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COMBINED MULTIVARIATE SHEWHART-CUMULATIVE SUM QUALITY CONTROL CHART PROCEDURE

by

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ABSTRACT

Quality control procedures are usually intended to monitor a manufacturing process and provide an indication of when that process ceases to produce its output according to predetermined specifications. A quality control procedure that has become very popular is the use of quality control charts. There are several different types of quality control charts, each of which has unique strengths and weaknesses. Seldom, however, are they used in combination to capitalize on their strengths or minimize their weaknesses.

The products of the manufacturing process are often described by several quality characteristics. Even though these characteristics are often correlated, many existing quality control chart procedures fail to fully utilize this information in determining whether the products are within specifications. A multivariate control chart procedure, however, uses this correlation information to provide the basis for more accurate decisions regarding the quality of the product.

This paper presents a discussion of two of the primary control chart procedures - the Shewhart control chart and the cumulative sum (CUSUM) control chart. A new methodology is then presented which uses a combined Shewhart-CUSUM procedure applicable to multivariate quality control. This methodology is then applied to an actual problem, the quality control of beer can pop tops.

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SECTION 1.0
INTRODUCTION

This paper presents a new methodology for applying multivariate statistical methods to quality control chart techniques. Currently, many applications of quality control chart procedures use primarily one type of control chart and univariate statistical methods. This paper presents a method which combines two of the most commonly used types of control charts into a procedure applicable to multivariate quality control problems. The combined procedure is simple to understand, versatile, easy to use, and should result in better decisions regarding the quality of a product.

The remainder of this section: explains quality control procedures and their purposes; describes statistical quality control charts and their uses; and presents an overview of the methodology developed in this paper.

QUALITY CONTROL PROCEDURES

Quality control procedures are usually intended for monitoring a manufacturing process. In general, a manufacturing process is intended to produce a product according to predetermined specifications. The product can be anything from beer can pop tops to aircraft bombsights. The process can get "out of control" in that the process ceases to produce the product according to the desired specifications. Therefore, feedback regarding the quality of the products is required in order to provide guidance to keep the process "in control." This manufacturing system, therefore, can be thought of as a cybernetic system where: the process results in an output; the quality of the output results in feedback to the process; and control entities adjust the process as dictated by the feedback to maximize the efficiency of the entire system.

The intent of the quality control procedures discussed in this paper is to monitor a manufacturing process and provide an indication to appropriate control entities of when that process is "out of control," where "out of control" is defined as a state where the number or fraction of defective products exceeds a previously specified limit. The control entities may then adjust the process to correct for the manufacture of defective products. The procedures also identify trends in the characteristics of interest so that adjustments may be made to the process before an out of control condition occurs.

The product is usually described by several important quality characteristics with which are associated certain desired quality standards. A commonly used approach is to reject the product if any one of the characteristics fails to fall within its prescribed limits. However, this approach fails to take into consideration that the quality of a product is often determined by a combination of correlated characteristics. In fact, there may be groups of characteristics within which the characteristics are correlated but the groups themselves are not. Therefore, some of the characteristics may be allowed to vary to some extent as long as the combined effect of that variation does not detrimentally affect the performance of the product.

If, however, the characteristics are highly correlated, measuring only one of the characteristics could be adequate for determining the quality of the product. The methodology presented here is specifically intended to deal with products where some correlations may exist between characteristics or groups of characteristics. It is intended to make greater use of this information in determining the quality of the product.

Quality control testing procedures depend on the characteristics of the process to be monitored, the quality characteristics to be monitored, historical data available on the various quality characteristics, and the objective of the test procedure. Cost of the testing procedure, however, is generally the primary determinant of what method is to be used and how it is to be implemented. The cost usually depends on the cost of sampling and testing, the cost of the product, and the cost of shutting down and adjusting the process. The methodology developed here is a statistical design and not a cost design. However, by increasing the efficiency of control chart procedures for products that are characterized by correlated quality characteristics, the cost of controlling the process should be reduced. The procedures can also be modified to meet the requirements of the specific application in order to further reduce costs.

Test measurements of samples can be in terms of counts or proportion of defective products (an attribute measure), or measurements of specific quality characteristics (a measure of variables). Often, the mean value or the variance of a characteristic is of interest. The quality control procedures can be univariate or multivariate in nature. Testing can be sequential for a continuous or incremental process or it can be aimed only at determining the acceptability of a lot or batch of products. The characteristics of the problem of monitoring the quality of products of a process, therefore, lends itself to statistical methods. Statistical quality control testing procedures that have been developed include acceptance sampling procedures, quality control charts, and other methods.

QUALITY CONTROL CHARTS

A common and very useful statistical procedure is the use of control charts to monitor a process. A control chart is a graphic representation of a control statistic. In general, the chart includes historical data to indicate trends in the statistic, and control values to provide limits for the statistic. A statistic which remains within the limits indicates that the process is still in control. If the statistic fails to fall within predetermined limits the control entities are notified so that the cause may be determined and the process may be adjusted, if required.

Some commonly used charts are the Shewhart chart ($\overline{X}$ chart to measure mean, $\overline{R}$ chart to measure range, and others), acceptance chart, geometric moving average chart, and cumulative sum (CUSUM) chart. These charts may be either one-sided or two-sided, depending on the desired test statistic and the characteristic of concern, and are intended to measure the mean value or dispersion of a variable. The two-sided procedure is generally aimed at determining not only whether a value of a variable is out of limits but whether it went out of limits above or below the target value for that variable. One-sided charts, generally, do not indicate direction but rely on subsequent tests to determine the direction of the deviation of the variable if that information is important.

This paper deals with two of the most commonly used control chart procedures, the Shewhart chart ($\overline{X}$ chart for measuring the mean) and the CUSUM chart. Often, a Shewhart range chart ($\overline{R}$) is also used in conjunction with the $\overline{X}$ chart to provide an indication of the dispersion (variance) of a variable. This is because the variance of a variable may increase while the mean value does not change significantly. It is important for a quality control procedure to detect this change in variance because it may be an indication that the process is no longer in control. The development of a multivariate analog to the Shewhart $\bar{R}$ chart, however, is beyond the scope of this paper and is left as an additional topic to be pursued by interested persons.

OVERVIEW OF THE METHODOLOGY

A primary consideration in selecting a control chart technique is the average run length (ARL) [although Woodall (1985) recommends that a percentage point of the run length distribution is a more appropriate measure for some applications]. The run length, in general, is a random variable and ARL is often used as a measure of the performance of a control chart procedure. It is defined as the average number of samples of the output of a process that occur before an out of control signal is given by the chart.

A good procedure has small ARL when the process is out of control and a large ARL when the process is in control. The large ARL for the in control condition minimizes the introduction of variability to the manufacturing process and to the quality control procedures by keeping the false alarm rate low. This helps maintain a high confidence level in the control chart procedure. The low ARL for the out of control condition allows for rapid adjustment of the process and minimizes the manufacture of defective products and, thus, helps reduce costs.

All control chart techniques will give an out of control signal at some time with probability one even if the process remains in control. This is because the control chart is a statistical sampling procedure which is based on a desired level of significance. Even if a process is in control, the control procedures will give an out of control signal a percentage of the time based on the significance level. Therefore, the cause of the out of control signal must be identified to determine if it is probably a statistical false alarm or whether the process is actually out of control.

For the Shewhart chart, ARL is determined by the desired significance level of the test. For example, if a significance level (α) is chosen to be .0025, the ARL for the in control condition is $1/\alpha$ or 400. The significance level is the probability of a Type I error. In the case of a control chart to determine whether a product is within specifications, this corresponds to the case where the product is rejected even though it is not defective. Statistically, this corresponds to rejection of the null hypothesis when it is true. For the CUSUM chart, the ARL is determined by the significance level, sample size, and the size of the deviation of the variable(s) that the chart is designed to detect. The level of significance should be selected based on the correlation of the variables, the acceptability of false alarms, and other economic factors.

To date, no discussions have been found which present a multivariate CUSUM chart analogous to the multivariate Shewhart chart. Woodall (August 1985), however, presents a procedure for the simultaneous use of several univariate CUSUM charts in a multivariate CUSUM (MCUSUM) procedure. His method requires determination of the ARL for each individual CUSUM chart and his discussion is based on a case utilizing only bivariate data and integer-valued variables. The methodology developed later in this paper inherently utilizes the shortest run length of the two types of control charts, and is applicable to multivariate data and continuous variables.

The purpose of this paper is to present a methodology for the development and use of a multivariate control chart procedure which incorporates the best qualities of the multivariate Shewhart chart and MCUSUM charts while compensating for their individual weaknesses. The methodology is simple yet robust. It is simple in that it does not require the difficult interpretation of the multivariate Shewhart statistic to determine which variable(s) caused the out of control signal. It is robust in that it is applicable to most types of processes where the quality of the product is determined by several correlated quality characteristics.

The purposes of the quality control chart methods utilizing this methodology are: to identify trends in the manufacturing process which could lead to an out of control condition; to identify when the process is out of control; to identify the variable(s) which put the process out of control; and to help identify the relationships between the variables which contributed to the out of control condition. The methodology is then applied to an actual quality control problem where there are four (4) correlated quality characteristics.

SECTION 2.0

MULTIVARIATE QUALITY CONTROL

Quality characteristics of a product are usually described as random variables. [However, nonrandom behavior may be caused by such factors as outside influences, serial correlation, and others. For the purposes of this paper, the quality characteristics are assumed to be random variables.] Originally, although several quality characteristics of a product might be observed, each was treated with statistical methods independent of the other characteristics. However, in many cases the random variables representing the quality characteristics are correlated. Statistical quality control procedures, therefore, generally should be based on a multivariate approach in order to consider the interrelationships among the quality characteristics and to make the best use of this information in determining the quality of a product. However, to date few quality control procedures that are currently in use utilize a multivariate approach.

Most statistical methods applied to quality control utilize univariate methods to determine if individual variables meet prescribed standards. A product is rejected if any of several variables fails to meet its standards. This method has a major detrimental effect on the probabilities of accepting a bad product and of rejecting a good product if there were more than one important quality characteristic. As a very simplistic example, a sampling scheme for a process may be designed to allow a 0.05 probability of rejecting a batch of products if any one of the independently treated quality characteristics fails to meet prescribed standards (α = 0.05). In a case where there are two variables, this would result in an overall probability of rejecting the lot (Type I error) of 0.0975 if the variables are treated independently. [Since the failure of any one of the characteristics to meet standards results in rejection of the product, this is equal to 0.05 probability based on the first variable plus (0.05 * (1.0-0.05)) for the second variable, given the first variable was within limits.] Similarly, if an alternative sampling scheme is used wherein there is a 0.9 probability of accepting a lot based on both quality characteristics being within prescribed limits, the actual probability of acceptance is only 0.81 (0.9^2) if the variables are treated independently. The effect becomes worse as additional quality characteristics are considered.

