MATRIX DECOMPOSITION ALGORITHMS FOR MODIFIED QUADRATIC SPLINE COLLOCATION FOR HELMHOLTZ PROBLEMS

by

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ABSTRACT

New one-step modified quadratic spline collocation methods are developed to numerically solve the Helmholtz equations subject to Dirichlet, Neumann, mixed (Neumann-Dirichlet and Dirichlet-Neumann), and periodic boundary conditions. The new methods are constructed so that (a) optimal order global accuracy and superconvergence are obtained and (b) the quadratic spline collocation equations can be solved by matrix decomposition algorithms. Appropriate quadratic spline collocation methods for second-order two-point boundary value problems are first formulated and then extended to the two-dimensional problem.

For Dirichlet boundary conditions, a new one-step modified quadratic spline collocation method is formulated and implemented for solving a second-order two-point boundary value problem to demonstrate numerically that its accuracy is optimal and comparable to existing optimal modified quadratic spline methods. The collocation points are taken to be the midpoints of the subintervals of the partition. Optimal accuracy is shown to be of order $3-k$ globally and superconvergence of order $4-k$ at certain points, for the k^{th} derivative, $k=0,1,2$. This method is then extended to a Dirichlet Helmholtzproblem and it is shown how the collocation equations can be solved using matrix decomposition algorithms.

Using a similar approach, new one-step modified quadratic spline collocation methods are developed for Neumann, mixed, and periodic boundary conditions. In each case, extensive numerical experimentation is conducted to demonstrate the optimality and superconvergence of the quadratic spline collocation methods and the efficacy of the matrix decomposition algorithms.

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Chapter 1

INTRODUCTION

The goal of this thesis is to formulate and implement one-step modified quadratic spline collocation (QSC) methods for solving second-order two-point boundary value problems (TPBVPs) and their extensions to Helmholtz boundary value problems (BVPs) in the unit square.

A collocation method involves the determination of an approximate solution in a suitable set of functions, sometimes called trial functions, by requiring the approximate solution to satisfy the boundary conditions (BCs) and the differential equation at certain points, called the collocation points. The collocation approach dates back to the 1930's when the trial functions were commonly taken to be polynomials. With the upsurge of interest in spline functions in the 1960's, attention turned to their use as trial functions. Over the last several decades, spline collocation methods have evolved as valuable techniques for the solution of a broad class of problems involving ordinary and partial differential equations (PDEs) (see [5] and [15] for comprehensive surveys).

The first spline collocation method proposed for the solution of second-order TPBVPs was a nodal cubic spline collocation (NCSC) method. In this method, the approximate solution lies in the space of cubic splines and satisfies the BCs and the differential equation at the nodes of the partition on which the cubic splines are defined. For linear equations, this approach leads to an algebraic problem similar to that arising in the basic finite difference method and it is easy to show computationally that it is no more accurate than this method. In fact, de Boor [8] proved that the NCSC method is second-order accurate in the L^∞ norm and no better. This method is thus suboptimal, that is, its rate of convergence is of lower order than is possible by approximation in the spline space. In fact, if, in a finite-element type method, one seeks an approximation in a spline space of order r, then for optimal accuracy, one would expect accuracy of order $r+1$. Thus, in NCSC, one would expect the error in the approximation to be of order 4.

This suboptimality of nodal spline collocation led researchers to seek cubic spline collocation (CSC) methods of optimal accuracy. In their fundamental work, de Boor and Swartz [9] showed that optimal rates of convergence can be attained by collocating at certain Gauss points in each subinterval of the partition on which the spline space is defined. This approach is called orthogonal spline collocation, and can be formulated for spline spaces of degree $r \geq 3$ with C^1 continuity. In addition, several researchers succeeded in modifying NCSC so that optimal rates of convergence were achievable on uniform partitions. This class of methods is commonly known as modified CSC methods, of which there are two approaches. The first is a two-step method (TSM) formulated by Fyfe [16] in which the NCSC solution is first determined and then improved using a deferred correction-type procedure. The second is a one-step method (OSM), developed independently by Archer [2, 3] and Daniel and Swartz [13], in which the optimal order approximation is determined directly by collocating a high-order perturbation of the differential equation. Similar approaches were adopted by Houstis et al., [17] in the development of optimal QSC methods for second-order TPBVPs, in which the collocation points are taken to be the midpoints of the subintervals of the partition, and in [19] for quintic spline collocation methods for general linear fourth-order TPBVPs.

A practical advantage of modified NCSC methods over orthogonal spline collocation methods is that, for a given partition, there are fewer unknowns with the same degree of piecewise polynomials, thereby reducing the size of the linear systems. However, in contrast to modified NCSC methods, orthogonal spline collocation methods do not require a uniform partition of the domain and are also applicable to high degree splines.

In [18], Houstis et al., extended the OSM of Archer [2, 3] and the TSM of Fyfe [16] for second-order TPBVPs to elliptic BVPs in the unit square, devoting particular attention to Dirichlet and Neumann Helmholtz problems. Recently, the analysis of the OSM for the

Dirichlet problem presented in [18] was shown to be incorrect [1]. Moreover, the claims in [18] that the OSM is of optimal order for the Neumann problem are unsubstantiated. One would not expect optimality for the Neumann (as well as the mixed and periodic) problem as the method of Archer applied to Neumann, mixed, and periodic TPBVPs is easily shown to be suboptimal, in fact, third-order. Christara [10] extended the work of Houstis et al., [17] to develop biquadratic OSM and TSM for elliptic problems with Dirichlet and Neumann BCs and obtained optimal global accuracy and superconvergence results. Superconvergence is the occurrence at certain points of a higher rate of convergence than is possible globally.

A challenge in both the CSC and QSC methods for elliptic problems is the efficient solution of the collocation equations. In [6], matrix decomposition algorithms (MDAs) were formulated and implemented for the OSM and TSM of [18] applied to the Helmholtz problem. MDAs are fast direct methods which involve fast Fourier transforms (FFTs) at a cost of $O(N^2 \log N)$ operations on a uniform $N \times N$ partition. For the OSM, it is possible to formulate an MDA for Dirichlet BCs only whereas for the TSM, MDAs were developed for Dirichlet, Neumann, mixed, and periodic BCs. Unlike the TSM, the OSM is not superconvergent. In [6], it is also demonstrated numerically that, as expected, the TSM is less efficient than the OSM since the TSM requires twice as many FFTs as the OSM. In [7], Bialecki et al., formulated and implemented optimal OSMs for all four BCs, for which the collocation equations can be solved using MDAs. These OSMs differ from earlier OSMs in that they are constructed by perturbing judiciously both the differential operator and the right hand side of the differential equation.

Constas [12] implemented the two-step QSC method for Helmholtz problems with Dirichlet, Neumann, and periodic BCs, and solved the linear systems using MDAs; see also [11]. However, no consideration was given to the OSM. This is the motivation for this thesis, to develop a new OSM for modified QSC in order to numerically solve Helmholtz equations satisfying Dirichlet, Neumann, mixed, and periodic BCs so that (a) optimal order global accuracy and superconvergence are obtained and (b) the QSC equations can be solved by MDAs. Following the approach of [7], we formulate appropriate QSC methods for second-order TPBVPs and then extend these to the two-dimensional problem. In Chapter 2 of this thesis, we consider a second-order TPBVP with Dirichlet BCs and derive the QSC methods analogous to the methods of Archer [2, 3] and Daniel and Swartz [13] for the cubic case. We then formulate a new one-step modified QSC method which involves perturbations of the right hand sides of the differential equations as well as the differential operator. We demonstrate numerically that its accuracy is optimal and comparable to the other optimal modified methods. Additionally, the numerical results also demonstrate the superconvergence of all of the modified methods at certain points. We then extend the new method to the Dirichlet Helmholtz problem and show how the collocation equations are solved using an MDA. In Chapter 3, for the second-order TPBVP with Neumann BCs, we derive the QSC methods analogous to the methods of Archer [2, 3] and Daniel and Swartz [13] for the cubic case. We then develop a new one-step modified QSC method for the second-order TPBVP problem and extend the method to the Neumann Helmholtz problem using an MDA. Here the QSC approximation satisfies perturbed BCs. Drawing upon our results for the Dirichlet and Neumann problems, we present the mixed Helmholtz problem in Chapter 4, including Neumann-Dirichlet and Dirichlet-Neumann BCs. The periodic problem is tackled in a similar way and is discussed in Chapter 5. In each case, numerical experimentation is conducted to demonstrate the optimality and superconvergence of the QSC methods and the efficacy of the MDAs. Chapter 6 comprises concluding remarks and a discussion of future research.

It should be noted that convergence analyses have not yet been derived for the methods developed in this thesis or for the methods of [7], although an analysis of a method closely related to the Dirichlet OSM in [7] has recently been developed in [1].

Chapter 2

DIRICHLET BOUNDARY CONDITIONS

2.1 Two-Point Boundary Value Problem

Consider the second-order TPBVP

$$Lu \equiv u'' + cu = f(x), \quad x \in [0,1], \tag{2.1}$$

subject to the homogeneous Dirichlet BCs,

$$u(0) = u(1) = 0, \tag{2.2}$$

where $c \leq 0$ is a constant.

2.1.1 Piecewise Quadratic Splines

Let $\rho_x = \{x_i\}_{i=0}^{N+1}$ be a uniform partition of $[0,1]$, where $x_i = ih,\ i = 0, 1, ..., N+1$, and $h = 1/(N+1)$. The space of quadratic splines is defined by

$$S_2 = \left\{v : v \in C^1[0,1],\ v|_{I_i} \in P_2,\ 1 \leq i \leq N+1\right\}, \tag{2.3}$$

where $I_i = [x_{i-1}, x_i]$ and P_2 is the set of polynomials of degree ≤ 2. To define a basis for S_2, we extend the partition ρ_x by introducing $x_i = ih,\ i = -2,\ -1,\ N+2,\ N+3$. As in Constas [12], we choose the functions $\{\mathcal{B}_m\}_{m=0}^{N+2}$ as a basis for S_2, where

$$\mathcal{B}_m(x) = \frac{1}{2}\xi\left(\frac{x}{h} - m + 2\right), \tag{2.4}$$

and ξ is the quadratic spline function defined by

$$\xi(x) = \begin{cases} x^2, & x \in [0,1], \\ -3 + 6x - 2x^2, & x \in [1,2], \\ 9 - 6x + x^2, & x \in [2,3], \\ 0, & \text{otherwise}. \end{cases} \tag{2.5}$$

From (2.4), we have

$$\mathcal{B}'_m(x) = \frac{1}{2h}\xi'\left(\frac{x}{h} - m + 2\right), \quad \mathcal{B}''_m(x) = \frac{1}{2h^2}\xi''\left(\frac{x}{h} - m + 2\right), \tag{2.6}$$

and, from (2.5),

$$\xi'(x) = \begin{cases} 2x, & x \in [0,1], \\ 6 - 4x, & x \in [1,2], \\ -6 + 2x, & x \in [2,3], \\ 0, & \text{otherwise}, \end{cases}, \quad \xi''(x) = \begin{cases} 2, & x \in [0,1], \\ -4, & x \in [1,2], \\ 2, & x \in [2,3], \\ 0, & \text{otherwise}. \end{cases} \tag{2.7}$$

In all of the collocation methods for the TPBVP (2.1) discussed in this thesis, the collocation points $\{\tau_i\}_{i=1}^{N+1}$ are taken to be $\tau_i = (x_i + x_{i-1})/2$, the midpoints of the subintervals I_i, $i = 1, ..., N+1$. Since $\tau_i/h - j + 2 = i - j + 3/2$, using (2.4)–(2.7), for $i = 1, ..., N+1$, $j = 0, ..., N+2$, we have

$$\mathcal{B}_j(\tau_i) = \begin{cases} 1/8, & i = j \pm 1, \\ 3/4, & i = j, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{B}'_j(\tau_i) = \begin{cases} \mp 1/(2h), & i = j \pm 1, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{B}''_j(\tau_i) = \begin{cases} 1/h^2, & i = j \pm 1, \\ -2/h^2, & i = j, \\ 0, & \text{otherwise}. \end{cases} \tag{2.8}$$

Let S_2^D be the space of quadratic splines satisfying the Dirichlet BCs (2.2), that is, $S_2^D = S_2 \cap \{v : v(0) = v(1) = 0\}$. Then a basis for S_2^D is defined in terms of $\{\mathcal{B}_m\}_{m=0}^{N+2}$ of (2.4) by

$$\{\mathcal{B}_m^D\}_{m=1}^{N+1} = \{\mathcal{B}_1 - \mathcal{B}_0, \mathcal{B}_2, ..., \mathcal{B}_N, \mathcal{B}_{N+1} - \mathcal{B}_{N+2}\}. \tag{2.9}$$

Therefore, from (2.8) and (2.9), we have

$$\mathcal{B}_j^D(\tau_i) = \begin{cases} 5/8, & i,j=1, \text{ or } i,j=N+1, \\ 1/8, & i,j=1,2,...,N+1;\ i=j\pm 1, \\ 3/4, & i,j=2,3,...,N;\ i=j, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{B}_j^{D\,\prime\prime}(\tau_i) = \begin{cases} -3/h^2, & i,j=1, \text{ or } i,j=N+1, \\ 1/h^2, & i,j=1,2,...,N+1;\ i=j\pm 1, \\ -2/h^2, & i,j=2,3,...,N;\ i=j, \\ 0, & \text{otherwise.} \end{cases} \tag{2.10}$$

As in [17], let w denote the quadratic spline interpolant of u such that

$$w_i = w(\tau_i) = u(\tau_i) = u_i,\ i = 1,2,...,N+1,$$

$$w(0) = u(0) - \frac{h^4 u^{(4)}(0)}{128}, \quad w(1) = u(1) - \frac{h^4 u^{(4)}(1)}{128}.$$

Two important properties of w used in this thesis are proved in [17]: if $u \in C^6[0,1]$, then, for $i = 1,2,...,N+1$,

$$w_i'' = u_i'' - \frac{h^2}{24}u_i^{(4)} + O(h^4), \tag{2.11}$$

and

$$Lw_i = f_i - \frac{h^2}{24}u_i^{(4)} + O(h^4), \tag{2.12}$$

where $u_i^{(k)} = u^{(k)}(\tau_i)$ and $f_i = f(\tau_i)$.

2.1.2 Modified Quadratic Spline Collocation Methods

Archer Quadratic Spline Collocation

This method is analogous to the modified CSC method proposed in [2, 3] for the cubic case. In this method, we seek $u_h^D \in S_2^D$ such that

$$\begin{cases} {u_h^D}''(\tau_i) + c u_h^D(\tau_i) = f_i, & i = 1, \\ \frac{1}{24}\left[{u_h^D}''(\tau_{i-1}) + 22{u_h^D}''(\tau_i) + {u_h^D}''(\tau_{i+1})\right] + c u_h^D(\tau_i) = f_i, & i = 2, 3, ..., N, \\ {u_h^D}''(\tau_i) + c u_h^D(\tau_i) = f_i, & i = N+1, \end{cases} \tag{2.13}$$

with $f_i = f(\tau_i),\ i = 1, 2, ..., N+1$.

