ, *we have*

$$\lambda_1 = -1 \qquad \textit{and} \qquad \lambda_2 = 1.$$

Thus, a complementary solution is

$$y_c(x) = c_1e^{-x} + c_2e^x.$$

Step 2: *To look at the derivatives of* $f(x)$, *it is easiest to look at the individual terms of* $f(x)$. *Why can we do this?*

Let $f_1(x) = 2e^x$ *and* $f_2(x) = x$. *Then the derivatives are*

$$f_1'(x) = 2e^x \qquad f_2'(x) = 1$$
$$f_1''(x) = 2e^x \qquad f_2''(x) = 0$$
$$f_1'''(x) = 2e^x \qquad f_2'''(x) = 0$$

$$\vdots \qquad \vdots$$

$$f_1^{(n)}(x) = 2e^x \qquad f_1^{(n)}(x) = 0$$

So, we guess that $y_{p_1}(x)$ has the form Ae^x and $y_{p_2}(x)$ has the linear form $Bx + C$.

Thus, our guess for y_p is the sum of these. Why?

Then our guess for y_p is the sum of $y_{p_1}(x)$ and $y_{p_2}(x)$ which is

$$y_p(x) = \underbrace{Ae^x}_{y_{p_1}} + \underbrace{Bx + C}_{y_{p_2}}.$$

Step 3: *Comparing y_c and y_p, we see that they both have an e^x term.*

What is wrong with our guess?

We try multiplying our guess for y_p by x and get

$$y_p(x) = Axe^x + Bx^2 + Cx.$$

Comparing again, we see that y_p and y_c no longer have terms in common.

What does this tell you about y_p and y_c. Can you prove it?

Now, find y_p' and y_p''.

Upon simplifying, you should obtain

$$y_p'(x) = Axe^x + Ae^x + 2Bx + C$$

and

$$y_p''(x) = Axe^x + 2Ae^x + 2B.$$

Then, substituting into our original equation, we have

$$(Axe^x + 2Ae^x + 2B) - (Axe^x + Bx^2 + Cx) = 2e^x + x.$$

What are the next two steps?

Thus, $2A = 2$, $-B = 0$, $-C = 1$, *and* $2B = 0$, *which give us* $A = 1$, $B = 0$, *and* $C = -1$.

Hence a particular solution is

$$y_p(x) = (1)xe^x + (0)x^2 + (-1)x = xe^x - x.$$

Then the general solution $y(x) = y_p(x) + y_c(x)$ *is*

$$y(x) = c_1e^{-x} + c_2e^x + xe^x - x.$$

Now we use our given initial conditions that $y(0) = 0$ *and* $y'(0) = 0$.

What is $y'(x)$*?*

Then $y'(0) = 0$ *implies*

$$0 = -c_1e^0 + c_2e^0 + (0)e^0 + e^0 - 1,$$

which simplifies to

$$0 = -c_1 + c_2.$$

Using $y(0) = 0$, *what do we obtain?*

Now, we solve the two equations together and obtain

$$c_1 = 0 \qquad \textit{and} \qquad c_2 = 0.$$

Hence the solution to our initial value problem is

$$y(x) = 0e^x + 0e^{-x} + xe^x - x,$$

which simplifies to

$$y(x) = xe^x - x.$$

Does this look like a reasonable solution to this initial value problem?

Justify your answer.

Graphing Calculator

Using fixed values of $\{-2, 2\}$ for c_1 and c_2, respectively, graph the solution to the homogeneous equation (the complementary solution) and the general solution to Example 6.1 on the same graph. How does the addition of the particular solution affect the general solution?

Do the same for Example 6.5. What is the result?

Which part of the original differential equation does the particular solution come from?

Try repeating the above for various fixed c_1 and c_2 values.

Try to make some general conjectures about how $f(x)$ changes the solution to the homogeneous equation.

Mathematica

Mathematica's **DSolve** function can also solve nonhomogeneous equations. For example, we can find the general solution of the differential equation in Example 6.1 using the following commands:

```
Clear[y]
DSolve[{y"[x]+4*y'[x]+4*y[x]==8*x^2+2},y[x],x]
```

Do you see the particular solution that was given in Example 6.1?
What is the complementary solution?
If you didn't know which part of the solution was the particular solution, how could you find it using Mathematica?

We investigate another differential equation and the effect that nonhomogeneity has on it. For the initial value problem

$$x''(t) + 4x = f(t); \qquad x(0) = 0, \qquad x'(0) = 0,$$

investigate the effects that changing $f(t)$ has on the solution. Let $f(t)$ take on the values:

1. $\cos t$
2. $\sin t$
3. 1
4. 0.

To solve and plot the solution to the IVP with $f(t) = \cos t$, we use the commands:

```
Clear[x]
sol1=DSolve[{x''[t]+4*x[t]==Cos[t],x[0]==0, x'[0]==0},x[t],t]
plot1=Plot[sol1[[1,1,2]],{t,0,2 Pi}, PlotStyle->Dashing[{0.04}]]
```

What does the **sol1[[1,1,2]]** do? (Spiral)

Notice the **Dashing** command which allows you to draw the curve using dotted lines.

We now do the second function $f(t) = \sin t$.

```
Clear[x]
sol2=DSolve[{x''[t]+4*x[t]==Sin[t],x[0]==0, x'[0]==0},x[t],t]
plot2=Plot[sol2[[1,1,2]],{t,0,2 Pi}, PlotStyle->Dashing[{0.02}]]
Show[plot1,plot2]
```

The last function allows us to see both plots on the same graph.

Try doing the third and fourth functions, and graphing all four on the same graph.

What is the solution to the differential in the fourth part?

Why is that? If you think of $x'(0)$ as the initial velocity, what does it mean graphically for $x'(0)$ to be equal to zero?

Chapter 7

APPLICATIONS; MECHANICAL VIBRATIONS AND UNDAMPED, FREE MOTION

One of the most important reasons to study differential equations is their widespread application to physical processes. Two areas of common application are in the motion of a mechanical system and in the flow of electric current in a simple series circuit. These, and many other physical problems, can be modeled by the solution of an initial value problem of the form

$$ax'' + kx' + cx = f(t); \qquad x(0) = x_0, \qquad x'(0) = v_0.$$

(Take note of the change of the dependent variable from y to x and the change of the independent variable from x to t. What is the reason for doing this?)

7.1 Application I: Mass-Spring Systems

To study mechanical vibrations, we start with the simple system consisting of a spring suspended from a rigid support with a mass attached to the end of the spring. Figure 7.1 shows a typical mass-spring system.

In order to investigate this system, we need two laws from physics: Hooke's [1]

[1]Robert Hooke (1635-1703) was an English scientist who worked on optics, simple harmonic motion and stress in stretched springs. Hooke first published his law of elastic behavior in 1676 as an anagram: *ceiiinosssttuv*; in 1678 he gave the solution *ut tensio sic vis*, which means, roughly, "as the force so is the displacement" [48].

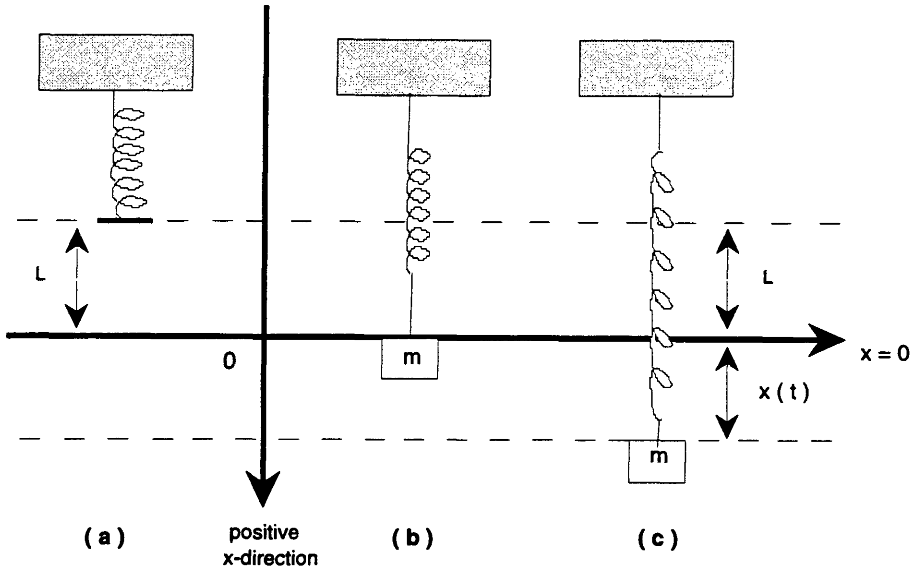

FIG. 7.1. A typical mass-spring system with the spring (a) in natural position, (b) in equilibrium position, and (c) in motion.

law and Newton's[2] second law of motion. Hooke's law says that, as long as the stretch of a spring is not close to its elastic limit, it will exert a force opposite to the direction of elongation of the spring with a magnitude directly proportional to the elongation L. (Note: In Figure 7.1, we are considering part (b) only, where the mass is in equilibrium.)

Why wouldn't the law apply when the stretch is close to the elastic limit?

What does "directly proportional to" mean?

[2]Sir Isaac Newton (1643-1727) was knighted in 1708, the first scientist to be so honored for his work. Newton laid the foundation for differential and integral calculus, several years before its independent discovery by Leibniz. He also made great contributions in physics and celestial mechanics, developing the theory of universal gravitation. His publication *Philosophiae naturalis principia mathematica* or *Principia* as it is usually known, is considered by some to be the greatest scientific book ever written [50].

In symbols, Hooke's law can be written as

$$F_s = kL,$$

where we are considering the magnitude (not the direction) of the force. The constant of proportionality k is called the *spring constant.*

Questions

If a 30-lb weight stretches a spring 6 ft., then what is the spring constant k?

What are the units of k?

What system of measurement are we working in?

What are the basic units in that system?

If a 30-lb weight is stretched a length of 6 *in*, what is the spring constant? (Discovery)

Newton's second law says $F = ma$, where m is the mass of the object, and a is the acceleration of the mass at time t. What is the relationship between acceleration, velocity, and position of an object?

Then, if $x(t)$ is the position of the object at time t, we have

$$F = m\frac{d^2x}{dt^2} = mx''(t).$$

Now, we consider the forces acting on the mass m in Figure 7.1. Notice that we take the downward direction to be the positive direction of the x-axis. We denote the distance of the body from its *equilibrium position* at time t by $x(t)$. We take $x > 0$ when the spring is stretched (from its equilibrium length) and $x < 0$ when it is compressed.

Force 1: Gravity

The force of gravity is the weight of the object

$$F_g = ma = mg,$$

where g is the acceleration due to gravity.

Questions

What are the units of g in the metric and English systems?