Bonferroni's inequality, however, may be used to bound the probabilities. [Bonferroni's inequality states that the probability of event E and event F occurring is greater than or equal to the probability of E occurring plus the probability of F occurring minus 1 ($P(EF) \geq P(E) + P(F) - 1.0$).] This, however, does not fully capitalize on the available information regarding the correlations to better evaluate the quality of the products of a process.

The use of univariate control charts goes back to the 1930's. An early statistical procedure uses the Shewhart control chart and observations are made of a process on a single quality characteristic of the product. This procedure requires an assumption that the quality characteristic is a normally distributed random variable. A technique often used is to notify control entities if an observation falls more than three standard deviations from the previously prescribed control value for that control characteristic ($\alpha = 0.0027$). Usually, then, the process is interrupted so that corrections can be made. For a product which has a large number of important quality characteristics, a corresponding number of control charts must be constructed for testing observations of those characteristics. Correlations between the different quality characteristics are not treated, and it is usually stated that this type of chart is not sensitive to small deviations from the control value or in identifying trends indicated by minor deviations.

H. Hotelling (1947) was the first to widely publicize the use of multivariate statistical methods to monitor a process. In this case he measured the performance of a complicated product, aircraft bombsights. He measured two correlated quality characteristics for a sample from a certain lot of bombsights, range and deflection error, to determine the acceptability of that lot. As he states, for complicated products it is usually necessary to measure performance characteristics of the product whereas for simple components, it is preferrable to use attributes. He introduced the use of the T^2 control chart as a method for monitoring the quality of the bombsights and he established a sampling scheme which allowed the monitoring of quality according to several types of groupings, specifically: flight, lot, and sight.

Hotelling's procedures emphasized monitoring the mean values of a parameter and the dispersion around that mean.

According to Alt (1984, p.110), the computational problems which resulted from the multivariate treatment of a large number of quality characteristics caused this field of quality control analysis to remain relatively dormant until the late 1950's with the increased availability of computers. Jackson and Morris (1957) applied multivariate quality control procedures to photographic processing. Their procedures utilized the measurement of red, blue, and green densities on a test photographic strip. They applied one of Hotelling's statistical developments, principal components, to the procedure to obtain a new set of mutually independent variables through a linear transformation of the original variables. The new variables, however, do not have close physical analogs and may be difficult to interpret. Hotelling's T^2 statistic was then applied to establish a set of T^2 control charts.

A later development in quality control statistical testing is the use of cumulative sum control charts (CSCC). According to Johnson and Leone, (June 1962, p.15), the CSCC is based on ideas put foreward by E.S. Page in 1954 but the chart was not applied extensively until the late 1950's to early 1960's. Johnson and Leone published a two part article on "Mathematical Principles Applied to their Construction and Use" in Industrial Quality Control in June and July, 1962. The CSCC procedure does not use a single observation to indicate deviation from process norms but, rather, uses a cumulative sum (CUSUM) of observations as the test statistic. As stated by Johnson (June 1962, p.15), the CSCC is especially well adapted to detecting small, abrupt changes in a value of a measured variable such as the mean or the proportion defective. This is because it is based on the cumulative sum of deviations as compared to the Shewhart chart which uses only the most current sample to indicate deviation from control norms. Therefore, the CSCC is especially sensitive to determination of when a process is going out of control. Being a cumulative sum procedure, it is often used as a sequential testing procedure for monitoring ongoing processes.

No writers on the topic of control chart applications recommend that one type of control chart be used to the exclusion of the other. They each have their applications. A method for determining the most efficient control procedure for a continuous process is the determination of average run length (ARL). The average run length is the average number of samples required in a sequential sampling procedure before an "out of control" signal is given. In general, a good control procedure has a large ARL when the process is "in control" and a small ARL when the process is "out of control." The large ARL for the in control condition corresponds to having a low false alarm rate. The small ARL for the out of control condition corresponds to being quick to detect that the quality of the products is not within quality control limits.

Reynolds (1975) derives an approximation of the run length of CUSUM control charts. Woodall (1985) discusses the design of quality control charts and presents a method for designing charts based on their statistical performance. He provides a rationale for choosing between Shewhart and CUSUM charts. Although his discussions are centered on univariate charts, his procedures can easily be applied to the comparison of multivariate Shewhart and CUSUM control charts.

Lucas (1982) provides a combined CUSUM-Shewhart control scheme intended to realize the strengths of each. This composite scheme is intended to give an out of control signal if the most current sample is outside the Shewhart control limits or if the CUSUM control limits are exceeded. Again, this discussion is centered on the use in univariate cases but the extrapolation to the multivariate case is only limited by the computational capabilities available.

The only available discussion of the use of multivariate CUSUM quality control procedures is Woodall and Ncube (1985). In fact, they state that "only the properties of univariate cumulative sum (CUSUM) procedures have been discussed in the literature." (Woodall and Ncube, 1985, p.285) In this article, they describe the simultaneous use of several CUSUM procedures as a single multivariate CUSUM (MCUSUM) procedure.

This paper draws from all of the previously discussed articles for ideas on developing a methodology for the multivariate quality control of a process. Both multivariate Shewhart charts and MCUSUM procedures are combined into a methodology which is intended to realize the strengths of both while compensating for their individual weaknesses. The results are then applied to a specific quality control problem. The remainder of Section 2 discusses the specifics of the design of multivariate Shewhart and CUSUM charts and the combined methodology.

SECTION 2.1

SHEWHART CONTROL CHARTS

The use of Shewhart control charts has long been an accepted method for monitoring the mean levels and variance of quality characteristics of the products of a manufacturing process. It is essentially a simple method of performing sequential statistical tests of the hypothesis that current sample(s) of the product meet the specified quality standards. An out of control signal indicates that only the most current sample exceeds control limits. The cause of this signal should then be investigated to determine if the sample is actually defective or whether the signal was a false alarm. Since the false alarm rate is determined by the significance level (α), a false alarm should be expected at a rate equal to $1/\alpha$.

The following general steps can be used to set up the Shewhart control chart(s):

1. Determine an appropriate sample statistic for the quality characteristic to be monitored. In measuring attributes, this can be the mean value of proportion defective, mean value of number of defective items, or

others. In measuring variables, this can be the mean, range, or other measurement of the variable.

2. Determine a value that the statistic should cluster around. This should be a target value or standard for the characteristic. In the design of Shewhart charts, it is common to use historical data on the measurements of the variable to determine the target value. If, in fact, the process produces a product with characteristics that cluster around a value different than the target value, either the target value needs to be reevaluated or the process needs to be adjusted. [It is important to understand the differences between specification values and control values for mean and range. Specification values are, generally, engineering specifications for a product. Control values are the statistically derived values used in setting up the control chart. Specification values and control values are not the same and may not correspond exactly.]

3. Establish control limits. If the sample statistic exceeds this control limit, control entities are notified to correct the process. The control limit should be based on the desired significance level of the test statistic.

Typically the Shewhart chart is designed with upper and lower control limits (UCLs and LCLs) set at plus or minus three standard deviations of the test statistic ($\pm 3\sigma$ where the variance is known or assumed known) assuming that the quality characteristic is a normally distributed random variable. For a process under control, these limits establish a probability of 0.27% that a value will fall outside of them. The control limits, however, may be established at whatever levels of significance are desired for the test.

UNIVARIATE SHEWHART CONTROL CHARTS

In the generalized univariate case, the quality characteristic of interest (X) is assumed to be normally distributed with known mean value μ_0 and variance σ_0^2 ($X \sim N(\mu_0, \sigma_0^2)$). If successive samples of size n are taken from the products of the process, the sample means $\bar{X}_1, \bar{X}_2, \ldots$ are also normally distributed with known variance σ_0^2/n and are assumed to be independent. [However, it is common for serial correlations to exist between samples. For the purposes of this paper, samples are assumed to be independent.] The univariate Shewhart control chart is then designed to maintain the condition

$E(\overline{X}_i) = \mu_o$, i = 1,2, ... The generalized Shewhart control chart for the mean value, therefore, has the following control limits:

desired value = μ_o

range = $2\, z_{\alpha/2}(\sigma_o/\sqrt{n})$

upper control limit = $\mu_o + z_{\alpha/2}(\sigma_o/\sqrt{n})$

lower control limit = $\mu_o - z_{\alpha/2}(\sigma_o/\sqrt{n})$

where z_α is the point on the standard normal distribution where $P(z \geq z_\alpha) = \alpha$.

In the sampling of a continuous process where successive random samples of size n are taken, the resultant application of this control chart methodology can be considered repeated tests of significance of the form H_O: $\mu = \mu_o$ versus H_1: $\mu \neq \mu_o$. It also provides continuous documentation of those tests. FIGURE 2.1 presents a typical Shewhart control chart for a mean value $(\overline{X})$.

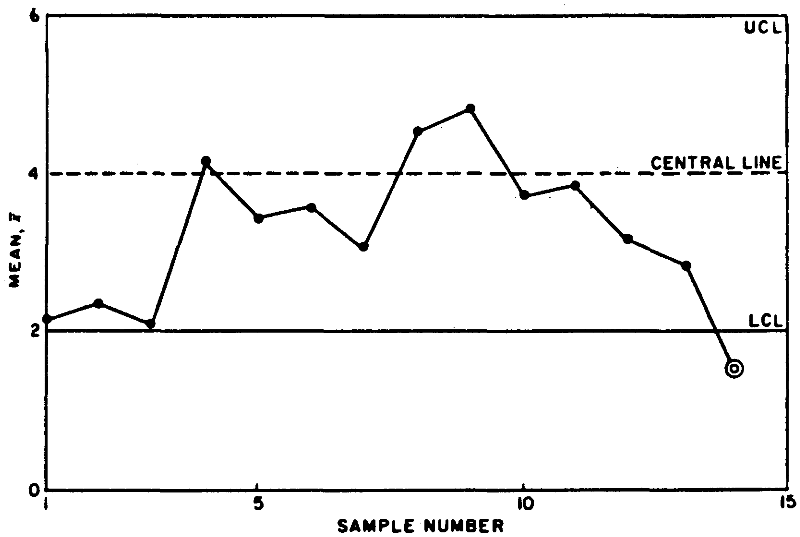

FIGURE 2.1. Typical Shewhart Control Chart (Crow, 1960, p.197)

There are several interpretations that can be made based on the data presented on the chart:

1. Statistic values that indicate a constantly increasing or decreasing trend should signal that the process may soon go out of control. Adjustment may be justified before the control limit is exceeded.

2. Statistic values which constantly fall close to the standard value may indicate either excellent production or nonrandom sampling.

3. Statistic values which fall consistently above or below the standard value but which do not exceed

control limits may require reevaluation of the standard value or examination of the production process.

4. Individual samples which exceed control limits, such as the last point plotted in FIGURE 2.1, may or may not signify an out of control situation. The cause of the out of control signal should be investigated.

