MATHEMATICAL METHODS FOR THE DETERMINATION OF SURFACE TENSION EFFECTS ON WATER WAVES

by

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ABSTRACT

This thesis studies the effects of surface tension and viscosity on water waves using two methods. The first method is based on the work of M.S. Longuet-Higgins (15). It makes use of complex variables and a perturbation method of approximation. In the zero-order approximation the surface tension is neglected. The first-order approximation allows the surface tension to be introduced near the crest of the limiting gravity waves.

The second method which we study involves a fifth-order partial differential equation proposed by Warren Ferguson, Philip Saffman and Henry Yuen (8). This model equation was formulated based on the physics of the water waves and on the need to obtain an equation which would be amenable to a numerical treatment.

The two methods are introduced by examining the physics of the phenomenon. The mathematics involved in the models is examined and analyzed. The first method yields qualitative results, whereas the second method gives rise to both qualitative and quantitative results. In the study of the second method, an independent verification of the numerical scheme makes use of the DEVRK and FFTCC(ISML) subroutines. The results obtained verify

those of Ferguson, Saffman and Yuen.

The second method seems to hold the most promise for further development which would lead to a model whose results would be in close agreement with real water waves.

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To all my family, I am deeply indebted for their encouragement and support.

CHAPTER 1

INTRODUCTION

Many studies (24),(18),(16) have been undertaken on water waves neglecting surface tension and viscosity effects because the equations including these effects become nonlinear differential equations which are too difficult, if not impossible, to solve in closed form. In addition, computers necessary to perform numerical methods of solution for these equations were not available when these studies were undertaken. We shall try to include the effects of these two terms, surface tension and viscosity, in our study.

Surface tension effects tend to generate capillary waves near the crests of gravity waves. The capillary waves which are large enough to be seen are nonlinear, although the viscosity will damp the short capillary waves. Thus, an indepth study of water waves should include both viscosity and surface tension. In including these two terms in this study, we shall apply the methods available for solving nonlinear problems. Our aim is to give a better approximation to the solution and to examine in detail the mathematics involved in modeling this physical phenomenon.

Examples of Surface Tension

Surface tension is defined as a force with the dimensions of dynes per centimeter. It is a measure of the work required to increase the area of a surface by one square centimeter. Surface tension is a measurable quantity existing in all liquid surfaces, and it occurs as a result of the imbalance of cohesive forces existing between the molecules of the liquid. The following examples will illustrate various surface tension phenomena.

1) A very small quantity of liquid not subject to outside disturbance will assume a spherical shape. Falling raindrops tend to be spherical in shape. A drop tends to form itself into a shape with minimum surface area. For a given volume, a sphere is a geometric shape which has the smallest surface area.

2) A small paintbrush dipped into water exhibits a phenomenon due to surface tension. While immersed in water, the hairs of the brush are seen to stand apart. When the brush is raised out of the water, the hairs cling to-

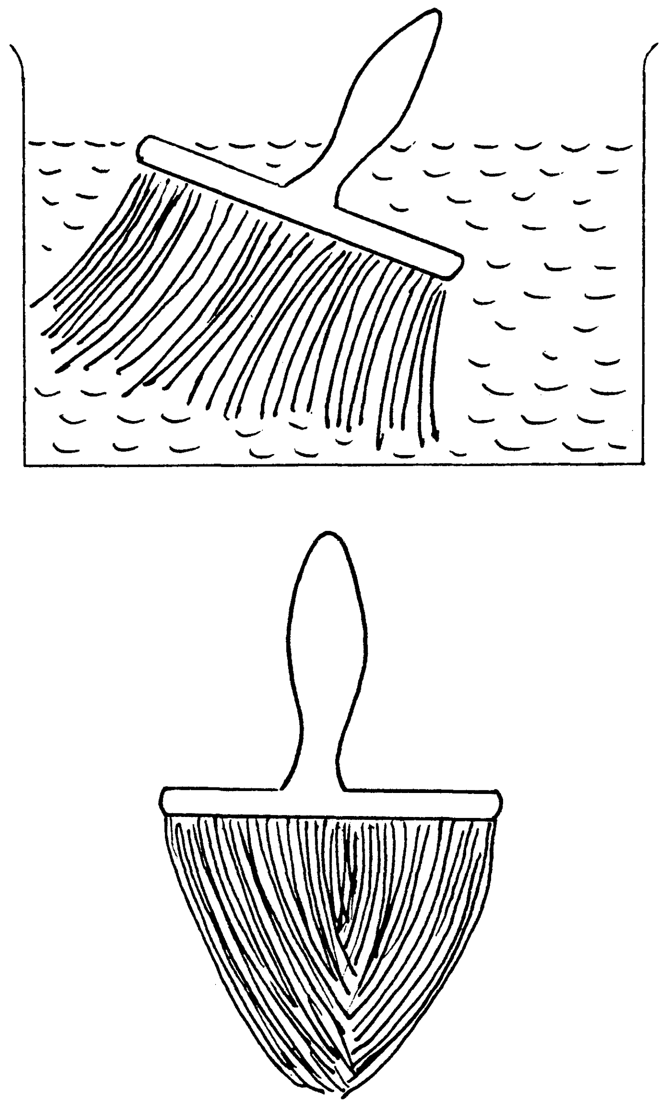

Figure 1. Small paintbrush dipped into water.

gether, due to the tendency of the water to shrink (Figure 1).

3) If a wire ring is dipped into a soap solution and then removed, a film is produced. Consider the situation as illustrated in Figure 2. Let AB and CD be two threads previously tied to the ring. These threads will be slack and will not interfere with the film, but will merely slide about in it. On piercing the film between CD and the ring, this part of the film disappears, but the rest remains; and it will be seen that the thread CD is now drawn tightly into a circular form. This clearly indicates that the film tends to shrink; that is, a tension exists in it. The thread AB remains loose, because this tension exists equally on both sides of it.

4) If two clean pieces of plate glass are put face to face (Figure 3), there is no difficulty in separating them. But if there is a small drop of water between them, which is squeezed into a thin layer, it requires

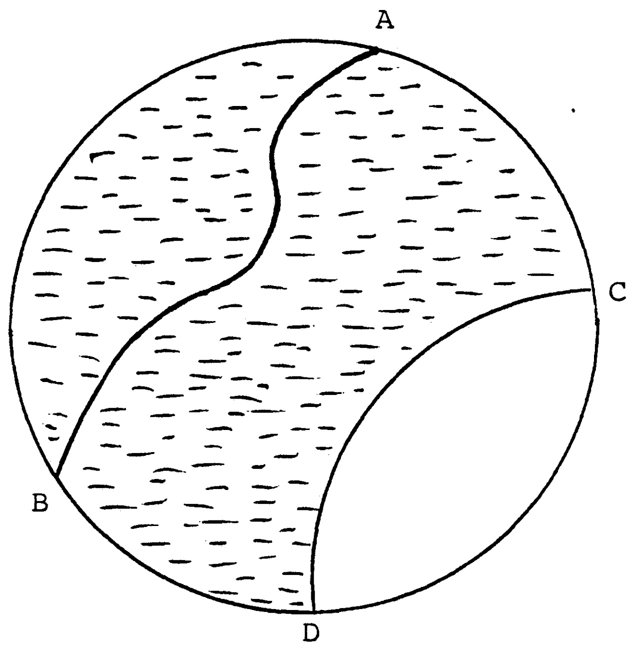

Figure 2. Wire ring dipped into a soap solution.

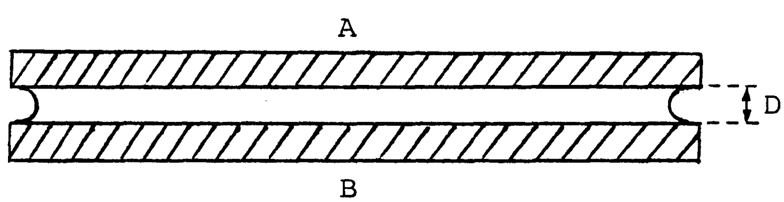

Figure 3. Drops of water between two flat sheets of glass.

a considerable force to pull the plates apart. The reason is that the surface tension at the outer edge of the film of water causes a difference in pressure between the water and the outer air. This difference is

$$p - p_0 = \frac{2T}{d}, \qquad (1.1)$$

where d is the distance between the plates, T is the surface tension, and p and p_0 are the pressures in the two media, respectively.

A Brief History and Survey of Early Studies of Surface Tension

Leonardo da Vinci (1452-1519) is believed to have been the first to observe and record the rise of a liquid in a tube of small bore (Figure 4A). This phenomenon became known as capillarity, because the tubes that were used possessed a bore "as fine as" a hair (from the Latin word "capillus", meaning hair). The phenomenon of capillarity is one of the most important and striking effects of surface tension. In fact, surface tension is sometimes called capillarity.

Sir Isaac Newton (1642-1727) referred to the forces of cohesion and adhesion that produce the rise of a liquid in a capillary tube. He recognized that the forces were intermolecular in origin and that the mutual attraction gave rise to a pressure inside the liquid. At the end of the 17th century, Hawksbee performed the first "accurate" observations of the ascent of liquids in capillary tubes and between glass plates. He observed that attraction forces arose from the matter near the inside surface of his glass tubes. Some 50 years later, Von Segner brought together Newton's concept of cohesive forces and Hawksbee's idea of surface matter giving rise to them, by proposing the first theory of capillarity. It was that cohesive forces created a pressure which was resisted by a uniform tension in the surface. This tension was called surface tension and was thought to explain capillarity. Viscompte Pierre S. de Laplace (1749-1827) gave the first theoretical treatment of the phenomena which occur near the surface separating two continuous media. He surmised that if the surface of separation is curved, then the pressure near it in the two media are different. He said the pressure across a curved surface was a consequence of two radii of curvature, and not just one radius as suggested by Von Segner. He developed an equation connecting curvature with the change in pressure, Δp, and surface tension, T. This equation made it possible to relate experimentally the height of capillary rise to the radius of the capillary tube. It can be shown that the difference in pressure at the interface is given by

$$\Delta p = T(\frac{1}{R_1} + \frac{1}{R_2}) \tag{1.2}$$

where

$$\Delta p = p_1 - p_2 . \tag{1.3}$$

Equation (1.2) is Laplace's formula, where R_1 and R_2 are the principal radii of curvature at the interface, and in equation (1.3) p_1 and p_2 are the pressures in medium one and medium two, respectively. The quantity Δp is called the surface pressure, that is, the pressure difference between the two media.