What is mass measured in? (Spiral)

In what direction is this force being applied?

Thus, we consider the force to have a positive sign.

Force 2: Resisting Force

The spring exerts a force resistive to the stretch, or restoring force whose magnitude is proportional to the spring's elongation beyond its natural length. From Figure 7.1(c), we see the spring's elongation is $x + L$, and using Hooke's law, we have the resistive force of the spring is

$$F_s = -k(x + L),$$

or

$$F_s = -kx - kL,$$

where $k > 0$ is the spring constant.

In what direction is the spring force?

Observe that when $x = 0$, the system is in equilibrium, which means the forces are in balance.

Thus, we have the equation $mg = kL$. Why?

Substituting mg for kL, we can now represent the spring's restoring force as

$$F_s = -kx - mg.$$

Force 3: Frictional (Damping) Force

There is also a frictional force acting on the mass, such as the viscosity of a fluid the object is moving through. Can you think of other examples? (Discovery)

Such an object is termed a *dashpot.* Experiments have shown that the resistive (frictional) force F_r of an object that is not "moving too fast" is proportional to the velocity of the mass and in the opposite direction.

How do we represent the velocity of the mass, if $x(t)$ is its position at time t? (Spiral)

Then we have

$$F_r = -\gamma x'(t),$$

where γ is the damping coefficient.

What are the units for γ?

Why is the sign on F_r negative?

Force 4: External Forces

There can also be a force $F(t)$ directed downward or upward on the object the spring is mounted on or on the object itself. For example, a car going on a road hits a bump. We can consider the car to be the mass of the system, with the springs of the car corresponding to the spring in our system, and the shock absorbers representing the dashpot. The bump in the road is an external force applied to the system. Can you think of some other examples of such a force?

Total Force

From our previous discussion, the total force F acting on the mass is the sum of the four forces F_g, F_s, F_r, and $F(t)$:

$$F(t, x, x') = mg + (-kx - mg) + (-\gamma x') + F(t).$$

See Figure 7.2.

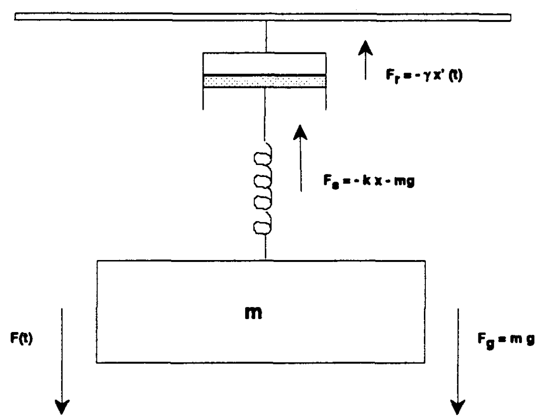

FIG. 7.2. Spring-mass system showing the sum of the four forces.

Why is the force a function of t, x, and x', and not of m, g, k, and γ?
What do you get when you simplify the right hand side?
Using Newton's second law, what we can we write for the left hand side of the equation?
Then, we have

$$mx'' = -kx - \gamma x' + F(t),$$

which is equivalent to

$$mx'' + \gamma x' + kx = F(t). \tag{7.1}$$

Keep in mind that x is a function of t, and that $x' = \dfrac{dx}{dt}$ and $x'' = \dfrac{d^2x}{dt^2}$.

If the system is *undamped*, what will γ equal?

Then, we have that

$$mx'' + kx = F(t).$$

If $\gamma \neq 0$, the system is said to be *damped.*

When $F(t) = 0$, we say the motion is *free*, or unforced, and we have the equation

$$mx'' + \gamma x' + kx = 0.$$

What is this type of equation called? (Spiral)

If $F(t) \neq 0$, we say the motion is *forced*, and we call $F(t)$ the forcing function.

A simple spring-mass system with damping is shown in Figure 7.3.

7.2 Undamped, free motion

In this section we consider the *undamped*, *free* motion case. Then, what are γ and $F(t)$ equal to? (Spiral)

Thus, equation (7.1) reduces to

$$mx'' + kx = 0, \tag{7.2}$$

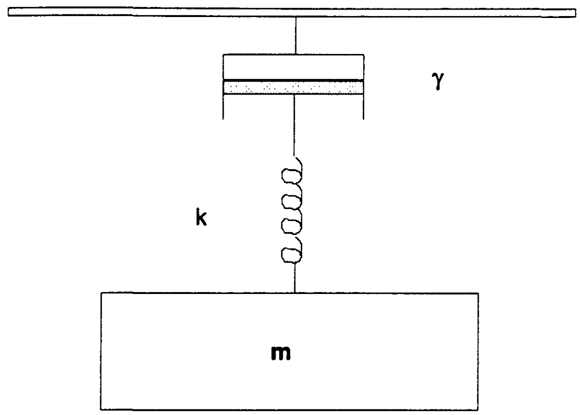

FIG. 7.3. Spring-mass system with spring constant k, damping coefficient γ, and mass m.

or $x'' + \frac{k}{m}x = 0$, which is equivalent to

$$x'' + \omega^2 x = 0, \qquad \text{where } \omega = \sqrt{\frac{k}{m}}. \tag{7.3}$$

Why would we want to let $\omega = \sqrt{\frac{k}{m}}$? (Discovery)

What are the units of ω?

What is the characteristic equation for equation (7.3)?

The roots of this characteristic equation are

$$\lambda = \pm i\omega.$$

Hence a general solution to (7.3) is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t. \tag{7.4}$$

Try graphing this function for various values of c_1, c_2, and ω. You should see graphs that are similar to the graph of $x = \cos t$. In order to see where the maxima and minima of the graphs occur, we can look at the derivative of the function, as we did in calculus. (Spiral)

Then, taking the first derivative and setting it equal to zero, we have

$$-\omega c_1 \sin \omega t + \omega c_2 \cos \omega t = 0,$$

which yields,

$$\tan \omega t_c = \frac{c_2}{c_1},$$

where t_c are the critical points of the function. Now, we let $\phi = \omega t_c$, so that $t_c = \frac{\phi}{\omega}$. Thus, we have

$$\tan \phi = \frac{c_2}{c_1},$$

and we sketch the corresponding right triangle, with $A = \sqrt{c_1^2 + c_2^2}$ for the hypotenuse:

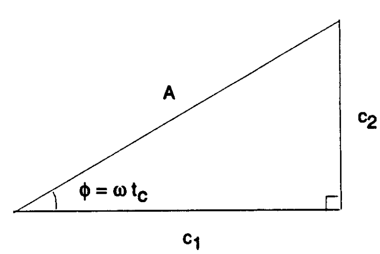

FIG. 7.4. A diagram of the angle ϕ.

From the diagram, what is the value of $\cos \phi$?

What is $\sin\phi$?

Then, we have $c_1 = A\cos\phi$ and $c_2 = A\sin\phi$. Now we substitute these values for c_1 and c_2 in equation (7.4), and obtain

$$x(t) = A\cos\omega t\cos\phi + A\sin\omega t\sin\phi.$$

But, what is $\cos\omega t\cos\phi + \sin\omega t\sin\phi$ equal to?

Then, factoring out A, and using a trigonometric identity, we have

$$x(t) = A\cos(\omega t - \phi). \tag{7.5}$$

Thus, we have transformed our solution for $x(t)$ into a more compact form. If we factor out ω, we obtain another equivalent form of $x(t) = A\cos\omega(t - \frac{\phi}{\omega})$. How is this form more useful than (7.4)? The value $\frac{\phi}{\omega}$ is called the *phase shift.* (See Figure 7.5.) Another way of deriving the previous relations, which is done in many texts, is to assume that $x(t) = A\cos(\omega t - \phi)$, and work backwards.

We call A the *amplitude* of the function, and ϕ is called the *phase angle.*

The term *period* refers to the time required for the system to complete one full oscillation, and is given by

$$\tau = \frac{2\pi}{\omega}. \tag{7.6}$$

The reciprocal of the period is called the *frequency* or *natural frequency* of the system.

Questions

Where does the maximum value of this curve occur? (Spiral)

What is the value of the curve at that point?

What are two ways of writing the frequency?

The value $\omega = \sqrt{\frac{k}{m}}$ is called the *circular frequency* or *angular frequency* of the system. Since $\tan\phi = \frac{c_2}{c_1}$, we have

$$\tan^{-1}\frac{c_2}{c_1} = \phi.$$

From the previous equation only, if c_1 and c_2 are both negative, in which quadrant(s) could ϕ be located in?

Since $A > 0$, what do we know about the sign of c_1 in relation to $\cos\phi$?

We also know that c_2 must have the same sign as $\sin\phi$. Thus, if $c_1 < 0$, and $c_2 < 0$, we have $\cos\phi < 0$ and $\sin\phi < 0$, which tells us that ϕ must be in what quadrant?

From equation (7.5), we see that we have a cosine wave for the motion of a mass in an undamped, free system. This is called *simple harmonic motion*, and a typical graph of it is shown in Figure 7.5.

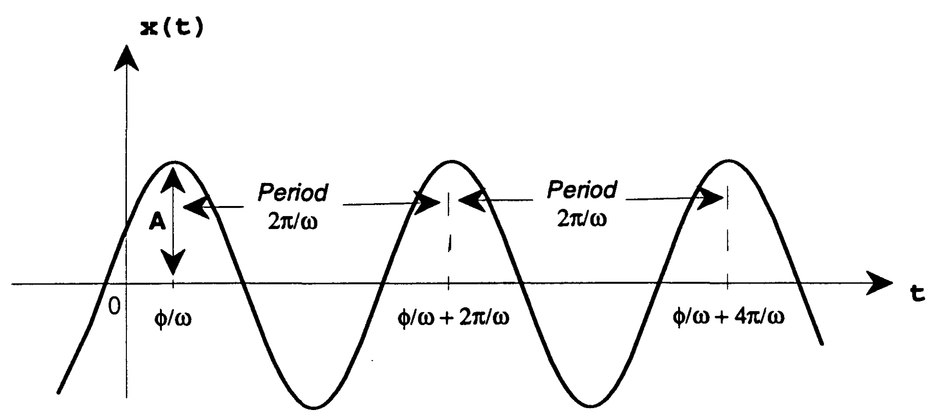

FIG. 7.5. Simple harmonic motion; $x = A\cos(\omega t - \phi)$.

Questions

Try graphing $x(t)$ versus ωt, instead of t. The maximum will occur at ϕ, instead of $\frac{\phi}{\omega}$. Where else do maxima occur? (Discovery)

Observing equation (7.6), what happens to the period τ as the mass m is increased? What does that mean in terms of how long it takes the mass to return to its starting point?

Does the mass vibrate more rapidly or more slowly? Why?