MULTIVARIATE SHEWHART CONTROL CHARTS

In the multivariate case where there are p quality characteristic, the methodology again utilizes sequential samples of size n from the process. The p-variable random vectors, $\mathbf{X} = (X_1, X_2, \ldots, X_p)'$, are assumed to be p-variate normally distributed with mean vector corresponding to the target values for the quality characteristics. In the case where the p x p covariance matrix Σ_o is also known or assumed known based on historical data, the test statistic is $n(\bar{\mathbf{X}} - \boldsymbol{\mu}_o)' \Sigma_o^{-1} (\bar{\mathbf{X}} - \boldsymbol{\mu}_o)$ where $\bar{\mathbf{X}}$ is the p x 1 vector of sample means and $\boldsymbol{\mu}_o$ is the p x 1 vector of target values for the means. For our purposes, $(\bar{\mathbf{X}} - \boldsymbol{\mu}_o)' \Sigma_o^{-1} (\bar{\mathbf{X}} - \boldsymbol{\mu}_o) = D^2$ so the test statistic used is denoted nD^2. This statistic has a χ^2 (chi-squared) distribution with p degrees of freedom when $\mu = \mu_0$ and $\Sigma = \Sigma_0$. (Alt, 1984)

This is derived from the generalized case where, if $X_1, X_2, \ldots, X_n$ are independent random variables having identical normal distributions with mean μ and variance σ^2, then the random variable $Y = \sum_{i=1}^{n} (X_i - \mu)^2/\sigma^2$ has a chi-squared distribution with n degrees of freedom. [Hotelling, 1947, presents the case where the covariance matrix has to be estimated based on the sample and, therefore, he uses the T^2 statistic.]

The significance test is of the form $H_0\colon \mu = \mu_0$ versus $H_1\colon \mu \neq \mu_0$ and the null hypothesis is rejected if $nD^2 > \chi^2_{p,\alpha}$, where $\chi^2_{p,\alpha}$ is a percentage point on a chi-squared distribution with p degrees of freedom corresponding to the desired significance level (α) of the test. The resulting control limits for the one-sided multivariate control chart are:

$UCL = \chi^2_{p,\alpha}$

$LCL = 0$ (The lower control limit is, in fact, a lower bound since the statistic cannot be less than zero. This is because a value of zero occurs for the statistic when μ is exactly equal to μ_0 .)

A typical multivariate Shewhart control chart is presented in FIGURE 2.2.

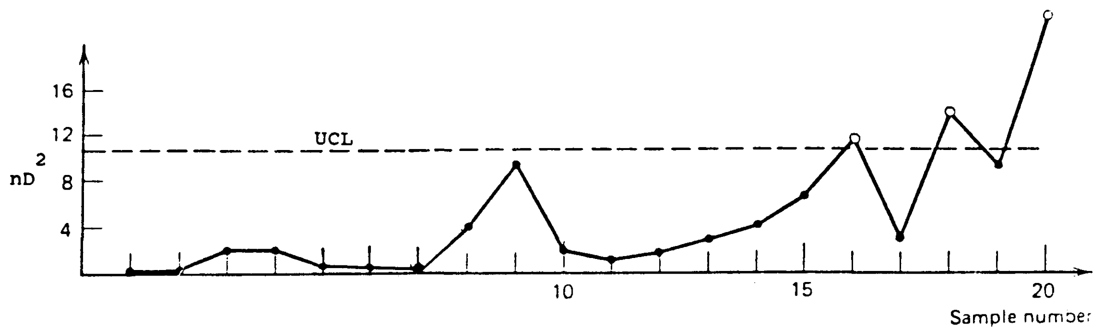

FIGURE 2.2. Typical Multivariate Shewhart Control Chart (Alt, 1984, p.114)

Deviations of the statistic above the UCL require determination of the cause and examination of the individual p-variables.

Alt (1974, p.111) discusses plotting an elliptical control region centered at μ_0 for bivariate data which can be used to replace the χ^2 bivariate chart for an individual sample. An example is shown in FIGURE 2.3. Note that, as compared to the bivariate χ^2 chart, this chart presents data for only the sample being examined but it is useful in identifying the quality characteristic(s) which led to the out of control condition.

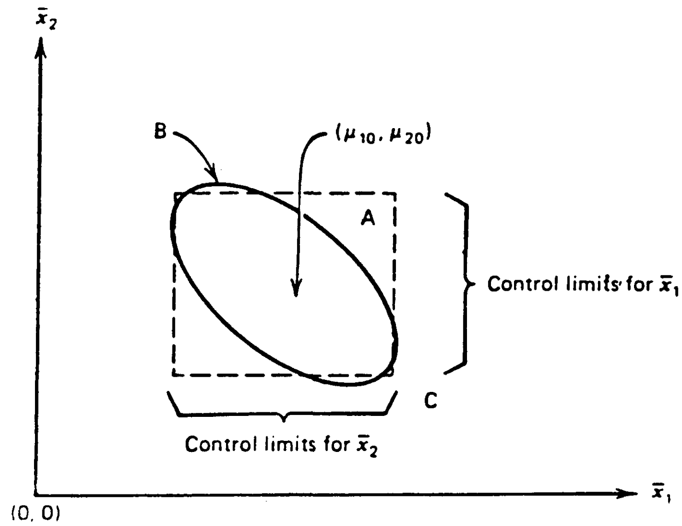

FIGURE 2.3. Bivariate Elliptical Control Region (Alt, 1984, p.112)

As can be seen in the chart, that portion of region A outside of the ellipse represents the portion of the Bonferroni adjusted rectangle in which both variables fall outside of the χ^2 bivariate control region. The use of individual Shewhart charts for the case where both means plotted in region A but outside of the ellipse would indicate that both means are within control limits while, in fact, they were out of limits if $\mu = \mu_0$. The probability that the vector of sample means ($\bar{\mathbf{X}}$) plots within the elliptical region is exactly $1-\alpha$, while the probability that the vector of sample means plots within the Bonferroni rectangle is at least $1-\alpha$, where α is the level of significance of the test.

In the multivariate case, Alt (1984, p.113) suggests that Bonferroni's inequality be used as the basis for the determination of which of the p quality characteristics are responsible for the out of control indication. This approach requires letting A_i denote the event in which the sample mean for the i^{th} quality characteristic plots within the specified univariate control limits with $z_{\alpha/2}$ replaced by $z_{\alpha/2p}$. Bonferroni's inequality, then, states that $P(A_1 \cap A_2 \cap \ldots \cap A_p) \geq 1-\alpha$. p individual univariate charts would then be constructed to determine which quality characteristic(s) caused the out of control signal. Each chart, therefore, would be constructed with type I error equal to α/p. (In fact, it is possible to use different levels of significance (α_i) for each of the individual variables. In that case, the univariate control limits would be z_{α_i} . Bonferonni's inequality, in this case, would be $P(A_1 \cap A_2 \cap \ldots \cap A_p) \geq 1 - (\sum_{i=1}^{p} \alpha_i/p)$.

Investigation of the design of multivariate quality control charts using different levels of significance for the individual variables, however, is beyond the scope of this paper.)

SECTION 2.2

CUMULATIVE SUM CONTROL CHARTS

Cumulative sum quality control procedures were first presented by E.S. Page in his article on "Continuous Inspection Schemes" in 1954. [Biometrika, Vol. 41, pp.100-115] According to Johnson and Leone (June 1962, p.15) these procedures only became extensively used in recent years [possibly due to the increased availability of large computers]. The procedures were not intended to replace the Shewhart schemes since the cumulative sum control chart (CSCC) was better at indicating small, abrupt changes in a process since it is based on a cumulative sum of deviations, while the Shewhart chart was better at signaling large changes. Univariate CSCCs can be designed using statistics based on the same types of data as the univariate Shewhart charts and can be based on measurements of attributes or variables. The statistic can be based on mean value, range, variance, and other measures.

While the test statistic used in the univariate Shewhart and CSCC are closely related, at this time, no multivariate CSCC has been found that is analogous to the multivariate Shewhart chart. Woodall and Ncube (1985), however, discuss a procedure for simultaneous use of several univariate CUSUM control charts in a multivariate CUSUM (MCUSUM) procedure. The remainder of this section discusses the univariate CSCC and MCUSUM control chart procedures developed by Woodall and Ncube.

UNIVARIATE CSCC

In the univariate case, the CSCC presents a sequence of points plotted on the chart representing a cumulative sum of deviations from a predetermined target value. The most current point plotted on the chart represents data not only from the most current sample of the process but includes data from all previous samples. In general, the value of the most current point plotted on the chart is the sum of the value of the previously plotted point and the statistic calculated based on the current process sample. The CUSUM chart, therefore, presents an excellent indication of trends in the variance of the variable.

Interpretation of the CSCC is dependent on the results of the comparison of the plotted statistics with predetermined control limits that are noted on the chart.

As opposed to the Shewhart charts where the control limits are fixed horizontal lines, the CSCC limits vary in position based on the most current sample statistic plotted. This is because the control limit lines are drawn on the chart based on the most current sample statistic which can change on a sample by sample basis.

In the case of a univariate CSCC for the mean of a sample, the value of the plot for sample m with mean value $\bar{x}_m$ is calculated $1/\sigma_{\bar{x}} \sum_{i=1}^{m} (\bar{x}_i - \mu_o)$ where:

$\sigma_{\bar{x}}$ is the standard deviation of the sample mean - the population standard deviation divided by the square root of sample size ($\sigma_o/\sqrt{n}$). (Assuming a constant sample size.)

$\bar{x}_i$ is the mean of the i^{th} sample.

μ_o is the target population mean (control) value

To determine if the statistic exceeds control limits, "V-Mask control schemes" have been devised. FIGURE 2.4 presents a V-mask for an example CSCC. The origin of the V-mask, the point labeled O, coincides with the last plotted point on the chart.