Let

$$u_h^D(x) = \sum_{j=1}^{N+1} u_j \mathcal{B}_j^D(x), \tag{2.14}$$

so that

$$ {u_h^D}''(x) = \sum_{j=1}^{N+1} u_j {\mathcal{B}_j^D}''(x). \tag{2.15}$$

Subtituting (2.14) and (2.15) into (2.13) and using (2.10), we obtain a system of linear equations

$$(A + cB)\mathbf{u} = \mathbf{f}, \tag{2.16}$$

where

$$\mathbf{u} = [u_1,\ u_2, ...,\ u_{N+1}]^T, \tag{2.17}$$

$$\mathbf{f} = [f_1,\ f_2, ...,\ f_{N+1}]^T, \tag{2.18}$$

$$A = \frac{1}{24h^2}\begin{bmatrix} -72 & 24 & 0 & & & & \\ 19 & -42 & 20 & 1 & & & \\ 1 & 20 & -42 & 20 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 20 & -42 & 20 & 1 \\ & & & 1 & 20 & -42 & 19 \\ & & & & 0 & 24 & -72 \end{bmatrix}_{(N+1)\times(N+1)}$$

and

$$B = \frac{1}{8}\begin{bmatrix} 5 & 1 & & & \\ 1 & 6 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 6 & 1 \\ & & & 1 & 5 \end{bmatrix}_{(N+1)\times(N+1)} \tag{2.19}$$

Modified Quadratic Spline Collocation

Houstis et al., [17] introduced a method which is an analog of the Daniel and Swartz method [13] for the cubic case. In this method, we seek $u_h^D \in S_2^D$ such that

$$\begin{cases} \frac{1}{24}\left[26u_h^{D\,\prime\prime}(\tau_i) - 5u_h^{D\,\prime\prime}(\tau_{i+1}) + 4u_h^{D\,\prime\prime}(\tau_{i+2}) - u_h^{D\,\prime\prime}(\tau_{i+3})\right] + cu_h^D(\tau_i) = f_i, & i = 1, \\ \frac{1}{24}\left[u_h^{D\,\prime\prime}(\tau_{i-1}) + 22u_h^{D\,\prime\prime}(\tau_i) + u_h^{D\,\prime\prime}(\tau_{i+1})\right] + cu_h^D(\tau_i) = f_i, & i = 2, 3, ..., N, \\ \frac{1}{24}\left[26u_h^{D\,\prime\prime}(\tau_i) - 5u_h^{D\,\prime\prime}(\tau_{i-1}) + 4u_h^{D\,\prime\prime}(\tau_{i-2}) - u_h^{D\,\prime\prime}(\tau_{i-3})\right] + cu_h^D(\tau_i) = f_i, & i = N+1. \end{cases} \tag{2.20}$$

Substituting (2.14)–(2.15) into (2.20) and using (2.10), we obtain the system of linear equations (2.16) where in this case, B, $\mathbf{u}$, and $\mathbf{f}$ are as in (2.19)–(2.18), respectively, and

$$A = \frac{1}{24h^2}\begin{bmatrix} -83 & 40 & -14 & 6 & -1 & & & & \\ 19 & -42 & 20 & 1 & 0 & & & & \\ 1 & 20 & -42 & 20 & 1 & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & 1 & 20 & -42 & 20 & 1 \\ & & & & 0 & 1 & 20 & -42 & 19 \\ & & & & -1 & 6 & -14 & 40 & -83 \end{bmatrix}_{(N+1)\times(N+1)}$$

<u>**New Modified Quadratic Spline Collocation**</u>

We develop a new modified QSC method in which both the differential operator and the right hand side of (2.1) are perturbed. In this method, we seek $u_h^D \in S_2^D$ such that

$$(L_h + c)u_h^D(\tau_i) = f_i^D, \quad i = 1, 2, ..., N+1, \tag{2.21}$$

where

$$L_h u_h^D(\tau_i) = \begin{cases} \frac{1}{24}\left[21 u_h^{D\,\prime\prime}(\tau_i) + u_h^{D\,\prime\prime}(\tau_{i+1})\right], & i = 1, \\ \frac{1}{24}\left[u_h^{D\,\prime\prime}(\tau_{i-1}) + 22 u_h^{D\,\prime\prime}(\tau_i) + u_h^{D\,\prime\prime}(\tau_{i+1})\right], & i = 2, 3, ..., N, \\ \frac{1}{24}\left[21 u_h^{D\,\prime\prime}(\tau_i) + u_h^{D\,\prime\prime}(\tau_{i-1})\right], & i = N+1, \end{cases}$$

and

$$f_i^D = \begin{cases} f_i - \dfrac{f(0)}{12} - \dfrac{h^2\left[f''(0) - cf(0)\right]}{144}, & i = 1, \\ f_i, & i = 2, 3, ..., N, \\ f_i - \dfrac{f(1)}{12} - \dfrac{h^2\left[f''(1) - cf(1)\right]}{144}, & i = N+1. \end{cases}$$

As before, (2.21) can be written as a linear system of the form (2.16) with B and $\mathbf{u}$ are as in (2.19) and (2.17), respectively,

$$\mathbf{f} = \left[f_h^D(\tau_1),\ f_h^D(\tau_2), ...,\ f_h^D(\tau_{N+1})\right]^T, \tag{2.22}$$

and

$$A = \frac{1}{24h^2}\begin{bmatrix} -62 & 19 & 1 & & & & \\ 19 & -42 & 20 & 1 & 0 & & \\ 1 & 20 & -42 & 20 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 20 & -42 & 20 & 1 \\ & & 0 & 1 & 20 & -42 & 19 \\ & & & & 1 & 19 & -62 \end{bmatrix}_{(N+1)\times(N+1)} \tag{2.23}$$

If T is the $(N+1) \times (N+1)$ tridiagonal matrix given by

$$T = \begin{bmatrix} -3 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -3 \end{bmatrix}_{(N+1)\times(N+1)} \tag{2.24}$$

then

$$A = \frac{1}{24h^2} T(T + 24I), \qquad B = \frac{1}{8}(T + 8I), \tag{2.25}$$

where I is the identity matrix. Note that (2.25) denotes important properties of the matrices A and B for deriving the MDAs for the Helmholtz problem.

Since the new modified QSC (2.21) differs from the Archer QSC (2.13) and the modified QSC (2.20) in only the first and last equations, we now show how these equations are derived. From (2.12) with $i = 1$, it follows that

$$\begin{aligned} \frac{1}{24}(21w_1'' + w_2'') + cw_1 &= \frac{1}{24}(-3w_1'' + w_2'') + w_1'' + cw_1 \\ &= \frac{1}{24}(-3w_1'' + w_2'') + Lw_1 \\ &= \frac{1}{24}(-3w_1'' + w_2'') + f_1 - \frac{h^2}{24}u_1^{(4)} + O(h^4). \end{aligned} \tag{2.26}$$

From (2.11), we have

$$-3w_1'' + w_2'' = -3u_1'' + u_2'' - \frac{h^2}{24}\left[-3u_1^{(4)} + u_2^{(4)}\right] + O(h^4). \tag{2.27}$$

Using Taylor series expansions about $x_0 = 0$, we have, with $u_0^{(k)} = u^{(k)}(0)$ and $u_i^{(k)} = u^{(k)}(\tau_i)$, $i = 1, 2$,

$$\begin{aligned} &\text{(a)} \quad u_1'' &&= u_0'' + \frac{h}{2}u_0^{(3)} + \frac{h^2}{8}u_0^{(4)} + O(h^3), \\ &\text{(b)} \quad u_2'' &&= u_0'' + \frac{3h}{2}u_0^{(3)} + \frac{9h^2}{8}u_0^{(4)} + O(h^3), \\ &\text{(c)} \quad u_1^{(4)} &&= u_0^{(4)} + O(h), \\ &\text{(d)} \quad u_2^{(4)} &&= u_0^{(4)} + O(h). \end{aligned} \tag{2.28}$$

Then (2.28(a)) and (2.28(b)) imply

$$-3u_1'' + u_2'' = -2u_0'' + \frac{3}{4}h^2 u_0^{(4)} + O(h^3),$$

and, on using (2.28(c)) and (2.28(d)), (2.27) becomes

$$\begin{aligned}-3w_1'' + w_2'' &= -2u_0'' + \frac{3}{4}h^2u_0^{(4)} + \frac{h^2}{12}u_0^{(4)} + O(h^3)\\ &= -2u_0'' + \frac{5}{6}h^2u_0^{(4)} + O(h^3).\end{aligned}$$

Thus, the right hand side of (2.26) becomes

$$\begin{aligned}f_1 - \frac{1}{12}u_0'' + \frac{5h^2}{144}u_0^{(4)} - \frac{h^2}{24}u_0^{(4)} + O(h^3) &= f_1 - \frac{1}{12}u_0'' - \frac{h^2}{144}u_0^{(4)} + O(h^3)\\ &= f_1 - \frac{1}{12}f(0) - \frac{h^2}{144}\left[f''(0) - cf(0)\right] + O(h^3),\end{aligned}$$

since

$$u_0'' = f(0), \qquad u_0^{(4)} = f''(0) - cf(0),$$

on using (2.1) and the Dirichlet BC at $x = 0$ in (2.2).

Similarly, we have

$$\frac{1}{24}(21w_{N+1}'' + w_N'') + cw_{N+1} = f_{N+1} - \frac{1}{12}f(1) - \frac{h^2}{144}\left[f''(1) - cf(1))\right] + O(h^3).$$

Replacing w by u_h^D and dropping the $O(h^3)$ term, we obtain (2.21) with $i = 1, N+1$.

2.1.3 Numerical Results

In this section, we show that the new modified QSC (2.21) possesses optimal accuracy and superconvergence and produces results which are almost identical to those of the Archer QSC (2.13) and the modified QSC (2.20). Moreover, we consider a simplified version of the new modified QSC in which the $O(h^2)$ terms are omitted in the perturbations of the right hand sides of the collocation equations for $i = 1, N+1$ [4]. As the numerical results indicate, this does not lead to degradation in the accuracy or superconvergence.

We consider three test problems for the second-order TPBVP (2.1) satisfying the Dirichlet BCs (2.2):

- Problem D-1: $c = -9, \quad u(x) = e^x(x^2 - x)^2.$
- Problem D-2: $c = -9, \quad u(x) = e^x(x^2 - x).$
- Problem D-3: $c = -9, \quad u(x) = \cosh\left(3x - \frac{3}{2}\right) - \cosh\left(\frac{3}{2}\right).$

Numerical results are presented in Tables 2.1–2.12. In each table, for several values of N, eight quantities are given for errors in approximations to u, u', and u'', evaluated at the nodal points $\{x_i\}_{i=0}^{N+1}$, the collocation points $\{\tau_i\}_{i=1}^{N+1}$, the Gauss points $\{\eta_i\}_{i=1}^{2(N+1)}$ with $\eta_i = x_i + h\left(3 - \sqrt{3}\right)/6,\ \eta_{i+1} = x_i + h\left(3 + \sqrt{3}\right)/6,\ i = 1, 3, ..., 2N+1$, and an estimate of global errors computed at 10 equally spaced points in each subinterval $I_i,\ i = 1, 2, ..., N+1$:

- $E_n(N)$ – the maximum absolute error at the nodal points, the points of the uniform partition ρ_x,
- $E_c(N)$ – the maximum absolute error at the collocation points,
- $E_G(N)$ – the maximum absolute error at the Gauss points,
- $E_g(N)$ – the maximum absolute global error,
- $R_n(N)$ – the experimental rate of convergence of the error at the nodal points computed from

$$R_n(N) = \frac{\log[E_n(N/2)/E_n(N)]}{\log[(N+1)/((N/2)+1)]}, \tag{2.29}$$

- $R_m(N)$ – the experimental rate of convergence of the error at the collocation points computed as in (2.29), with E_m replacing E_n,
- $R_G(N)$ – the experimental rate of convergence of the error at the Gauss points computed as in (2.29), with E_G replacing E_n,

- $R_g(N)$ – the experimental rate of convergence of the global error computed as in (2.29), with E_g replacing E_n.

The numerical results for the Archer QSC (2.13), the modified QSC (2.20), the new modified QSC (2.21) and its simplification exhibit the expected orders of convergence, namely, $O(h^{3-k})$ global accuracy for the k^{th} derivative, $k = 0, 1, 2$, and for superconvergence, we obtain fourth-order accuracy at the nodal points and the collocation points, third-order accuracy for the first derivative at the Gauss points, and second-order accuracy for the second derivative at the collocation points. Although the modified QSC methods produce similar numerical results exhibiting optimal rates of convergence and superconvergence, the new modified QSC method and its simplified version have desirable additional properties shown in (2.25), which can be employed for MDAs in Helmholtz problems.