On the other hand, if we increase the spring constant k, what happens to the period τ?

Does a stiffer spring cause the system to vibrate more rapidly or more slowly?

Example 7.1 *A mass weighing* 10 *lb stretches a spring* 2 *in. If the mass is displaced an additional* 2 *in. and then given an upward velocity of one foot per second, determine the equation of the motion of mass along with its amplitude, period, and phase angle. How long after release does the mass pass through the equilibrium position?*

Solution *The normal unit of force in the English system is ft · lb, so we convert inches to feet. What does Hooke's law state?* *(Spiral)*

Then, we have

$$10lb = k(\frac{1}{6}ft).$$

Thus, $k = 60\ lb/ft$ *is the spring constant. We are given the weight of the object. What is the formula for weight?*

Then, the mass m *is given by*

$$m = \frac{w}{g} = \frac{10\ lb}{32\ \frac{ft}{sec^2}} = \frac{5}{16}\frac{lb \cdot sec^2}{ft}.$$

Hence, the equation of motion reduces to

$$\frac{5}{16}x'' + 60x = 0,$$

or

$$x'' + 192x = 0.$$

The characteristic equation is then

$$\lambda^2 + 192 = 0,$$

which has roots

$$\lambda = \pm\sqrt{-192} = \pm i8\sqrt{3}.$$

Hence, the general solution is

$$x(t) = c_1 \cos(8\sqrt{3}t) + c_2 \sin(8\sqrt{3}t).$$

From the problem, the mass is displaced an additional 2 *in. which can be written as*

$$x(0) = \frac{1}{6},$$

and the initial upward velocity is one foot per second can be expressed as

$$x'(0) = -1.$$

We find

$$x'(t) = -8c_1\sqrt{3}\sin(8\sqrt{3}t) + 8c_2\sqrt{3}\cos(8\sqrt{3}t),$$

which gives

$$-1 = -8c_1\sqrt{3}\sin 0 + 8c_2\sqrt{3}\cos 0,$$

and thus,

$$c_2 = \frac{-1}{8\sqrt{3}}.$$

Also,

$$\frac{1}{6} = c_1 \cos 0 + c_2 \sin 0,$$

which yields

$$c_1 = \frac{1}{6}.$$

Hence, the solution satisfying the initial conditions is

$$x(t) = \frac{1}{6}\cos(8\sqrt{3}t) + \frac{-1}{8\sqrt{3}}\sin(8\sqrt{3}t).$$

The amplitude (maximum displacement of the wave) is given by

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{\frac{1}{36} + (\frac{-1}{8\sqrt{3}})^2} = \sqrt{\frac{1}{36} + \frac{1}{192}} = \sqrt{\frac{19}{576}} \approx 0.18162 \ \mathit{ft}.$$

The period is

$$\tau = \frac{2\pi}{w} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{\frac{5}{16}}{60}} = \frac{2\pi}{8\sqrt{3}} = \frac{\pi}{4\sqrt{3}} \approx 0.45345 \ \mathit{sec.}$$

The phase angle is

$$\phi = \tan^{-1}(\frac{c_2}{c_1}) = tan^{-1}(\frac{\frac{-1}{8\sqrt{3}}}{\frac{1}{6}}) = tan^{-1}(\frac{-3}{4\sqrt{3}}) = tan^{-1}(\frac{-\sqrt{3}}{4}).$$

From the tangent equation, there are two solutions for ϕ*, one in the second quadrant, and one in the fourth quadrant. We look at* $c_1 = A\cos\phi$ *and* $c_2 = A\sin\phi$ *and see that since* $c_1 > 0$ *and* $A > 0$*,* $\cos\phi$ *must be positive also. Likewise, since* $c_2 < 0$*, we must have* $\sin\phi < 0$*. These two conditions are satisfied simultaneously in the fourth quadrant. Thus,*

$$\phi = -\tan^{-1}\left(\frac{\sqrt{3}}{4}\right) \approx -0.40864 \text{ radians.}$$

Graphing Calculator

Graph the solution to the initial value problem on your graphing calculator. Change the domain and range until about two oscillations appear in the window. Use your calculator's tracing ability to approximate the maximum of the function that occurs just to the left of the y-axis. What is the approximate value of that local maximum? How do we find that value from the problem? Hint: It is a critical value for t.

Use your calculator to get an approximation for the value.

Now use your calculator's evaluation capabilities to evaluate the graph at the point we just found. Does it appear to be a maximum?

Try using your calculator's ability to find a local maximum to see if it agrees with our findings.

What is the period of the function? When should the graph be at a maximum again? Most graphing calculators will have the outputs stored in the variable that appears on the screen. Try using the evaluation function to evaluate the function at $x_{max} + \frac{2\pi}{\omega}$. Does it confirm our hypothesis?

How can you use your calculator to check the initial conditions?

Try using your calculator to find the zeroes of this function.

Example 7.2 *A mass of* 2 *kg is suspended from a spring with a known spring constant of* 10 *N/m and allowed to come to rest. It is then set in motion by giving it an* *initial velocity of* 150 *cm/sec. Find an expression for the motion of the mass, assuming no air resistance. Also, determine the circular frequency, natural frequency, and period of the system.*

Solution *There is no external force applied to the mass, so* $F(t) = 0$*, and no resistance from the surrounding medium implies* $\gamma = 0$*. We are given* $m = 2$ *kg and* $k = 10$ *N/m. Thus, the differential equation is*

$$2x'' + 10x = 0,$$

or equivalently,

$$x'' + 5x = 0.$$

The characteristic equation is

$$\lambda^2 + 5 = 0,$$

which has roots

$$\lambda = \pm i\sqrt{5}.$$

Thus, a general solution is

$$x(t) = c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t.$$

Since the object is in the equilibrium position initially, we have $x(0) = 0$*. This gives us*

$$0 = c_1 \cos 0 + c_2 \sin 0,$$

which yields $c_1 = 0$. *Also, we have*

$$x'(t) = -c_1\sqrt{5}\sin\sqrt{5}t + c_2\sqrt{5}\cos\sqrt{5}t,$$

and the initial velocity is $150cm/sec = 1.5m/sec$ *can be expressed as* $x'(0) = 1.5$. *This gives us*

$$1.5 = -c_1\sqrt{5}\sin 0 + c_2\sqrt{5}\cos 0,$$

which gives

$$c_2 = \frac{1.5}{\sqrt{5}} = \frac{3\sqrt{5}}{10}.$$

Thus, the position of the mass at any time t is given by the equation

$$x(t) = \frac{3\sqrt{5}}{10}\sin\sqrt{5}t.$$

For the circular frequency, *we find*

$$\omega = \sqrt{5} \approx 2.236 \ radians/sec,$$

and the natural frequency *is*

$$f = \frac{\omega}{2\pi} = \frac{\sqrt{5}}{2\pi} \approx 0.3559 \ Hz.$$

The period τ *is given by*

$$\tau = \frac{1}{f} = \frac{2\pi}{\sqrt{5}} \approx 2.81 \ sec.$$

Mathematica

Using the **DSolve** command learned previously, we can solve the differential equation and plot the solution to Example 8.2, and look at the phase angle:

```
Clear[x]
DSolve[{x"[t] + 5*x[t] == 0,x[0]==0,x'[0]==1.5},x[t],t]
soln1=%[[1,1,2]] //ComplexExpand//Simplify
plot1=Plot[soln1,{t,-1,3}]
```

What did the **ComplexExpand** and **Simplify** commands do?

What did the **%[[1,1,2]]** allow us to do? Why did we need to do that? (Spiral)

About where does the graph cross the t-axis for $t > 0$?

Approximately where does a relative maximum of the solution occur?

We can use Mathematica to answer both of these questions more precisely:

```
FindRoot[soln1==0,{t,1}]
```

The **FindRoot** command used Newton's Method to find a zero of the function starting with an initial value of $t = 1$.

How does that compare with your original estimate of the root?

What is Newton's Method? (Spiral)

Try using initial values of $t = 3$, $t = 2.2$, and $t = 2$ in the above?

What do you find?

Explain what is happening, in terms of the graph. (Discovery)

Mathematica can often solve equations exactly using the **Solve** command.

Try the following:

```
Solve[soln1==0,t]
```

What do you find? What does Mathematica's message mean?

Use the term "one-to-one" in your answer.

In order to find other solutions, what would we need to do to the function? (Discovery)

How do we find the location of a maximum value for the function?

We can use Mathematica to find a numerical approximation for t_c, by finding the derivative, setting it equal to zero, and solving for t:

```
soln1p=D[soln1,t]
Solve[soln1p==0,t]
```

What did you find for the critical value approximation?

What does Mathematica's warning message mean?

What is ϕ in terms of c_1 and c_2?

Let's try to use Mathematica to find it:

```
c2=3*Sqrt[5]
c1=0
phi=ArcTan[c2/c1]
```

What does Mathematica give you? Why?

Of what are we trying to take the inverse tangent?

Why not try that on Mathematica, using Mathematica's special "constant" **Infinity**?

Then we find the phase shift by dividing ϕ by ω, and look at the numerical value:

```
phi=ArcTan[Infinity]
phaseshift=phi/Sqrt[5]
N[phaseshift]
```

Questions

What kind of units does ϕ have? Why is it called the "circular" frequency?

How does the ratio of the phase shift to the period relate to what fraction ϕ is of a complete circle? Investigate how the changing of the initial conditions affects the amplitude and phase angle.

Does the period (and frequency) change when the initial conditions change?

What constant in the original equation will affect the period?

Chapter 8

DAMPED, UNFORCED VIBRATIONS AND ELECTRICAL CIRCUITS APPLICATION

In the last lesson we examined vibrations of a spring-mass system with undamped, unforced motion. Thus we had the equation

$$mx'' + \gamma x' + kx = F(t) \qquad (7.1).$$

What was $F(t)$? (Spiral)

If there is no damping, $\gamma = 0$ and the equation then reduced to

$$mx'' + kx = 0 \qquad (7.2).$$

In most applications, however, there is some kind of frictional or damping force present, such as a shock absorber, which we termed a dashpot. Thus we consider the motion of a system described by

$$mx'' + \gamma x' + kx = 0, \quad \text{where} \quad \gamma > 0. \tag{8.1}$$

What is the characteristic equation associated with (8.1)?

When we solve it what do we get for λ?

We find

$$\lambda = -\frac{\gamma}{2m} \pm \frac{1}{2m}\sqrt{\gamma^2 - 4mk} \tag{8.2}$$

What is the discriminant of the characteristic equation ? (Spiral)

What does it tell us about the roots of the characteristic equation ?