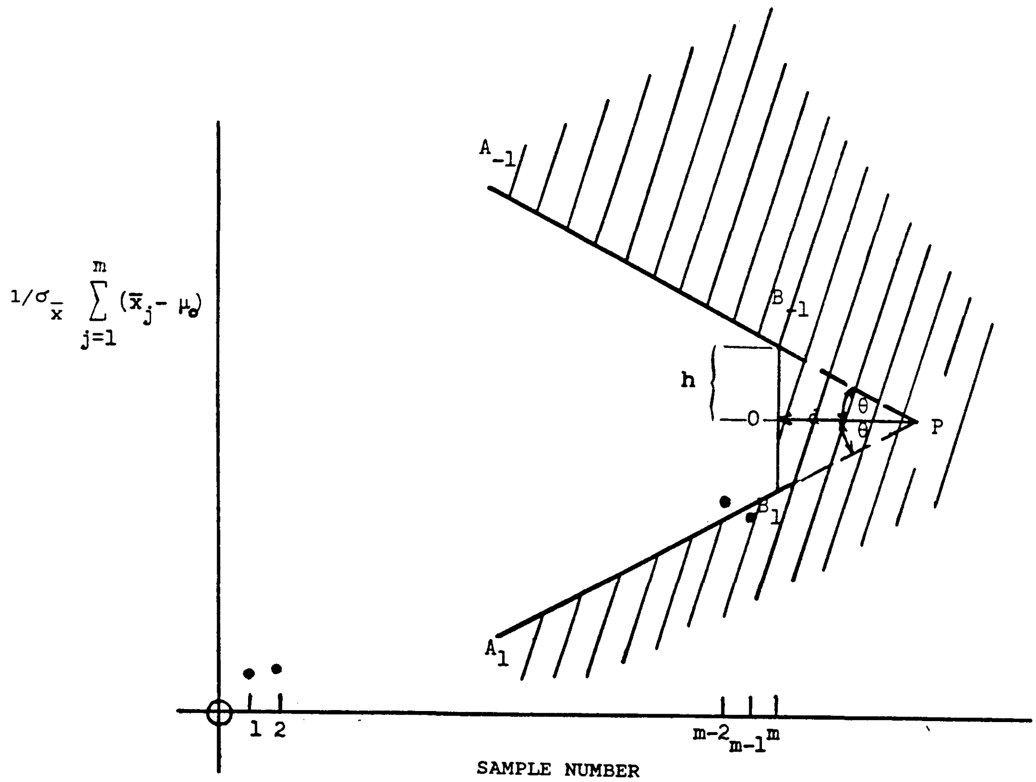

FIGURE 2.4. V-Mask for a CSCC
(From Johnson and Leone, June 1962)

Several Authors present procedures for designing the V-mask. Johnson and Leone (June 1962) present procedures based on calculations of d and θ. Lucas (1976) presents procedures for the design of the V-mask based on two parameters, k and h, which is essentially equivalent to the procedure of Johnson and Leone ($k = 2\sigma_{\bar{x}} \tan\theta$ and $h = 2\sigma_{\bar{x}} d \tan\theta$). His procedures are summarized here:

1. Determine μ_o , the target value for the variable. Determine the deviation of the variable that the chart is designed to give an out of control signal at, . Determine the standard deviation of that variable for the process, σ_o. In designing the chart, it is suggested that one sample unit on the abscissa equal 2σ on the ordinate. (This makes plotting easy since the ordinate will be in multiples of the sample variance.)

2. Calculate $k = \Delta/2$. ($k = 2\sigma_{\bar{x}} \tan\theta$ when one unit on the abscissa equals $2\sigma_{\bar{x}}$ units on the ordinate) This provides, in general, the quickest detection but may be adjusted based on the specific needs of the process being examined. Normalize k to use the ARL curves provided by Lucas. k is the allowable slack.

3. Choose h^* based on the ARL curves provided by Lucas. In general, four hundred (400) is a good initial ARL value. Calculate h^*, $h = h^* \sigma_{\bar{x}}$. ($h = 2\sigma_{\bar{x}} d \tan\theta$ when one unit on the abscissa equals $2\sigma_{\bar{x}}$ units on the ordinate)

h is the vertical decision interval.

4. Place the mask on the points plotted on the CUSUM chart to determine that it meets the needs of the test.

Interpretation of the procedure is as follows:

- An out of control condition exists if any point plotted on the CSCC falls in the shaded region of the V-mask.

- If any plotted point falls below line A_1B_1, an increase in the process mean is indicated.

- If any plotted point falls above line $A_{-1}B_{-1}$, a decrease in the process mean is indicated.

FIGURE 2.5 presents an example of a CSCC with control lines for different levels of deviation from the standard value of the test statistic.

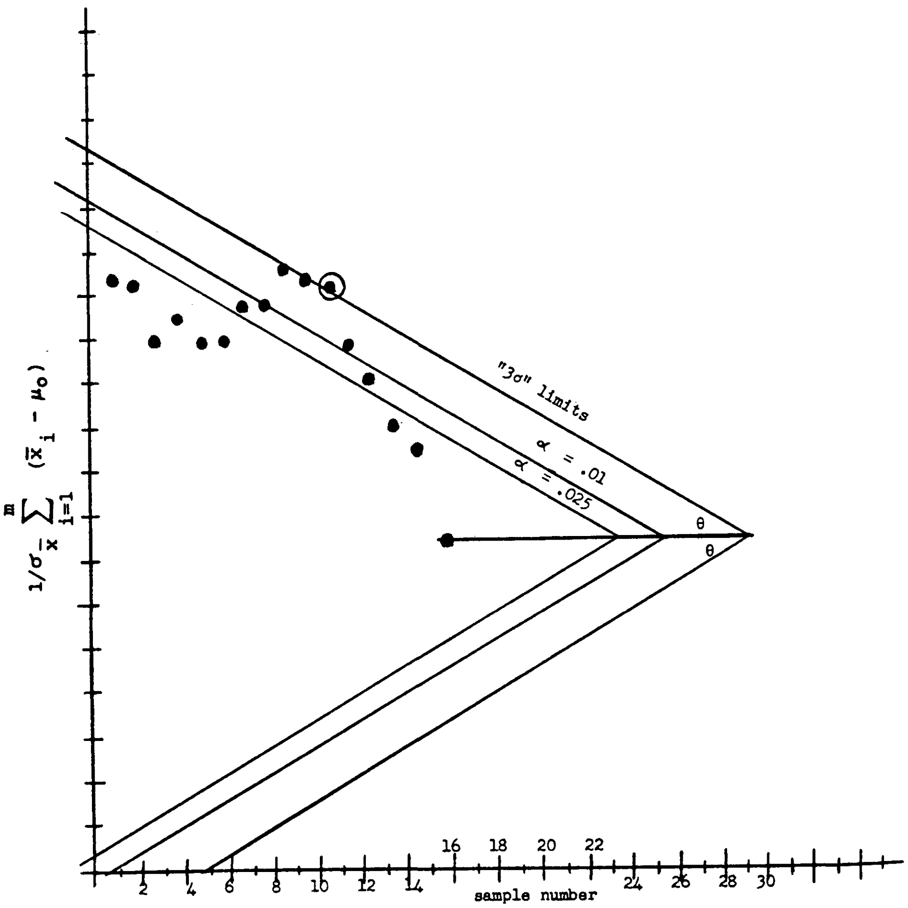

FIGURE 2.5. CSCC with Various Control Limits (Johnson and Leone, June 1962)

MULTIVARIATE CSCC

No discussions of multivariate CUSUM charts analogous to the multivariate Shewhart charts has been found to date. In order to deal with the interdependency of product quality characteristics, multivariate cumulative sum (MCUSUM) procedures have been developed. The only thorough discussion of these types of procedures found to date, however, is in Woodall and Ncube (1985). In that article they claim to be the first to simultaneously use several CUSUM procedures as a single multivariate CUSUM procedure. Their discussion is restricted to application to the bivariate case but establishes the basis for extrapolation to multivariate quality control problems.

In the generalized case, it is assumed that the quality of a product is described by p characteristics. Each observation, therefore, provides a p x 1 vector of values. Each of the p variables is assumed to be normally distributed with a mean vector corresponding to the target values for the quality characteristics. The covariance matrix Σ_o is known or assumed known.

For the multivariate CUSUM (MCUSUM) procedure, univariate CUSUM procedures are applied to the observations of each of the individual p variables. Interpretation of the out of control signal is simple in that an out of control signal can easily be attributed to the CSCC corresponding to the variable which caused the out of control signal. This procedure is quite simple except for determination of the ARL which is necessary for the design of the procedure for specific quality control applications. Woodall and Ncube present the procedures for using principal components to achieve independent variables and for determining the ARL for integer-valued bivariate data. Lucas (1976) presents a set of nomograms which facilitates the selection of k and h based on the desired ARLs for the in control and out of control situations. The next section discusses the combined multivariate Shewhart and MCUSUM methodology and how it realizes the strengths of each procedure.

SECTION 2.3

COMBINED MULTIVARIATE SHEWHART-MCUSUM PROCEDURES

This section discusses a new technique based on the use of combined multivariate Shewhart-MCUSUM quality control chart procedures. The combined technique is simple to understand. For a process, a multivariate Shewhart chart is designed to monitor the mean of the quality control characteristics of interest. Lucas' MCUSUM procedures are then used to construct individual univariate CUSUM charts for those same quality characteristics of interest. The process is then monitored using these control charts.

SETTING UP THE PROCEDURE

The procedure described here is not a cost design but, rather, a statistical design. It should be adapted according to the specific needs of the manufacturer and to the characteristics of the manufacturing process it is designed to monitor. The general steps in setting up the procedure are:

1. Determine the quality characteristics to be monitored. Attempt to determine their relationships in

order to facilitate interpretation of the charts.

2. Determine the measures of the quality characteristics that are to be used. Usually it is the sample mean for continuous variables and mean or proportion defective for attribute measures.

3. Determine target values and control limits. This can be done in several ways. For small sample sizes in a multivariate case (where sample size will be less than the number of quality characteristics of interest) an estimate of population mean and variance must be made. For sample sizes larger than the number of quality characteristics, approximately 25 samples taken from when the process was known to be in control can serve to determine this information. This will provide the data for determination of the population mean vector and the covariance matrix used in the Shewhart chart. Asymmetries between specification limits and the performance of the variables should be resolved in the determination of quality control limits. [In a cost design, the cost of the product, the cost of sampling and testing, and the cost of interrupting the process must be considered in the selection of sample size, sampling interval, and quality control limits.]

4. Build a Shewhart chart as described in Section 2.1. Pick a level of significance and compute the UCL considering the resulting ARLs for the in control and out of control conditions.

5. Build individual CUSUM charts for the individual quality characteristics as described in Section 2.2. Design V-masks for the charts if it is important for the CUSUM charts to detect small deviations in the individual quality chatacteristics.

6. Monitor the samples as they are plotted on the charts. The statistics may be put in tabular form to provide a concise history of the performance of the process.

An out of control signal given by the Shewhart will result in examination of the variables to determine whether the cause was due to an out of control condition or whether it was a false alarm. The multivariate statistic used in the Shewhart chart, however, is difficult to interpret regarding the identification of the variable(s) which caused the out of control signal. The individual CUSUM charts, however, can then be referenced to determine which variable or combination of variables contributed to the out of control condition and the process can be adjusted based on the results. In the situation where the out of control signal is a result of a linear combination of variables, the individual charts will provide information to help determine what combination was probably the cause.

An out of control signal given by any one or combination of the individual CUSUM charts will cause the process to be interrupted so that the cause can be determined and adjustments can be made if necessary. The individual charts indicate the specific variable(s) responsible for the out of control condition. The Shewhart chart can also be referenced to determine if there are any unfavorable trends indicated by the multivariate statistic. An additional benefit provided by the use of the individual CUSUM charts is that they provide graphic representation of trends in the individual characteristics. These can be monitored and the process adjusted, if required, prior to an out of control signal being given.

This paper does not develop an accompanying multivariate procedure to monitor the variance of the process. As mentioned earlier, it is possible for the means of the variables to remain unchanged while the variances change, and most existing univariate procedures utilize some method to monitor variance. (This is an excellent topic for future consideration.)