If the surface is spherical of radius R, as is the case for a column of fluid in a capillary tube (Figure 4A), then

$$\Delta p = \frac{2T}{R} . \tag{1.4}$$

For equilibrium (i.e., the fluid is at rest in the tube), we must have

$$\Delta p = g\rho h , \tag{1.5}$$

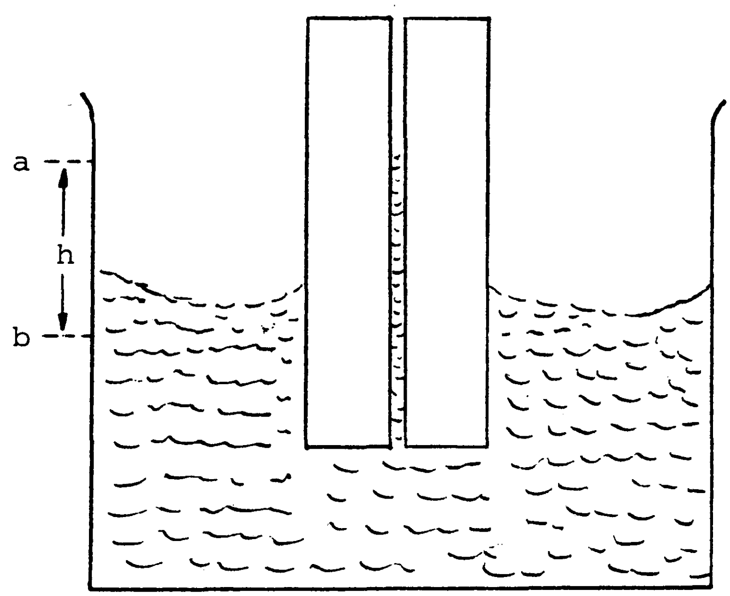

Figure 4A. Rise of liquid in a capillary tube.

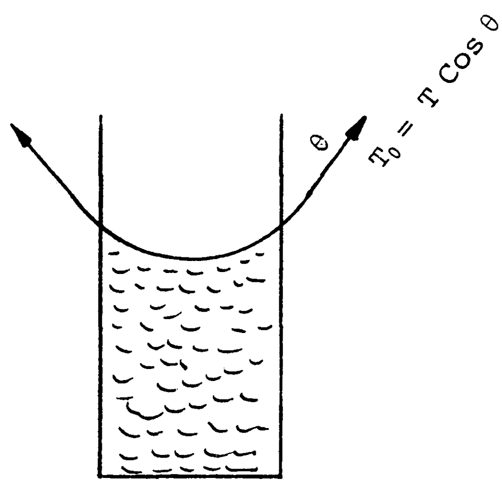

Figure 4B. The curvature surface makes an angle θ with the tube wall.

where g is the acceleration due to gravity in centimeters per second squared, ρ is the density of the fluid, and h is the height of the column of the fluid. Thus, we have from equations (1.4) and (1.5) that the surface tension is

$$T_0 = \frac{h}{2} \rho g R \qquad (1.6)$$

for a liquid in a capillary tube with radius of curvature "R". If the surface makes an angle θ with the tube wall (Figure 4B), it can be shown that the surface tension T is

$$T = \frac{hg\rho R}{2 \cos\theta} . \qquad (1.7)$$

<u>Gravity Waves and Capillary Waves</u>

Our study will concentrate on gravity and capillary waves; also, we shall include the surface tension and viscosity effects. Capillary waves are generated by surface tension on gravity waves. The free surface of a liquid in equilibrium in a gravitational field is a plane. If, under the action of some external perturbation, the surface is moved from its equilibrium position at some point, motion will occur in the liquid. This motion will be propagated over the whole surface in the form of waves, which are called gravity waves since they are due to the action of the gravitational field. Gravity waves appear mainly on the surface of the liquid; they affect the interior also, but this effect diminishes rapidly as depth increases. If gravity waves are of length small compared with depth of the fluid, they are called small waves. If the waves have length large compared with depth, they are called long waves.

Fluid surfaces tend to assume an equilibrium shape, both under the action of the force of gravity and the surface tension forces. We did not take surface forces into account for the gravity wave. However, these forces have an important effect on gravity waves of small wave length. In fact, for the case of short wave length, surface tension is the more important force and the effect of gravity in this case may be neglected, and the waves are called capillary waves or ripples.

The dispersion relation derived from the wave equation can be used to classify the gravity and capillary waves. First, we shall explain the meaning of a dispersion relation. For linear problems, dispersive waves are recognized by the existence of elementary solutions in the form of sinusoidal wavetrains; that is, the solution ϕ will have the form

$$\phi(x,t) = A\, e^{iKx-i\omega t} , \tag{1.8}$$

where x is the horizontal axis, y is the vertical axis, K is the wave number, ω is the frequency, and A is the amplitude. In the elementary solution, K, ω, and A are constants. Since the equations are linear, A factors out and is arbitrary. Thus, in order to satisfy the equations, K and ω must be related by an equation

$$G(\omega,K) = 0. \tag{1.9}$$

The function $G(\omega,K)$ is determined by the particular equations of the problem. For example, if $\phi(x,t)$ satisfies the beam equation

$$\phi_{tt} + \gamma^2\phi_{xxxx} = 0 , \tag{1.10}$$

we require

$$\omega^2 = \gamma^2 K^2.$$

The relation between ω and K is called the dispersion relation. For the model equation of deep water waves, we have the dispersion relation

$$\omega = \frac{\alpha K + \sigma K^5}{1 + K^2} , \tag{1.11}$$

which we shall derive in equation (3.5) (page 40).
The phase velocity C is given by $\frac{\omega}{K}$, and the group velocity C_g is given by $\frac{d\omega}{dK}$. In the gravity wave limit $(K \to 0)$,

the phase velocity is greater than the group velocity $C > C_g$, whereas in the capillary wave limit $(K \to \infty)$, $C < C_g$.

Much work has been done to study gravity waves and capillary waves. Stokes (24) made a study of gravity waves in deep water when the wave was near breaking. Michel (18) calculated the numerical value of the steepness of the highest waves in water. Recently (1975), Longuet-Higgins and Cokelet (16) made a major advance by constructing a numerical scheme to calculate the unsteady motion of large amplitude gravity waves with no surface tension. Effects of surface tension have been considered by Cox (4), Munk (19), and Longuet-Higgins (15), and some others. They note that the sharp corner at the crest induces locally a very large surface tension effect in the form of a local pressure source. As the gravity wave propagates, the crest acts like a traveling pressure source which generates capillary waves. We will examine and compare two papers concerning the effects of surface tension and viscosity on the generation of capillary waves. Our aim is to give a brief description of the physics involved and to examine some of the mathematical techniques used. Wherever possible, we will compare the two different approaches.

CHAPTER 2

THE GENERATION OF CAPILLARY WAVES BY STEEP GRAVITY WAVES

A Method of Approximation

In 1962, M.S. Longuet-Higgins (15) published a paper, the purpose of which was to study a mechanism for the generation of capillary waves by steep gravity waves. Gravity waves in deep water are considered at the point of breaking. These waves, where gravity alone is taken into account, develop sharp crests, and for the highest wave the curvature of the surface becomes quite large; in fact, it is infinite at a sharp crest. However, where the curvature is great, the surface tension becomes locally important. Its effect is to produce an increase in normal stress near the crest. This traveling stress is responsible for generating the capillary waves. Longuet-Higgins in this study calculates the amplitude and wave length of the capillary waves based on these hypotheses. His treatment of the capillarity differs from the analysis done by Lamb (11), in that surface tension terms are treated on a similar footing to the gravitational terms, and expansion of the solution is made in powers of a small parameter corresponding roughly to the maximum steepness of the wave.

In the first approximation, the effect of capillarity is to modify the wave length of the waves. Higher approximations in powers of steepness have been investigated by Wilton (1915). This treatment has a limitation which is typical for many nonlinear problems. When the amplitude of the perturbation parameter used in the expansion becomes unsuitably large, the curvature of the surface, and hence the surface tension, becomes very unequally distributed over the surface. Thus, instead of affecting the wave uniformly, the surface tension produces a local disturbance near the crests. Such a local effect is not well represented by higher order Fourier expansions. Moreover, viscosity acts to damp the short waves produced by the local disturbance.

The work by M.S. Longuet-Higgins introduces surface tension terms in an altogether different way. In the first approximation the waves are treated as pure gravity waves, of nearly the maximum steepness, and then the surface tension is introduced as a perturbation on the basic flow. The perturbation is still treated as small, and this limits the validity of this work to a certain range of wave lengths and steepnesses, roughly those over which the phenomenon is observed.

The amplitude of the capillary waves calculated in this way is compared with Cox's (4) experimental observations and is found to be in agreement. The effective transfer of energy from the gravity waves to the capillary waves results in appreciable damping of the gravity waves, which may exceed considerably the direct damping of the short waves by viscosity.

The problem is formulated in the following manner. Suppose that two-dimensional, irrotational waves in a perfect fluid are traveling with velocity $-C$, in the direction of x, increasing negatively. Let the motion be reduced to a steady state by superposing on it a uniform positive velocity C (Figure 5). Define a velocity potential $\phi(x,y)$ and a stream function $\psi(x,y)$ by the relations

$$\begin{aligned} d\phi &= u dx + v dy \\ d\psi &= -v dx + u dy \ , \end{aligned} \tag{2.1}$$

where u and v are the x and y components of velocity (the y-axis is taken to be vertically upward (Figure 5)). If we write

$$\begin{aligned} Z &= x + iy \\ \chi &= \phi + i\psi \ , \end{aligned} \tag{2.2}$$