8.1 Underdamped or Oscillatory Motion

When $\gamma^2 - 4mk < 0$, there are two complex conjugate roots to (8.2), which are

$$\lambda_1 = \frac{-\gamma}{2m} + i\frac{\sqrt{4mk - \gamma^2}}{2m} \quad \text{and} \quad \lambda_2 = \frac{-\gamma}{2m} - i\frac{\sqrt{4mk - \gamma^2}}{2m}. \tag{8.3}$$

Why did $\gamma^2 - 4mk$ become $4mk - \gamma^2$?

Then, we let

$$\alpha = \frac{-\gamma}{2m} \quad \text{and} \quad \beta = \frac{\sqrt{4mk - \gamma^2}}{2m}.$$

So, what is a general solution to (8.1)?

In factored form, we have

$$x(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t),$$

where

$$\alpha = \frac{-\gamma}{2m} \quad \text{and} \quad \beta = \frac{\sqrt{4mk - \gamma^2}}{2m}.$$

Then as we did in the damped case, we can express $x(t)$ in the alternate form

$$x(t) = Ae^{\alpha t} \cos(\beta t - \phi), \quad \text{where} \quad A = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \tan \phi = \frac{c_2}{c_1}. \tag{8.4}$$

Thus, we see that $x(t)$ is the product of an exponential factor $Ae^{\alpha t} = Ae^{(\frac{-\gamma}{2m})t}$ and a cosine factor $\cos(\beta t - \phi)$.

What does the graph of $Ae^{\alpha t}$ look like for various values of α?

What does the graph of $\cos(\beta t - \phi)$ look like? (Spiral)

The factor $\cos(\beta t - \phi)$ describes oscillatory motion.

As t grows, which factor in (8.4) is the "dominant" one?

We call $Ae^{\alpha t} = Ae^{(\frac{-\gamma}{2m})t}$ the *damping factor.*

Since γ and m are positive, the damping factor $Ae^{\alpha t} = Ae^{(\frac{-\gamma}{2m})t}$ goes to zero as $t \to \infty$. Why?

The system is called *underdamped* since there is not enough damping (γ is too small) to prevent the system from oscillating. Why is that?

In Figure 8.1, we see the graph of a typical solution $x(t)$. Describe what is happening

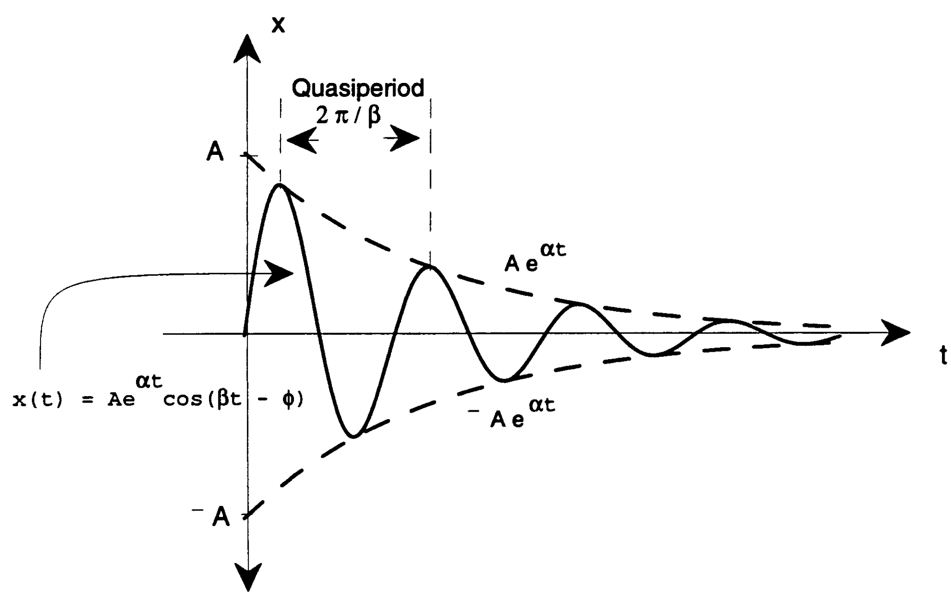

FIG. 8.1. Underdamped or oscillatory motion.

in Figure 8.1, and why. (Discovery)

The distance between two successive local maxima of the curve is called the *quasiperiod*, or *pseudoperiod*, and is given by

$$p = \frac{2\pi}{\beta} = \frac{4m\pi}{\sqrt{4mk - \gamma^2}}.$$

Try to derive this.

How does this compare with our definition of the *period* of a function? (Spiral)

Also, $\frac{1}{p}$ is called the *quasifrequency* and $Ae^{\alpha t}$ is known as the *time-varying amplitude.* How do these compare with our previous definitions of *frequency* and *amplitude*?

If we let the damping coefficient γ approach zero, what happens to $Ae^{\alpha t} = Ae^{(\frac{-\gamma}{2m})t}$?

Then, our curve becomes bounded above by A instead of by $Ae^{\alpha t}$ and what kind of motion do we have?

Also note that values of t where the graph of $x(t)$ touches $\pm Ae^{\alpha t}$ are *not* the same values at which $x(t)$ attains its relative maximum and minimum values. Why do we note this?

How can you prove that? (Discovery)

8.2 Critically Damped Motion

When $\gamma^2 = 4mk$, the discriminant is zero. What kind of roots will the characteristic equation (8.1) have in this case? (Spiral)

From (8.3) we see that

$$\lambda_1 = \lambda_2 = -\frac{\gamma}{2m}$$

is the repeated root. Hence, a general solution to (7.1) is

$$x(t) = c_1 e^{(\frac{-\gamma}{2m})t} + c_2 t e^{(\frac{-\gamma}{2m})t}. \qquad (8.5)$$

If we wish to examine what happens as $t \to \infty$, what kind of form do we obtain for (8.5)?

What rule from calculus can we use to help evaluate the limit? (Spiral)

What forms are we allowed to apply it to?

Our function is not in that form, but we can rewrite it as

$$x(t) = \frac{c_1 + c_2 t}{e^{(\frac{\gamma}{2m})t}}. \qquad (8.6)$$

Now, we can apply *L'Hôpital's Rule*, since the equation is in the form $\frac{\infty}{\infty}$.

L'Hôpital's Rule
If $\lim_{t\to\infty} f(t) = \infty$ and $\lim_{t\to\infty} g(t) = \infty$,
then $\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{f'(t)}{g'(t)}$.

What do we obtain when we apply l'Hôpital's rule to equation (8.6)?

Thus, we apply l'Hôpital's rule, and we have

$$\lim_{t\to\infty} \frac{c_2}{(\frac{\gamma}{2m})e^{(\frac{\gamma}{2m})t}} = 0. \qquad (\gamma \text{ and } m \text{ are both positive})$$

Thus, $x(t)$ decays to zero as $t \to \infty$. Rewriting equation (8.5), we get

$$x(t) = (c_1 + c_2 t)e^{(\frac{-\gamma}{2m})t}.$$

How do we find where this function crosses the t-axis? (Spiral)

Is $e^{(\frac{-\gamma}{2m})t}$ ever zero? How do we know that?

Thus, we find that $c_1 + c_2 t = 0$. What kind of an equation is this?

How many positive zeroes can it have?

Then we know that for positive t, $x(t)$ is zero at one point at most. Thus the body passes through its equilibrium point at most once. Some graphs of critically damped motion are shown in Figure 8.2. The graphs illustrate a fixed positive x_0 value, with three different initial velocities v_0. How does the initial velocity affect the shape of the curve?

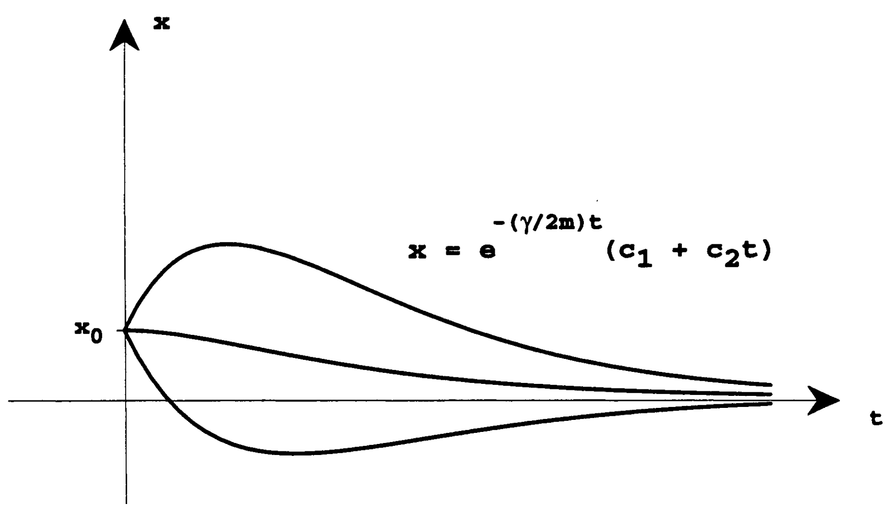

FIG. 8.2. Critically damped motion with various initial velocities.

8.3 Overdamped Motion

When $\gamma^2 - 4mk > 0$, what kind of roots will equation (8.2) produce?
In this case, the roots are

$$\lambda_1 = \frac{-\gamma}{2m} + \frac{\sqrt{\gamma^2 - 4mk}}{2m} \qquad \text{and} \qquad \lambda_2 = \frac{-\gamma}{2m} - \frac{\sqrt{\gamma^2 - 4mk}}{2m}.$$

What is the difference in this form compared to the case of *underdamped* motion? (Spiral)

Since γ, m, and k are all positive constants, what must the sign of λ_2 be?
Also, looking at λ_1, we see that $m > 0$, and $k > 0$, gives us that $\gamma^2 - 4mk < \gamma^2$.
Then, $\sqrt{\gamma^2 - 4mk} < \sqrt{\gamma^2}$. So,

$$\frac{\sqrt{\gamma^2 - 4mk}}{2m} < \frac{\sqrt{\gamma^2}}{2m} = \frac{\gamma}{2m}.$$

Thus, we have

$$\lambda_1 = \frac{-\gamma}{2m} + \frac{\sqrt{\gamma^2 - 4mk}}{2m} < -\frac{\gamma}{2m} + \frac{\gamma}{2m} = 0.$$

Our general solution is then

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

with λ_1 and λ_2 both negative. Thus, as $t \to \infty$, what happens to $x(t)$?
Some typical graphs of overdamped motion are shown in Figure 8.3. How do the graphs for overdamped motion compare with those for critically damped motion and underdamped motion? Explain.