CHARACTERISTICS

The run length of this procedure for the out of control condition will be the minimum of the run lengths of the individual CUSUM charts and the Shewhart chart. The specific objectives of the quality control procedure, as applied to the manufacturing process, should determine the levels of the variations that the individual CUSUM charts and Shewhart chart are designed to detect. These can be calculated. However, it is highly likely that the control limits will have to be adjusted as the process runs to give the optimum combination of lowest false alarm rate, quickest determination that $\mu \neq \mu_0$, and lowest cost for conducting the quality control procedures. (This is a good topic for future consideration.)

The univariate CSCC is approximately four times as fast as a univariate Shewhart chart for detecting 1σ shifts in a mean. (Lucas, 1976, p.3) The new procedure, therefore, combines the strengths of the two types of charts. Small shifts can easily be detected quickly by the CUSUM charts while large shifts will be more quickly detected by the Shewhart chart. The resulting ARL for the out of control condition for the combined procedure should also be very small since the signal will be given by the chart with the smallest run length.

When the process is in control, both control charts will maintain a relatively large ARL. Although the ARL for the combined procedure will be relatively large, it will be less than or equal to the ARL for an individual CUSUM or Shewhart control chart procedure. This should not be significant and can be adjusted by adjusting the control limits of the chart(s) which are affecting the ARL. Also, Lucas (1982) states that if the procedure is designed correctly, the ARL for the univariate combined procedure for the in control condition may be larger than the ARL of either the univariate CUSUM or Shewhart procedure. This is also likely to be true for the multivariate case since the ARL for the multivariate Shewhart chart should be larger than its univariate analog.

One of the strengths of univariate Shewhart charts is that they provides excellent information on trends. The statistic of the multivariate Shewhart chart, however, is difficult to interpret. The combined method provides for interpretation of trends indicated by the multivariate Shewhart chart by examination of the data provided by the individual CUSUM charts and attempting to correlate it with the multivariate Shewhart data.

This procedure can be adapted by applying the Fast Initial Response (FIR) methods as described by Lucas and Crosier (1982). FIR methods do not reset the CUSUM statistic to zero at initial start up or after an out of control signal is given. Rather, the CUSUM statistic is set to a "head start" value. If the process starts up in control, the head start value will have little effect on the performance of the chart. However, if the process starts up out of control, an out of control signal will be given faster than if a CUSUM value of zero is used. This modification provides improved sensitivity for the CUSUM portion of the combined method but should be applied only if the manufacturing process is prone to be out of control at start up or soon after. If the process is prone to be in control at start up, this modification has little effect and complicates the procedure. (This is an excellent topic for future examination.)

The combined procedure is easily adapted to tabular form to provide concise summaries of the history of the manufacturing process.

SECTION 3.0

APPLICATION OF COMBINED PROCEDURES TO QUALITY CONTROL OF BEER CAN POP TOPS

The Adolph Coors Company produces its own cans for its beer. Coors Container Operations is responsible, as part of its operations, for the pop tops for the beer cans. These can be the older pop tops in which the tab comes off the can, or the newer stay on tabs (SOTs or "ecology" tops) which are intended to reduce the litter problems associated with the older pop tops. Quality control of these pop tops is very important in order to produce a pop top which is easy to open but which does not allow release of the pressure within the can. This section describes the application of the combined multivariate Shewhart-CUSUM procedures to the problem of quality control of the SOT type of pop top, although this methodology is also applicable to quality control of the other type of pop top.

PROBLEM DESCRIPTION

The SOT pop top is characterized by four (4) quality characteristics which are called Rivet Diameter, Rivet

Residual, Score, and Pop. They are described as follows:

- Rivet Diameter (RD) - the diameter of the rivet that holds the tab on the top of the can (end). The rivet is actually a protruding portion of the end and not a separate piece. (x_1)

- Rivet Residual (RR) - the thickness of the rivet. (x_2)

- Score - the depth of the aluminum in the cut in the aluminum end. This cut outlines the portion of the end that is pushed into the can when the pop top is opened. (x_3)

- Pop - the force required to open the pop top. (x_4) This variable is characterized by variability introduced by human differences in the measurement of this force.

There is a definite physical relationship between Rivet Diameter, Rivet Residual, and Score. As the Rivet Diameter is expanded, the Rivet Residual is reduced (assuming that the amount of aluminum used to make the rivet is held constant). In addition, if additional aluminum is used to make the rivet, this results in a thinner end which affects the Score (assuming that the amount of aluminum used to make the end is held constant). As a result, a negative correlation should exist between Rivet Diameter and Rivet Residual. Also, a negative correlation with Score should be evident if both Rivet Diameter and Rivet Residual are changed in the same direction.

Coors specification limits are presented in TABLE 3.1. Coors specifies ranges for the variables but not target means. For the purposes of this example problem, the target means are taken as the means of the individual variables computed from the data in Appendix A, with the exception of Rivet Diameter (x_1). In order to test the procedure's ability to detect variation, the target value for Rivet Diameter was set at 201.54, approximately one standard deviation below the actual mean value computed from the data.

Since the target mean is determined by the population mean (with the exception of Rivet Diameter), it is statistically derived. This is an important method to determine target values for variables when the actual performance of the product is described by functional characteristics that are hard to measure and the relationships between the quality characteristics and the product performance are not fully understood. In this case, a pop top must prevent pressure loss under a variety of conditions but be easy to open. The application of the methodology to this problem should help explain some of the relationships between the four quality characteristics and the functional utility of the SOT. In the case where the population mean does not correspond to target mean, the manufacturing process should be adjusted or the target mean should be reevaluated.

Currently, the Coors procedures utilize univariate Shewhart $\overline{X}$ charts and $\overline{R}$ charts for monitoring each variable individually. It is likely, however, that the variables are correlated. This is evidenced by the following occurrences in the existing quality control procedures:

1. All four variables fall within limits but the pop top fails. (FIGURE 3.1 presents the acceptance region for bivariate data and, since a corresponding figure cannot be drawn in 4-space, it is used to demonstrate some of the problems experienced by Coors.) This occurrence corresponds to that region outside of ellipse B but inside rectangle A in FIGURE 3.1 for

bivariate data.

2. One or more of the variables is out of limits but the pop top works adequately. This corresponds to that area outside of rectangle A that is inside of ellipse B in FIGURE 3.1.

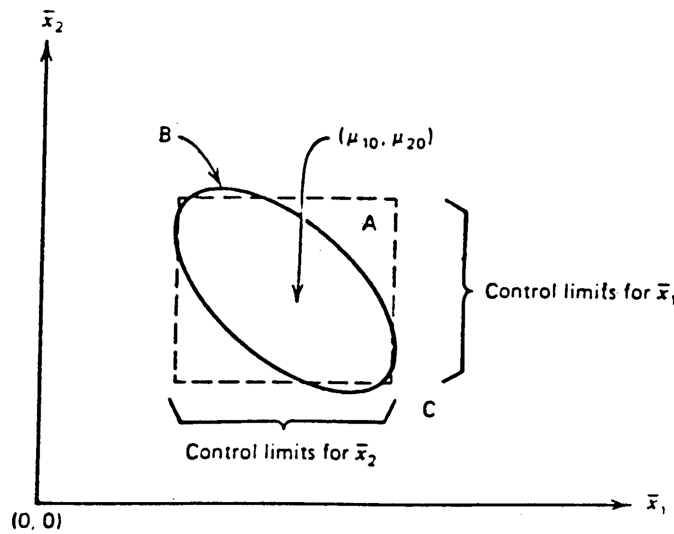

FIGURE 3.1. Bivariate Elliptical Control Region (Alt, 1984, p.112)

The problem, therefore, is to create a quality control procedure that more accurately measures the quality of the SOT as a functional entity and helps explain their interrelationships. This requires a multivariate approach in order to consider the information available by considering correlations which exist between the four quality characteristics.

DATA AND REQUIRED STATISTICS

Sampling was performed on the pop tops over a period of time and the individual parameters were measured. This resulted in 428 measurements of each variable from which the mean (μ_{oi}) and standard deviation (σ_{oi}) was determined. (The data is provided in appendix A.) For the purposes of this paper, these are assumed to be the population means and standard deviations. μ_o, therefore, represents the 4 x 1 vector of population target means. The 4 x 4 covariance matrix Σ_o was determined from this data and, for the purposes of this paper, it is assumed to represent the population covariance matrix.

It is possible to determine Σ_o in several ways. For larger sample sizes, the sample covariance matrix or variances and covariances can be determined for each sample. These can then be averaged over a range of samples to determine the covariance matrix. However, for small samples it may be preferrable to compute the covariance matrix directly from a sufficiently large number of samples to ensure that it is positive definite. (The determinant of the covariance matrix $\left|\Sigma_0\right|$ is greater than zero.) This was done here since the Coors sampling is based on sample size (n) of two. Σ_0^{-1} was then determined by inverting the covariance matrix. This provided the data necessary to calculate the D^2 portion of the nD^2 multivariate statistic. This statistic is distributed χ^2_4 (4 degrees of freedom) assuming multivariate normality and assuming $\mu = \mu_0$ and $\Sigma = \Sigma_0$. The values are presented:

$$\mu_0 = \begin{bmatrix} \mu_{01}^{*} \\ \mu_{02} \\ \mu_{03} \\ \mu_{04} \end{bmatrix} = \begin{bmatrix} 201.54 \\ 59.24 \\ 30.79 \\ 4.04 \end{bmatrix} \qquad \begin{aligned} \sigma_{01} &= 2.28 \\ \sigma_{02} &= 3.48 \\ \sigma_{03} &= 2.45 \\ \sigma_{04} &= 0.24 \end{aligned}$$

* Set at target value.

$$\Sigma_o = \begin{bmatrix} 5.20 & -1.51 & -1.74 & -0.04 \\ -1.51 & 12.13 & -1.90 & -0.13 \\ -1.74 & -1.90 & 6.02 & 0.14 \\ -0.04 & -0.13 & 0.14 & 0.06 \end{bmatrix}$$

$$\Sigma_o^{-1} = \begin{bmatrix} 0.23 & 0.04 & 0.08 & 0.04 \\ 0.04 & 0.09 & 0.04 & 0.13 \\ 0.08 & 0.04 & 0.21 & -0.36 \\ 0.04 & 0.13 & -0.36 & 17.83 \end{bmatrix}$$

The determinant of Σ_o is 16.9.

The upper control limit (UCL) for the multivariate Shewhart chart, therefore, can be determined $nD^2 = \chi^2_{p,\alpha}$ where α is the level of significance of the test. This can be taken from Table B.3 in Appendix B. The lower control limit (LCL) for the multivariate chart based on the statistic is always zero since this represents the case where μ is exactly equal to μ_o. Therefore, this is also the lower bound.

The last 20 samples in the data are used to demonstrate the use of the control charts. Since Coors takes two measurements at four hour intervals, n = 2.