Table 2.1. Archer QSC for Dirichlet 2-point BVP, Problem D-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_c(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_c(N)$	$R_G(N)$	$R_g(N)$
u								
16	4.0147(-5)	1.9085(-5)	9.5301(-5)	9.5711(-5)				
32	3.2040(-6)	1.6400(-6)	1.2267(-5)	1.2222(-5)	3.8115	3.7000	3.0908	3.1028
64	2.2730(-7)	1.2096(-7)	1.5284(-6)	1.5145(-6)	3.9032	3.8458	3.0723	3.0804
128	1.5153(-8)	8.2262(-9)	1.8955(-7)	1.8720(-7)	3.9509	3.9218	3.0453	3.0502
256	9.7846(-10)	5.3655(-10)	2.3556(-8)	2.3221(-8)	3.9753	3.9606	3.0254	3.0281
u'								
16	1.2479(-2)	6.8158(-3)	8.5637(-4)	6.6648(-3)				
32	3.5075(-3)	1.8488(-3)	1.3025(-4)	1.8057(-3)	1.9134	1.9670	2.8392	1.9688
64	9.3306(-4)	4.8021(-4)	1.8042(-5)	4.6869(-4)	1.9534	1.9887	2.9162	1.9897
128	2.4086(-4)	1.2226(-4)	2.3769(-6)	1.2055(-4)	1.9758	1.9959	2.9571	1.9811
256	6.1202(-5)	3.0839(-5)	3.0511(-7)	3.0741(-5)	1.9877	1.9984	2.9783	1.9825
u''								
16		2.6409(-2)	7.5258(-1)	1.0850				
32		7.9699(-3)	4.0688(-1)	5.8163(-1)		1.8062	9.2719(-1)	9.3999-1)
64		2.1968(-3)	2.1180(-1)	3.0147(-1)		1.9010	9.6308(-1)	9.6945(-1)
128		5.7725(-4)	1.0809(-1)	1.5351(-1)		1.9499	9.8141(-1)	9.8459(-1)
256		1.4799(-5)	5.4605(-2)	7.7468(-2)		1.9748	9.9068(-1)	9.9226(-1)

Table 2.2. Modified QSC for Dirichlet 2-point BVP, Problem D-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_c(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_c(N)$	$R_G(N)$	$R_g(N)$
u								
16	1.0956(-6)	1.8705(-5)	8.3468(-5)	8.1123(-5)				
32	8.6009(-8)	1.4177(-6)	1.1220(-5)	1.0963(-5)	3.8363	3.8893	3.0255	3.0175
64	5.8868(-9)	9.7237(-8)	1.4502(-6)	1.4215(-6)	3.9561	3.9530	3.0182	3.0135
128	3.8240(-10)	6.3584(-9)	1.8420(-7)	1.8087(-7)	3.9887	3.9791	3.0104	3.0079
256	2.4351(-11)	4.0633(-10)	2.3206(-8)	2.2808(-8)	3.9955	3.9903	3.0055	3.0042
u'								
16	1.4003(-2)	6.1482(-3)	6.8328(-4)	7.3575(-3)				
32	3.7377(-3)	1.7438(-3)	9.7494(-5)	1.9286(-3)	1.9913	1.8998	2.9355	2.0186
64	9.6468(-4)	4.6547(-4)	1.3038(-5)	4.9234(-4)	1.9980	1.9484	2.9679	2.0142
128	2.4500(-4)	1.2031(-4)	1.6866(-6)	1.2429(-4)	1.9996	1.9739	2.9838	2.0083
256	6.1732(-5)	3.0587(-5)	2.1450(-7)	3.1221(-5)	1.9999	1.9869	2.9918	2.0044
u''								
16		2.9242(-2)	7.8170(-1)	1.1141				
32		8.2737(-3)	4.1514(-1)	5.8989(-1)		1.9034	9.5411(-1)	9.5865(-1)
64		2.1945(-3)	2.1400(-1)	3.0366(-1)		1.9578	9.7754(-1)	9.7957(-1)
128		5.6462(-4)	1.0866(-1)	1.5408(-1)		1.9806	9.8885(-1)	9.8982(-1)
256		1.4316(-4)	5.4749(-2)	7.7611(-2)		1.9908	9.9444(-1)	9.9491(-1)

Table 2.3. New Modified QSC for Dirichlet 2-point BVP, Problem D-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_c(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_c(N)$	$R_G(N)$	$R_g(N)$
u								
16	1.2724(-6)	1.9409(-5)	8.3833(-5)	8.1529(-5)				
32	8.9610(-8)	1.4329(-6)	1.1228(-5)	1.0969(-5)	4.0000	3.9290	3.0310	3.0241
64	5.9486(-9)	9.7517(-8)	1.4503(-6)	1.4216(-6)	4.0012	3.9644	3.0191	3.0143
128	3.8344(-10)	6.3631(-9)	1.8420(-7)	1.8088(-7)	4.0001	3.9822	3.0106	3.0080
256	2.4367(-11)	4.0640(-10)	2.3206(-8)	2.2808(-8)	3.9984	3.9911	3.0056	3.0042
u'								
16	1.4037(-2)	6.1336(-3)	6.7945(-4)	7.3874(-3)				
32	3.7391(-3)	1.7431(-3)	9.7297(-5)	1.9299(-3)	1.9943	1.8967	2.9301	2.0237
64	9.6473(-4)	4.6544(-4)	1.3030(-5)	4.9238(-4)	1.9985	1.9479	2.9659	2.0150
128	2.4500(-4)	1.2031(-4)	1.6863(-6)	1.2429(-4)	1.9996	1.9738	2.9832	2.0084
256	6.1732(-5)	3.0587(-5)	2.1449(-7)	3.1221(-5)	1.9999	1.9869	2.9916	2.0044
u''								
16		2.9877(-2)	7.8234(-1)	1.1147(-1)				
32		8.3230(-3)	4.1519(-1)	5.8994(-1)		1.9269	9.5516(-1)	9.5939(-1)
64		2.1980(-3)	2.1400(-1)	3.0366(-1)		1.9642	9.7769(-1)	9.7968(-1)
128		5.6484(-4)	1.0866(-1)	1.5408(-1)		1.9823	9.8887(-1)	9.8983(-1)
256		1.4318(-4)	5.4749(-2)	7.7611(-2)		1.9912	9.9444(-1)	9.9491(-1)

Table 2.4. New Modified QSC (simplified version) for Dirichlet 2-point BVP, Problem D-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_c(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_c(N)$	$R_G(N)$	$R_g(N)$
u								
16	8.4014(-6)	1.2529(-5)	8.0269(-5)	7.8318(-5)				
32	6.4216(-7)	9.2176(-7)	1.0965(-5)	1.0739(-5)	3.8766	3.9341	3.0012	2.9954
64	4.4492(-8)	6.2600(-8)	1.4325(-6)	1.4060(-6)	3.9380	3.9675	3.0025	2.9993
128	2.9297(-9)	4.0800(-9)	1.8304(-7)	1.7986(-7)	3.9689	3.9839	3.0017	3.0001
256	1.8798(-10)	2.6043(-10)	2.3131(-8)	2.2743(-8)	3.9844	3.9920	3.0010	3.0002
u'								
16	1.3711(-2)	6.2761(-3)	7.1641(-4)	7.0953(-3)				
32	3.6928(-3)	1.7643(-3)	1.0389(-4)	1.8881(-3)	1.9778	1.9132	2.9111	1.9959
64	9.5854(-4)	4.6833(-4)	1.4010(-5)	4.8679(-4)	1.9896	1.9566	2.9556	1.9996
128	2.4420(-4)	1.2069(-4)	1.8197(-6)	1.2357(-4)	1.9950	1.9783	2.9778	2.0002
256	6.1630(-5)	3.0636(-5)	2.3189(-7)	3.1129(-5)	1.9975	1.9891	2.9889	2.0002
u''								
16		2.5073(-2)	7.7612(-1)	1.1085				
32		7.6448(-3)	4.1353(-1)	5.8828(-1)		1.7907	9.4918(-1)	9.5521(-1)
64		2.1147(-3)	2.1357(-1)	3.0323(-1)		1.8958	9.7474(-1)	9.7760(-1)
128		5.5638(-4)	1.0855(-1)	1.5397(-1)		1.9480	9.8740(-1)	9.8880(-1)
256		1.4271(-4)	5.4721(-2)	7.7584(-2)		1.9740	9.9371(-1)	9.9440(-1)

Table 2.5. Archer QSC for Dirichlet 2-point BVP, Problem D-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	8.3395(-6)	3.9541(-6)	4.3314(-5)	4.3015(-5)				
32	6.3745(-7)	3.2580(-7)	5.7335(-6)	5.6711(-6)	3.8765	3.7634	3.0487	3.0547
64	4.4251(-8)	2.3530(-8)	7.3396(-7)	7.2418(-7)	3.9352	3.8768	3.0324	3.0361
128	2.9181(-9)	1.5835(-9)	9.2697(-8)	9.1334(-8)	3.9668	3.9372	3.0187	3.0208
256	1.8740(-10)	1.0274(-10)	1.1642(-8)	1.1462(-8)	3.9832	3.9682	3.0101	3.0112
u'								
16	6.7306(-3)	3.4891(-3)	1.7629(-4)	3.4063(-3)				
32	1.8260(-3)	9.3220(-4)	2.5805(-5)	9.1117(-4)	1.9668	1.9899	2.8970	1.9880
64	4.7637(-4)	2.4087(-4)	3.5052(-6)	2.3887(-4)	1.9822	1.9964	2.9450	1.9750
128	1.2171(-4)	6.1211(-5)	4.5725(-7)	6.1191(-5)	1.9908	1.9986	2.9715	1.9870
256	3.0767(-5)	1.5428(-5)	5.8407(-8)	1.5488(-5)	1.9953	1.9994	2.9855	1.9933
u''								
16		5.7698(-3)	4.0027(-1)	5.7084(-1)				
32		1.6268(-3)	2.0995(-1)	2.9849(-1)		1.9087	9.7284(-1)	9.7748(-1)
64		4.3324(-4)	1.0760(-1)	1.5273(-1)		1.9518	9.8608(-1)	9.8845(-1)
128		1.1188(-4)	5.4479(-2)	7.7268(-1)		1.9752	9.9296(-1)	9.9415(-1)
256		2.8435(-5)	2.7412(-2)	3.8863(-2)		1.9874	9.9646(-1)	9.9706(-1)

Table 2.6. Modified QSC for Dirichlet 2-point BVP, Problem D-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	8.8283(-7)	3.6786(-6)	4.0679(-5)	3.9823(-5)				
32	6.1148(-8)	2.7282(-7)	5.5179(-6)	5.4140(-6)	4.0251	3.9221	3.0118	3.0084
64	4.0439(-9)	1.8583(-8)	7.1847(-7)	7.0583(-7)	4.0068	3.9632	3.0074	3.0055
128	2.6038(-10)	1.2125(-9)	9.1658(-8)	9.0108(-8)	4.0016	3.9823	3.0041	3.0031
256	1.6658(-11)	7.7429(-11)	1.1575(-8)	1.1383(-8)	3.9887	3.9913	3.0021	3.0016
u'								
16	7.0397(-3)	3.3533(-3)	1.4055(-4)	3.6080(-3)				
32	1.8711(-3)	9.1158(-4)	1.9343(-5)	9.5184(-4)	1.9976	1.9637	2.9900	2.0089
64	4.8247(-4)	2.3802(-4)	2.5366(-6)	2.4438(-4)	1.9994	1.9809	2.9969	2.0058
128	1.2251(-4)	6.0837(-5)	3.2475(-7)	6.1909(-5)	1.9999	1.9902	2.9989	2.0032
256	3.0866(-5)	1.5380(-5)	4.1082(-8)	1.5580(-5)	2.0000	1.9951	2.9995	2.0017
u''								
16		5.9177(-3)	4.0616(-1)	5.7673(-1)				
32		1.6198(-3)	2.1156(-1)	3.0011(-1)		1.9533	9.8330(-1)	9.8482(-1)
64		4.2343(-4)	1.0802(-1)	1.5316(-1)		1.9792	9.9161(-1)	9.9234(-1)
128		1.0822(-4)	5.4587(-2)	7.7376(-2)		1.9904	9.9579(-1)	9.9615(-1)
256		2.7352(-5)	2.7440(-2)	3.8890(-2)		1.9954	9.9789(-1)	9.9807(-1)

Table 2.7. New Modified QSC for Dirichlet 2-point BVP, Problem D-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	8.6168(-7)	3.7189(-6)	4.0699(-5)	3.9842(-5)				
32	6.0744(-8)	2.7366(-7)	5.5184(-6)	5.4143(-6)	3.9985	3.9338	3.0124	3.0090
64	4.0371(-9)	1.8598(-8)	7.1848(-7)	7.0584(-7)	3.9995	3.9665	3.0075	3.0056
128	2.6026(-10)	1.2128(-9)	9.1658(-8)	9.0108(-8)	3.9998	3.9832	3.0041	3.0031
256	1.6656(-11)	7.7433(-11)	1.1575(-8)	1.1383(-8)	3.9882	3.9916	3.0021	3.0016
u'								
16	7.0416(-3)	3.3525(-3)	1.4033(-4)	3.6097(-3)				
32	1.8712(-3)	9.1155(-4)	1.9332(-5)	9.5191(-4)	1.9980	1.9634	2.9885	2.0095
64	4.8248(-4)	2.3802(-4)	2.5361(-6)	2.4438(-4)	1.9995	1.9809	2.9963	2.0059
128	1.2251(-4)	6.0837(-5)	3.2473(-7)	6.1909(-5)	1.9999	1.9902	2.9987	2.0032
256	3.0866(-5)	1.5380(-5)	4.1081(-8)	1.5580(-5)	2.0000	1.9951	2.9995	2.0017
u''								
16		5.9539(-3)	4.0620(-1)	5.7677(-1)				
32		1.6226(-3)	2.1157(-1)	3.0011(-1)		1.9600	9.8341(-1)	9.8490(-1)
64		4.2362(-4)	1.0802(-1)	1.5316(-1)		1.9811	9.9163(-1)	9.9235(-1)
128		1.0823(-4)	5.4587(-2)	7.7376(-1)		1.9909	9.9579(-1)	9.9615(-1)
256		2.7353(-5)	2.7440(-2)	3.8890(-2)		1.9955	9.9789(-1)	9.9807(-1)

Table 2.8. New Modified QSC (simplified version) for Dirichlet 2-point BVP, Problem D-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	1.9076(-6)	2.4045(-6)	4.0019(-5)	3.9245(-5)				
32	1.3399(-7)	1.7617(-7)	5.4683(-6)	5.3705(-6)	4.0040	3.9404	3.0008	2.9985
64	8.8779(-9)	1.1943(-8)	7.1507(-7)	7.0286(-7)	4.0039	3.9702	3.0010	2.9998
128	5.7130(-10)	7.7774(-10)	9.1436(-8)	8.9914(-8)	4.0025	3.9851	3.0007	3.0001
256	3.6233(-11)	4.9622(-11)	1.1560(-8)	1.1370(-8)	4.0013	3.9926	3.0004	3.0001
u'								
16	6.9795(-3)	3.3798(-3)	1.4750(-4)	3.5539(-3)				
32	1.8624(-3)	9.1558(-4)	2.0597(-5)	9.4395(-4)	1.9918	1.9690	2.9681	1.9987
64	4.8130(-4)	2.3857(-4)	2.7233(-6)	2.4332(-4)	1.9961	1.9840	2.9847	1.9999
128	1.2236(-4)	6.0909(-5)	3.5018(-7)	6.1771(-5)	1.9981	1.9919	2.9925	2.0001
256	3.0847(-5)	1.5389(-5)	4.4399(-8)	1.5562(-5)	1.9991	1.9959	2.9963	2.0001
u''								
16		5.5001(-3)	4.0501(-1)	5.7558(-1)				
32		1.5631(-3)	2.1125(-1)	2.9980(-1)		1.8968	9.8127(-1)	9.8339(-1)
64		4.1736(-4)	1.0794(-1)	1.5308(-1)		1.9479	9.9054(-1)	9.9159(-1)
128		1.0788(-4)	5.4566(-2)	7.7355(-2)		1.9739	9.9524(-1)	9.9576(-1)
256		2.7426(-5)	2.7434(-2)	3.8885(-2)		1.9869	9.9761(-1)	9.9787(-1)