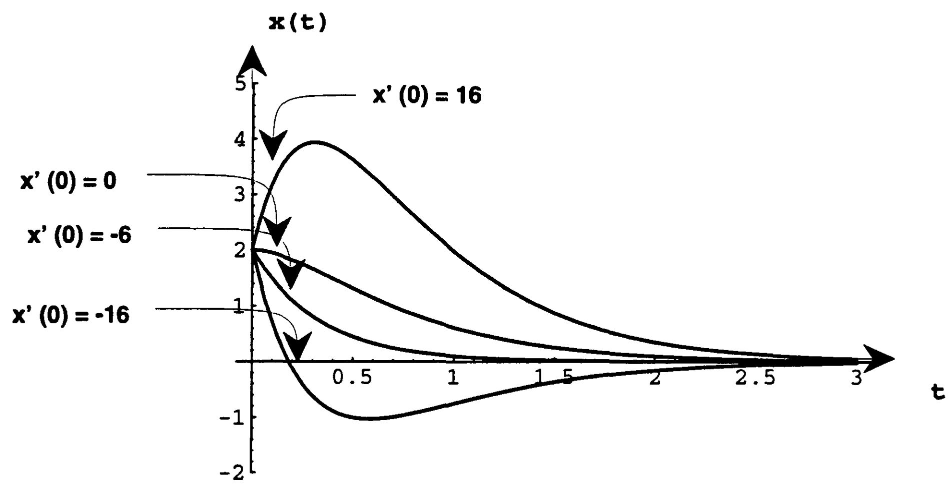

FIG. 8.3. Overdamped motion; solutions of the differential equation $x''(t) + 5x'(t) + 6x(t) = 0; x(0) = 2$, with $x'(0) = \{16, 0, -6, -16\}$, respectively.

Example 8.1 *A* 16-*lb weight is attached to a* 5-*ft-long spring. At equilibrium the spring measures* 8.2 *ft. If the weight is pushed up* 2 *ft from the equilibrium position and released from rest, find the displacement* $x(t)$ *if it is also known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity.*

Solution *The elongation due to the weight is* $L = 8.2 - 5 = 3.2$ *ft.*
Then Hooke's law gives $16 = k(3.2) \Rightarrow k = 5 lb/ft$. *Since* $m = \frac{w}{g}$, *we also have*

$$m = \frac{16 lb}{32 \frac{ft}{sec^2}} = \frac{1}{2} slug.$$

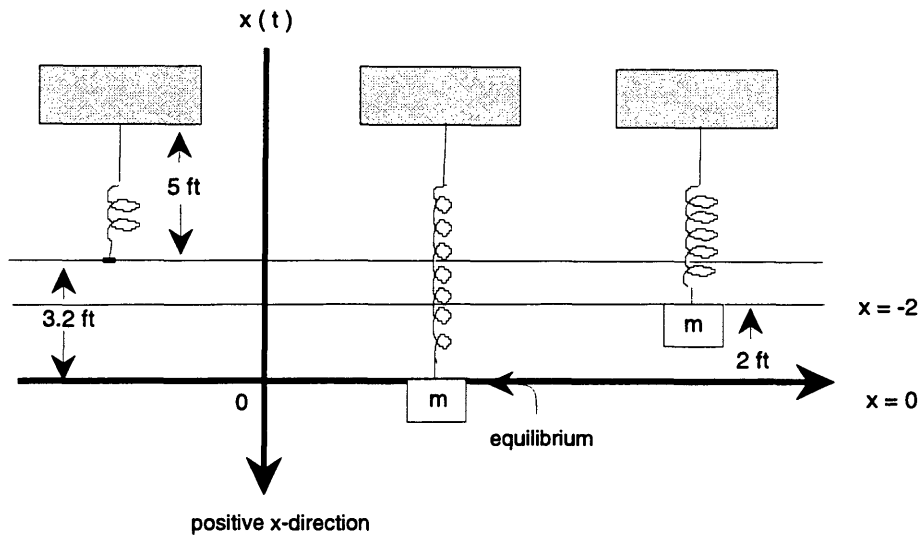

FIG. 8.4. A diagram for Example 8.1

Also, the fact that the medium's resistive force is equal to the velocity implies that $\gamma = 1$. *Thus, our differential equation is*

$$\frac{1}{2}x'' + 1x' + 5x = 0,$$

or

$$x'' + 2x' + 10x = 0.$$

Since the weight is pushed up 2ft from the equilibrium position, we have one initial condition namely $x(0) = -2$. *Also, the fact that the mass is released from rest implies that the initial velocity is* $x'(0) = 0$. *The characteristic equation* $\lambda^2 + 2\lambda + 10 = 0$ *has roots*

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(10)}}{2(1)} = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3i.$$

Then, $\alpha = -1$ *tells us we will have exponential factors, but* $\beta = 3$ *tells us that we will have sine and cosine terms (oscillations), so the system is* underdamped. *The general solution is*

$$x(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t = e^{-t}(c_1 \cos 3t + c_2 \sin 3t).$$

Then,

$$x'(t) = e^{-t}(-3c_1 \sin 3t + 3c_2 \cos 3t) + (c_1 \cos 3t + 3c_2 \sin 3t).$$

Applying the initial conditions, we find that $c_1 = -2$, *and* $c_2 = -\frac{2}{3}$. *So, the IVP solution is*

$$x(t) = e^{-t}(-2\cos 3t - \frac{2}{3}\sin 3t). \tag{8.7}$$

We can also look at the other form of the solution (8.7), which is

$$x(t) = Ae^{\alpha t}\cos(\beta t - \phi), \qquad \textit{where} \quad A = \sqrt{c_1^2 + c_2^2} \quad \textit{and} \quad \tan\phi = \frac{c_2}{c_1}.$$

For our example, we have $\alpha = -1$, $\beta = 3$, $c_1 = -2$, *and* $c_2 = \frac{-2}{3}$. *Thus, we have*

$$A = \sqrt{(-2)^2 + (\frac{-2}{3})^2} = \sqrt{\frac{40}{9}} = \frac{2\sqrt{10}}{3}.$$

To find ϕ, *we have* $\tan\phi = \frac{-2/3}{-2}$. *Thus,* $tan\phi = \frac{1}{3}$. *Since the tangent function is positive in* both *the first and third quadrants, we need to check the signs of* $\cos\phi$ *and* $\sin\phi$ *to find the correct quadrant. Since,* $c_1 = A\cos\phi$ *and* A *is positive, we have* $\cos\phi < 0$. *Similarly, since* $c_2 = A\sin\phi$, *we have* $\sin\phi < 0$. *This implies that* ϕ *is in Quadrant III. Do we need to look at both the sine and cosine of* ϕ*? Why, or why not? Is that always the case?*

Then we have that the phase angle is $\phi = \pi + \tan^{-1}\left(\frac{1}{3}\right) \approx 3.4633$ *radians.*

Thus the alternate form of our solution is

$$x(t) = \frac{2\sqrt{10}}{3} e^{-t} \cos(3t - 3.4633). \tag{8.8}$$

The circular frequency is $\omega = 3$ *rad/sec, the quasifrequency is* $\frac{3}{2\pi} \approx 0.4774$ *Hz (cycles/sec), and the quasiperiod is* $\frac{2\pi}{3} \approx 2.0944$ *seconds.*

Graphing Calculator

Use your graphing calculator to graph both forms of the solution found in equations (8.7) and (8.8), in the interval $-1 \le x \le 4$. Do they agree?

Does the graph agree with the initial conditions?

On that same graph plot the functions $x(t) = \pm\frac{2\sqrt{10}}{3}e^{-t}$. What do you see occurring?

Use your calculator's ability to find local maxima and find three of the local maxima of the function that are to the right of the $x(t)$-axis. (You will probably have to change your domain and range (zoom in) to look at the second and third maxima).

Using the quasiperiod, explain how you know that your maxima are correct.

8.4 Electrical Circuits

There are many other applications of linear second order differential equations with constant coefficients, including the motion of a linearized damped unforced pendulum $ml\theta'' + \beta l\theta' + mg\theta = 0$, buoyancy problems $x'' + \frac{\pi r^2 \rho}{m}x = 0$, and electrical circuits, which we now consider.

Consider a simple electric circuit as shown in Figure 8.5 consisting of an electromotive force, such as a battery or generator, a resistor, an inductor, and a capacitor all connected in series. These circuits are called *RLC* series circuits, where the resistance R(ohms), the inductance L(henrys) and the capacitance C(farads) are all known, positive values. The electromotive force (emf) $E(t)$, given in volts, is called the impressed voltage. The current I flowing through the circuit is measured in amperes and the charge Q on the capacitor at time t is measured in coulombs. The two

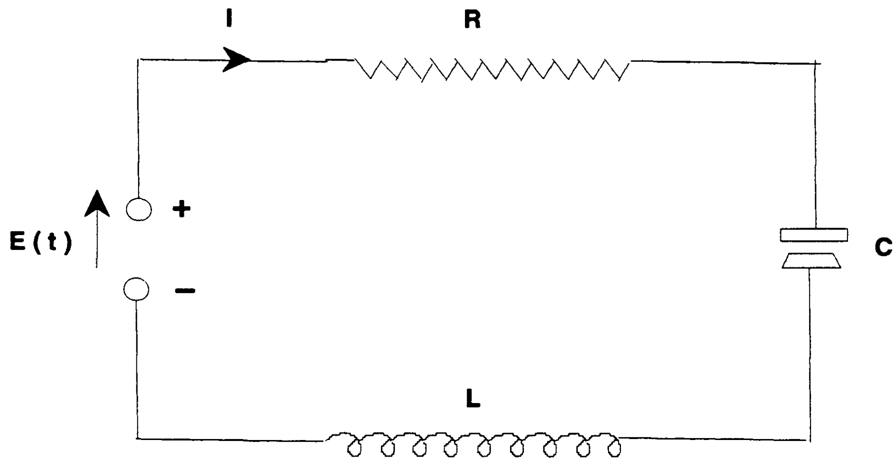

FIG. 8.5. A simple electric circuit.

physical principles governing RLC series circuits are related to conservation of charge and conservation of energy. G.R. Kirchoff[1] formulated these two laws in 1859. These laws say that the current passing through each of the elements in the circuit must be the same and the sum of the voltage drops around a closed circuit must be zero. By observing resistance, capacitance, and inductance it is known that:

1. the voltage drop across a resistor is proportional to the current passing through

[1]Gustav Robert Kirchoff (1824 - 1877) was a student of Carl Friedrich Gauss. He was a physicist who made important contributions to the theory of circuits, the study of elasticity, black body radiation, and spectrum analysis. In 1875 he was appointed to the chair of mathematical physics at Berlin [49].

the resistor ($E_R = RI$). The constant of proportionality R is called the resistance.

2. the voltage drop across the capacitor is proportional to the electric charge Q on the capacitor ($E_C = \frac{Q}{C}$), where C is the capacitance.