As discussed earlier, it is often important to also monitor the variance of the process. A multivariate procedure is not developed here to perform this analysis. However, in general it is necessary to have sample sizes (n) larger than the value of the number of variables examined (p) in order to have valid estimates of the sample variances in the multivariate case. For the Coors problem, this would require a change in Coors procedures in order to provide sample sizes of five or more.

MULTIVARIATE SHEWHART CONTROL CHART

FIGURE 3.2 presents the multivariate Shewhart statistic plotted for the last 20 samples in the data. α was selected at .005 in order to ensure that some of the points would plot above the UCL for demonstration purposes while still resulting in an ARL of 200 for the in control conditions. This value can be set as required by the particular process. The UCL is 14.86.

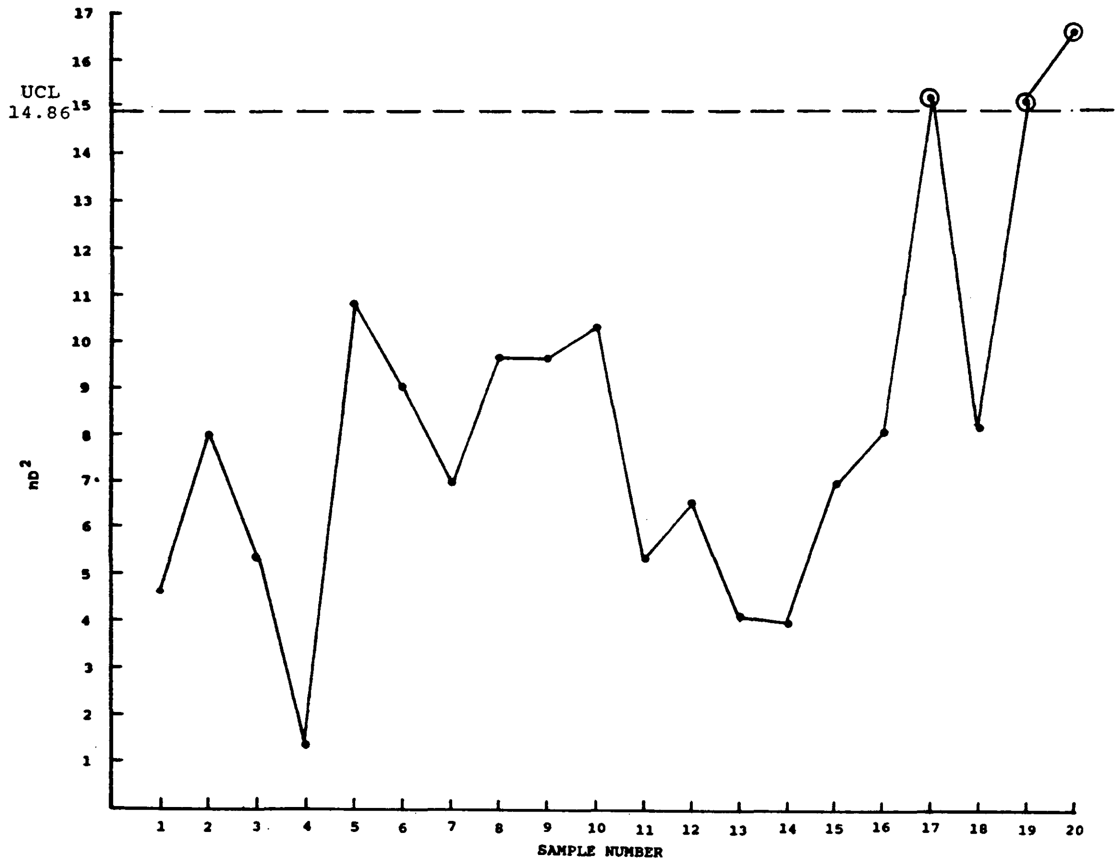

FIGURE 3.2. Multivariate Shewhart Chart

The following observations can be made from the chart:

1. Samples 5,6,7,8,9, and 10 plot consistently high. The individual CUSUM charts should probably be examined to determine which variable(s) caused this condition.

2. The entire set of samples plots consistently well above the LCL, possibly indicating a problem either in the process or in the determination of the target value for one or more of the variables. The CUSUM charts should be examined to determine which of the variables caused this trend. (In fact, the arbitrary setting of the target value for Rivet Diameter is probably a primary contributor to this condition.)

3. Samples 17,19, and 20 plotted above the UCL, thus signaling an out of control condition. The CUSUM charts should be examined to determine the variable(s) which caused the out of control condition since, statistically, only one false alarm should occur in every 200 samples when the process is in control.

CUSUM CONTROL CHARTS

The primary purposes of the CUSUM charts in this example are to demonstrate trends in any of the individual variables, to identify the variable(s) responsible for trends or out of control signals indicated by the Shewhart chart, and to help in understanding the relationships between the variables. Because of this, it was not important to use the V-mask to indicate minor deviations. However, the design of V-masks to indicate 2σ deviations in the individual variables is presented in the following paragraph so the process can be better understood.

Lucas' method (1976) was used to determine the values of k and h for the V-masks for the different charts. These were selected in order to provide ARLs of around 400-500 for in control conditions and small ARLs for out of control conditions. Since the primary purpose of the CUSUM charts in this problem is to indicate trends rather than give an out of control signal, these ARL values can be allowed to vary based on the requirements of the control procedures. The following data was used in the design of the V-masks for the individual charts:

	x_1	x_2	x_3	x_4
$\sigma_{\bar{x}}$	1.61	2.46	1.73	0.17
k	1.61	2.46	1.73	0.17
h	4.43	6.77	4.76	0.47
θ	26.6°	26.6°	26.6°	26.6°

Figures 3.3 through 3.6 present the CUSUM charts for variables Rivet Diameter (x_1), Rivet Residual (x_2), Score (x_3), and Pop (x_4), respectively.