the functions $\phi = \phi(x,y)$ and $\psi = \psi(x,y)$ are analytic and

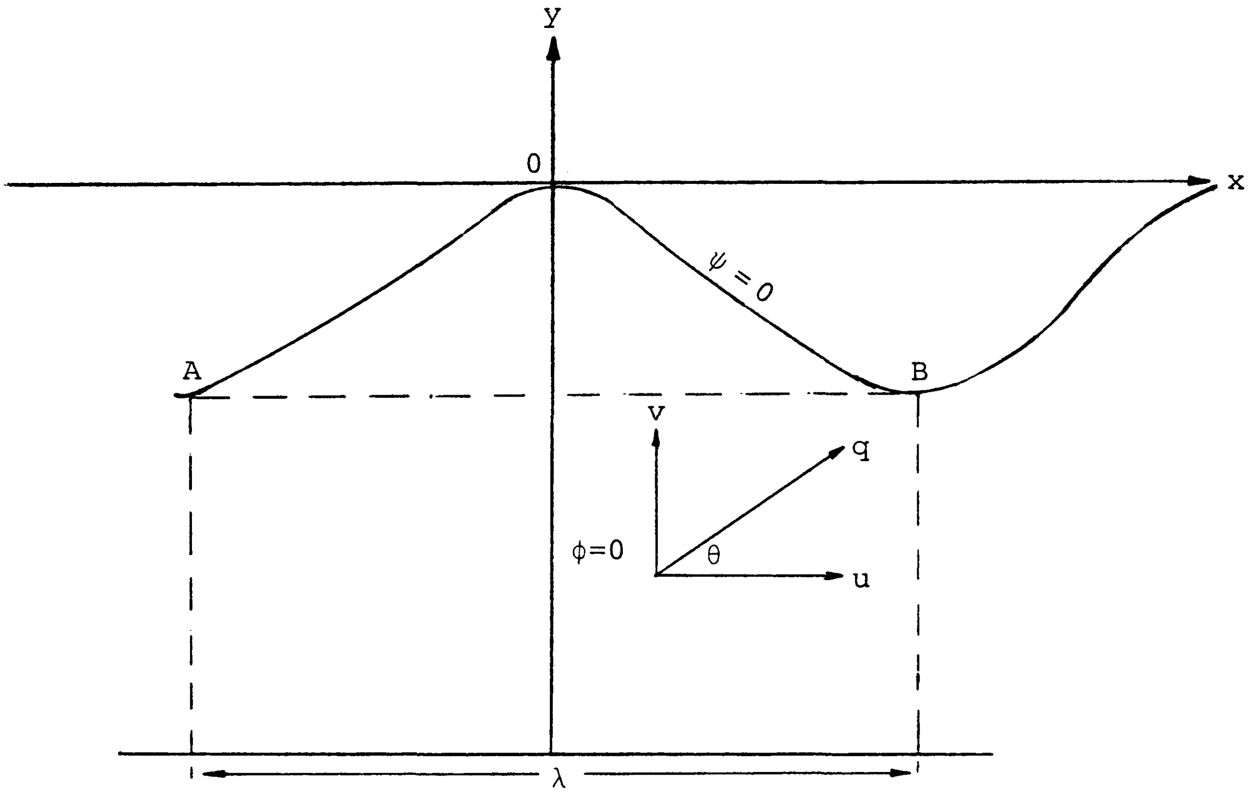

Figure 5. Fluid motion relative to xy-plane.

must satisfy the Cauchy-Riemann equations. Then the following relations hold:

$$u = \frac{\partial \phi}{\partial x} ; \qquad v = \frac{\partial \phi}{\partial y}$$

$$-v = \frac{\partial \psi}{\partial x} ; \qquad u = \frac{\partial \psi}{\partial y} \tag{2.3}$$

$$u - iv = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} - i \frac{\partial \phi}{\partial y} .$$

It can also be shown that

$$\frac{d\chi}{dZ} = u - iv = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} . \tag{2.4}$$

Let q and θ denote the magnitude and direction of the velocity,

$$(u,v) = (q \cos\theta, q \sin\theta), \tag{2.5}$$

and define τ by

$$q = C e^{\tau} , \tag{2.6}$$

where C is the propagation velocity of the wave. So, we have

$$u - iv = q e^{-i\theta} = C e^{\tau} e^{-i\theta} = C e^{\xi} , \tag{2.7}$$

where $\xi = \tau - i\theta$. ξ is a regular function of (u,v) and hence of χ. We take ϕ and ψ as coordinates, and attempt to find Z and ξ in terms of ϕ and ψ. The curvature K along any streamline ψ = Const is given by

$$K = \frac{\partial\theta}{\partial S} = \frac{\partial\theta}{\partial\phi} \cdot \frac{\partial\phi}{\partial S} = q\,\frac{\partial\theta}{\partial\phi} \qquad (2.8)$$

where ∂S is an elementary arc of the stream line. We note that

$$\frac{\partial\theta}{\partial\phi} = \frac{\partial\tau}{\partial\psi} \quad \text{and} \quad \frac{\partial\theta}{\partial\psi} = -\frac{\partial\tau}{\partial\phi}\ , \qquad (2.9)$$

and from (2.9), (2.8) K becomes

$$K = C\,e^{\tau}\,\frac{\partial\tau}{\partial\psi} = C\,\frac{\partial}{\partial\psi}\,(e^{\tau}) = \frac{\partial q}{\partial\psi}\ . \qquad (2.10)$$

The boundary conditions are determined in the following manner. It is assumed that the depth of the water is effectively infinite so that as $y \to -\infty$, $(u - iv) \to C$ and $\xi \to 0$. The free surface, being a streamline, may be chosen as $\psi = 0$, and we also take $\phi = 0$ at the crest of the wave (Figure 5). In a steady flow the surface is a stream line, say $\psi = 0$, and Bernoulli's equation holds (Crapper, (5)), so we have

$$\frac{P}{\rho} + \frac{q^2}{2} + gy = \text{Const} \qquad \text{when } \psi = 0, \qquad (2.11)$$

where P is the pressure difference across the surface created by surface tension which is given by equation (1.4). At the free surface the pressure P is given by

$$P = \text{Const} - TK,$$

where T is the surface tension and K is the curvature. Thus, writing $\frac{T}{\rho} = T'$ and substituting in (2.11), using (2.10), we have

$$\frac{q^2}{2} + gy - T' \frac{\partial q}{\partial \psi} = \text{Const}, \qquad \psi = 0; \tag{2.12}$$

that is,

$$\frac{1}{2} e^{2\tau} + \frac{g}{C^2} y - \frac{T'}{C} \frac{\partial}{\partial \psi} (e^{\tau}) = \text{Const}. \tag{2.13}$$

Differentiating the left-hand side with respect to ϕ yields

$$\frac{\partial}{\partial \phi} [\frac{1}{2} e^{2\tau} - \frac{T'}{C} \frac{\partial}{\partial \psi} (e^{\tau})] + R\{\frac{g}{iC^3} e^{-(\tau - i\theta)}\} = 0, \tag{2.14}$$

since we can show

$$\frac{\partial y}{\partial \phi} = \frac{1}{q^2} \frac{\partial \phi}{y} = \frac{1}{q} \sin\theta = R\{\frac{1}{iC^3} e^{-(\tau - i\theta)}\}, \tag{2.15}$$

where R means the real part.

For studying this problem with these boundary conditions, we shall give the approximation methods used in (15). In the zero-order approximation of equations (2.12) and (2.14), the surface tension is neglected entirely and one assumes that the flow corresponds to a pure gravity wave of finite amplitude. Let all quantities referring to this basic flow be denoted by a suffix (0). Thus, the boundary condition (2.12) and the differential boundary condition (2.14) become

$$\frac{1}{2} q_0^2 + gy_0 = \text{Const}, \qquad \psi = 0 \tag{2.16}$$

and

$$\frac{\partial}{\partial\phi}\left(\frac{1}{2} e^{2\tau_0}\right) + \mathcal{R}\left\{\frac{g}{iC^3} e^{-(\tau - i\theta)}\right\} = 0, \qquad \psi = 0. \tag{2.17}$$

Dividing by $e^{2\tau_0}$, we have

$$\frac{\partial\tau_0}{\partial\phi} + \frac{g}{C^3} e^{-3\tau_0} \sin\theta_0 = 0, \qquad \psi = 0, \tag{2.18}$$

which is the surface condition obtained by Levi-Civita (12) for a pure gravity wave. Also, we have

$$\tau_0 - i\theta_0 \to 0 \quad \text{as} \quad \psi \to -\infty.$$

In the next stage Longuet-Higgins takes into account the perturbation due to the surface tension by writing

$$\xi = \xi_0 + \xi_1, \qquad q = q_0 + q_1, \ . \ . \ ., \text{ etc.}$$

in all equations (2.12) through (2.14) where they occur. Note that ξ_0, q_0, y_0, . . . etc. represent the basic gravity wave and ξ_1, q_1, y_1, . . . etc. represent the perturbation due to surface tension. Squares and higher powers of perturbation terms will be neglected. The boundary condition (2.12) becomes

$$(\frac{1}{2}\, q_0^2 + q_0 q_1) + g(y_0 + y_1) - T' \left(\frac{\partial q_0}{\partial \psi} + \frac{\partial q_1}{\partial \psi}\right) = \text{Const}, \tag{2.19}$$

and subtracting (2.16) from (2.19) yields

$$q_0 q_1 + g y_1 - T' \frac{\partial q_1}{\partial \psi} = T' \frac{\partial q_0}{\partial \psi} + \text{Const}. \tag{2.20}$$

It will be noticed that the term $T' \frac{\partial q_0}{\partial \psi}$, which was neglected in the zero-order approximation, becomes a forcing function for the perturbation. Also, from the differentiated form of the boundary condition, equation (2.14), we have

$$\frac{\partial}{\partial \phi}\left[e^{2\tau_0}\tau_1 - \frac{T'}{C}\frac{\partial}{\partial \psi}(e^{\tau_0}\tau_1)\right] - \mathcal{R}\left\{\frac{g}{iC^3}\, e^{-(\tau_0 - i\theta_0)}(\tau_1 - i\theta_1)\right\}$$

$$= \frac{T'}{C}\frac{\partial^2}{\partial \phi}\frac{(e^{\tau_0})}{\partial \psi}, \qquad \psi = 0. \tag{2.21}$$

Also,

$$\tau_1 - i\theta_1 \to 0 \quad \text{as} \quad \psi \to -\infty.$$

Since the perturbations are taken to be small relative to the basic flow, it is necessary that the term $T' \frac{\partial q_0}{\partial \psi}$ be small compared to other terms in the boundary condition (2.16). So, in particular, we must have

$$T' \frac{\partial q_0}{\partial \psi} << \frac{1}{2}\, q_0^2\, ,$$

that is,

$$\frac{T'}{C}\,\frac{\partial \tau_0}{\partial \psi} \ll \frac{1}{2}\, e^{\tau_0}. \tag{2.23}$$