Table 2.9. Archer QSC for Dirichlet 2-point BVP, Problem D-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_c(N)$	$R_G(N)$	$R_g(N)$
u								
16	3.4796(-5)	1.6940(-5)	1.0761(-4)	1.0756(-4)				
32	2.7217(-6)	1.4080(-6)	1.3987(-5)	1.3891(-5)	3.8418	3.7502	3.0762	3.0858
64	1.9132(-7)	1.0232(-7)	1.7630(-6)	1.7436(-6)	3.9167	3.8677	3.0553	3.0615
128	1.2699(-8)	6.9105(-9)	2.2045(-7)	2.1748(-7)	3.9573	3.9320	3.0334	3.0370
256	8.1823(-10)	4.4923(-10)	2.7529(-8)	2.7121(-8)	3.9783	3.9655	3.0183	3.0203
u'								
16	1.5194(-2)	8.1064(-3)	7.3833(-4)	7.9210(-3)				
32	4.2001(-3)	2.1815(-3)	1.1039(-4)	2.1297(-3)	1.9385	1.9790	2.8650	1.9803
64	1.1071(-3)	5.6511(-4)	1.5169(-5)	5.5257(-4)	1.9670	1.9926	2.9279	1.9903
128	2.8441(-4)	1.4375(-4)	1.9908(-6)	1.4265(-4)	1.9828	1.9973	2.9627	1.9757
256	7.2091(-5)	3.6245(-5)	2.5509(-7)	3.6249(-5)	1.9912	1.9989	2.9810	1.9876
u''								
16		2.3122(-2)	9.1063(-1)	1.3060				
32		6.7897(-3)	4.8523(-1)	6.9185(-1)		1.8474	9.4908(-1)	9.5789(-1)
64		1.8507(-3)	2.5073(-1)	3.5642(-1)		1.9175	9.7398(-1)	9.7844(-1)
128		4.8390(-4)	1.2748(-1)	1.8094(-1)		1.9571	9.8685(-1)	9.8909(-1)
256		1.2377(-4)	6.4281(-2)	9.1165(-2)		1.9781	9.9339(-1)	9.9451(-1)

Table 2.10. Modified QSC for Dirichlet 2-point BVP, Problem D-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_c(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_c(N)$	$R_G(N)$	$R_g(N)$
u								
16	9.8902(-7)	1.5658(-5)	9.6864(-5)	9.4463(-5)				
32	4.0404(-8)	1.1861(-6)	1.3082(-5)	1.2806(-5)	4.4821	3.8902	3.0184	3.0126
64	2.2881(-9)	8.1281(-8)	1.6967(-6)	1.6648(-5)	4.2356	3.9543	3.0131	3.0097
128	1.4040(-10)	5.3111(-9)	2.1594(-7)	2.1216(-7)	4.0718	3.9802	3.0075	3.0056
256	9.2626(-12)	3.3925(-10)	2.7235(-8)	2.6776(-8)	3.9441	3.9910	3.0039	3.0030
u'								
16	1.6491(-2)	7.5317(-3)	5.8040(-4)	8.5631(-3)				
32	4.3941(-3)	2.0925(-3)	8.2026(-5)	2.2523(-3)	1.9939	1.9309	2.9499	2.0134
64	1.1336(-3)	5.5271(-4)	1.0922(-5)	5.7652(-4)	1.9987	1.9639	2.9743	2.0102
128	2.8788(-4)	1.4211(-4)	1.4101(-6)	1.4578(-4)	1.9997	1.9816	2.9867	2.0059
256	7.2534(-5)	3.6035(-5)	1.7917(-7)	3.6650(-5)	1.9999	1.9907	2.9932	2.0031
u''								
16		2.4651(-2)	9.3518(-1)	1.3306				
32		6.9409(-3)	4.9216(-1)	6.9878(-1)		1.9108	9.6779(-1)	9.7093(-1)
64		1.8364(-3)	2.5257(-1)	3.5825(-1)		1.9615	9.8415(-1)	9.8557(-1)
128		4.7184(-4)	1.2795(-1)	1.8141(-1)		1.9826	9.9211(-1)	9.9279(-1)
256		1.1955(-4)	6.4400(-2)	9.1285(-2)		1.9918	9.9605(-1)	9.9639(-1)

Table 2.11. New Modified QSC for Dirichlet 2-point BVP, Problem D-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_c(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_c(N)$	$R_G(N)$	$R_g(N)$
u								
16	4.5441(-7)	1.6342(-5)	9.7217(-5)	9.4772(-5)				
32	3.2147(-8)	1.2011(-5)	1.3089(-5)	1.2813(-5)	3.9932	3.9357	3.0230	3.0168
64	2.1383(-9)	8.1558(-8)	1.6968(-6)	1.6650(-6)	3.9982	3.9677	3.0139	3.0104
128	1.3785(-10)	5.3159(-9)	2.1595(-7)	2.1216(-7)	3.9998	3.9838	3.0076	3.0057
256	9.2222(-12)	3.3933(-10)	2.7235(-8)	2.6776(-8)	3.9238	3.9919	3.0040	3.0030
u'								
16	1.6523(-2)	7.5174(-3)	5.7648(-4)	8.5920(-3)				
32	4.3955(-3)	2.0918(-3)	8.1829(-5)	2.2535(-3)	1.9963	1.9285	2.9434	2.0177
64	1.1337(-3)	5.5269(-4)	1.0915(-5)	5.7657(-4)	1.9990	1.9635	2.9718	2.0109
128	2.8788(-4)	1.4211(-4)	1.4098(-6)	1.4578(-4)	1.9998	1.9815	2.9859	2.0060
256	7.2534(-5)	3.6034(-5)	1.7916(-7)	3.6650(-5)	1.9999	1.9907	2.9930	2.0032
u''								
16		2.5261(-2)	9.3579(-1)	1.3312				
32		6.9891(-3)	4.9221(-1)	6.9883(-1)		1.9372	9.6862(-1)	9.7152(-1)
64		1.8398(-3)	2.5257(-1)	3.5826(-1)		1.9689	9.8427(-1)	9.8566(-1)
128		4.7207(-4)	1.2795(-1)	1.8141(-1)		1.9846	9.9212(-1)	9.9280(-1)
256		1.1957(-4)	6.4400(-2)	9.1285(-2)		1.9923	9.9606(-1)	9.9639(-1)

Table 2.12. New Modified QSC (simplified version) for Dirichlet 2-point BVP, Problem D-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_c(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_c(N)$	$R_G(N)$	$R_g(N)$
u								
16	7.2935(-6)	1.0526(-5)	9.4212(-5)	9.2138(-5)				
32	5.4553(-7)	7.7187(-7)	1.2869(-5)	1.2620(-5)	3.9092	3.9392	3.0013	2.9971
64	3.7443(-8)	5.2331(-8)	1.6819(-6)	1.6519(-6)	3.9519	3.9701	3.0019	2.9996
128	2.4549(-9)	3.4077(-9)	2.1497(-7)	2.1131(-7)	3.9753	3.9852	3.0013	3.0001
256	1.5719(-10)	2.1742(-10)	2.7173(-8)	2.6721(-8)	3.9874	3.9927	3.0007	3.0001
u'								
16	1.6249(-2)	7.6389(-3)	6.0985(-4)	8.3457(-3)				
32	4.3567(-3)	2.1097(-3)	8.7507(-5)	2.2185(-3)	1.9845	1.9399	2.9271	1.9975
64	1.1285(-3)	5.5511(-4)	1.1744(-5)	5.7189(-4)	1.9927	1.9695	2.9627	1.9998
128	2.8721(-4)	1.4243(-4)	1.5219(-6)	1.4518(-4)	1.9965	1.9847	2.9812	2.0002
256	7.2449(-5)	3.6075(-5)	1.9373(-7)	3.6573(-5)	1.9983	1.9923	2.9905	2.0002
u''								
16		2.1981(-2)	9.3060(-1)	1.3260				
32		6.5160(-3)	4.9082(-1)	6.9744(-1)		1.8331	9.6450(-1)	9.6863(-1)
64		1.7819(-3)	2.5221(-1)	3.5790(-1)		1.9127	9.8220(-1)	9.8420(-1)
128		4.6645(-4)	1.2786(-1)	1.8132(-1)		1.9554	9.9109(-1)	9.9207(-1)
256		1.1936(-4)	6.4377(-2)	9.1262(-2)		1.9775	9.9554(-1)	9.9603(-1)

2.2 Helmholtz Problem

Consider the Helmholtz equation

$$\Delta u(x,y) + cu(x,y) = g(x,y), \quad (x,y) \in \Omega, \tag{2.30}$$

where Δ is the Laplacian operator, $\Delta \equiv D_x^2 + D_y^2$, $D_x^k = \partial^k/\partial x^k$ and $D_y^k = \partial^k/\partial y^k$, Ω is the unit square, $\Omega = [0,1] \times [0,1]$, and $c \leq 0$ is a constant. For homogeneous Dirichlet BCs, we have $u = 0$ on $\partial\Omega$, specifically,

$$\begin{cases} u(0,y) = u(1,y) = 0, & y \in [0,1], \\ u(x,0) = u(x,1) = 0, & x \in [0,1]. \end{cases} \tag{2.31}$$

We seek an approximate solution u_h of (2.30) and (2.31) in the space of biquadratic splines subject to Dirichlet BCs, that is, $S_2^D \otimes S_2^D$, where, for two spaces V and W of functions, $V \otimes W$ denotes the space of functions on Ω consisting of all finite linear combinations of products $v(x)w(y)$, $v \in V$ and $w \in W$.

Let $\rho_x = \{x_i\}_{i=0}^{N+1}$ and $\rho_y = \{y_j\}_{j=0}^{N+1}$ be, respectively, uniform partitions of $[0,1]$ in the x and y directions such that $x_i = ih$ and $y_j = jh$, where $h = 1/(N+1)$. Let $\{\tau_i^x\}_{i=1}^{N+1}$ and $\{\tau_j^y\}_{j=1}^{N+1}$ be, respectively, the sets of midpoints of the subintervals $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$, $\tau_i^x = (x_i + x_{i-1})/2$ and $\tau_j^y = (y_j + y_{j-1})/2$. Then $\{(\tau_i^x, \tau_j^y)\}_{i,j=1}^{N+1}$ is the set of collocation points in Ω.

2.2.1 New Modified Quadratic Spline Collocation

In QSC methods for solving (2.30) and (2.31), we seek an approximate solution $u_h^D \in S_2^D \otimes S_2^D$ of the form

$$u_h^D(x,y) = \sum_{m=1}^{N+1} \sum_{n=1}^{N+1} u_{m,n} \mathcal{B}_m^D(x) \mathcal{B}_n^D(y). \tag{2.32}$$

Similar to the modified QSC for second-order TPBVPs [4], we have

$$(L_x + L_y + c)u_h^D(\tau_i^x, \tau_j^y) = g_{i,j}^D, \tag{2.33}$$

where

$$g_{i,j}^D = \begin{cases} g_{i,j}, & i,j = 2,3,...,N, \\ g_{i,j} - \frac{1}{12} g_{0,j}, & i = 1;\ j = 2,3,...,N, \\ g_{i,j} - \frac{1}{12} g_{i,0}, & i = 2,3,...,N;\ j = 1, \\ g_{i,j} - \frac{1}{12} g_{N+2,j}, & i = N+1;\ j = 2,3,...,N, \\ g_{i,j} - \frac{1}{12} g_{i,N+2}, & i = 2,3,...,N;\ j = N+1, \\ g_{i,j} - \frac{1}{12}\left[g_{0,j} + g_{i,0}\right], & i,j = 1, \\ g_{i,j} - \frac{1}{12}\left[g_{0,j} + g_{i,N+2}\right], & i = 1;\ j = N+1, \\ g_{i,j} - \frac{1}{12}\left[g_{N+2,j} + g_{i,0}\right], & i = N+1;\ j = 1, \\ g_{i,j} - \frac{1}{12}\left[g_{N+2,j} + g_{i,N+2}\right], & i,j = N+1, \end{cases}$$

with $g_{i,j} = g(\tau_i^x, \tau_j^y),\ i, j = 1, 2, ..., N+1$, and, for each $j = 1, 2, ..., N+1$,

$$L_x u_h^D(\tau_i^x, \tau_j^y) = \begin{cases} \frac{1}{24}\left[21 D_x^2 u_h^D(\tau_i^x, \tau_j^y) + D_x^2 u_h^D(\tau_{i+1}^x, \tau_j^y)\right], & i = 1, \\ \frac{1}{24}\left[D_x^2 u_h^D(\tau_{i-1}^x, \tau_j^y) + 22 D_x^2 u_h^D(\tau_i^x, \tau_j^y) \right. \\ \quad \left. + D_x^2 u_h^D(\tau_{i+1}^x, \tau_j^y)\right], & i = 2, 3, ..., N, \\ \frac{1}{24}\left[21 D_x^2 u_h^D(\tau_i^x, \tau_j^y) + D_x^2 u_h^D(\tau_{i-1}^x, \tau_j^y)\right], & i = N+1, \end{cases}$$

and, for each $i = 1, 2, ..., N+1$,

$$L_y u_h^D(\tau_i^x, \tau_j^y) = \begin{cases} \frac{1}{24}\left[21 D_y^2 u_h^D(\tau_i^x, \tau_j^y) + D_y^2 u_h^D(\tau_i^x, \tau_{j+1}^y)\right], & j = 1, \\ \frac{1}{24}\left[D_y^2 u_h^D(\tau_i^x, \tau_{j-1}^y) + 22 D_y^2 u_h^D(\tau_i^x, \tau_j^y) \right. \\ \quad \left. + D_y^2 u_h^D(\tau_i^x, \tau_{j+1}^y)\right], & j = 2, 3, ..., N, \\ \frac{1}{24}\left[21 D_y^2 u_h^D(\tau_i^x, \tau_j^y) + D_y^2 u_h^D(\tau_i^x, \tau_{j-1}^y)\right], & j = N+1. \end{cases}$$

Therefore, the collocation equations (2.33) (as do the methods of subsequent chapters) give rise to a system of linear equations of the form

$$(A \otimes B + B \otimes A + cB \otimes B)\mathbf{u} = \mathbf{g}, \tag{2.34}$$

where $\otimes$ denotes the matrix tensor product[1],

$$\mathbf{u} = \left[u_{1,1}, ..., u_{1,N+1}, ..., u_{N+1,1}, ..., u_{N+1,N+1}\right]^T,$$

[1]If the matrix $A = (a_{i,j})$ is $M_A \times N_A$ and B is $M_B \times N_B$, then the matrix $A \otimes B$ is the $M_A M_B \times N_A N_B$ block matrix whose (i, j) block is $a_{i,j} B$.

and

$$\mathbf{g} = [g_{1,1}, ..., g_{1,N+1}, ..., g_{N+1,1}, ..., g_{N+1,N+1}]^T.$$

To solve such a system, suppose there exists a nonsingular matrix Z such that

$$Z^{-1}AZ = \Lambda_A, \qquad Z^{-1}BZ = \Lambda_B, \tag{2.35}$$

where Λ_A and Λ_B are diagonal. Since $(Z \otimes I)(Z^{-1} \otimes I) = I$, the system (2.34) is equivalent to

$$(Z^{-1} \otimes I)(A \otimes B + B \otimes A + cB \otimes B)(Z \otimes I)(Z^{-1} \otimes I)\mathbf{u} = (Z^{-1} \otimes I)\mathbf{g}. \tag{2.36}$$

Using (2.35), (2.36) can be written as

$$[\Lambda_A \otimes B + \Lambda_B \otimes (A + cB)]\,\mathbf{v} = \mathbf{w}, \tag{2.37}$$

where

$$\mathbf{v} = (Z^{-1} \otimes I)\mathbf{u}, \qquad \mathbf{w} = (Z^{-1} \otimes I)\mathbf{g}.$$

If $\Lambda_A = \text{diag}(\lambda_i^A)_{i=1}^{N+1}$ and $\Lambda_B = \text{diag}(\lambda_i^B)_{i=1}^{N+1}$, the solution of (2.37) reduces to the solution of the independent systems

$$\left[\lambda_i^A B + \lambda_i^B (A + cB)\right] \mathbf{v}_i = \mathbf{w}_i, \quad i = 1, 2, ..., N+1,$$

where

$$\mathbf{v_i} = [v_{i,1}, v_{i,2}, ..., v_{i,N+1}]^T, \qquad \mathbf{w_i} = [w_{i,1}, w_{i,2}, ..., w_{i,N+1}]^T.$$

We thus have the following **Matrix Decomposition Algorithm (MDA)**:

Step 1. Compute $\mathbf{w} = (Z^{-1} \otimes I)\mathbf{g}$.