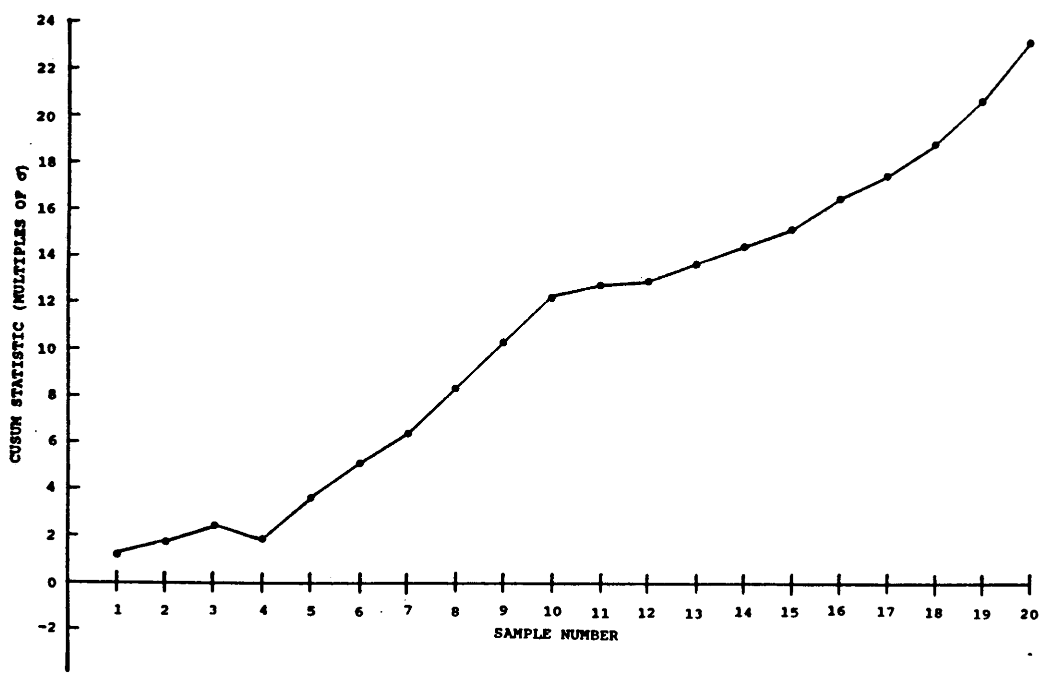

FIGURE 3.3. Rivet Diameter CUSUM Chart

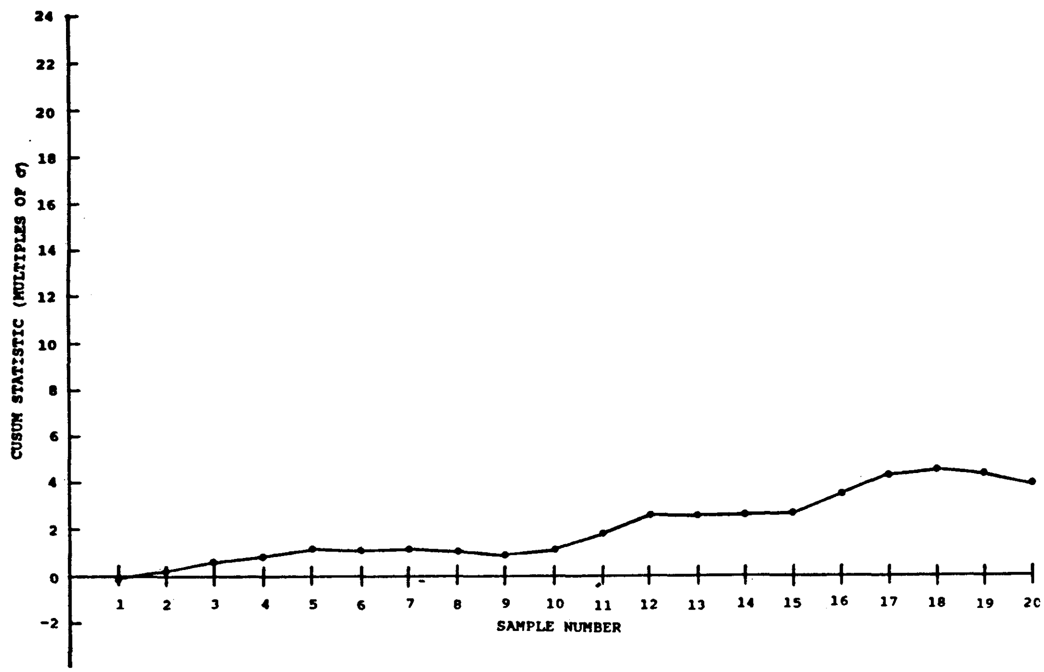

FIGURE 3.4 Rivet Residual CUSUM Chart

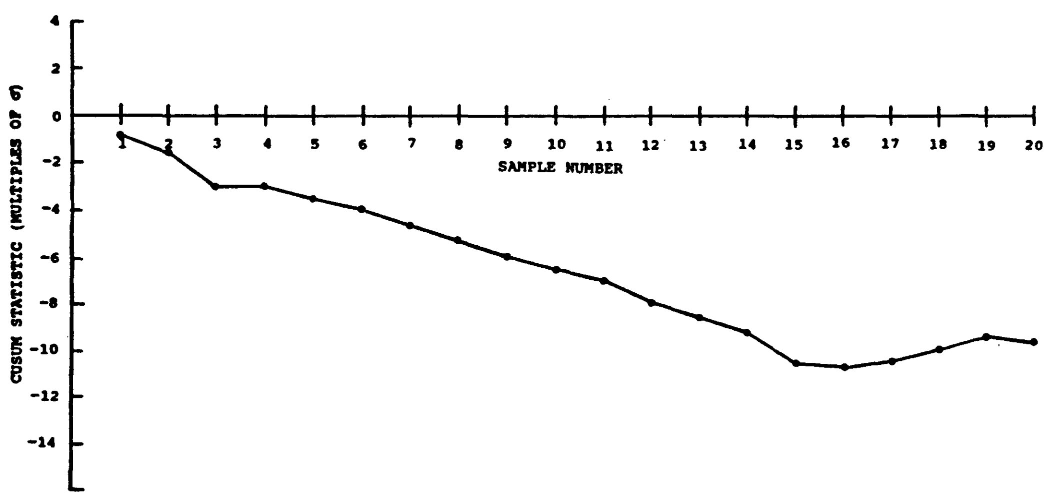

FIGURE 3.5 Score CUSUM Chart

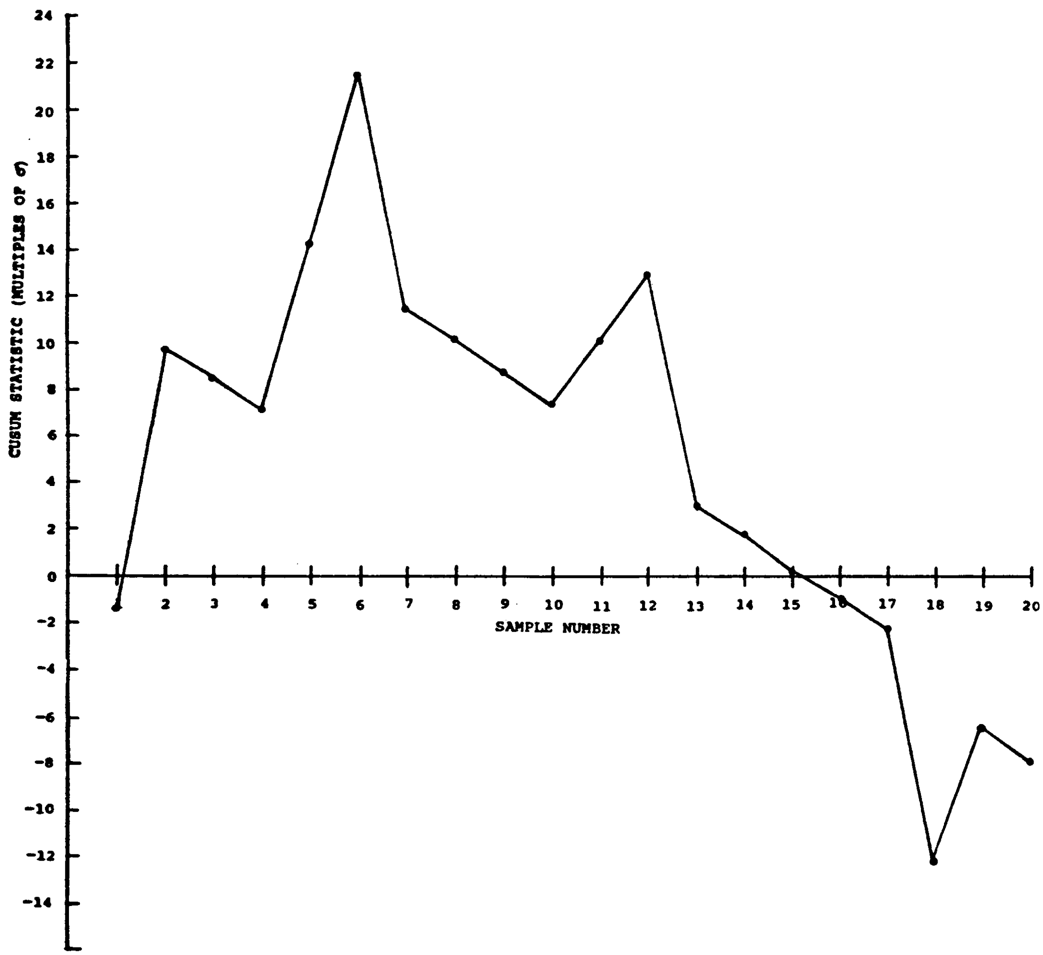

FIGURE 3.6 Pop CUSUM Chart

The following observations can be made based on the individual CUSUM charts and on correlating the individual CUSUM chart with the Shewhart chart:

Rivet Diameter (x_1) - FIGURE 3.3:

1. A major positive trend is obvious. This indicates that either there is a problem in the

manufacturing process related to rivet diameter, the target value for this variable needs to be reevaluated, or there is a problem in the sampling. Since the target value was set arbitrarily below the statistical mean, this target value should be reevaluated or the manufacturing process adjusted to cause this variable to more closely approach the target value.

2. This variable is, at least in part, responsible for the positive trend in the Shewhart statistic.

3. Samples 5-10 and 16-20 indicate increasing positive deviation of this variable. This variable is a major contributor to the out of control condition in samples 17, 19, and 20. The other variables should be examined to determine their contribution to this condition.

Rivet Residual (x_2) - FIGURE 3.4:

1. This variable exhibits a slight positive trend. This trend contributes slightly to the positive trend indicated by the Shewhart chart.

2. This variable is not the cause of the high Shewhart statistic for samples 5-10 nor does it cause the

out of control condition in samples 17, 19, and 20.

Score (x_3) - FIGURE 3.5:

1. This variable exhibits a negative trend. This may be due to problems in the process, it may indicate that the target value needs to be reevaluated, or there may be problems in the sampling.

2. The variable indicates increasing negative deviation until sample 16, at which time the variable appears to have stabilized.

3. This variable contributes some to the high Shewhart values for samples 5-10 but is not a reason for the out of control condition in samples 17, 19, and 20.

Pop (x_4) - FIGURE 3.6:

1. This variable is characterized by major changes in the direction of deviations. The general trend, however, is in a negative direction.

2. The variable is probably a major contributor to the high Shewhart value for samples 5-10 as well as to the out of control condition of samples 19 and 20.

3. Considering the erratic behavior of this

variable, it might be interesting to throw it out and perform a similar analysis on only the first three variables using this procedure.

Rivet Diameter, Rivet Residual, and Score:

Both Rivet Diameter and Rivet Residual exhibit positive trends throughout the range of samples. Considering that a negative correlation exists between these variables, this should be examined to determine the cause. One possibility is that the process is changing to allow for the increase in the amount of aluminum used in the rivet, thus allowing both to increase despite their negative correlation.

In addition, Score shows a general negative trend throughout the range of samples. This is consistent with the trends indicated by Rivet Diameter and Rivet Residual considering their physical interrelationships.

CONCLUSIONS REGARDING THE SAMPLE PROBLEM

While the methodology was not used to set up quality control chart procedures optimized for the needs of Coors, it appears to have promise for simplifying and possibly reducing the cost of their quality control procedures. The methodology could easily be adapted to other quality control needs of Coors also, such as the quality control of the remaining portion of the aluminum beer cans.

It is evident from the Shewhart and CUSUM statistics for the Coors variables that it is necessary for Coors to determine target values for the variables monitored by their quality control procedures. Only one variable, Rivet Diameter, ever exceeded Coors' specification limits and this was in samples 9, 10, and 20. Only in sample 20 did this contribute to the out of control signal given by the Shewhart chart. The out of control signal given for samples 17 and 19 resulted from a linear combination of variables which, individually, did not exceed specification limits. This indicates that that a multivariate approach may provide more complete information regarding the quality of a product and this information can be used to make better decisions regarding the product's acceptability.

The CUSUM charts were very useful in identifying trends in the individual variables and in helping to determine which combination of variables contributed to conditions evident on the Shewhart chart. There are many additional ways to further utilize these charts in this procedure, such as by using warning limits on the CUSUM charts. These should be investigated for this specific problem.

The procedures described in this paper did not include the development of charts to determine the performance of the variance of the variables. A procedure to monitor either range or variance is required for this specific problem in order to provide a greater amount of information on the manufacturing process and the quality of its products.

TABLE 3.1

COORS SPECIFICATION LIMITS

RE-HIT TOOLING - STANDARD - .075 BUBBLE-UP PUNCH HEIGHT
TENTATIVE SOT CONVERSION (207.5, .0104 GAGE) 5042-H19 - SPEC SHEET

Characteristic	OPERATING LIMIT	SAMPLE SIZE AND FREQUENCEY	
		PRODUCTION	QUALITY CONTROL
Opening Force Quality Control Production	 4.0 - 8.0 lb. 3.25 - 8.0 lb.	1 end each hour	5 ends 4 times per shift
Rivet *Diameter Residual	 .197 - .207 .0055 - .0075	5 ends at tooling change and 1 end every hour	1 end 2 times per shift at the beginning and mid-shift
Score Residual 12:00 o'clock 3:00, 6:00 & 9:00 o'clock	 .0035 - .0047 .0026 - .0036	1 end each shift or as required	1 end 2 times per shift at the beginning and mid-shift
Bubble-Down Depth Bubble-Up Height *Coin Residual Re-Hit Bubble-Up Height *Re-Hit Coin Residual Secondary Score Width	.095 - .098 .089 - .101 .0080 - .0090 .092 - .104 .0080 - .0090 .003 - .004	5 ends at tooling change or as required Set-up .0084 - .0086 Set-up .0084 - .0086	1 end at tooling change or as required 5 consecutive checks spaced 15/20 minutes apart must fall within .0085 ± .001 2 outside tolerance, reset tooling
Primary Score Width	.0015 - .0025		1 end at tooling change or as required
Panel Deboss Embossed Letter depth Tear Strip Bead Height	.019 - .023 .004 - .006 .013 - .017		1 end at tooling change or as required
Visual Defects		2 ends per tube	6 ends each hour
Buckle Strength	50 lb. minimum		6 ends for trouble-shooting
Metal Exposure	none allowed except in score		5 ends following embossing tooling changes
Helium Test			1 end per tube

*NOTES: Rivet Diameter = Measure high/low, record average
Coin Residual = Measure 3:00, 12:00, 9:00, record average ($\bar{X}$) & Range (R) on finished ends

SECTION 4.0

CONCLUSIONS

The methodology discussed in this paper appears to be very versatile, adaptable, and easy to use. It significantly increases the amount of meaningful information available on which decisions regarding the quality of the products of a manufacturing process can be made. It, therefore, potentially may reduce the cost of many quality control procedures while increasing their effectiveness. In addition, the application of this methodology to specific multivariate quality control problems can aid in understanding the relationships between quality characteristics which jointly determine the performance or acceptability of a product.