A Zero-Order Approximation

Different methods of approximation have been used for solving the equation (2.18). Levi-Civita (1925) approximates (2.18) by using the linear boundary condition

$$\frac{\partial \theta}{\partial \psi} = -\frac{g}{C^3}\,\theta\ , \tag{2.24}$$

which is obtained from (2.18) by assuming τ and θ are small everywhere. This approximation is suitable only for waves of small steepness. The approximation by T.V. Davies (6) for waves of finite steepness is more suitable. We shall use this approximation for solving (2.18). This approximation replaces the term $\sin\theta_0$ by $\frac{L}{3}\sin\theta_0$, where L is a constant. Then (2.18) becomes

$$R\{\frac{\partial \xi_0}{\partial \chi} + \frac{Lg}{3iC^3}\, e^{-3\xi_0}\} = 0, \qquad \psi = 0, \tag{2.25}$$

of which the solution symmetrical about $\phi = 0$ is

$$e^{3\xi_0} = 1 - A\, e^{-im\chi}, \tag{2.26}$$

where A is a real constant lying between zero and one, and

$$m = \frac{Lg}{C^3} . \tag{2.27}$$

Thus,

$$\frac{d\chi_0}{dZ} = u_0 - iv_0 = C(1 - A\, e^{im\chi})^{1/3} . \tag{2.28}$$

This approximate expression is also the first term in a series solution found by Havelock (1918). In the limiting case when A = 1 and in the neighborhood of the wave crest, where $m\chi$ is small, equation (2.28) becomes

$$\frac{d\chi_0}{dZ} \cong C(im\chi)^{1/3} \tag{2.29}$$

so that

$$CZ \cong \int_0^{\chi} \frac{d\chi}{(im\chi)^{1/3}} = \frac{3}{2}\,\frac{\chi^{2/3}}{(im)^{1/3}} ,$$

and

$$\chi = (im)^{1/2}\,[\tfrac{2}{3}\,CZ]^{3/2} . \tag{2.30}$$

This is equivalent to Stoke's 120° angle solution, satisfying the exact boundary condition

$$\frac{1}{2}\,\left|\frac{d\chi}{dZ}\right|^2 + gy = 0, \qquad (\arg Z = \mp \frac{\pi}{2} \pm \frac{\pi}{3}),$$

provided that we take

$$m = \frac{3}{2}\,\frac{g}{C^3} . \tag{2.31}$$

To find the constant L, one equates equations (2.27) and (2.31) and finds $L = \frac{3}{2}$.

For the waves which are just lower than the limiting wave, θ_0 generally lies between $\pm \frac{\pi}{6}$, so the appropriate value of L should lie somewhere between 1 and $\frac{3}{2}$. The value 1 is appropriate for low waves when θ_0 is uniformly small. On the other hand, since we are dealing with steep waves -- height from trough to crest divided by wave length -- where $\theta_0 = \pm \frac{\pi}{6}$ at the two extremes of the flow, it seems preferable to take $L = \frac{3}{2}$. The relative error introduced by this assumption is expected to be of order 20%.

Unfortunately, the series representation by T.V. Davies (6) and also the Havelock series (9) do not give an adequate representation of the flow in the neighborhood of the crest of a steep wave. Thus, we shall now examine a different method carried out by M.S. Longuet-Higgins (15).

If we let

$$A = 1 - \delta \,, \tag{2.32}$$

where δ is a very small positive quantity, then (2.26) becomes

$$e^{\tau_0 - i\theta_0} \cong [1 - (1-\delta)\, e^{-im\chi}]^{1/3} \,. \tag{2.33}$$

In the neighborhood of the crest, where $m\chi$ is comparable to δ, we can expand $e^{im\chi}$ in a Taylor series and use only the first-order term so that

$$e^{\tau_0 - i\theta_0} \cong [\delta + im\chi]^{1/3} , \tag{2.34}$$

where $\chi = \phi + i\psi$.

At the free surface $\psi = 0$, we have

$$e^{\tau_0} \cong [\delta^2 + m^2\phi^2]^{1/3} \tag{2.35}$$

and

$$\theta_0 = -\frac{1}{3} \tan^{-1} (\frac{m\phi}{\delta}) . \tag{2.35}$$

Also, from the relation $\frac{\partial\theta}{\partial\phi} = \frac{\partial\tau}{\partial\psi}$ we obtain

$$\frac{\partial\tau_0}{\partial\psi} \cong \frac{m\delta}{3(\delta^2 + m^2\phi^2)}$$

$$K_0 = C\, e^{\tau_0} \frac{\partial\tau_0}{\partial\psi} \cong \frac{Cm\delta}{3(\delta^2 + m^2\phi^2)^{5/6}} . \tag{2.36}$$

At the crest of the wave, where $\phi = 0$, we have

$$q_0 = C\, e^{\tau_0} \cong C\, \delta^{1/3}$$

$$K_0 \cong -\frac{Cm}{3\delta^{2/3}} = -\frac{g}{2C^2\delta^{2/3}} . \tag{2.37}$$

Hence, the vertical acceleration of the fluid at the crest is given by

$$K_0^2 q_0^2 \cong \frac{g}{2} , \tag{2.38}$$

which is the limiting value for the Stokes 120° angle solution (see Appendix A). When $m\chi$ is no longer small, that is, at distances from the crest which are comparable to a wave length, we use (2.28) with $\psi = 0$ to obtain

$$e^{\tau_0} \cong (1 - 2A \cos m\phi + A^2)^{1/6} \tag{2.39}$$

and

$$\theta_0 \cong -\frac{1}{3} \tan^{-1} \left(\frac{A \sin m\phi}{1 - A \cos m\phi}\right). \tag{2.40}$$

In particular, if $A = 1 - \delta$, we have

$$e^{\tau_0} \cong [\delta^2 + 4(1-\delta) \sin^2 \frac{m\phi}{2}]^{1/6}. \tag{2.41}$$

The Perturbation Solution

At the stage near the crest, as we said before, we shall introduce the surface tension as a small perturbation where ξ_1, q_1, . . ., etc. represent the perturbation, which is the method used by Longuet-Higgins for solving the perturbation equations (2.20) and (2.21). If the gravitational terms in these equations are neglected, equation (2.20) becomes

$$q_0 q_1 - T' \frac{\partial q_1}{\partial \psi} = T' \frac{\partial q_0}{\partial \psi} \tag{2.42}$$

(a constant term on the right-hand side is omitted since that constant does not affect the motion). Since $q_0 = C\, e^{\tau_0}$ and $q_1 = C\, e^{\tau_0}\tau_1$, (2.42) becomes

$$[e^{\tau_0} - \frac{T'}{C}\frac{\partial\tau_0}{\partial\psi}]\,\tau_1 - \frac{T'}{C}\frac{\partial\tau_1}{\partial\psi} = \frac{T'}{C}\frac{\partial\tau_0}{\partial\psi}\ . \qquad (2.43)$$

If we let

$$P(\phi) = e^{\tau_0} - \frac{T'}{C}\frac{\partial\tau_0}{\partial\psi} \quad \text{and} \quad Q(\phi) = \frac{\partial\tau_0}{\partial\psi},$$

then equation (2.43) may be written as

$$P(\phi)\,\tau_1 - \frac{T'}{C}\frac{\partial\tau_1}{\partial\psi} = \frac{T'}{C}\,Q(\phi). \qquad (2.44)$$

The function τ_1 must also satisfy Laplace's equation and the condition

$$\tau_1 \to 0 \quad \text{as} \quad \psi \to -\infty\ . \qquad (2.45)$$

Now consider the conformal transformation

$$\alpha + i\beta = \int_0^{\alpha+i\beta} P(\chi)\; d\chi, \qquad (2.46)$$

where $P(\chi)$ is a function of the complex variable $\chi = \phi + i\psi$, which is equal to $P(\phi)$ on the real axis ($\psi = 0$). Then on $\psi = 0$ we have $\beta = 0$ for all ϕ, and therefore $\frac{\partial\beta}{\partial\phi} = 0$. Hence, from the previous relation of analytic functions we have

$$\frac{\partial\tau_1}{\partial\psi} = P(\phi)\,\frac{\partial\tau_1}{\partial\beta}, \qquad \psi = 0. \qquad (2.47)$$

The condition (2.44) reduces to

$$\frac{C}{T'}\,\tau_1 - \frac{\partial\tau_1}{\partial\beta} = R(\alpha), \tag{2.48}$$

where $R(\alpha) = Q(\phi)/P(\phi)$. Also, τ_1 must satisfy Laplace's equation in the coordinates α, β, and

$$\tau_1 \to 0 \quad \text{as} \quad \beta \to -\infty. \tag{2.49}$$

Note that $R(\alpha)$ is an even function of ϕ and hence of α. Now, define

$$r(t) = \int_{-\infty}^{\infty} R(\alpha)\, e^{i\alpha t}\, d\alpha\ . \tag{2.50}$$

Then $r(t)$ is a real, even function of t, and by inverse transformation

$$R(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r(t)\, e^{-i\alpha t}\, dt. \tag{2.51}$$

Substituting this expression in (2.48), we get

$$\tau_1 = -\mathcal{R}\{\frac{1}{\pi}\int_{0}^{\infty} \frac{r(t)\ e^{-it(\alpha+i\beta)}}{t - \frac{C}{T'}}\, dt\}\ . \tag{2.52}$$

Since $\tau_1 - i\theta_1$ is a regular function of $\alpha+i\beta$, we have

$$\tau_1 - i\theta_1 = -\frac{1}{\pi}\int_{0}^{\infty} \frac{r(t)\ e^{-i(\alpha+i\beta)t}}{t - \frac{C}{T'}}\, dt, \tag{2.53}$$

where $r(t)$ is given by (2.50). In (2.52) the integration is taken to pass above the singularity $t = \frac{C}{T'}$. We are interested especially in the behavior of this solution for large values of $|\xi|$ and $|\alpha|$. If the function $r(t)$ is suitably bounded at infinity, the chief contribution to the integrand will arise from the residue at $t = \frac{C}{T'}$, and will be given by

$$\tau_1 - i\theta_1 \simeq \begin{cases} 2ir(\frac{C}{T'})\ e^{-i(\frac{C}{T'})(\alpha+i\beta)}, & \alpha < 0 \\ 0, & \alpha > 0 \end{cases} . \tag{2.54}$$

In particular, when $\alpha < 0$ we have

$$\tau_1 - i\theta_1 \simeq -ib\ e^{-i(\frac{C}{T'})(\alpha+i\beta)}, \tag{2.55}$$

where

$$\begin{aligned} b &= -2r(\tfrac{C}{T'}) \\ &= -4\int_0^{\infty} \frac{Q(\phi)}{P(\phi)} \cos(\tfrac{C}{T'})\ \frac{\partial\alpha}{\partial\phi}\ d\phi. \end{aligned} \tag{2.56}$$