This is equivalent to computing

$$\widehat{\mathbf{w}}_j = Z^{-1}\widehat{\mathbf{g}}_j, \qquad j = 1, 2, ..., N+1,$$

where

$$\widehat{\mathbf{g}}_j = [g_{1,j}, g_{2,j}, ..., g_{N+1,j}]^T, \qquad \widehat{\mathbf{w}}_j = [w_{1,j}, w_{2,j} ..., w_{N+1,j}]^T.$$

In this and subsequent methods for other BCs, the elements of the matrix Z are sines or cosines, hence Z^{-1} can be expressed in terms of Z or Z^T. Therefore, this step can be done using FFTs at a cost of $O(N^2 \log N)$ operations.

Step 2. Solve the $(N+1)$ systems

$$\left[\lambda_i^A B + \lambda_i^B (A + cB)\right] \mathbf{v}_i = \mathbf{w}_i, \quad i = 1, 2, ..., N+1.$$

These will be tridiagonal or pentadiagonal and can be solved at a total cost of $O(N^2)$ operations.

Step 3. Compute $\mathbf{u} = (Z \otimes I)\mathbf{v}$.

This is equivalent to computing

$$\widehat{\mathbf{u}}_j = Z\widehat{\mathbf{v}}_j, \qquad j = 1, 2, ..., N+1,$$

where

$$\widehat{\mathbf{u}}_j = [u_{1,j}, u_{2,j} ..., u_{N+1,j}]^T, \qquad \widehat{\mathbf{v}}_j = [v_{1,j}, v_{2,j}, ..., v_{N+1,j}]^T.$$

As in step 1, this step can be performed at a cost of $O(N^2 \log N)$ operations.

The total cost of the algorithm is thus $O(N^2 \log N)$.

For (2.33), the matrices A and B are defined in (2.23) and (2.19), respectively. Therefore, from (2.25), for $i = 1, 2, ..., N + 1$, we have

$$\lambda_i^A = \frac{1}{24h^2}\lambda_i^T(\lambda_i^T + 24), \qquad \lambda_i^B = \frac{1}{8}(\lambda_i^T + 8), \tag{2.38}$$

where by [12], λ_i^T is an eigenvalue of the tridiagonal matrix T defined by (2.24) and given by

$$\lambda_i^T = -4\sin^2\frac{i\pi}{2(N+1)}, \quad i = 1, 2, ..., N + 1,$$

and the columns of the matrix Z are the corresponding eigenvectors of T, so that

$$Z_{i,j} = \begin{cases} \sqrt{\dfrac{2}{N+1}}\sin\left[\dfrac{(2i-1)j\pi}{2(N+1)}\right], & i = 1, 2, ..., N+1;\ j = 1, 2, ..., N, \\ \\ \sqrt{\dfrac{1}{N+1}}\sin\left[\dfrac{(2i-1)\pi}{2}\right], & i = 1, 2, ..., N+1;\ j = N+1. \end{cases} \tag{2.39}$$

From (2.39), we have

$$Z^{-1} = Z^T.$$

It should be noted that this method is more straightforward than OSMs for the cubic case as it has no boundary or corner collocation equations that require special attention.

2.2.2 Numerical Results

We consider four test problems for the Helmholtz equation (2.30) satisfying the Dirichlet BCs (2.31):

- Helmholtz D-1: $c = -1, \quad u(x, y) = e^{xy}(x^2 - x)^2(y^2 - y)^2.$
- Helmholtz D-2: $c = -1, \quad u(x, y) = e^{x-y}(x^2 - x)(y^2 - y).$
- Helmholtz D-3: $c = 0, \quad u(x, y) = e^{x+y}(x^2 - x)(y^2 - y).$

- Helmholtz D-4: $c = 0, \quad u(x,y) = e^{x+y}(x^{5.5} - x^{4.5})(y^{5.5} - y^{4.5}).$

Numerical results are presented in Tables 2.13–2.16. In each table, for several values of N, eight quantities are given for errors in approximations to u, $D_x u$, $D_y u$, $D_x^2 u$, $D_y^2 u$, and $D_{xy}^2 u$, evaluated at the nodal points $\{(x_i,\ y_j)\}_{i,j=0}^{N+1}$, the collocation points $\{(\tau_i^x,\ \tau_j^y)\}_{i,j=1}^{N+1}$, the Gauss points $\{(\eta_i^x,\ \eta_j^y)\}_{i,j=1}^{2(N+1)}$, and an estimate of global errors computed at 25 equally spaced points on a uniform grid in each cell defined by $[(x_{i-1},\ y_{j-1}),(x_{i-1},\ y_j),(x_i,\ y_{j-1}),(x_i,\ y_j)]$, $i,j = 1,2,...,N+1$.

The numerical results show optimal global accuracy and superonvergence behavior of the approximate solution, very similar to the numerical results for second-order TPBVPs. For optimal rates of convergence, we obtain $O(h^{3-k})$ global accuracy for the k^{th} derivative, $k = 0,1,2$. For superconvergence, we obtain fourth-order accuracy at the nodal points and the collocation points, third-order for the first derivatives ($D_x u$ and $D_y u$) and the mixed derivative ($D_{xy}^2 u$) at the Gauss points, and second-order for the second derivatives ($D_x^2 u$ and $D_y^2 u$). Note that for symmetric problems, results for derivatives with respect to x and y are identical.

Table 2.13. Helmholtz D-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_c(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_c(N)$	$R_G(N)$	$R_g(N)$
u								
5	2.0362(-5)	1.7548(-5)	5.8202(-5)	5.4369(-5)				
10	1.7773(-6)	1.7670(-6)	9.7903(-6)	9.2685(-6)	4.0231	3.7874	2.9408	2.9188
15	4.0116(-7)	4.1384(-7)	3.2012(-6)	3.0457(-6)	3.9726	3.8739	2.9835	2.9702
20	1.3543(-7)	1.4311(-7)	1.4196(-6)	1.3570(-6)	3.9934	3.9050	2.9903	2.9730
25	5.7438(-8)	6.1978(-8)	7.4975(-7)	7.1737(-7)	4.0161	3.9181	2.9889	2.9846
$D_x u$								
5	3.3254(-3)	1.4616(-3)	4.0071(-4)	1.4616(-3)				
10	1.0764(-3)	4.9131(-4)	7.2017(-5)	4.9131(-4)	1.8609	1.7986	2.8316	1.7986
15	5.1965(-4)	2.4262(-4)	2.4080(-5)	2.4262(-4)	1.9435	1.8831	2.9238	1.8831
20	3.0429(-4)	1.4406(-4)	1.0796(-5)	1.4406(-4)	1.9680	1.9169	2.9498	1.9169
25	1.9941(-4)	9.5286(-5)	5.7349(-6)	9.5286(-5)	1.9788	1.9353	2.9622	1.9353
$D_y u$								
5	3.3254(-3)	1.4616(-3)	4.0071(-4)	1.4616(-3)				
10	1.0764(-3)	4.9131(-4)	7.2017(-5)	4.9131(-4)	1.8609	1.7986	2.8316	1.7986
15	5.1965(-4)	2.4262(-4)	2.4080(-5)	2.4262(-4)	1.9435	1.8831	2.9238	1.8831
20	3.0429(-4)	1.4406(-4)	1.0796(-5)	1.4406(-4)	1.9680	1.9169	2.9498	1.9169
25	1.9941(-4)	9.5286(-5)	5.7349(-6)	9.5286(-5)	1.9788	1.9353	2.9622	1.9353
$D_x^2 u$								
5		4.1801(-3)	6.5556(-2)	7.5584(-2)				
10		1.5388(-3)	3.9182(-2)	4.5242(-2)		1.6487	8.4910(-1)	8.4669(-1)
15		8.0139(-4)	2.7776(-2)	3.2148(-2)		1.7412	9.1822(-1)	9.1190(-1)
20		4.8913(-4)	2.1546(-2)	2.4928(-2)		1.8156	9.3403(-1)	9.3542(-1)
25		3.2899(-4)	1.7583(-2)	2.0331(-2)		1.8569	9.5158(-1)	9.5432(-1)
$D_y^2 u$								
5		4.1801(-3)	6.5556(-2)	7.5584(-2)				
10		1.5388(-3)	3.9182(-2)	4.5242(-2)		1.6487	8.4910(-1)	8.4669(-1)
15		8.0139(-4)	2.7776(-2)	3.2148(-2)		1.7412	9.1822(-1)	9.1190(-1)
20		4.8913(-4)	2.1546(-2)	2.4928(-2)		1.8156	9.3403(-1)	9.3542(-1)
25		3.2899(-4)	1.7583(-2)	2.0331(-2)		1.8569	9.5158(-1)	9.5432(-1)
$D_{xy}^2 u$								
5	1.4366(-2)	8.1964(-3)	3.0470(-3)	8.1964(-3)				
10	5.0719(-3)	2.3975(-3)	4.6561(-4)	2.3975(-3)	1.7177	2.0281	3.0992	2.0281
15	2.3140(-3)	1.1486(-3)	1.4585(-4)	1.1486(-3)	2.0944	1.9638	3.0980	1.9638
20	1.3607(-3)	6.6421(-4)	6.3293(-5)	6.6421(-4)	1.9527	2.0143	3.0698	2.0143
25	8.7926(-4)	4.3904(-4)	3.4053(-5)	4.3904(-4)	2.0445	1.9384	2.9023	1.9384

Table 2.14. Helmholtz D-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_c(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_c(N)$	$R_G(N)$	$R_g(N)$
u								
5	3.4954(-5)	2.2337(-5)	1.2730(-4)	1.2261(-4)				
10	3.1526(-6)	2.1637(-6)	2.2191(-5)	2.1408(-5)	3.9691	3.8513	2.8820	2.8793
15	6.9786(-7)	4.9918(-7)	7.3883(-6)	7.1401(-6)	4.0245	3.9141	2.9352	2.9304
20	2.3651(-7)	1.7101(-7)	3.3095(-6)	3.1939(-6)	3.9790	3.9393	2.9533	2.9584
25	1.0060(-7)	7.3585(-8)	1.7560(-6)	1.6934(-6)	4.0028	3.9484	2.9675	2.9709
$D_x u$								
5	8.5574(-3)	4.0128(-3)	5.2411(-4)	4.0128(-3)				
10	2.6517(-3)	1.2654(-3)	1.0721(-4)	1.2654(-3)	1.9329	1.9040	2.6182	1.9040
15	1.2657(-3)	6.1115(-4)	3.8243(-5)	6.1115(-4)	1.9738	1.9425	2.7510	1.9425
20	7.3772(-4)	3.5878(-4)	1.7793(-5)	3.5848(-4)	1.9851	1.9587	2.8137	1.9587
25	4.8229(-4)	2.3589(-4)	9.6793(-6)	2.3589(-4)	1.9901	1.9635	2.8506	1.9635
$D_y u$								
5	8.5574(-3)	4.0128(-3)	5.2411(-4)	4.0128(-3)				
10	2.6517(-3)	1.2654(-3)	1.0721(-4)	1.2654(-3)	1.9329	1.9040	2.6182	1.9040
15	1.2657(-3)	6.1115(-4)	3.8243(-5)	6.1115(-4)	1.9738	1.9425	2.7510	1.9425
20	7.3772(-4)	3.5878(-4)	1.7793(-5)	3.5848(-4)	1.9851	1.9587	2.8137	1.9587
25	4.8229(-4)	2.3589(-4)	9.6793(-6)	2.3589(-4)	1.9901	1.9635	2.8506	1.9635
$D_x^2 u$								
5		5.4559(-3)	1.6927(-1)	1.9562(-1)				
10		1.9516(-3)	9.8422(-2)	1.1401(-1)		1.6961	8.9452(-1)	8.9074(-1)
15		9.8885(-4)	6.8988(-2)	7.9783(-1)		1.8145	9.4832(-1)	9.5263(-1)
20		5.9519(-4)	5.3012(-2)	6.1270(-1)		1.8669	9.6866(-1)	9.7089(-1)
25		3.9729(-4)	4.3020(-2)	4.9707(-1)		1.8925	9.7792(-1)	9.7930(-1)
$D_y^2 u$								
5		5.4559(-3)	1.6927(-1)	1.9562(-1)				
10		1.9516(-3)	9.8422(-2)	1.1401(-1)		1.6961	8.9452(-1)	8.9074(-1)
15		9.8885(-4)	6.8988(-2)	7.9783(-1)		1.8145	9.4832(-1)	9.5263(-1)
20		5.9519(-4)	5.3012(-2)	6.1270(-1)		1.8669	9.6866(-1)	9.7089(-1)
25		3.9729(-4)	4.3020(-2)	4.9707(-1)		1.8925	9.7792(-1)	9.7930(-1)
$D_{xy}^2 u$								
5	1.0721(-1)	3.5410(-2)	1.8398(-3)	3.5410(-2)				
10	3.2890(-2)	1.3111(-2)	3.0278(-4)	1.3111(-2)	1.9495	1.6391	2.9769	1.6391
15	1.5690(-2)	6.7093(-3)	9.8238(-5)	6.7093(-3)	1.9753	1.7880	3.0041	1.7880
20	9.1475(-3)	4.0583(-3)	4.4647(-5)	4.0583(-3)	1.9842	1.8487	2.9000	1.8487
25	5.9821(-3)	2.7150(-3)	2.3900(-5)	2.7150(-3)	1.9886	1.8821	2.9260	1.8821