3. the voltage drop across an inductor is proportional to the instantaneous rate of change of the current I ($E_L = L\frac{dI}{dt}$). The constant of proportionality L is called the inductance.

What did Kirchoff's law state?

Thus, we have

$$LI' + RI + \frac{1}{C}Q = E(t), \tag{8.9}$$

where the units are

$$1\text{volt} = 1\text{ohm} \cdot 1\text{ampere} = \frac{1\text{coulomb}}{1\text{farad}} = \frac{1\text{henry} \cdot 1\text{ampere}}{1\text{second}}.$$

How do these correspond to the units in the mass-spring system?

Current is the derivative of charge with respect to time; what is the equation relating the current I to the charge Q?

Then, substituting into (8.9), we have

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) \tag{8.10}$$

for the differential equation in terms of the charge Q. To obtain an IVP, we still need two initial conditions; it is possible to measure the initial charge on the capacitor $Q(0)$ and the initial current $I(0) = Q'(0)$.

Sometimes in applications we are interested in determining the current $I(t)$. If we differentiate (8.9) with respect to t, what do we obtain?

Then, substituting I for $\frac{dQ}{dt}$, we obtain a differential equation for I:

$$LI'' + RI' + \frac{1}{C}I = E'(t).$$

Can you think of a similar equation we could find using the mass-spring system? (Discovery)

Example 8.2 *Find the charge on the capacitor in an RLC series circuit when it has a resistor of 10 ohms, a capacitor of 0.001 farad, an inductance of 0.25 henries, and no impressed electromotive force . Also, assume the initial charge on the capacitor is 6 coulombs, and the initial current is zero.*

Solution *From the statement of the problem, we see that* $L = 0.25$, $R = 10$, $C = 0.001$, $E(t) = 0$, *and the initial conditions at* $t = 0$ *are* $Q(0) = 6$ *and* $I(0) = Q'(0) = 0$. *Then, from equation (8.10), we obtain*

$$0.25Q'' + 10Q' + \frac{1}{0.001}Q = 0; \qquad Q(0) = 6, \qquad Q'(0) = 0,$$

which simplifies to

$$Q'' + 40Q' + 4000Q = 0; \qquad Q(0) = 6, \qquad Q'(0) = 0.$$

Thus, the characteristic equation is $\lambda^2 + 40\lambda + 4000 = 0$, *which has roots*

$$\lambda_1 = -20 + 60i \qquad \text{and} \qquad \lambda_2 = -20 - 60i.$$

Since $\gamma^2 - 4mk$ *is negative, we have sine and cosine factors in the general solution.*

Thus, our system is underdamped. Applying the initial conditions, we find

$$6 = c_1 e^0 \cos 0 + c_2 e^0 \sin 0,$$

which yields $c_1 = 6$. *Then,*

$$Q(t) = 6e^{-20t} \cos 60t + c_2 e^{-20t} \sin 60t,$$

which has as its derivative

$$Q'(t) = 6e^{-20t}(-60 \sin 60t) + \cos 60t(-120e^{-20t}) c_2 e^{-20t}(60 \cos 60t) + \sin 60t(-20c_2 e^{-20t}).$$

The initial condition $Q'(0) = 0$ *gives us* $c_2 = 2$. *Hence, the solution to the IVP is given by*

$$Q(t) = 6e^{-20t} \cos 60t + 2e^{-20t} \sin 60t.$$

A second form using equation (8.4), is

$$x(t) = 2\sqrt{10}e^{-20t} \cos(60t - \tan^{-1}(\frac{1}{3})).$$

The quasiperiod is $\frac{2\pi}{\beta} = \frac{2\pi}{60} = \frac{\pi}{30} \approx 0.1047$ *radians, and the quasifrequency is* $\frac{30}{\pi} \approx 9.5493$ *Hz.*

We now compare the mass-spring system found in Chapter 7 with the RLC series circuit system found in Chapter 8 in the form of a table:

Comparison of Mechanical and Electric Systems			
Mechanical Spring-Mass with Damping		Electric RLC Series Circuit	
$mx'' + \gamma x' + kx = f(t)$		$LQ'' + RQ' + (1/C)Q = E(t)$	
Displacement	x	Charge	Q
Velocity	x'	Current	$Q' = I$
Mass	m	Inductance	L
Damping constant	γ	Resistance	R
Spring constant	k	Elastance	$1/C$
External force	$f(t)$	Electromotive force	$E(t)$

Mathematica

We can look at damping in detail using Mathematica. For example, suppose we want to solve the initial value problem:

$$y'' + 4y' + 5y = 0; \qquad y(0) = 1, \qquad y'(0) = 0.$$

We use **DSolve** as before:

```
Clear[y]
sollist=DSolve[{y"[x]+4 y'[x]+5 y[x]==0,y[0]==1,y'[0]==0},y[x],x]
sol1=sollist[[1,1,2]]
```

Then, we plot it in the domain of $-1 \leq x \leq 8$:

```
Plot[sol1,{x,-1,8}]
```

What does the graph appear to be doing?

Is this correct from the form of the solution?

What is the discriminant of the characteristic equation?

What does that tell you about the damping?

What is wrong with the graph?

We attempt to get a better look by changing the domain and also specifying the range:

Plot[**sol1**,**{x,2,8}**,**PlotRange**– >**{-0.00001,0.00001}**]

Now, what do you see occurring? Does this agree with the form of the analytic solution?

Another thing we can do is use the **FindRoot** function again:

FindRoot[**sol1==0,{x,2}**]

FindRoot[**sol1==0,{x,4}**]

FindRoot[**sol1==0,{x,5}**]

FindRoot[**sol1==0,{x,10}**]

What are these outputs telling us?

Try making up three examples of differential equations that include a case of each type of damping.

Then, using the **Plot** and **Show** commands, plot them on the same graph.

Now, try changing the initial conditions, so that first the position then the velocity vary. What do you find?

Chapter 9

FORCED VIBRATIONS

Now we consider what happens to a spring-mass system when an external force is applied. In Chapter 7, we derived the differential equation

$$mx'' + \gamma x' + kx = F(t). \qquad (7.1)$$

Machines with rotating components commonly involve mass-spring systems in which the external force is simple harmonic:

$$F(t) = F_0 \cos \omega t \qquad \text{or} \qquad F(t) = F_0 \sin \omega t,$$

where F_0 is the amplitude of the periodic force and ω is its circular frequency. Can you think of an example? (Discovery)

We will let the external force have the form $F_0 \cos \omega t$, where F_0 and ω are nonnegative constants. Then our equation of motion is

$$mx'' + \gamma x' + kx = F_0 \cos \omega t,$$

or, equivalently,

$$x'' + \frac{\gamma}{m}x' + \frac{k}{m}x = \frac{F_0}{m} \cos \omega t.$$

What kind of equation is this? How do we solve it? (Spiral)

9.1 Undamped Forced Vibrations

If we assume there is no damping, what is the value of γ?

Then, we have

$$x'' + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t, \tag{9.1}$$

In Lesson 6, we solved nonhomogeneous equations.

The corresponding homogeneous equation yields the characteristic equation

$$\lambda^2 + \frac{k}{m} = 0$$

which has roots

$$\lambda_1 = i\sqrt{\frac{k}{m}} \qquad \text{and} \qquad \lambda_2 = -i\sqrt{\frac{k}{m}}.$$

Thus, the homogeneous or complementary solution is

$$x_c(t) = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t = c_1\cos\omega_0 t + c_2\sin\omega_0 t,$$

where we have defined $\omega_0 = \sqrt{\frac{k}{m}}$.

Is this the same as ω was defined in Lesson 7? (Spiral)

Then ω_0 is called the *natural frequency* of the system.

Why is the word "natural" used?

What are the steps for finding a particular solution of equation (9.1)? (Spiral)

Looking at the derivatives of $F_0\cos\omega t$, we guess a particular solution of the form

$$x_p(t) = A\cos\omega t + B\sin\omega t, \qquad \text{where} \qquad \omega \neq \omega_0 = \sqrt{\frac{k}{m}}$$

and A and B are constants to be determined.

Then, differentiating and substituting into (9.1), we find that

$$A(\frac{k}{m} - \omega^2)\cos\omega t + B(\frac{k}{m} - \omega^2)\sin\omega t = \frac{F_0}{m}\cos\omega t.$$

Equating coefficients, we have

$$A(\frac{k}{m} - \omega^2) = \frac{F_0}{m} \qquad \text{and} \qquad B(\frac{k}{m} - \omega^2) = 0.$$

Then, since $\omega \neq \sqrt{\frac{k}{m}}$, we obtain

$$A = \frac{F_0/m}{k/m - \omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2} \qquad \text{and} \qquad B = 0.$$

Thus,

$$x_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t$$

is a particular solution. Then how do we find the general solution?

Hence our general solution is

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t. \tag{9.2}$$

9.2 Beats

Now suppose that the mass is initially at rest, $x'(0) = 0$, and assume the initial position is $x(0) = 0$. What do we get when we substitute $x(0) = 0$ into equation (9.2)?

Then, solving for A, we

$$A = \frac{-F_0}{m(\omega_0^2 - \omega^2)}.$$

Also, what is $x'(t)$?

Then, substituting in $x'(0) = 0$, we have $0 = \omega_0 B(1)$, which gives us

$$B = 0.$$

Why can't $\omega_0 = 0$?

Hence, equation (9.2) becomes

$$x(t) = \frac{-F_0}{m(\omega_0^2 - \omega^2)} \cos \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t,$$

which gives

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t). \tag{9.3}$$

The trigonometric identity for $\cos A - \cos B$ in terms of the sine function is

$$\cos A - \cos B = 2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(B - A).$$

Then equation (9.3) can simplify to

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cdot 2 \sin \frac{\omega t + \omega_0 t}{2} \cdot \sin \frac{\omega_0 t - \omega t}{2},$$

or

$$x(t) = 2 \frac{F_0}{m(\omega_0^2 - \omega^2)} \cdot \sin \frac{\omega_0 - \omega}{2} t \cdot \sin \frac{\omega_0 + \omega}{2} t. \tag{9.4}$$

If $|\omega_0 - \omega|$ is small, then $\omega_0 + \omega \approx 2\omega$. Thus, $(\omega_0 - \omega)/2$ is small in comparison to $(\omega_0 + \omega)/2$.

What does that tell us about the period of $\sin \frac{\omega_0 + \omega}{2} t$ and the period of $\sin \frac{\omega_0 - \omega}{2} t$?

What does it tell us about the frequency of their oscillations?