The intent of the specific procedure discussed here is to realize the strengths of the individual types of quality control charts while avoiding their unique weaknesses. It succeeds at this. The multivariate Shewhart chart is good at detecting large deviations and indicating major trends caused by one or more variables. The difficult interpretation of the multivariate Shewhart chart is avoided by the use of the individual CUSUM charts. They are very good at explaining the trends and identifying the variables which contributed to out of control signals given by the Shewhart chart. In addition, they provide excellent indications of trends and minor deviations in the individual variables and are especially helpful in explaining their relationships.

The methodology is easily adaptable to most types of quality control problems which previously might have been monitored using only one of the types of control charts. It is applicable to problems involving several correlated quality characteristics, whether they are measured as variables or attributes. The data provided on the charts can easily be put in tabular form, thus providing concise tracking of the statistics of interest and providing a history of the performance of the manufacturing process.

It is evident from the example problem that target values and variability must be specified based on the functional requirements of the product as well as on historical statistical performance of the quality characteristics in the manufacturing process. In the optimal case, the target values and the performance of the individual characteristics should be identical. The failure to determine target values that are meaningful and which can easily be correlated with the performance of the quality characteristics will make interpretation of the charts difficult and reduce the amount of meaningful information which they provide. Therefore, if specification limits of a product are used initially to determine quality limits, they must be modified with consideration to the actual statistical values of the characteristics. The result should be quality limits which are related to the functioning of the product.

TOPICS FOR FURTHER INVESTIGATION

There are many further applications of this type of combined procedure which have not been approached in this paper. Some possible applications that warrant investigation include: analysis of the multivariate variance-covariance matrix; further investigation of possible multivariate CUSUM procedure; the use of principal components in multivariate quality control; determination of the ARL for the combined procedure; the use of different significance levels for the individual variables; economic design considerations of the combined procedure; modification of the combined procedure by utilizing FIR; modification of the multivariate Shewhart chart to make it more sensitive to smaller deviations or trends; and others.

The areas for further investigation discussed in the previous paragraph could greatly expand the applicability of the this approach to multivariate quality control. The procedures discussed in this paper are only the initial step in what promises to be a very interesting and potentially valuable approach to multivariate quality control using combined multivariate control charts.

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APPENDIX A

DATA

This data is provided by the Quality Engineering/R&D Task Force, headed by Mr. Tom Repoff at the Adolph Coors Company, Golden, Colorado. The first line on the data file provides the parameters in the file:

- 5 columns
- 480 lines of data
- Time
- Rivet diameter
- Rivet residual
- Score
- Pop

Lines where -1E+18 appear represent times where data is unavailable.

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APPENDIX B

REFERENCE TABLES

TABLE B.1. Cumulative Normal Distribution
(From Crow, 1960, p.229)

TABLE B.2. Percentiles of the Normal Distribution
(From Crow, 1960, p.230)

TABLE B.3. Upper Percentage Points of the χ^2 Distribution
(From Crow, 1960, pp.232-233)

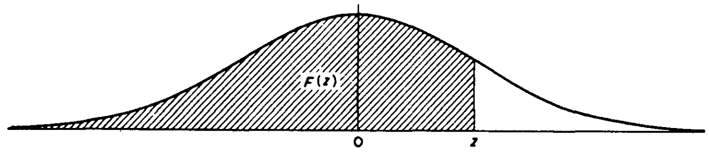

TABLE B.1. CUMULATIVE NORMAL DISTRIBUTION *

$$F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

* Use explained in Sec. 1.2.1. For more extensive tables, see National Bureau of Standards, *Tables of Normal Probability Functions*, Washington, U. S. Government Printing Office, 1953 (Applied Mathematics Series 23). Note that they show

$$\int_{-z}^{z} f(z)\,dz, \quad \text{not} \quad \int_{-\infty}^{z} f(z)\,dz$$

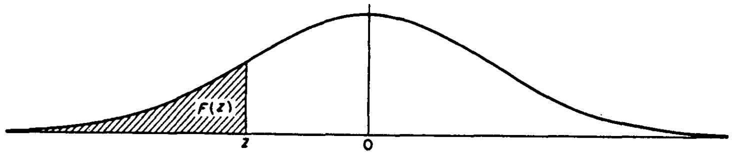

TABLE B.2. PERCENTILES OF THE NORMAL DISTRIBUTION*

$$F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$F(z)$	z	$F(z)$	z
.0001	−3.719	.500	.000
.0005	−3.291	.550	.126
.001	−3.090	.600	.253
.005	−2.576	.650	.385
.010	−2.326	.700	.524
.025	−1.960	.750	.674
.050	−1.645	.800	.842
.100	−1.282	.850	1.036
.150	−1.036	.900	1.282
.200	−.842	.950	1.645
.250	−.674	.975	1.960
.300	−.524	.990	2.326
.350	−.385	.995	2.576
.400	−.253	.999	3.090
.450	−.126	.9995	3.291
.500	.000	.9999	3.719

* Use explained in Sec. 1.2.1. For a normally distributed variable x, we have $x = \mu + z\sigma$, where μ = mean of x and σ = standard deviation of x. For more extensive tables, see R. A. Fisher and F. Yates, *Statistical Tables*, 4th rev. ed., Edinburgh, Oliver & Boyd, Ltd., 1953, pp. 39, 60–62.

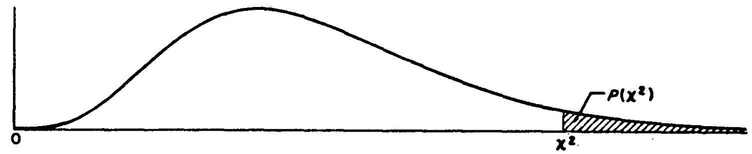

TABLE B. 3. UPPER PERCENTAGE POINTS OF THE χ^2 DISTRIBUTION *

$$P(\chi^2) = \int_{\chi^2}^{\infty} \frac{1}{\left(\frac{f-2}{2}\right)!\,2^{f/2}} (\chi^2)^{(f-2)/2} e^{-\chi^2/2} d(\chi^2)$$

f \ $P(\chi^2)$	.995	.990	.975	.950	.900	.750	.500	.250	.100	.050	.025	.010	.005
1	3927×10^{-8}	1571×10^{-7}	9821×10^{-7}	3932×10^{-6}	0.01579	0.1015	0.4549	1.323	2.706	3.841	5.024	6.635	7.879
2	0.01003	0.02010	0.05064	0.1026	.2107	.5754	1.386	2.773	4.605	5.991	7.378	9.210	10.60
3	.07172	.1148	.2158	.3518	.5844	1.213	2.366	4.108	6.251	7.815	9.348	11.34	12.84
4	.2070	.2971	.4844	.7107	1.064	1.923	3.357	5.385	7.779	9.488	11.14	13.28	14.86
5	.4117	.5543	.8312	1.145	1.610	2.675	4.351	6.626	9.236	11.07	12.83	15.09	16.75
6	.6757	.8721	1.237	1.635	2.204	3.455	5.348	7.841	10.64	12.59	14.45	16.81	18.55
7	.9893	1.239	1.690	2.167	2.833	4.255	6.346	9.037	12.02	14.07	16.01	18.48	20.28
8	1.344	1.646	2.180	2.733	3.490	5.071	7.344	10.22	13.36	15.51	17.53	20.09	21.96
9	1.735	2.088	2.700	3.325	4.168	5.899	8.343	11.39	14.68	16.92	19.02	21.67	23.59
10	2.156	2.558	3.247	3.940	4.865	6.737	9.342	12.55	15.99	18.31	20.48	23.21	25.19
11	2.603	3.053	3.816	4.575	5.578	7.584	10.34	13.70	17.28	19.68	21.92	24.72	26.76
12	3.074	3.571	4.404	5.226	6.304	8.438	11.34	14.85	18.55	21.03	23.34	26.22	28.30
13	3.565	4.107	5.009	5.892	7.042	9.299	12.34	15.98	19.81	22.36	24.74	27.69	29.82
14	4.075	4.660	5.629	6.571	7.790	10.17	13.34	17.12	21.06	23.68	26.12	29.14	31.32
15	4.601	5.229	6.262	7.261	8.547	11.04	14.34	18.25	22.31	25.00	27.49	30.58	32.80
16	5.142	5.812	6.908	7.962	9.312	11.91	15.34	19.37	23.54	26.30	28.85	32.00	34.27
17	5.697	6.408	7.564	8.672	10.09	12.79	16.34	20.49	24.77	27.59	30.19	33.41	35.72
18	6.265	7.015	8.231	9.390	10.86	13.68	17.34	21.60	25.99	28.87	31.53	34.81	37.16
19	6.844	7.633	8.907	10.12	11.65	14.56	18.34	22.72	27.20	30.14	32.85	36.19	38.58
20	7.434	8.260	9.591	10.85	12.44	15.45	19.34	23.83	28.41	31.41	34.17	37.57	40.00
21	8.034	8.897	10.28	11.59	13.24	16.34	20.34	24.93	29.62	32.67	35.48	38.93	41.40
22	8.643	9.542	10.98	12.34	14.04	17.24	21.34	26.04	30.81	33.92	36.78	40.29	42.80
23	9.260	10.20	11.69	13.09	14.85	18.14	22.34	27.14	32.01	35.17	38.08	41.64	44.18
24	9.886	10.86	12.40	13.85	15.66	19.04	23.34	28.24	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	19.94	24.34	29.34	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	20.84	25.34	30.43	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	21.75	26.34	31.53	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	22.66	27.34	32.62	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	23.57	28.34	33.71	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	24.48	29.34	34.80	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	33.66	39.34	45.62	51.80	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	42.94	49.33	56.33	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	52.29	59.33	66.98	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	61.70	69.33	77.58	85.53	90.53	95.02	100.42	104.22
80	51.17	53.54	57.15	60.39	64.28	71.14	79.33	88.13	96.58	101.88	106.63	112.33	116.32
90	59.20	61.75	65.65	69.13	73.29	80.62	89.33	98.65	107.56	113.14	118.14	124.12	128.30
100	67.33	70.06	74.22	77.93	82.36	90.13	99.33	109.14	118.50	124.34	129.56	135.81	140.17
z_P	−2.576	−2.326	−1.960	−1.645	−1.282	−0.6745	0.0000	+0.6745	+1.282	+1.645	+1.960	+2.326	+2.576

* Use explained in Sec. 3.2.2, 4.2, 4.3, 4.6, and 4.7.1. For a number of degrees of freedom $f > 100$, take

$$\chi_f^2(P) = f\left(1 - \frac{2}{9f} + z_P\sqrt{\frac{2}{9f}}\right)^3$$

where z_P is negative for $P < 0.5$. This table was adapted, with the permission of the author and the editor, from C. M. Thompson, BIOMETRIKA, Vol. 32 (1941–42), pp. 188–89.