Table 2.15. Helmholtz D-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_c(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_c(N)$	$R_G(N)$	$R_g(N)$
u								
5	9.2957(-5)	6.1728(-5)	4.7192(-4)	4.7248(-4)				
10	8.4103(-6)	5.9330(-6)	6.6568(-5)	7.5417(-5)	3.9639	3.8642	3.0000	3.0273
15	1.8640(-6)	1.3649(-6)	2.4672(-5)	2.4144(-5)	4.0212	3.9216	3.0231	3.0398
20	6.3058(-7)	4.6693(-7)	1.0839(-5)	1.0569(-5)	3.9857	3.9447	3.0247	3.0379
25	2.6843(-7)	2.0074(-7)	5.6929(-6)	5.5348(-6)	3.9988	3.9527	3.0149	3.0288
$D_x u$								
5	2.3276(-2)	1.0898(-3)	2.2073(-3)	1.0898(-2)				
10	7.2094(-3)	3.4388(-3)	4.0684(-4)	3.4388(-3)	1.9336	1.9030	2.7899	1.9030
15	3.4409(-3)	1.6610(-3)	1.3981(-4)	1.6610(-3)	1.9740	1.9421	2.8507	1.9421
20	2.0054(-3)	9.7518(-4)	6.3591(-5)	9.7518(-4)	1.9853	1.9585	2.8971	1.9585
25	1.3110(-3)	6.4117(-4)	3.4072(-5)	6.4117(-4)	1.9901	1.9634	2.9217	1.9634
$D_y u$								
5	2.3276(-2)	1.0898(-3)	2.2073(-3)	1.0898(-2)				
10	7.2094(-3)	3.4388(-3)	4.0684(-4)	3.4388(-3)	1.9336	1.9030	2.7899	1.9030
15	3.4409(-3)	1.6610(-3)	1.3981(-4)	1.6610(-3)	1.9740	1.9421	2.8507	1.9421
20	2.0054(-3)	9.7518(-4)	6.3591(-5)	9.7518(-4)	1.9853	1.9585	2.8971	1.9585
25	1.3110(-3)	6.4117(-4)	3.4072(-5)	6.4117(-4)	1.9901	1.9634	2.9217	1.9634
$D_x^2 u$								
5		1.4853(-2)	4.8127(-1)	5.5563(-1)				
10		5.3081(-3)	2.7089(-1)	3.1302(-1)		1.6976	9.4818(-1)	9.4668(-1)
15		2.6888(-3)	1.8835(-1)	2.1764(-1)		1.8152	9.6988(-1)	9.6992(-1)
20		1.6182(-3)	1.4435(-1)	1.6678(-1)		1.8673	9.7847(-1)	9.7878(-1)
25		1.0801(-3)	1.1701(-1)	1.3518(-2)		1.8928	9.8318(-1)	9.8355(-1)
$D_y^2 u$								
5		1.4853(-2)	4.8127(-1)	5.5563(-1)				
10		5.3081(-3)	2.7089(-1)	3.1302(-1)		1.6976	9.4818(-1)	9.4668(-1)
15		2.6888(-3)	1.8835(-1)	2.1764(-1)		1.8152	9.6988(-1)	9.6992(-1)
20		1.6182(-3)	1.4435(-1)	1.6678(-1)		1.8673	9.7847(-1)	9.7878(-1)
25		1.0801(-3)	1.1701(-1)	1.3518(-2)		1.8928	9.8318(-1)	9.8355(-1)
$D_{xy}^2 u$								
5	2.9152(-1)	9.6212(-2)	9.1590(-3)	9.6212(-2)				
10	8.9412(-2)	3.5635(-2)	2.1584(-3)	3.5635(-2)	1.9498	1.6386	2.3845	1.6386
15	4.2653(-2)	1.8236(-2)	7.9551(-4)	1.8235(-2)	1.9754	1.7878	2.6639	1.7878
20	2.4866(-2)	1.1031(-2)	3.7457(-4)	1.1031(-2)	1.9842	1.8486	2.7698	1.8486
25	1.6261(-2)	7.3800(-3)	2.0484(-4)	7.3800(-3)	1.9886	1.8820	2.8260	1.8820

Table 2.16. Helmholtz D-4

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_c(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_c(N)$	$R_G(N)$	$R_g(N)$
u								
5	7.7457(-4)	5.6275(-4)	1.4527(-3)	1.5446(-3)				
10	8.2029(-5)	5.1622(-5)	2.4292(-4)	2.5379(-4)	3.7042	3.9412	2.9506	2.9796
15	1.6529(-5)	1.2506(-5)	7.6077(-5)	7.7189(-5)	4.2754	3.7837	3.0985	3.1766
20	5.6904(-6)	4.3479(-6)	3.3842(-5)	3.3830(-5)	3.9213	3.8852	2.9788	3.0335
25	2.3759(-6)	1.8808(-6)	1.7596(-5)	1.7455(-5)	4.0894	3.9237	3.0623	3.0982
$D_x u$								
5	6.4492(-2)	2.3475(-2)	1.3987(-2)	2.3475(-2)				
10	2.2157(-2)	9.8139(-3)	2.8626(-3)	9.8139(-3)	1.7626	1.4388	2.6173	1.4388
15	1.0796(-2)	5.0234(-3)	1.0006(-3)	5.0234(-3)	1.9188	1.7873	2.8053	1.7873
20	6.4374(-3)	3.0131(-3)	4.7437(-4)	3.0131(-3)	1.9015	1.8796	2.7447	1.8796
25	4.2612(-3)	2.0002(-3)	2.6379(-4)	2.0002(-3)	1.9317	1.9184	2.7477	1.9184
$D_y u$								
5	6.4492(-2)	2.3475(-2)	1.3987(-2)	2.3475(-2)				
10	2.2157(-2)	9.8139(-3)	2.8626(-3)	9.8139(-3)	1.7626	1.4388	2.6173	1.4388
15	1.0796(-2)	5.0234(-3)	1.0006(-3)	5.0234(-3)	1.9188	1.7873	2.8053	1.7873
20	6.4374(-3)	3.0131(-3)	4.7437(-4)	3.0131(-3)	1.9015	1.8796	2.7447	1.8796
25	4.2612(-3)	2.0002(-3)	2.6379(-4)	2.0002(-3)	1.9317	1.9184	2.7477	1.9184
$D_x^2 u$								
5		9.4245(-2)	1.2945	1.5204				
10		3.7989(-2)	7.9824(-1)	9.3009(-1)		1.4990	7.9769(-1)	8.1079(-1)
15		2.0503(-2)	5.8773(-1)	6.7758(-1)		1.6459	8.1704(-1)	8.4535(-1)
20		1.3085(-2)	4.6016(-1)	5.3171(-1)		1.6516	8.9982(-1)	8.9151(-1)
25		9.0287(-3)	3.7590(-1)	4.3465(-1)		1.7374	9.4695(-1)	9.4374(-1)
$D_y^2 u$								
5		9.4245(-2)	1.2945	1.5204				
10		3.7989(-2)	7.9824(-1)	9.3009(-1)		1.4990	7.9769(-1)	8.1079(-1)
15		2.0503(-2)	5.8773(-1)	6.7758(-1)		1.6459	8.1704(-1)	8.4535(-1)
20		1.3085(-2)	4.6016(-1)	5.3171(-1)		1.6516	8.9982(-1)	8.9151(-1)
25		9.0287(-3)	3.7590(-1)	4.3465(-1)		1.7374	9.4695(-1)	9.4374(-1)
$D_{xy}^2 u$								
5	2.0671	3.1760(-1)	7.8499(-2)	3.2656(-1)				
10	7.0923(-1)	1.9018(-1)	2.4346(-2)	1.9018(-1)	1.7648	8.4601(-1)	1.9314	8.9195(-1)
15	3.4962(-1)	1.1394(-1)	1.2097(-2)	1.1394(-1)	1.8878	1.3673	1.8666	1.3673
20	2.0683(-1)	7.4567(-2)	6.5473(-3)	7.4567(-2)	1.9304	1.5591	2.2576	1.5591
25	1.3635(-1)	5.2304(-2)	3.8816(-3)	5.2304(-2)	1.9510	1.6605	2.4479	1.6605

Chapter 3

NEUMANN BOUNDARY CONDITIONS

3.1 Two-Point Boundary Value Problem

Consider the TPBVP comprising (2.1) subject to the homogeneous Neumann BCs,

$$u'(0) = u'(1) = 0. \tag{3.1}$$

Let $\{\mathcal{B}_m^N\}_{m=0}^{N+2}$ be a basis for the quadratic spline space S_2 such that

$$\{\mathcal{B}_m^N\}_{m=0}^{N+2} = \{\mathcal{B}_0, \mathcal{B}_1 + \mathcal{B}_0, \mathcal{B}_2, ..., \mathcal{B}_N, \mathcal{B}_{N+1} + \mathcal{B}_{N+2}, \mathcal{B}_{N+2}\}, \tag{3.2}$$

where $\mathcal{B}_m$, $m = 0, 1, ..., N+2$, is defined in (2.4). Note that $\{\mathcal{B}_m^N\}_{m=1}^{N+1}$ is a basis for the space of quadratic splines satisfying the Neumann BCs (3.1), that is, $S_2^N = S_2 \cap \{v : v'(0) = v'(1) = 0\}$. From (2.8) and (3.2), we have

$$\mathcal{B}_j^N(\tau_i) = \begin{cases} 7/8, & i,j=1, \text{ or } i,j=N+1, \\ 1/8, & i=1,2,...,N+1; \ j=0,1,...,N+2; \ i=j\pm 1, \\ 3/4, & i,j=2,3,...,N; \ i=j, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{B}_j^{N\,''}(\tau_i) = \begin{cases} -1/h^2, & i,j=1, \text{ or } i,j=N+1, \\ 1/h^2, & i=1,2,...,N+1; \ j=0,1,...,N+2; \ i=j\pm 1, \\ -2/h^2, & i,j=2,3,...,N; \ i=j, \\ 0, & \text{otherwise}. \end{cases} \tag{3.3}$$

3.1.1 Modified Quadratic Spline Collocation Methods

Archer Quadratic Spline Collocation

In this method, we seek $u_h^N \in S_2^N$ such that

$$\begin{cases} u_h^{N\,''}(\tau_i) + cu_h^N(\tau_i) = f_i, & i=1, \\ \frac{1}{24}\left[u_h^{N\,''}(\tau_{i-1}) + 22u_h^{N\,''}(\tau_i) + u_h^{N\,''}(\tau_{i+1})\right] + cu_h^N(\tau_i) = f_i, & i=2,3,...,N, \\ u_h^{N\,''}(\tau_i) + cu_h^N(\tau_i) = f_i, & i=N+1, \end{cases} \tag{3.4}$$

cf., the modified CSC method proposed in [2, 3] for the cubic case, with

$$u_h^N(x) = \sum_{j=1}^{N+1} u_j \mathcal{B}_j^N(x). \tag{3.5}$$

Using (3.3) and (3.5), the collocation equations (3.4) can be written as the system of linear equations (2.16), where in this case, $\mathbf{u}$, and $\mathbf{f}$ are as in (2.17) and (2.18), respectively,

$$A = \frac{1}{24h^2}\begin{bmatrix} -24 & 24 & 0 & & & & \\ 21 & -42 & 20 & 1 & & & \\ 1 & 20 & -42 & 20 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 20 & -42 & 20 & 1 \\ & & & 1 & 20 & -42 & 21 \\ & & & & 0 & 24 & -24 \end{bmatrix}_{(N+1)\times(N+1)}$$

and

$$B = \frac{1}{8}\begin{bmatrix} 7 & 1 & & & \\ 1 & 6 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 6 & 1 \\ & & & 1 & 7 \end{bmatrix}_{(N+1)\times(N+1)} \tag{3.6}$$

Modified Quadratic Spline Collocation

In the method of [17] for the Neumann problem, the approximation solution does not satisfy the Neumann BCs but instead the BCs (3.1) are perturbed so that the desired approximation $u_h^N \in S_2$ satisfies

$$\begin{cases} u_h^{N\,\prime}(0) + \dfrac{1}{24}\left[5\Delta_x^2 u_h^{N\,\prime}(\tau_2) - 3\Delta_x^2 u_h^{N\,\prime}(\tau_3)\right] & = 0, \\ u_h^{N\,\prime}(1) + \dfrac{1}{24}\left[5\Delta_x^2 u_h^{N\,\prime}(\tau_N) - 3\Delta_x^2 u_h^{N\,\prime}(\tau_{N-1})\right] & = 0, \end{cases} \tag{3.7}$$

where

$$\Delta_x^2 u_h^{N\prime}(\tau_i) = \frac{1}{h^2}\left[u_h^{N\prime}(\tau_{i-1}) - 2u_h^{N\prime}(\tau_i) + u_h^{N\prime}(\tau_{i+1})\right],\ i = 2, 3, ..., N-1, N.$$

Therefore, in this method, we have the following collocation equations:

$$\begin{cases} u_h^{N\prime}(0) + \frac{h^2}{24}\left[5u_h^{N\prime}(\tau_{i+1}) - 13u_h^{N}(\tau_{i+2}) + 11u_h^{N\prime}(\tau_{i+3}) - 3u_h^{N\prime}(\tau_{i+4})\right] = 0, & i = 0, \\ \frac{1}{24}\left[26u_h^{N\prime\prime}(\tau_i) - 5u_h^{N\prime\prime}(\tau_{i+1}) + 4u_h^{N\prime\prime}(\tau_{i+2}) - u_h^{N\prime\prime}(\tau_{i+3})\right] + cu_h^N(\tau_i) = f_i, & i = 1, \\ \frac{1}{24}\left[u_h^{N\prime\prime}(\tau_{i-1}) + 22u_h^{N\prime\prime}(\tau_i) + u_h^{N\prime\prime}(\tau_{i+1})\right] + cu_h^N(\tau_i) = f_i, & i = 2, 3, ..., N, \\ \frac{1}{24}\left[26u_h^{N\prime\prime}(\tau_i) - 5u_h^{N\prime\prime}(\tau_{i-1}) + 4u_h^{N\prime\prime}(\tau_{i-2}) - u_h^{N\prime\prime}(\tau_{i-3})\right] + cu_h^N(\tau_i) = f_i, & i = N+1, \\ u_h^{N\prime}(1) + \frac{h^2}{24}\left[5u_h^{N\prime}(\tau_{i-1}) - 13u_h^{N}(\tau_{i-2}) + 11u_h^{N\prime}(\tau_{i-3}) - 3u_h^{N\prime}(\tau_{i-4})\right] = 0, & i = N+2, \end{cases} \tag{3.8}$$

where the first and last equations (cases $i = 0$ and $i = N + 2$) come from (3.7).