Since $\frac{\partial\alpha}{\partial\phi} = P(\phi)$ when $\beta = 0$, and $Q(\phi) = \frac{\partial\tau_0}{\partial\psi}$, equation (2.56) becomes

$$b = -4\int_0^{\phi} \frac{\partial\tau_0}{\partial\psi} \cos(\tfrac{C}{T'})\ d\phi\ , \tag{2.57}$$

where

$$\alpha = \int_0^{\phi} P(\phi)\ d\phi, \qquad \psi = 0. \tag{2.58}$$

In the expression $P(\phi) = e^{\tau_0} - \frac{T'}{C}\frac{\partial\tau_0}{\partial\psi}$, the ratio of the second term to the first is small, by (2.23), so that to this approximation we have

$$P(\phi) \cong e^{\tau_0} \tag{2.59}$$

and hence

$$\alpha \cong \int_0^{\phi} e^{\tau_0} d\phi.$$

The solution (2.54) represents a wave upstream of the crest, that is, on the forward face of the gravity wave. The phase velocity with respect to the surrounding medium is equal to $-q_0$. Now, from equation (2.8) $K = \frac{\partial\theta}{\partial S} = q\frac{\partial\theta}{\partial\phi}$, then the capillary wave number K_C is given by

$$K_C = \frac{\partial}{\partial S}\left(\frac{\alpha C}{T'}\right) \cong q_0 \frac{\partial}{\partial\phi}\left(\frac{\alpha C}{T'}\right). \tag{2.60}$$

But by equation (2.58),

$$\frac{\partial\alpha}{\partial\phi} = P(\phi) \cong e^{\tau_0}.$$

Hence,

$$q_0 = (T'K_C)^{1/2}. \tag{2.61}$$

But $\pm(T'K_C)^{1/2}$ is the classical expression for the velocity of free capillary waves of small amplitude (Lamb, 1932). So the waves at some distance from the crest are free capillary waves for this approximation. The variation of the wave amplitude may be derived by considering its relation to the local energy E. If a_c denotes the capillary wave amplitude and K_c the wave number of a capillary wave on a locally uniform stream, the amplitude of the surface slope θ' is

$$|\theta'| = a_c K_c. \tag{2.62}$$

The total energy density (Lamb, 1932) is given by

$$E = \frac{T}{2} |\theta'|^2. \tag{2.63}$$

To interpret the variation of E, and hence $|\theta'|$, with distance S along the surface, consider the balance of capillary wave energy equation given by Longuet-Higgins (1961) of the form

$$\frac{\partial}{\partial S} [E(C_g + q_0)] + S_x \frac{\partial q_0}{\partial S} = 0, \tag{2.64}$$

where C_g is the group-velocity of the capillary waves and S_x is called the radiation stress. In the case of capillary waves, this stress is

$$S_x = \frac{3}{4} T |\theta'|^2 = \frac{3}{2} E. \tag{2.65}$$

Substituting $C_g = -\frac{3}{2} q_0$ and $S_x = \frac{3}{2} E$ in equation (2.64), we have

$$\frac{1}{E}\frac{\partial E}{\partial S} = \frac{2}{q_0}\frac{\partial q_0}{\partial S} \; . \qquad (2.66)$$

Hence, E is proportional to $q_0{}^2$, and this implies that $|\theta'|$ is proportional to q_0. In other words, there is a gradual variation of the steepness of free capillary waves along the surface; the wave steepness is proportional to the magnitude of the underlying current. But from equation (2.55), the steepness of the capillary waves is equal to -b, a constant. This discrepancy is due to our previous neglect of the gravitational terms in the free surface condition (2.20). We shall show this using the differentiated form of this condition (2.21) in order to eliminate y from (2.20) and considering only free waves in which the right-hand side of (2.21) is equal to zero. Thus, (2.21) becomes

$$\frac{\partial}{\partial \phi}\left[e^{2\tau_0}\tau_1 - \frac{T'}{C}\frac{\partial}{\partial \psi}(e^{\tau_0}\tau_1)\right] - \mathcal{R}\left\{\frac{g}{iC^3}\, e^{-(\tau_0 - i\theta_0)}(\tau_1 - i\theta_1)\right\} = 0,$$

$$\psi = 0. \qquad (2.67)$$

By the same transformation of coordinates as in (2.46), this becomes

$$\frac{\partial}{\partial \phi}\left[e^{\tau_0}P(\phi)\left(\tau_1 - \frac{T'}{C}\frac{\partial \tau_1}{\partial \beta}\right)\right] - \mathcal{R}\left\{\frac{g}{iC^3}\, e^{-(\tau_0 - i\theta_0)}(\tau_1 - i\theta_1)\right\} = 0.$$

$$(2.68)$$

Now let us write

$$\tau_1 - i\theta_1 = S\, e^{-i(\frac{C}{T'})(\alpha + i\beta)}, \qquad (2.69)$$

where S is assumed to be a complex amplitude. Then we have, when $\psi = \beta = 0$,

$$P(\phi)(\tau_1 - \frac{T'}{C}\frac{\partial\tau_1}{\partial\beta}) = -R\ p(\phi)\ \frac{T'}{C}\frac{\partial S}{\partial\beta}\ e^{-i(\frac{C}{T'})\alpha}\ . \qquad (2.70)$$

Substituting in (2.68) with $P(\phi)\ \frac{\partial S}{\partial\beta} = -\frac{\partial S}{i\partial\phi}$ gives

$$R\{\frac{\partial}{\partial\phi}[e^{\tau_0}\ \frac{T'}{iC}\ \frac{\partial S}{\partial\phi}\ e^{-i(\frac{C}{T'})\alpha}] - \frac{g}{iC^3}\ e^{-(\tau_0 - i\theta_0)}S\ e^{-i(\frac{C}{T'})\alpha}\} = 0. \qquad (2.71)$$

Since the exponent varies rapidly compared to S, we may carry out the $\frac{\partial}{\partial\phi}$, in the exponent only, giving

$$R\{[e^{\tau_0}P(\phi)\ \frac{\partial S}{\partial\phi} - \frac{g}{iC^3}\ e^{-(\tau_0 - i\theta_0)}S]\ e^{-i(\frac{C}{T'})\alpha}\} \cong 0.$$

Taking the real part of each side and replacing $P(\phi)$ by e^{τ_0}, we have

$$R\ \{\frac{\partial}{\partial\phi}\ (\ln S)\} \cong -\frac{g}{C^3}\ e^{-3\tau_0}\sin\theta_0 = \frac{\partial\tau_0}{\partial\phi}\ . \qquad (2.72)$$

Integration now gives

$$R\ (\ln S) \cong \tau_0\ + \text{Const},$$

hence $|S|$ is proportional to e^{τ_0}. But from equation (2.69), when $\beta = 0$, we have

$$|\theta_1| \propto e^{\tau_0} \propto q_0 , \tag{2.73}$$

so that in this higher approximation the wave steepness is indeed proportional to the stream velocity. So, with the absence of viscosity, the steepness of the capillary waves is given approximately by

$$|\theta_1| \doteq \frac{bq_0}{(q_0)_{crest}} , \tag{2.73a}$$

where b is given by (2.57) and $(q_0)_{crest}$ denotes the velocity at the crest of the wave and q_0 the inner velocity.

The Effect of Viscosity

So far, the viscosity, ν, of the fluid has been neglected. Now we shall include this effect, and we expect its direct effect is mainly on the short capillary waves where the loss of energy is comparatively great. The rate of energy dissipation by viscosity in a capillary wave in deep water (Lamb, 1932) is given by

$$\frac{\partial E}{\partial t} = 4 \nu K_C^2 E = 4 \nu \left(\frac{q_0{}^2}{T'}\right)^2 E, \tag{2.74}$$

where K_C is given by equation (2.61). Including this term in the energy equation (2.64) with the same values of S_x and C_g, we obtain

$$\frac{1}{E}\frac{\partial E}{\partial S} = \frac{2}{q_0}\left[\frac{\partial q_0}{\partial S} + \frac{4\nu}{(T')^2}\cdot q_0^4\right]. \tag{2.75}$$

Since $\frac{\partial}{\partial S} = q_0 \frac{\partial}{\partial \phi}$, equation (2.75) becomes

$$\frac{1}{E}\frac{\partial E}{\partial \phi} = \frac{2}{q_0}\frac{\partial q_0}{\partial \phi} + \frac{8\nu}{(T')^2} q_0{}^2. \tag{2.76}$$

On the forward face of the wave $\frac{\partial q_0}{\partial \phi}$ is negative, so for small values of q_0, E will tend to increase with distance away from the crest. But as q_0 increases, E may tend to decrease away from the crest. And the wave steepness gets its maximum where

$$\frac{\partial q_0}{\partial \phi} = \frac{-4\nu}{(T')^2} q_0{}^3. \tag{2.78}$$

Integrating (2.76) with respect to ϕ, we obtain

$$\ln(E) = 2\ln(q_0) + \frac{8\nu}{(T')^2}\int_0^\phi q_0{}^2 \, d\phi + C, \tag{2.79}$$

where C is the constant of integration, and so

$$E \;\alpha\; q_0{}^2 \exp\left[\frac{8\nu}{(T')^2}\int_0^\phi q_0{}^2 d\phi\right]. \tag{2.80}$$

But since E is proportional to $|\theta'|^2$, then

$$|\theta_1| \;\alpha\; q_0 \exp\left[\frac{4\nu}{(T')^2}\int_0^\phi q_0{}^2 d\phi\right]. \tag{2.81}$$

Adjusting the constant of proportionality so that $|\theta_1| = b$ at the crest, we finally have

$$|\theta_1| = \frac{bq_0}{(q_0)_{crest}} \exp\left[\frac{4\nu}{(T')^2}\int_0^{\phi} q_0^2\, d\phi\right]. \qquad (2.82)$$

This is the capillary steepness when the viscosity effect is included.

If we compare this result to (2.73a), we see that the viscosity has a pronounced effect on the steepness of the capillary wave. Further analysis by Longuet-Higgins showed that this effect would also depend on the wavelength which is brought on by ϕ in the expression above.