Then, their product will be the rapidly oscillating function $\sin\frac{\omega_0+\omega}{2}t$ "oscillating inside" the slowly oscillating function $2\frac{F_0}{m(\omega_0^2-\omega^2)}\sin\frac{\omega_0-\omega}{2}t$. The outer function is called a *sinusoidal envelope.*

Where else did we have the product of two functions? (Spiral)

Which function was the envelope in that case? (Discovery)

This phenomenon is referred to as *beats*, and occurs in acoustics when two tuning forks of nearly equal frequency are sounded simultaneously. The periodic variation of the amplitude can be clearly distinguished. What do you think it sounds like?

The graph of a typical beats occurrence is shown in Figure 9.1.

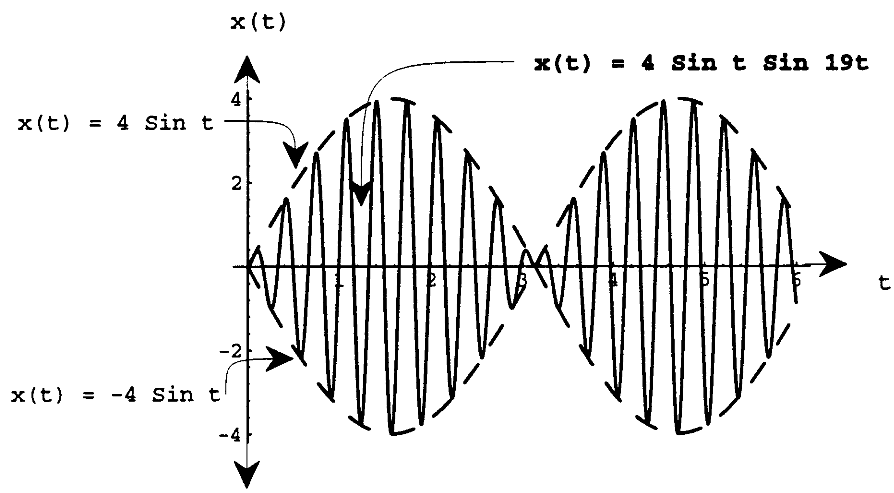

FIG. 9.1. Beats.

9.3 Resonance

Now, we consider the case where $\omega_0 = \omega$. That means the frequency of the forcing function is the same as the natural frequency of the system. In that case, the nonhomogeneous term $F_0 \cos \omega t$ will be a solution of the homogeneous equation . How do we know that? (Spiral)

Then, what must we do in order to get a linearly independent solution to equation (9.1)?

Since $\cos \omega_0 t$ is a term of the complementary function, using the method of undetermined coefficients, we try a particular solution of the form

$$x_p = t(A \cos \omega_0 t + B \sin \omega_0 t).$$

Substituting into (9.1), what do we find for A and B? (Discovery)

Hence, a particular solution is

$$x_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t,$$

and a general solution is

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t.$$

As $t \to \infty$, what happens to the term $\frac{F_0}{2m\omega_0} t$?

Thus $x(t)$ becomes unbounded, regardless of what c_1 and c_2 are.

(See Figure 9.2). This phenomenon is called *resonance* (actually, pure resonance), and in this case the function $\frac{F_0}{2m\omega_0} t \sin \omega_0 t$ "oscillates inside" the linear function $\frac{F_0}{2m\omega_0} t$.

Resonance can create serious problems in design structure stability. Why?

Natural systems can sometimes be modeled as mass-spring systems. Thus, they have a set of natural frequencies. Soldiers normally do not march in step across bridges. This is to avoid any possibility of resonance occur between the frequency of feet stomping in unison with one of the natural frequencies of the bridge. Another example of resonance occurs when a singer's acoustic vibrations cause a wine glass to explode. Resonance can cause tremendous damage in automobile suspension systems, aircraft, ships, electrical circuits, and space shuttle engines. Resonance is not always destructive, however. Can you think of an example where it might be useful? (Discovery)

Note: The solution appears to grow without bound; in reality the displacement is

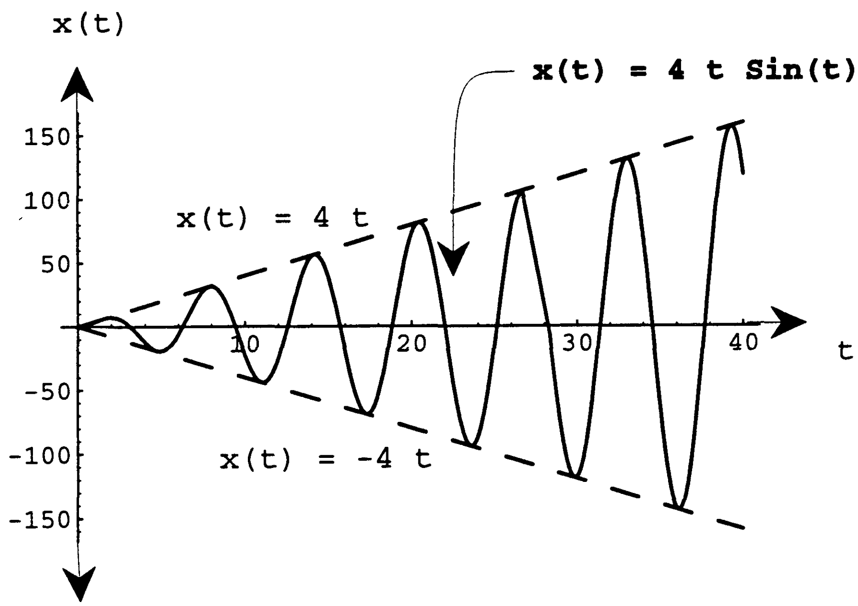

FIG. 9.2. Resonance.

limited by two factors. At large displacements, Hooke's law fails. Why is that? Also, at large displacements the linear differential equation must be replaced by one containing a nonlinear spring-displacement relationship, and usually, the spring will break when it is stretched too far.

9.4 Damped Forced Vibrations

In real applications, there is always some damping, possibly from friction if nothing else. We know the complementary (homogeneous) solution of the equation

$$mx'' + \gamma x' + kx = F_0 \cos \omega t \tag{9.5}$$

is given by one of the following forms, depending on the discriminant:

$$x_c(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \tag{9.6}$$

$$x_c(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_2 t} \tag{9.7}$$

$$x_c(t) = c_1 e^{\lambda_1 t} \cos \omega t + c_2 e^{\lambda_2 t} \sin \omega t. \tag{9.8}$$

What is the type of the roots for the characteristic equation in each case? (Spiral)

In the damped case, λ_1 and λ_2 are both negative. Why?

Then, as $t \to \infty$, each of these forms for $x_c(t)$ approaches zero.

To find $x_p(t)$, we proceed as we did in the undamped case. The method of undetermined coefficients indicates that we should guess

$$x_p = A \cos \omega t + B \sin \omega t.$$

Then, substituting into equation (9.5), collecting terms, and equating coefficients of $\cos\omega t$ and $\sin\omega t$, we obtain two equations in two unknowns, A and B.

$$\begin{aligned}(k - m\omega^2)A + \gamma\omega B &= F_0,\\ -\gamma\omega A + (k - m\omega^2)B &= 0\end{aligned}$$

How do we solve such a system of equations?

Upon solving, we find

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (\gamma\omega)^2} \qquad \text{and} \qquad B = \frac{(\gamma\omega)F_0}{(k - m\omega^2)^2 + (\gamma\omega)^2}.$$

To simplify the notation, we can let

$$\rho = \frac{k}{\sqrt{(k - m\omega^2)^2 + (\gamma\omega)^2}} \qquad \text{and} \qquad \alpha = \tan^{-1}\frac{\gamma\omega}{k - m\gamma^2}, \qquad 0 \leq \alpha \leq \pi.$$

Why do we do this? (Hint: Think back to Lesson 8) (Spiral & Discovery)

Then,

$$A = \rho\frac{F_0}{k}\cos\alpha \qquad \text{and} \qquad B = \rho\frac{F_0}{k}\sin\alpha.$$

How were those equations obtained? (Discovery)

Hence, the particular solution

$$x_p = A\cos\omega t + B\sin\omega t$$

becomes

$$x_p = \rho\frac{F_0}{k}(\cos\omega t\cos\alpha + \sin\omega t\sin\alpha),$$

which we can rewrite as

$$x_p = \rho \frac{F_0}{k} (\cos \omega t - \alpha).$$

How did we do that? (Spiral)

What kind of motion does $x_p(t)$ have? Will it die out?

Thus the particular solution is a periodic solution which remains after the transient solution dies away, and is called the *steady-state solution* or the *forced response* of the system. Then we rewrite $x(t)$ as

$$x(t) = \underbrace{Ae^{\lambda_1 t} + Be^{\lambda_2 t}}_{transientsolution} + \underbrace{\frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}} \cos{(\omega t - \phi)}}_{steady-statesolution}.$$

Notice that the frequency ω of the steady-state solution is the same as that of the external force $F_0 \cos \omega t$. What does that tell you about the long-term behavior of the system?

How do the initial conditions imposed on the system determine the transient solution's behavior?

As time passes, however, the energy put into the system by the initial velocity and displacement are dissipated through the damping force, and the motion settles into a steady-state. If there were no damping, the initial conditions would continue for all time. Figure 9.3 shows an example of this type of convergence. If we increase γ, how does it affect the system?

What if the damping constant is very small and the forcing function's frequency is near the resonance frequency of the system? How will such a structure behave? Will it be stable? (Discovery)

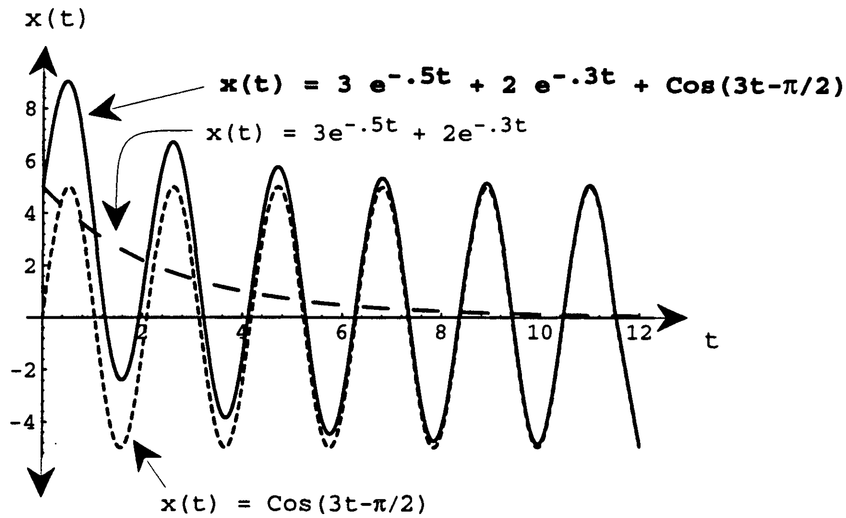

FIG. 9.3. Damped forced vibrations.