Using (2.4)–(2.7), we have

$$\mathcal{B}_j'(0) = \begin{cases} -\frac{1}{h}, & j = 0, \\ \frac{1}{h}, & j = 1, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{B}_j'(1) = \begin{cases} -\frac{1}{h}, & j = N+1, \\ \frac{1}{h}, & j = N+2, \\ 0, & \text{otherwise}. \end{cases} \tag{3.9}$$

Thus, if we let

$$u_h^N(x) = \sum_{j=0}^{N+2} u_j \mathcal{B}_j(x), \tag{3.10}$$

then from (3.8), using (2.8), (3.9) and (3.10), we obtain a system of linear equations of the form (2.16) with

$$\mathbf{u} = [u_0,\ u_1, ..., u_{N+2}]^T, \tag{3.11}$$

$$\mathbf{f} = [0,\ f_1,\ f_2, ...,\ f_{N+1},\ 0]^T, \tag{3.12}$$

$$A = \frac{1}{24h^2}\begin{bmatrix} -\frac{53h}{2} & \frac{61h}{2} & -3h & -5h & \frac{11h}{2} & -\frac{3h}{2} & & \\ 26 & -57 & 40 & -14 & 6 & -1 & & \\ 1 & 20 & -42 & 20 & 1 & 0 & & \\ & 1 & 20 & -42 & 20 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 20 & -42 & 20 & 1 \\ & & -1 & 6 & -14 & 40 & -57 & 26 \\ & & \frac{3h}{2} & -\frac{11h}{2} & 5h & 3h & -\frac{61h}{2} & \frac{53h}{2} \end{bmatrix}_{(N+3)\times(N+3)}$$

and

$$B = \frac{1}{8}\begin{bmatrix} 0 & & & & & \\ 1 & 6 & 1 & & & \\ & 1 & 6 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 6 & 1 \\ & & & & & 0 \end{bmatrix}_{(N+3)\times(N+3)}$$

New Modified Quadratic Spline Collocation

In this method, we perturb the Neumann BCs based on the work of [17]. According to Houstis et al., [17, Eqn (30b)],

$$\begin{cases} w'(0) &= -\dfrac{h^2}{12}u^{(3)}(0) + O(h^4), \\ \\ w'(1) &= -\dfrac{h^2}{12}u^{(3)}(1) + O(h^4). \end{cases} \tag{3.13}$$

From (2.1) and the Neumann BCs (3.1), we have

$$u^{(3)}(0) = f'(0), \qquad\qquad u^{(3)}(1) = f'(1),$$

so that (3.13) becomes

$$\begin{cases} w'(0) &= -\dfrac{h^2}{12}f'(0) + O(h^4), \\ \\ w'(1) &= -\dfrac{h^2}{12}f'(1) + O(h^4). \end{cases}$$

As before, we replace w by u_h^N and drop the $O(h^4)$ term. Thus, in this new modified QSC method, we seek $u_h^N \in S_2$ such that

$$u_h^N(x) = \sum_{j=0}^{N+2} u_j \mathcal{B}_j^N(x), \tag{3.14}$$

and

$$\begin{cases} {u_h^N}'(0) = -\dfrac{h^2}{12}f'(0), \\ (L_h + c)\, u_h^N(\tau_i) = f_h^N(\tau_i), \quad i = 1, 2, ..., N+1, \\ {u_h^N}'(1) = -\dfrac{h^2}{12}f'(1), \end{cases} \tag{3.15}$$

where

$$L_h u_h^N(\tau_i) = \begin{cases} \dfrac{1}{24}\left[23 u_h^{N\,\prime\prime}(\tau_i) + u_h^{N\,\prime\prime}(\tau_{i+1})\right], & i = 1, \\ \\ \dfrac{1}{24}\left[u_h^{N\,\prime\prime}(\tau_{i-1}) + 22 u_h^{N\,\prime\prime}(\tau_i) + u_h^{N\,\prime\prime}(\tau_{i+1})\right], & i = 2, 3, ..., N, \\ \\ \dfrac{1}{24}\left[23 u_h^{N\,\prime\prime}(\tau_i) + u_h^{N\,\prime\prime}(\tau_{i-1})\right], & i = N+1, \end{cases}$$

and

$$f_h^N(\tau_i) = \begin{cases} f_i + \dfrac{h}{24} f'(0), & i = 1, \\ \\ f_i, & i = 2, 3, ..., N, \\ \\ f_i - \dfrac{h}{24} f'(1), & i = N+1. \end{cases}$$

Using (3.2) and (3.9), we have

$$\mathcal{B}_j^{N\,\prime}(0) = \begin{cases} -\dfrac{1}{h}, & j = 0, \\ \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{B}_j^{N\,\prime}(1) = \begin{cases} \dfrac{1}{h}, & j = N+2, \\ \\ 0, & \text{otherwise}. \end{cases} \tag{3.16}$$

Thus, in this case, we obtain a system of linear equations

$$(\tilde{A} + c\tilde{B})\tilde{\mathbf{u}} = \tilde{\mathbf{f}}, \tag{3.17}$$

where $\tilde{\mathbf{u}}$ is as in (3.11),

$$\tilde{\mathbf{f}} = \left[f_h^N(0),\ f_h^N(\tau_1), ..., f_h^N(1)\right]^T, \tag{3.18}$$

$$\tilde{A} = \frac{1}{24h^2} \begin{bmatrix} -24h & 0 & 0 & & & & & & \\ 23 & -22 & 21 & 1 & & & & & \\ 1 & 21 & -42 & 20 & 1 & & & & \\ & 1 & 20 & -42 & 20 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & 20 & -42 & 21 & 1 \\ & & & & 1 & 21 & -22 & 23 \\ & & & & & 0 & 0 & 24h \end{bmatrix}_{(N+3)\times(N+3)}$$

and

$$\tilde{B} = \frac{1}{8} \begin{bmatrix} 0 & & & & & \\ 1 & 7 & 1 & & & \\ & 1 & 6 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 7 & 1 \\ & & & & & 0 \end{bmatrix}_{(N+3)\times(N+3)}$$

The new modified QSC method (3.15) differs from (3.8) in the first two and the last two equations. The first and last equations are due to the perturbed Neumann BCs. We now show how the second and the second to last equations in (3.15) are derived. From (2.12) with $i = 1$, it follows that

$$\begin{aligned} \frac{1}{24}(23w_1'' + w_2'') + cw_1 &= \frac{1}{24}(-w_1'' + w_2'') + w_1'' + cw_1 \\ &= \frac{1}{24}(-w_1'' + w_2'') + Lw_1 \\ &= \frac{1}{24}(-w_1'' + w_2'') + f_1 - \frac{h^2}{24}u_1^{(4)} + O(h^4). \end{aligned} \tag{3.19}$$

From (2.11), we have

$$-w_1'' + w_2'' = (-u_1'' + u_2'') - \frac{h^2}{24}\left[-u_1^{(4)} + u_2^{(4)}\right] + O(h^4). \tag{3.20}$$

Then (2.28(a)–(b)) imply

$$-u_1'' + u_2'' = hu_0^{(3)} + h^2u_0^{(4)} + O(h^3),$$

and, on using (2.28(c)–(d)), (3.20) becomes

$$-w_1'' + w_2'' = hu_0^{(3)} + h^2u_0^{(4)} + O(h^3).$$

Thus, the right hand side of (3.19) becomes

$$f_1 + \frac{1}{24}hu_0^{(3)} + O(h^3) = f_1 + \frac{1}{24}hf'(0) + O(h^3),$$

with

$$u_0^{(3)} = f'(0)$$

on using (2.1) and the Neumann BC at $x = 0$ in (3.1).

Similarly, we have

$$\frac{1}{24}(23w_{N+1}'' + w_N'') + cw_{N+1} = f_{N+1} - \frac{1}{24}hf'(1) + O(h^3).$$

Thus, replacing w by u_h^N and dropping the $O(h^3)$ terms, we obtain (3.15) with $i = 1, N+1$.

From the system of linear equations (3.15), we can solve for u_0 and u_{N+2}:

$$u_0 = \frac{h^3}{12}f'(0), \qquad u_{N+2} = -\frac{h^3}{12}f'(1).$$

Moving the terms with u_0 and u_{N+2} to the right hand side of the system, we obtain an equivalent system of the form (2.16), where

$$A = \frac{1}{24h^2}\begin{bmatrix} -22 & 21 & 1 & & & & \\ 21 & -42 & 20 & 1 & & & \\ 1 & 20 & -42 & 20 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 20 & -42 & 20 & 1 \\ & & & 1 & 20 & -42 & 21 \\ & & & & 1 & 21 & -22 \end{bmatrix}_{(N+1)\times(N+1)} \tag{3.21}$$

$$\begin{aligned} \mathbf{f} = \ & \left[f_h^N(\tau_1) - \left(\frac{23}{24h^2} + \frac{c}{8}\right) u_0,\ f_h^N(\tau_2) - \frac{u_0}{24h^2}, \ldots, \right. \\ & \left. f_h^N(\tau_N) - \frac{u_{N+2}}{24h^2},\ f_h^N(\tau_{N+1}) - \left(\frac{23}{24h^2} + \frac{c}{8}\right) u_{N+2} \right]^T, \end{aligned} \tag{3.22}$$

and B and $\mathbf{u}$ are as in (3.6) and (2.17), respectively.

If T is the $(N+1) \times (N+1)$ tridiagonal matrix given by

$$T = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}_{(N+1)\times(N+1)} \tag{3.23}$$

then A and B can be written as in (2.25).

3.1.2 Numerical Results

We consider three test problems for the second-order TPBVP (2.1) satisfying the Neumann BCs (3.1):

- Problem N-1: $c = -9, \quad u(x) = e^x(x^2 - x)^2.$
- Problem N-2: $c = -9, \quad u(x) = e^{2x}(x^2 - x)^2.$
- Problem N-3: $c = -9, \quad u(x) = \sinh\left(3x - \frac{3}{2}\right) - 3x\cosh\left(\frac{3}{2}\right).$

Numerical results are presented in Tables 3.1–3.9 in the same format as the Dirichlet second-order TPBVP. The numerical results for the Archer QSC (3.4) are suboptimal and pretty erratic: problem N-3 exhibits superconvergence for the first derivative (i.e. third-order accuracy at the nodal point in stead of the Gauss points as expected) while problems N-1 and N-3 do not. For the modified QSC (3.8) and the new modified QSC (3.15), the numerical results show similar convergence to the Dirichlet case, that is, in general, we have $O(h^{3-k})$ accuracy for the k^{th} derivative, $k = 0, 1, 2$. For superconvergence, we again obtain fourth-order accuracy at the nodal points and the collocation points, third-order accuracy for the first derivative at the Gauss points, and second-order accuracy for the second derivative at the collocation points.

Table 3.1. Archer QSC - Neumann BCs, Problem N-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	4.2509(-3)	3.8855(-3)	4.1652(-3)	4.2334(-3)				
32	1.1914(-3)	1.1383(-3)	1.1791(-3)	1.1889(-3)	1.9176	1.8509	1.9026	1.9146
64	3.1661(-4)	3.0944(-4)	3.1495(-4)	3.1628(-4)	1.9550	1.9215	1.9474	1.9534
128	8.1695(-5)	8.0762(-5)	8.1480(-5)	8.1652(-5)	1.9764	1.9597	1.9726	1.9756
256	2.0755(-5)	2.0636(-5)	2.0728(-5)	2.0750(-5)	1.9879	1.9796	1.9860	1.9875
u'								
16	3.6748(-3)	1.8263(-2)	1.2555(-2)	1.8204(-2)				
32	1.0297(-3)	5.2009(-3)	3.5233(-3)	5.1717(-3)	1.9181	1.8936	1.9158	1.8972
64	2.7381(-4)	1.3918(-3)	9.3553(-4)	1.3822(-3)	1.9540	1.9446	1.9562	1.9465
128	7.0639(-5)	3.6030(-4)	2.4120(-4)	3.5758(-4)	1.9767	1.9717	1.9776	1.9727
256	1.7945(-5)	9.1679(-5)	6.1247(-5)	9.0954(-5)	1.9880	1.9857	1.9887	1.9862
u''								
16		3.4969(-2)	7.2736(-1)	1.0499				
32		1.0245(-2)	4.0044(-1)	5.7137(-1)		1.8509	8.9983(-1)	9.1725(-1)
64		2.7849(-3)	2.1019(-1)	2.9868(-1)		1.9215	9.5085(-1)	9.5691(-1)
128		7.2686(-4)	1.0769(-1)	1.5279(-1)		1.9597	9.7571(-1)	9.7797(-1)
256		1.8572(-4)	5.4505(-2)	7.7282(-2)		1.9796	9.8793(-1)	9.8886(-1)

Table 3.2. Modified QSC - Neumann BCs, Problem N-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	2.9884(-4)	2.9227(-4)	3.7136(-4)	3.7052(-4)				
32	2.3908(-5)	2.4265(-5)	3.4674(-5)	3.4479(-5)	3.8078	3.7520	3.5749	3.5799
64	1.6975(-6)	1.7561(-6)	3.1312(-6)	3.1048(-6)	3.9020	3.8738	3.5472	3.5514
128	1.1320(-7)	1.1825(-7)	2.9684(-7)	2.9359(-7)	3.9505	3.9363	3.4373	3.4409
256	7.3100(-9)	7.6739(-9)	3.0498(-8)	3.0103(-8)	3.9751	3.9680	3.3014	3.3044
u'								
16	1.3108(-2)	6.9713(-3)	1.4646(-3)	6.8916(-3)				
32	3.6661(-3)	1.8123(-3)	1.6419(-4)	1.8576(-3)	1.9208	2.0311	3.2992	1.9765
64	9.5960(-4)	4.7043(-4)	1.7939(-5)	4.8728(-4)	1.9773	1.9896	3.2661	1.9741
128	2.4466(-4)	1.2065(-4)	2.0192(-6)	1.2396(-4)	1.9939	1.9853	3.1867	1.9972
256	6.1710(-5)	3.0609(-5)	2.3617(-7)	3.1199(-5)	1.9984	1.9899	3.1133	2.0015
u''								
16		2.6783(-2)	7.7924(-1)	1.1117				
32		8.0682(-3)	4.1493(-1)	5.8969(-1)		1.8089	9.5011(-1)	9.5585(-1)
64		2.1796(-3)	2.1398(-1)	3.0365(-1)		1.9307	9.7691(-1)	9.7913(-1)
128		5.6361(-4)	1.0865(-1)	1.5408(-1)		1.9733	9.8876(-1)	9.8975(-1)
256		1.4310(-4)	5.4749(-2)	7.7611(-2)		1.9889	9.9442(-1)	9.9490(-1)

Table 3.3. New Modified QSC - Neumann BCs, Problem N-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	2.3728(-5)	2.2872(-6)	7.5677(-5)	7.5716(-5)				
32	1.6729(-6)	1.6493(-7)	1.0661(-5)	1.0586(-5)	3.9983	3.9644	2.9548	2.9662
64	1.1118(-7)	1.1100(-8)	1.4130(-6)	1.3969(-6)	3.9996	3.9809	2.9812	2.9876
128	7.1671(-9)	7.2041(-10)	1.8181(-7)	1.7931(-7)	3.9999	3.9901	2.9916	2.9951
256	4.5494(-10)	4.5862(-11)	2.3055(-8)	2.2710(-8)	4.0001	3.9958	2.9961	2.9979
u'								
16	1.4109(-2)	6.0675(-3)	7.4473(-4)	7.4584(-3)				
32	3.7442(-3)	1.7383(-3)	1.0190(-4)	1.9349(-3)	2.0000	1.8846	2.9987	2.0342
64	9.6507(-4)	4.6511(-4)	1.3336(-5)	4.9272(-4)	2.0000	1.9449	2.9998	2.0179
128	2.4502(-4)	1.2029(-4)	1.7060(-6)	1.2432(-4)	2.0000	1.9731	3.0000	2.0091
256	6.1733(-5)	3.0586(-5)	2.1574(-7)	3.1222(-5)	2.0000	1.9867	3.0001	2.0046
u''								
16		3.0083(-2)	7.8254(-1)	1.1150				
32		8.3381(-3)	4.1520(-1)	5.8996(-1)		1.9345	9.5550(-1)	9.5963(-1)
64		2.1990(-3)	2.1400(-1)	3.0367(-1)		1.9662	9.7774(-1)	9.7971(-1)
128		5.6491(-4)	1.0866(-1)	1.5408(-1)		1.9828	9.8887(-1)	9.8983(-1)
256		1.4318(-4)	5.4749(-2)	7.7611(-1)		1.9913	9.9444(-1)	9.9491(-1)