CHAPTER 3

A MODEL EQUATION TO STUDY THE EFFECTS OF SURFACE TENSION, NONLINEARITY, AND VISCOSITY IN WATER WAVES

The Model Equation

The equation

$$\frac{\partial \eta}{\partial t} + \alpha \frac{\partial \eta}{\partial x} - \frac{\partial^2}{\partial x^2} \left(\frac{\partial \eta}{\partial t} + \eta \frac{\partial \eta}{\partial x}\right) + \mu \frac{\partial^4 \eta}{\partial x^4} + \sigma \frac{\partial^5 \eta}{\partial x^5} = 0 \qquad (3.1)$$

is proposed by Warren Ferguson, Philip Saffman, and Henry Yuen (1977) as a model equation for studying the effects of surface tension, viscosity and nonlinearity in water waves. Here, $\eta(x,t)$ models the elevation of the free surface, and x and t model the spatial and temporal coordinates. μ and σ are the viscosity and the surface tension, respectively. α is a constant. The first four terms model the propagation of gravity waves. The fourth term contains the nonlinearity. The fifth and sixth terms model the effects of viscosity (μ) and surface tension (σ), respectively. The constants μ and σ will be taken to be small. The guideline used in constructing the model equation was to find an equation with a linear dispersion relation having the same qualitative features as the real water wave dispersion relation. The model equation possesses steady, propagating symmetric gravity wave solutions similar to Stokes' waves for real water waves. It has a limiting waveform with a sharp corner at its crest, and it can be solved numerically. In the following, we shall discuss the properties of the model equation, the steady solutions, and the limiting waveform in the inviscid gravity wave limit. Finally, we shall give a numerical solution of the equation, including the surface tension and viscosity effects.

Properties of the Model Equation

From (3.1), the equation for the inviscid ($\mu=0$) infinitesimal waves about the mean level $\eta=0$ is

$$\frac{\partial \eta}{\partial t} + \alpha \frac{\partial \eta}{\partial x} - \frac{\partial^2}{\partial x^2}\left(\frac{\partial \eta}{\partial t}\right) + \sigma \frac{\partial^5 \eta}{\partial x^5} = 0. \tag{3.2}$$

Since the elementary solution for the linear wave equation is given by

$$\eta(x,t) = A\, e^{iKx - i\omega t}, \tag{3.3}$$

where K is the wave number, ω is the frequency, and A is the amplitude, we can obtain the dispersion relation

$$\omega = \frac{\alpha K + \sigma K^5}{1 + K^2}\ . \tag{3.4}$$

The phase velocity is

$$C^{(M)} = \frac{\omega}{K} = \frac{\alpha + \sigma K^4}{1 + K^2}, \tag{3.5}$$

where the upper subscript (M) means the quantity is obtained from the model equation. The group velocity is

$$C_g^{(M)} = \frac{d\omega}{dK} = \frac{\alpha(1-K^2) + \sigma(5K^4+3K^6)}{(1+K^2)^2}. \tag{3.6}$$

In the gravity wave limit $(K \to 0)$, $C > C_g$, whereas in the capillary wave limit $(K \to \infty)$, $C < C_g$ (see Table 1). The phase velocity attains its minimum when $K = K_{min}$, where

$$K_{min}^{(M)} = [(1+\frac{\alpha}{\sigma})^{1/2} - 1]^{1/2}. \tag{3.7}$$

At this value of K, C_g and C are equal (Figure 6A) for small values of K; these results are in qualitative agreement with those for deep water waves for which the dispersion relation is

$$\omega = (gK + \sigma K^3)^{1/2}. \tag{3.8}$$

The phase velocity is

$$C^{(W)} = \frac{\omega}{K} = (\frac{g}{K} + \frac{\sigma}{K})^{1/2}, \tag{3.9}$$

where the upper subscript (W) means it is derived from the water equation,

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\vec{\nabla}\rho - g\vec{j}, \quad \vec{\nabla} \cdot \vec{u} = 0,$$

and the group velocity is

$$C_g^{(W)} = \frac{d\omega}{dK} = \frac{1}{2} C^{(W)} \left[\frac{1 + (\frac{3\sigma}{g})K^2}{1 + (\frac{\sigma}{g})K^2}\right]. \tag{3.10}$$

The phase velocity has a minimum of $K = K_{min}$, where

$$K_{min}^{(M)} = (\frac{g}{\sigma})^{1/2}. \tag{3.11}$$

As usual, g is the acceleration due to gravity, and σ is the coefficient of surface tension. Plots of C and C_g for both the model equation and water waves are given in Figures 6A and 6B. In order to compare the qualitative behavior of C and C_g for the model and water waves, the values of g and σ were chosen to generate a value of $K_{min}^{(W)}$ which would be equal to $K_{min}^{(M)}$. The main differences are for the two limits $K \to 0$ and $K \to \infty$.

For capillaries ($K \to \infty$), the model equation has C and C_g approaching infinity faster than for water waves (Figure 6B). For gravity waves ($K \to 0$), C and C_g remain finite for the model equation and become infinite for deep water waves. However, this work examines the gravity wave and capillary wave interaction for finite values of K, and the discrepancies at the two limits have little effect on these phenomena.

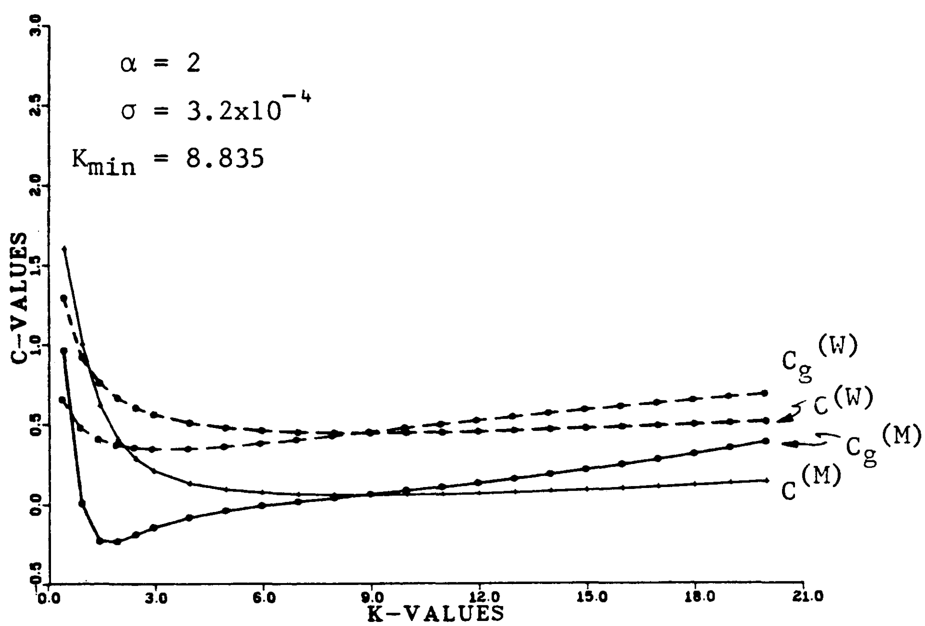

Figure 6A. Plot of the phase and group velocities for the model equation and water waves with same K_{min}.

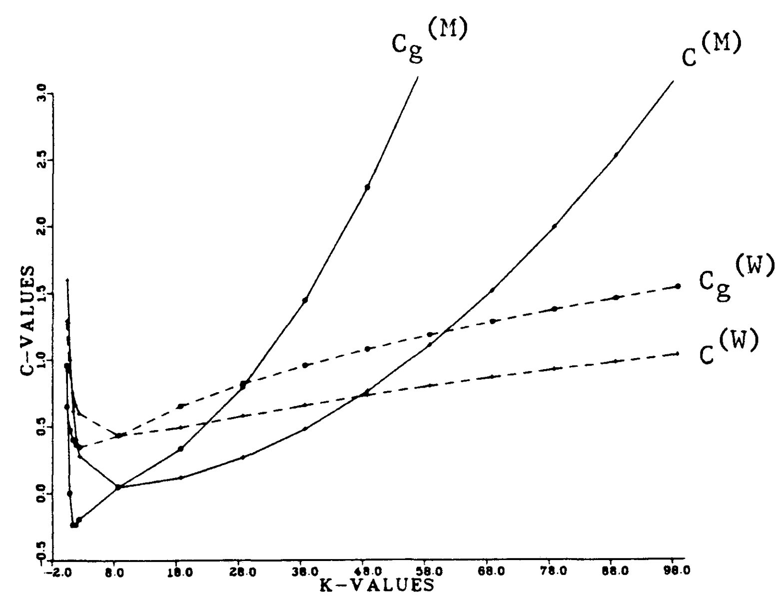

Figure 6B. Plot of the phase and group velocities for the model and water waves for K = 0 to 100.

Table 1. Values of C and C_g for both model and water waves for $K = .5$ to 98.

K	C Model	C_g Model	C Water	C_g Water
0.5	1.60002	0.96007	1.2920	0.6501
1.0	1.00016	0.00064	0.9180	0.4706
1.5	0.61588	-0.23488	0.7554	0.3989
2.0	0.40102	-0.23652	0.6613	0.3629
2.5	0.27759	-0.19411	0.5995	0.3442
8.0	0.05093	0.03129	0.4351	0.4135
18.0	0.10951	0.30460	0.4879	0.6372
28.0	0.25311	0.74978	0.5729	0.8074
38.0	0.46314	1.38454	0.6534	0.9466
48.0	0.73783	2.21065	0.7273	1.0671
58.0	1.07675	3.22853	0.7953	1.1750
68.0	1.47979	4.43829	0.8585	1.2735
78.0	1.94689	5.83999	0.9176	1.3648
88.0	2.47802	7.43366	0.9734	1.4503
98.0	3.07317	9.21931	1.0262	1.5310

We now examine the steady, inviscid, periodic gravity wave solution of equation (3.1). The solution is taken to be of the form

$$\eta(x,t) = f(x - Ut) = f(\xi), \tag{3.12}$$

where U is the constant speed of propagation. Substituting (3.12) into (3.1) and integrating twice, with the assumption that

$$h = f - U, \tag{3.13}$$

we obtain

$$h^2\left(\frac{dh}{d\xi}\right)^2 = \frac{2}{3}(\alpha-U)h^3 + Ah^2 + B. \tag{3.14}$$

Here, A and B are constants of integration. From equation (3.14) we obtain

$$\xi = \int \frac{hdh}{\sqrt{\frac{2}{3}(\alpha-U)h^3 + Ah^2 + B}} . \tag{3.15}$$

This shows that the waveforms can be expressed in terms of Jacobian elliptic functions, like the cnoidal waves of shallow water theory (26).