Example 9.1 *Assume that we have a spring-mass system with an object of mass* $m = 1$*, spring constant* $k = 9$*, no damping* $(\gamma = 0)$*, and an external forcing function of* $80 \cos 5t$*. Find* $x(t)$*, given the initial displacement* $x(0) = 0$ *and the initial velocity* $x'(0) = 0$*.*

Solution *The general differential equation we need to solve is*

$$mx'' + \gamma x' + kx = F_0 \cos \omega t.$$

For this example, we have

$$x'' + 9x = 80 \cos 5t. \tag{9.9}$$

First, we solve the homogeneous equation , which gives us

$$\lambda_1 = 3i \qquad \textit{and} \qquad \lambda_2 = -3i. \qquad (\alpha = 0, \beta = 3)$$

The complementary solution is

$$x_c(t) = c_1 \cos 3t + c_2 \sin 3t.$$

Then, $\omega_0 = 3$ *is the natural frequency of the system, while* $\omega = 5$ *is the external forcing frequency from (9.9). Now, we find the particular solution by assuming* $x_p = A\cos 5t$. *Notice there is no term* $B\sin 5t$ *in our guess, since* x'' *will be a cosine term again, and there is no* x' *term in equation (9.9).* *(Spiral)*
Thus, $x_p' = -5A\sin 5t$ *and* $x_p'' = -25A\cos 5t$, *which yields*

$$-25A\cos 5t + 9A\cos 5t = 80\cos 5t.$$

Then, $-16A = 80$ *and* $A = -5$.
Thus, a particular solution of (9.9) is

$$x_p(t) = -5\cos 5t.$$

Hence, the general solution is $x = x_c + x_p$, *or*

$$x(t) = c_1 \cos 3t + c_2 \sin 3t - 5\cos 5t,$$

which has as its derivative

$$x'(t) = -3c_1 \sin 3t + 3c_2 \cos 3t + 25 \sin 5t.$$

The initial conditions $x(0) = 0$ *and* $x'(0) = 0$ *yield* $c_1 = 5$ *and* $c_2 = 0$, *so the desired solution is*

$$x(t) = 5 \cos 3t - 5 \cos 5t,$$

which is an oscillating curve.

Example 9.2 *Find the IVP solution of the undamped, forced system*

$$\frac{1}{8}x''(t) + 2x(t) = \sin \omega t; \qquad x(0) = 0, \qquad x'(0) = 0. \tag{9.10}$$

then examine the behavior of the system for $\omega = 3$, $\omega = 3.9$, *and* $\omega = 4$.

Solution *The characteristic equation associated with the homogeneous equation is*

$$\frac{1}{8}\lambda^2 + 2 = 0,$$

or $\lambda^2 + 16 = 0$, *which has roots*

$$\lambda_1 = 4i, \qquad \lambda_2 = -4i.$$

Thus the complementary solution is

$$x_c(t) = c_1 \cos 4t + c_2 \sin 4t,$$

from which we see that the natural frequency of the system is $\omega_0 = 4$. *Next, we guess a particular solution of the form*

$$x_p = A \sin \omega t.$$

Then we substitute x_p *and* $x_p'' = -\omega^2 A \sin \omega t$ *into (9.10) and obtain*

$$\frac{-1}{8}\omega^2 A \sin \omega t + 2A \sin \omega t = \sin \omega t.$$

Thus,

$$(\frac{-1}{8}\omega^2 + 2)A = 1,$$

and solving for A *yields*

$$A = \frac{8}{16 - \omega^2}.$$

Here, we notice that ω *cannot be equal to* ± 4, *which we will investigate in more detail later. Then, our particular solution is*

$$x_p(t) = \frac{8}{16 - \omega^2} \sin \omega t, \qquad (\omega \neq \pm 4).$$

So, the general solution of (9.10) is

$$x(t) = c_1 \cos 4t + c_2 \sin 4t + \frac{8}{16 - \omega^2} \sin \omega t.$$

Using our initial conditions $x(0) = 0$ *and* $x'(0) = 0$, *we find*

$$c_1 = 0 \qquad \text{and} \qquad c_2 = \frac{-2\omega}{16 - \omega^2}.$$

Hence, the IVP solution is

$$x(t) = \frac{-2\omega}{16-\omega^2}\sin 4t + \frac{8}{16-\omega^2}\sin \omega t, \qquad \textit{where} \qquad \omega \neq \pm 4.$$

Now, letting ω *equal* 3 *and* 3.9, *respectively, we have*

$$x_3(t) = \frac{-6}{7}\sin 4t + \frac{8}{7}\sin 3t$$

and

$$x_{3.9}(t) = \frac{-7.8}{16-3.9^2}\sin 4t + \frac{8}{16-3.9^2}\sin 3.9t$$

When $\omega = 4$, *our solution was not valid. This was due to the fact that* $A\sin 4t$ *is a solution of the homogeneous equation . Thus, for this case we guess a new* x_p,

$$x_p = At\sin 4t + Bt\cos 4t.$$

Then,

$$x_p'' = 8A\cos 4t - 16Bt\cos 4t - 8B\sin 4t - 16At\sin 4t$$

and upon substituting into (9.10), we obtain $A = 0$ *and* $B = -1$. *Thus, the particular solution is*

$$x_p = -t\cos 4t.$$

Hence, a general solution is

$$x_4(t) = c_1\cos 4t + c_2\sin 4t - t\cos 4t,$$

and

$$x'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t + 4t \sin 4t - \cos 4t.$$

Then, applying the initial conditions, we get $c_1 = 0$ *and* $c_2 = 1/4$. *Thus, the IVP solution when* $\omega = 4$ *is*

$$x(t) = \frac{1}{4} \sin 4t - t \cos 4t.$$

Graphing Calculator

First, graph the solutions we obtained for $\omega = 3$ and $\omega = 3.9$ in an appropriate viewing rectangle. What happens to the amplitude as ω approaches the resonant frequency?

Try graphing other values of ω, such as $\omega = 3.5$, and $\omega = 4.2$.

What do you see on the graphs? Why is this occurring? (Discovery)

Now try graphing the solutions for $\omega = 3.9$ and $\omega = 4$, separately.

To what do you have to adjust your range in order to get a good viewing rectangle?

What phenomenon is being exhibited by each graph?

Can you tell from the graph of $\omega = 4$ that it is not exhibiting beats? Why?

Example 9.3 *Find the general solution of the damped, forced system*

$$x'' + x' + 16x = -8 \sin 4t \tag{9.11}$$

The homogeneous equation $x'' + x' + 16x = 0$ *has the solution*

$$x_c(t) = c_1 e^{-1/2t} \cos \frac{3\sqrt{7}}{2}t + c_2 e^{-1/2t} \sin \frac{3\sqrt{7}}{2}t$$

How was this obtained?

We guess a particular solution of the form

$$x_p = A\cos 4t + B\sin 4t.$$

Then, differentiating, substituting into (9.11), and solving for A and B, we find $A = 2$ *and* $B = 0$. *A particular solution is then* $x_p = 2\cos 4t$, *and the general solution is*

$$x(t) = c_1 e^{-1/2t}\cos\frac{3\sqrt{7}}{2}t + c_2 e^{-1/2t}\sin\frac{3\sqrt{7}}{2}t + 2\cos 4t.$$

As $t \to \infty$, *the first two terms approach zero. Thus the transient solution is*

$$c_1 e^{-1/2t}\cos\frac{3\sqrt{7}}{2}t + c_2 e^{-1/2t}\sin\frac{3\sqrt{7}}{2}t$$

and the steady-state solution is

$$2\cos 4t.$$

The solutions settle into a steady oscillation similar to Figure 9.3. The IVP solution is determined by c_1 *and* c_2, *which are determined from the initial displacement and the initial velocity.*

Mathematica

We can use Mathematica to investigate the effects of changing the frequency of the forcing function. First, we look at the undamped case. We use **DSolve** to solve the IVP

$$u''(t) + u(t) = 5\cos(\omega t); \qquad u(0) = 0, \qquad u'(0) = 0$$

for various values of ω. Mathematica allows us to solve it in general:

```
Clear[delist, omega, u]
delist=DSolve[{u"[t] + u[t]==5 Cos[omega t],
u[0]==0, u'[0]==0},u[t],t]
soln=delist[[1,1,2]]//Simplify
```

Next, we find the solution to the homogeneous equation :

```
helist=DSolve[{u"[t] + u[t]==0,
u[0]==0, u'[0]==0},u[t],t]
```

Confirm this solution by hand. What is the natural frequency of the system?
Now, we use the (ReplaceAll) function **/.** to evaluate the function for various values of ω:

```
soln1=soln /. omega- >0.5
soln2=soln /. omega- >0.7
soln3=soln /. omega- >0.9
```

What happens if we try to evaluate the solution at $\omega = 1$?
In order to find the solution when ω is 1, we go back to the original differential equation, and use:

```
newlist=DSolve[{u''[t] + u[t]==5 Cos[t],
u[0]==0, u'[0]==0},u[t],t]
newsoln=newlist[[1,1,2]]//Simplify
```

Then we can plot the solutions separately, and use **Show** to combine them, or we can plot them all together as a list:

```
p1=Plot[{soln1,soln2,soln3,newsoln},{t,0,80},PlotRange->{-120,120}]
```

What is occurring on the graph?

Now, try changing the initial values to $u(0) = 2$, and $u'(0) = 2$. What effect does it have on the curves?

Explain what happens as $t \to \infty$.

CONCLUSIONS AND FUTURE WORK

9.5 Conclusions

In this thesis, we have presented an innovative method for teaching linear second order differential equations. In the introduction, we looked at the strengths and weaknesses of several teaching methods in use today, including mastery and spiral learning, discovery learning, technology-assisted learning, written and oral communication, and the traditional lecture method. Each of these methods has been used successfully by college professors, and each appeals to different people. A possible combination of these methods was proposed, and presented in an actual textbook unit. It is conjectured that a combination of these methods will produce an increase in student learning, due to variations in preferred learning styles. Also, it is believed that the format may produce more active learning.

9.6 Future Work

A pilot testing program is needed to verify that an increase in learning will take place with this method. It will be necessary to set up a control group where students using the new method are scientifically compared to those using a traditional method, to test that a significant difference in learning does take place.

It is intended that the remainder of a differential equations text will be written in this format, in order to combine units of study. The spiral format will be incorporated to a greater degree by allowing a unit to be divided up and individual lessons placed throughout the text. This will allow for the necessary repetition of a concept before a related concept is introduced.

It is also conjectured that this method can be applied to subjects such as calculus, linear algebra, and others. Since this subject and unit were chosen arbitrarily, there is reason to believe that the method can be applied to a variety of subjects. If testing yields significant learning increases, the possibilities are great.

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