The infinitesimal waves ($f \rightarrow 0$) of wave number K and amplitude a are sinusoidal, with

$$U = \frac{\alpha}{(1+K^2)}, \qquad B = \frac{\alpha^4 K^2}{3(1+K^2)^4}, \qquad a = \left(\frac{A}{K^2} - \frac{B}{U^2}\right)^{1/2}. \tag{3.16}$$

As B decreases, the amplitude, a, generally increases, and there exists a limiting waveform with a sharp crest with height $f = U$, occurring when the constant $B = 0$. In this case, the waveform can be shown to be

$$h = \frac{\alpha-U}{6}(\xi^2 - \frac{\lambda^2}{4}), \qquad -\frac{\lambda}{2} < \xi \leq \frac{\lambda}{2}, \tag{3.17}$$

with crests at $\xi = \frac{\lambda}{2}$ and trough at $\xi = 0$. The wavelength λ is related to A by

$$\lambda = \frac{6\, A^{1/2}}{(\alpha-U)} . \tag{3.18}$$

The angle β subtended by the crest is given by

$$\beta = \pi - 2 \tan^{-1}[\frac{\lambda}{6}(\alpha-U)]. \tag{3.19}$$

If we fix the value A to be $\frac{1}{3}$, then from equations (3.18) and (3.19) the angle subtended by the crest is 120°. The wavelength and amplitude of the limiting wave are specified in terms of its velocity by

$$\lambda_{lim} = \frac{2\sqrt{3}}{(\alpha-U)}, \qquad a_{lim} = \frac{1}{2(\alpha-U)} . \tag{3.20}$$

The phase and group velocities of the model are in agreement with the corresponding velocities of water waves for finite wave number, K. The crest-limiting angle of the model equation agrees with the Stokes angle (24).

We now turn our attention to the development of a numerical scheme which will enable us to examine the phenomenon of capillary wave generation. We use the model equation for this endeavor, since it is extremely difficult to study this phenomenon using the real water wave equation. Confidence in our numerical predictions is strengthened by the agreement found between the solutions of the model equation and the real water wave equation for the special cases we have just studied.

Numerical Solutions

We shall attempt a numerical solution to the model equation (3.1) with periodic boundary conditions in x. There are four basic effects which control the amplitude and position of capillaries on gravity waves. They are:

(i) the gravity wave amplitude,

(ii) the surface tension σ,

(iii) the unsteadiness of the gravity wave,

(iv) the viscosity μ.

We shall test effects (ii) and (iv) with the following numerical scheme. The method employed was to transform the partial differential equation (3.1) in x-space, by using the transform

$$\eta = \sum_{-\infty}^{\infty} \hat{\eta}_n(t)\ e^{iw_n x} \tag{3.21}$$

$$\eta^2 = U = \sum_{-\infty}^{\infty} \hat{U}_n(t)\ e^{iw_n x}. \tag{3.22}$$

Here,

$$w_n = \frac{2\pi n}{\lambda}, \qquad \lambda = \frac{6\, A^{1/2}}{(\alpha - U)}\ . \tag{3.23}$$

These transformations give rise to an ordinary differential equation as follows. Equation (3.1) can be rewritten as

$$\frac{\partial^2}{\partial x^2}\left\{\frac{\partial \eta}{\partial t} + \frac{1}{2}\frac{\partial U}{\partial x}\right\} = \frac{\partial \eta}{\partial t} + \alpha\frac{\partial \eta}{\partial x} + \mu\frac{\partial^4 \eta}{\partial x^4} + \sigma\frac{\partial^5 \eta}{\partial x^5}\ . \tag{3.24}$$

We now use (3.21) and (3.22) to obtain the ordinary differential equation

$$(1+w_n^2)\ \frac{d\hat{\eta}_n(t)}{dt} = -(i\alpha w_n + \mu w_n^4 + i\sigma w_n^5)\ \hat{\eta}_n(t) - \frac{1}{2} i w_n^3\, \hat{U}_n(t). \tag{3.25}$$

Given $\eta(x,t)$ at $t=0$, we calculate $\hat{\eta}(t)$ for one period, $x = \lambda$, using the subroutine FFTCC(IMSL) (Appendix B). Next, we compute η^2 and use FFTCC to calculate $\hat{U}_n(t)$. Finally, we use a Runge-Kutta scheme named DVERK(IMSL) (Appendix C) to solve (3.25). Different cases were run in order to determine the effects of surface tension and viscosity. All cases have $\alpha = 2$, and we define a parameter relating the amplitude of the wave under consideration to the limiting wave's amplitude as

$$\varepsilon = \frac{a}{a_{lim}} . \tag{3.26}$$

We have run two cases with different values of the viscosity and same values of surface tension.

<u>Case 1</u>: In this case, we let $\sigma = 3.2x10^{-4}$, $\mu = 0$, and $\varepsilon = 0.99$. The result from the plot shows that the ripple waves appear at the gravity wave crest and spread to cover the whole wave (Figure 7). Also, the wave grows in amplitude.

<u>Case 2</u>: In this case, we let $\sigma = 3.2x10^{-4}$, $\mu = 3.2x10^{-3}$, and $\varepsilon = 0.99$. The result from the plot shows that the ripples are damped down by viscosity before the wave reaches the next crest (Figure 8). We suspect that the unusual steepness of the right side of the first full wave is due to irregularities in the plotting routine rather than physical reasons. A refinement of the plotter routine should give a more accurate picture of the wave.

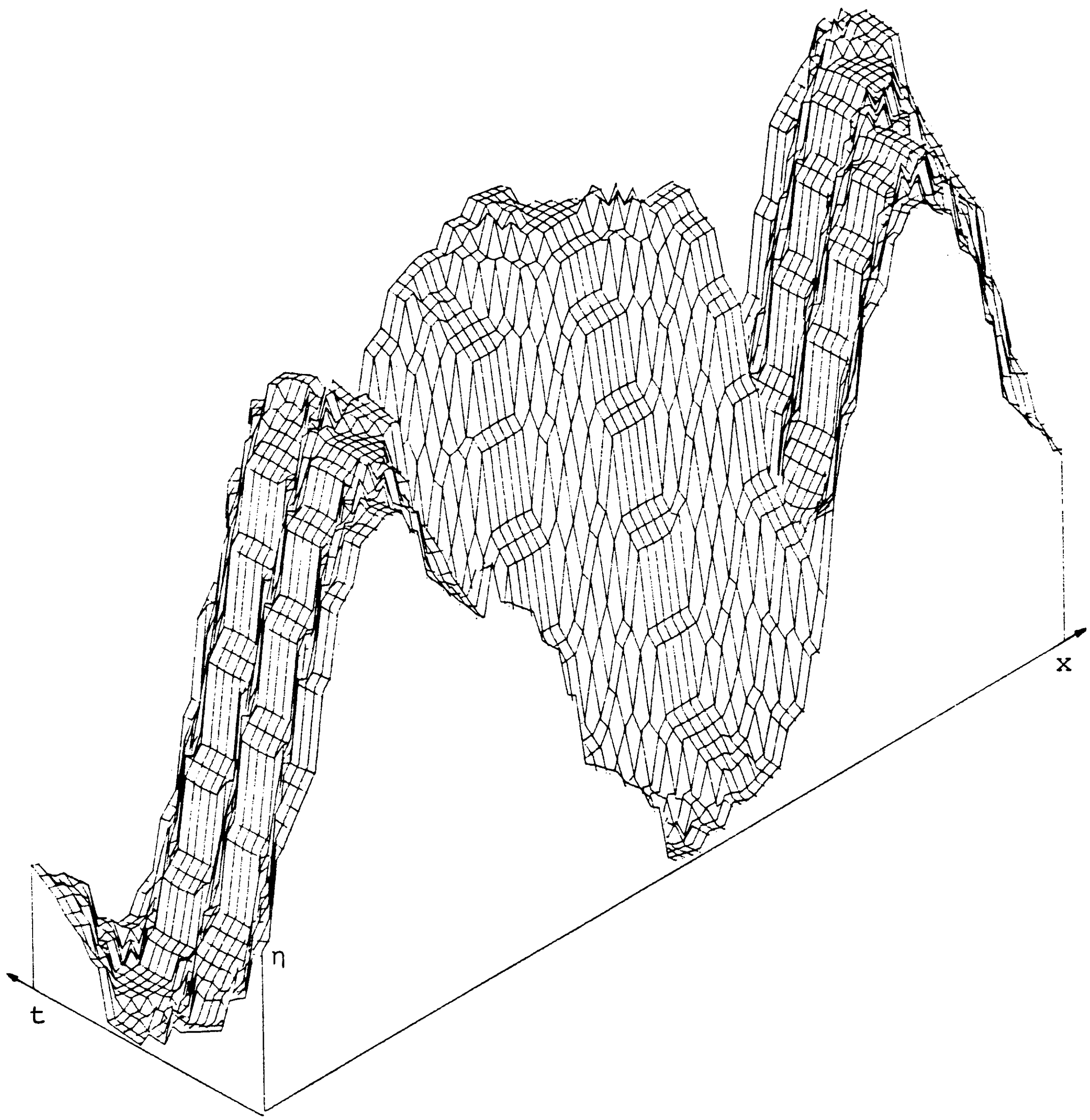

Figure 7. Plot for Case 1, where $\sigma = 3.2\text{x}10^{-4}$, $\mu = 0$, and $\varepsilon = 0.99$.

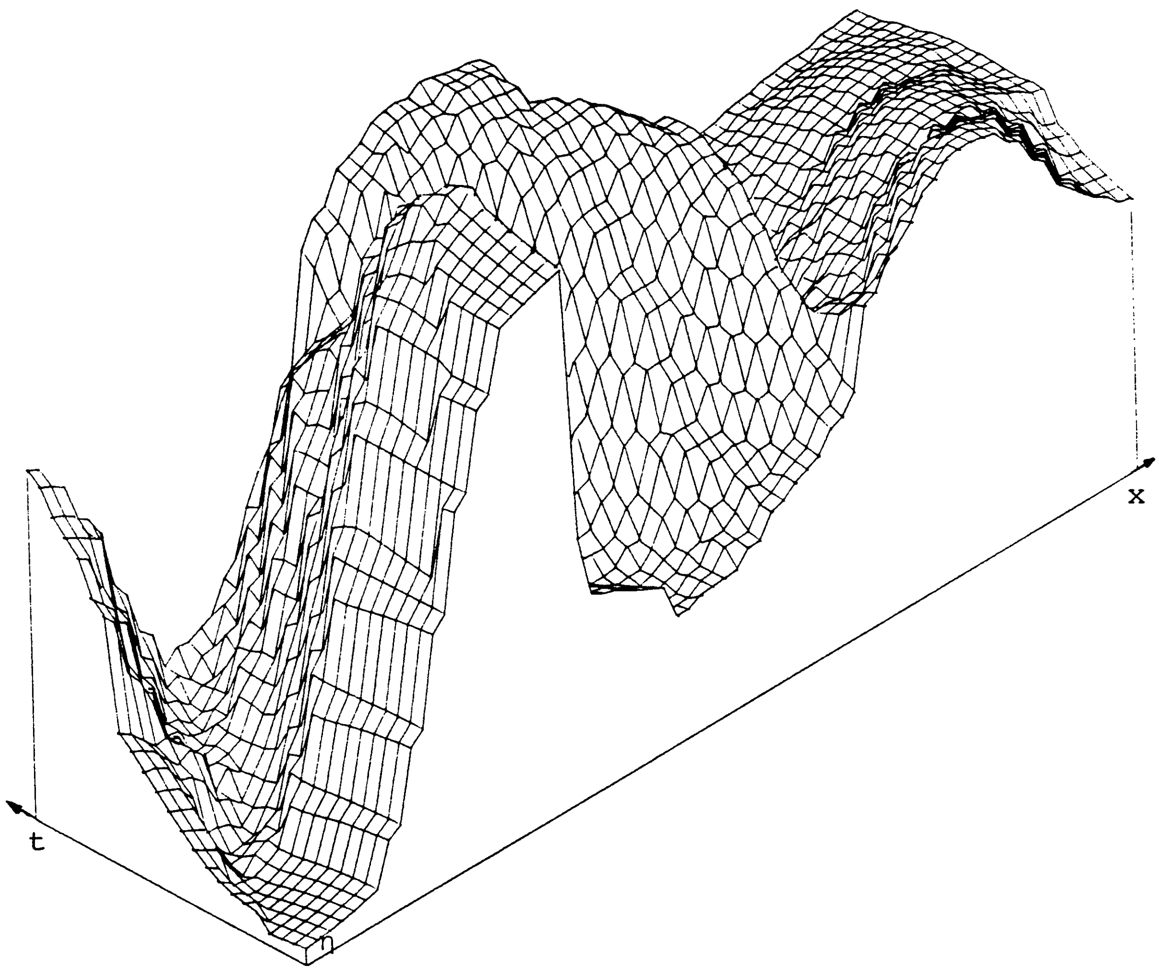

Figure 8. Plot for Case 2, where $\sigma = 3.2 \times 10^{-4}$, $\mu = 3.2 \times 10^{-3}$, and $\varepsilon = 0.99$.

CHAPTER 4

CONCLUSION

The two methods for studying surface tension effects on water waves, which we have analyzed in Chapters 2 and 3, predict that the effect of surface tension occurs at and near the crest of the water wave. The effect is in the form of short capillary waves or ripples. In addition, both models predict that if viscosity is present, then these short capillary waves will tend to be damped out. If little or no viscosity is present, then the ripples eventually spread over the whole wave.

The method treated in Chapter 2 clearly shows that the effect of surface tension on gravity waves approaching their maximum amplitude is strongly localized near the wave crest. The ripples gain energy not only from the surface tension at the crest, but also by interaction with the gravity wave on its forward face, through the radiation stress. At the same time, ripple energy is being drained away by viscosity. In fact, on the rear face of the wave both viscosity and radiation stress reduce the capillary energy. This model shows that the capillary energy depends on the curvature at the crest and that the amplitude of the capillary waves depends on the maximum curvature of the gravity waves.

The numerical method treated in Chapter 3 makes it possible for one to perform calculations which verify the predictions that the ripples will spread over the entire wave if viscosity is negligible and that the ripples tend to be damped out by increased viscosity.

The numerical method seems to hold the most promise for further study. In setting up our numerical routine, we corresponded with Professors Philip Saffman and Warren Ferguson in order to obtain some details which were not mentioned in their article (8).

Professor Warren Ferguson, in addition to answering our questions, also made some suggestions on how the numerical treatment could be improved. The improvements would be directed toward establishing a method which would be stable for large values of N in the fast Fourier transform. Also, the question of existence of periodic solutions to the equation for inviscid gravity waves with only a small value for the surface tension σ, is still unanswered. Finally, we expect that the agreement of the results from the model equation with real water waves could be improved by adjusting constants in the model equation so that one obtains similar phase and group velocities not only at low wave numbers (see Figures 6A and 6B), but also for higher wave numbers.

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APPENDIX A

THE STOKES 120° ANGLE

It has been shown both theoretically and experimentally by Taylor (25) that in a standing gravity wave of maximum amplitude, the vertical acceleration at the sharp crest is equal to -g. Also, in a wave of limiting height the acceleration near the crest is equal to $\frac{1}{2}g$ directed away from the crest. In Stokes' limiting angle (24), the velocity potential ϕ is given by

$$\phi + i\psi = C(x+iy)^{3/2} = Cq^{3/2}e^{3/2\, i\theta}, \tag{A.1}$$

where

$$C = \frac{2}{3}(-ig)^{1/2} \tag{A.2}$$

and $x = q\cos\theta$, and $y = q\sin\theta$. The lines $\theta = (-\frac{\pi}{2} \pm \frac{\pi}{3})$ are stream lines. Also, since

$$u - iv = \frac{d\chi}{dZ} = \frac{3}{2}\,C(x+iy)^{1/2} = \frac{3}{2}\,Cq^{1/2}e^{i\theta/2}, \tag{A.3}$$

we have

$$(u^2+v^2) = \frac{9}{4}\,|C|^2\,q = gq.$$

Thus, the Bernoulli Condition, $\frac{1}{2}(u^2+v^2) = -gy$, is satisfied on the free surface.

To obtain the acceleration, differentiate (A.3) with respect to x. This gives

$$\frac{\partial u}{\partial x} - i\,\frac{\partial v}{\partial x} = \frac{3}{4}\,C(x+iy)^{-1/2} = \frac{3}{4}\,Cq^{-1/2}\,e^{-i\theta/2}. \qquad \text{(A.4)}$$

Multiplying (A.3) by the complex conjugate of (A.4), we obtain

$$(u - iv)\,\frac{3}{4}\,Cq^{-1/2}\,e^{i\theta/2} = \left(\frac{3}{2}\,Cq^{1/2}\,e^{i\theta/2}\right)\left(\frac{3}{4}\,Cq^{-1/2}\,e^{i\theta/2}\right)$$

$$= \frac{9}{8}\,|C|^2\,e^{i\theta}. \qquad \text{(A.5)}$$

The expression on the left represents the vector acceleration a, say, then by (A.2) we have

$$a = \frac{1}{2}\,g\,e^{i\theta}. \qquad \text{(A.6)}$$

In other words, the acceleration has a magnitude of $\frac{g}{2}$ and is directed everywhere outwards from the vertex.

APPENDIX B

COMPUTER PROGRAM FOR FAST FOURIER TRANSFORM (FFTCC)

EXA.FOR

```
      INTEGER N,IWK(42)
      REAL WK(42)
      COMPLEX A(6),X(6)
      N=6
      A(1)=(2.0,1.0)
      A(2)=(1.0,4.0)
      A(3)=(3.0,3.0)
      A(4)=(1.0,0.0)
      A(5)=(4.0,2.0)
      A(6)=(0.0,3.0)
      WRITE(4,101)
101   FORMAT(1X,'THE FOURIER TRANSFORM IS',//)
      CALL FFTC(A,N,IWK,WK)
      PI=3.1415
      DO 100 K=1,N
      DO 200 J=0,N-1
      X(K)=A(J+1)*CEXP(CMPLX(0.0,(2.0*PI*J*K)/N))
200   CONTINUE
      WRITE(4,25)A(K)
 25   FORMAT(2F10.4)
100   CONTINUE
      WRITE(4,102)
102   FORMAT(1X,'THE INVERS FOURIER TRANSFORM IS',//)
      DO 10 I=1,N
      A(I)=CONJG(A(I))
 10   CONTINUE
      CALL FFTCC(A,N,IWK,WK)
      DO 20 I=1,N
      A(I)=(CONJG(A(I))/N
      WRITE(4,35)A(I)
 35   FORMAT(1X,2F10,4)
 20   CONTINUE
      END
```

APPENDIX C

DVERK SUBROUTINE FOR SOLVING DIFFERENTIAL EQUATIONS.
SEE ISML LIBRARY ROUTINE FOR ARGUMENT.

```
         EXTERNAL  FCN
         INTEGER  IER, IND ,K,N,NW
         REAL  C(24),TOL,W(10,9),T,TEND,Y(10)
         OPEN(UNIT=10,FILE='MAX.DAT')
         N=10
C        OUR VALUES FOR Y(1) to Y(10) ARE TAKEN FROM
C        THE VALUES OF ETA-HAT
         TOL=1.E-5
         IND=1
         NW=10
         T=0.0
         DO 300 K=1,10
         TEND=FLOAT(K)/10.0
         CALL DVERK(N,FCN,T,Y,TEND,TOL,IND,C,NW,W,IER)
         IF(IER.EQ.0) GO TO 50
         WRITE(4,35) IND,IER
   35    FORMAT(1X,'ERROR IN DVERK IND=',I5,'IER=',I5)
         STOP
   50    WRITE(10,25)TEND,Y(1),Y(2),C(24)
   25    FORMAT(1X,F3.1,2(E16.8,3X),F4.0)
  100    CONTINUE
         STOP
         END
         SUBROUTINE FCN(N,T,Y,YP)
         INTEGER N
         REAL T,Y(N),YP(N)
C        HERE  YP IN OUR PROBLEM MEANS  ETAPRIME-HAT AND
C        Y MEANS ETA-HAT.  WE SHALL SUPPLY IN EACH
C        EQUATION THE VALUE OF U-HAT AS B(I)
         YP(1)=A(1)*Y(1)+B(1)
C        WHERE THE CONSTANT A(J) IS:
C        A(J)=1.0/(1+W(N)**2)*(I*E*W(N)+M*W(N)**4
C        1 +I*S*W(N)**5)
C        THE CONSTANT B(J) IS
C        B(J)=1.0/(1+W(N)**2)*(-0.5*I*W(N)**3)*V(M)
C        WHERE V(M) IS THE VALUE OF U-HAT
0
         YP(2)=A(2)*Y(2)+B(2)
         YP(3)=A(3)*Y(3)+B(3)
         ..............
         YP(10)=A(10)*Y(10)+B(10)
         RETURN
         END
```