Table 3.4. Archer QSC - Neumann BCs, Problem N-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	1.4361(-2)	1.3091(-2)	1.4062(-2)	1.4299(-2)				
32	4.1309(-3)	3.9426(-3)	4.0870(-3)	4.1220(-3)	1.8785	1.8093	1.8629	1.8753
64	1.1146(-3)	1.0889(-3)	1.1087(-3)	1.1134(-3)	1.9325	1.8981	1.9246	1.9309
128	2.8999(-4)	2.8662(-4)	2.8921(-4)	2.8983(-4)	1.9644	1.9473	1.9605	1.9636
256	7.3991(-5)	7.3561(-5)	7.3892(-5)	7.3971(-5)	1.9817	1.9732	1.9797	1.9813
u'								
16	1.6979(-2)	6.3298(-2)	4.3719(-2)	6.3096(-2)				
32	4.9321(-3)	1.8421(-2)	1.2508(-2)	1.8318(-2)	1.8638	1.8610	1.8866	1.8646
64	1.3396(-3)	4.9858(-3)	3.3552(-3)	4.9515(-3)	1.9228	1.9279	1.9412	1.9298
128	3.4952(-4)	1.2982(-3)	8.6956(-4)	1.2884(-3)	1.9602	1.9632	1.9700	1.9642
256	8.9310(-5)	3.3129(-4)	2.2139(-4)	3.2867(-4)	1.9796	1.9814	1.9848	1.9819
u''								
16		1.1782(-1)	2.4772	3.6570				
32		3.5483(-2)	1.4022	2.0288		1.8093	8.5795(-1)	8.8829(-1)
64		9.8000(-3)	7.4807(-1)	1.0713		1.8981	9.2687(-1)	9.4205(-1)
128		2.5796(-3)	3.8666(-1)	5.5085(-1)		1.9473	9.6282(-1)	9.7043(-1)
256		6.6205(-4)	1.9661(-1)	2.7936(-1)		1.9732	9.8124(-1)	9.8505(-1)

Table 3.5. Modified QSC - Neumann BCs, Problem N-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	2.3315(-3)	2.2307(-3)	2.5523(-3)	2.5540(-3)				
32	2.0444(-4)	2.0295(-4)	2.4170(-4)	2.4117(-4)	3.6696	3.6139	3.5535	3.5579
64	1.5237(-5)	1.5417(-5)	2.0383(-5)	2.0290(-5)	3.8304	3.8023	3.6481	3.6517
128	1.0417(-6)	1.0643(-6)	1.7068(-6)	1.6948(-6)	3.9141	3.9000	3.6183	3.6219
256	6.8129(-8)	6.9943(-8)	1.5223(-7)	1.5077(-7)	3.9568	3.9497	3.5066	3.5103
u'								
16	4.3308(-2)	2.7112(-2)	9.3788(-3)	2.6935(-2)				
32	1.2911(-2)	6.6735(-3)	1.0677(-3)	6.5798(-3)	1.8247	2.1135	3.2760	2.1249
64	3.4496(-3)	1.7005(-3)	1.1259(-4)	1.7491(-3)	1.9469	2.0169	3.3184	1.9545
128	8.8479(-4)	4.3513(-4)	1.2079(-5)	4.4875(-4)	1.9851	1.9886	3.2568	1.9847
256	2.2353(-4)	1.1056(-4)	1.3583(-6)	1.1313(-4)	1.9960	1.9878	3.1703	1.9991
u''								
16		1.2435(-1)	2.7193	3.8992				
32		4.1296(-2)	1.4790	2.1056		1.6619	9.1820(-1)	9.2896(-1)
64		1.1566(-2)	7.6943(-1)	1.0926		1.8775	9.6397(-1)	9.6772(-1)
128		3.0325(-3)	3.9228(-1)	5.5646(-1)		1.9530	9.8288(-1)	9.8445(-1)
256		7.7430(-4)	1.9805(-1)	2.8079(-1)		1.9806	9.9159(-1)	9.9232(-1)

Table 3.6. New Modified QSC - Neumann BCs, Problem N-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	1.4036(-4)	2.3765(-5)	2.7465(-4)	2.7639(-4)				
32	9.9146(-6)	1.7074(-6)	3.8848(-5)	3.8728(-5)	3.9955	3.9700	2.9487	2.9629
64	6.5922(-7)	1.1470(-7)	5.1425(-6)	5.0958(-6)	3.9988	3.9835	2.9830	2.9919
128	4.2503(-8)	7.4381(-9)	6.6056(-7)	6.5232(-7)	3.9997	3.9913	2.9940	2.9991
256	2.6981(-9)	4.7356(-10)	8.3670(-8)	8.2473(-8)	4.0000	3.9957	2.9977	3.0004
u'								
16	5.1135(-2)	2.0153(-2)	4.0388(-3)	2.7653(-2)				
32	1.3570(-2)	6.0532(-3)	5.5369(-4)	7.0983(-3)	2.0000	1.8133	2.9958	2.0502
64	3.4978(-3)	1.6538(-3)	7.2498(-5)	1.7970(-3)	2.0000	1.9140	2.9991	2.0265
128	8.8805(-4)	4.3192(-4)	9.2751(-6)	4.5200(-4)	2.0000	1.9588	2.9999	2.0136
256	2.2374(-4)	1.1034(-4)	1.1729(-6)	1.1334(-4)	2.0000	1.9798	3.0001	2.0069
u''								
16		1.5385(-1)	2.7488	3.9287				
32		4.3904(-2)	1.4816	2.1082		1.8906	9.3181(-1)	9.3845(-1)
64		1.1761(-2)	7.6963(-1)	1.0928		1.9432	9.6619(-1)	9.6928(-1)
128		3.0459(-3)	3.9229(-1)	5.5647(-1)		1.9710	9.8320(-1)	9.8467(-1)
256		7.7518(-4)	1.9805(-1)	2.8080(-1)		1.9854	9.9163(-1)	9.9235(-1)

Table 3.7. Archer QSC - Neumann BCs, Problem N-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	5.1708(-3)	4.6763(-3)	5.0554(-3)	5.1473(-3)				
32	1.4137(-3)	1.3433(-3)	1.3974(-3)	1.4104(-3)	1.9551	1.8805	1.9386	1.9518
64	3.7077(-4)	3.6135(-4)	3.6859(-4)	3.7033(-4)	1.9744	1.9370	1.9660	1.9727
128	9.5022(-5)	9.3803(-5)	9.4741(-5)	9.4965(-5)	1.9863	1.9676	1.9821	1.9855
256	2.4058(-5)	2.3903(-5)	2.4022(-5)	2.4050(-5)	1.9929	1.9836	1.9908	1.9925
u'								
16	1.1495(-3)	2.4824(-2)	1.6948(-2)	2.4726(-2)				
32	1.7621(-4)	6.9111(-3)	4.6658(-3)	6.8698(-3)	2.8274	1.9278	1.9447	1.9308
64	2.4525(-5)	1.8282(-3)	1.2267(-3)	1.8153(-3)	2.9091	1.9617	1.9707	1.9633
128	3.2395(-6)	4.7049(-4)	3.1470(-4)	4.6689(-4)	2.9533	1.9802	1.9849	1.9811
256	4.1642(-7)	1.1936(-4)	7.9708(-5)	1.1841(-4)	2.9763	1.9900	1.9923	1.9904
u''								
16		4.2087(-2)	1.0165	1.4166				
32		1.2090(-2)	5.3920(-1)	7.5647(-1)		1.8805	9.5585(-1)	9.4579(-1)
64		3.2522(-3)	2.7787(-1)	3.9160(-1)		1.9370	9.7797(-1)	9.7129(-1)
128		8.4423(-4)	1.4107(-1)	1.9933(-1)		1.9676	9.8903(-1)	9.8520(-1)
256		2.1512(-4)	7.1075(-2)	1.0057(-1)		1.9836	9.9453(-1)	9.9248(-1)

Table 3.8. Modified QSC - Neumann BCs, Problem N-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	2.1607(-4)	2.0979(-4)	3.1364(-4)	3.1238(-4)				
32	1.7274(-5)	1.7489(-5)	3.1308(-5)	3.1069(-5)	3.8089	3.7458	3.4741	3.4796
64	1.2308(-6)	1.2730(-6)	3.0884(-6)	3.0560(-6)	3.8967	3.8652	3.4169	3.4211
128	8.2308(-8)	8.6053(-8)	3.2019(-7)	3.1614(-7)	3.9464	3.9307	3.3067	3.3099
256	5.3273(-9)	5.5998(-9)	3.5386(-8)	3.4885(-8)	3.9718	3.9640	3.1956	3.1978
u'								
16	1.7515(-2)	9.1395(-3)	1.1468(-3)	8.9841(-3)				
32	4.7978(-3)	2.3885(-3)	1.2717(-4)	2.4240(-3)	1.9522	2.0231	3.3155	1.9750
64	1.2484(-3)	6.1750(-4)	1.3785(-5)	6.3186(-4)	1.9861	1.9956	3.2778	1.9834
128	3.1777(-4)	1.5764(-4)	1.5415(-6)	1.6065(-4)	1.9962	1.9920	3.1963	1.9979
256	8.0117(-5)	3.9875(-5)	1.7947(-7)	4.0456(-5)	1.9990	1.9943	3.1200	2.0007
u''								
16		2.0154(-2)	1.0396	1.4788				
32		6.0763(-3)	5.4573(-1)	7.7464(-1)		1.8076	9.7164(-1)	9.7484(-1)
64		1.6436(-3)	2.7957(-1)	3.9650(-1)		1.9288	9.8670(-1)	9.8797(-1)
128		4.2536(-4)	1.4150(-1)	2.0060(-1)		1.9721	9.9349(-1)	9.9406(-1)
256		1.0804(-4)	7.1184(-2)	1.0090(-1)		1.9882	9.9676(-1)	9.9704(-1)

Table 3.9. New Modified QSC - Neumann BCs, Problem N-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	1.7682(-5)	1.4843(-6)	1.0021(-4)	9.9527(-5)				
32	1.2468(-6)	1.0754(-7)	1.3960(-5)	1.3806(-5)	3.9982	3.9572	2.9717	2.9780
64	8.2857(-8)	7.2584(-9)	1.8416(-6)	1.8168(-6)	3.9995	3.9767	2.9881	2.9917
128	5.3416(-9)	4.7191(-10)	2.3645(-7)	2.3296(-7)	3.9998	3.9875	2.9947	2.9966
256	3.4362(-10)	3.4614(-11)	2.9959(-8)	2.9496(-8)	3.9807	3.7903	2.9973	2.9983
u'								
16	1.8315(-2)	8.4110(-3)	5.6245(-4)	9.4941(-3)				
32	4.8604(-3)	2.3290(-3)	7.6967(-4)	2.4860(-3)	2.0000	1.9359	2.9986	2.0202
64	1.2528(-3)	6.1322(-4)	1.0073(-5)	6.3624(-4)	2.0000	1.9686	2.9998	2.0105
128	3.1807(-4)	1.5735(-4)	1.2886(-6)	1.6095(-4)	2.0000	1.9845	3.0000	2.0053
256	8.0136(-5)	3.9856(-5)	1.6296(-7)	4.0475(-5)	2.0000	1.9923	3.0000	2.0027
u''								
16		2.2579(-2)	1.0420	1.4812				
32		6.2775(-3)	5.4593(-1)	7.7484(-1)		1.9298	9.7460(-1)	9.7692(-1)
64		1.6582(-3)	2.7959(-1)	3.9651(-1)		1.9638	9.8717(-1)	9.8830(-1)
128		4.2634(-4)	1.4150(-1)	2.0060(-1)		1.9816	9.9356(-1)	9.9411(-1)
256		1.0810(-4)	7.1184(-2)	1.0090(-1)		1.9907	9.9677(-1)	9.9705(-1)

3.2 Helmholtz Problem

Consider the Helmholtz equation (2.30) subject to the homogeneous Neumann BCs

$$\begin{cases} D_x u(0,y) = D_x u(1,y) = 0, & y \in [0,1], \\ D_y u(x,0) = D_y u(x,1) = 0, & x \in [0,1]. \end{cases} \tag{3.24}$$

We seek an approximate solution of the form

$$u_h^N(x,y) = \sum_{m=0}^{N+2}\sum_{n=0}^{N+2} u_{m,n}\mathcal{B}_m^N(x)\mathcal{B}_n^N(y). \tag{3.25}$$

Note that from (2.4) and (3.2), we have

$$\mathcal{B}_j^N(0) = \begin{cases} \frac{1}{2}, & j=0, \\ 1, & j=1, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{B}_j^N(1) = \begin{cases} \frac{1}{2}, & j=N+2, \\ 1, & j=N+1, \\ 0, & \text{otherwise}. \end{cases} \tag{3.26}$$

We first use the perturbations of the BCs to find $u_{m,n}$ on $\partial\Omega$, that is, $u_{k,0}, u_{k,1}, ..., u_{k,N+2}$ and $u_{1,k}, u_{2,k}, ..., u_{N+1,k}$, $k = 0, N+2$. Then, the QSC method gives rise to a system of linear equations of the form (2.34). Finally, applying the MDA given in Section 2.2.1, we find $u_{m,n}$, $m,n = 1,2,...,N+1$.

3.2.1 New Modified Quadratic Spline Collocation

Similar to the new modified QSC for second-order TPBVPs, the perturbed Neumann BCs are

$$D_x u_h^N(\tau_i^x, \tau_j^y) = -\frac{h^2}{12} D_x g(\tau_i^x, \tau_j^y), \quad i = 0, N+2; \quad j = 0,1,...,N+2, \tag{3.27}$$

and

$$D_y u_h^N(\tau_i^x,\tau_j^y) = -\frac{h^2}{12} D_y g(\tau_i^x,\tau_j^y), \quad i=1,2,...,N+1; \quad j=0,N+2, \tag{3.28}$$

where

$$\tau_0^x=\tau_0^y=0, \qquad \tau_{N+2}^x=\tau_{N+2}^y=1.$$

From (3.25), we have

$$D_x u_h^N(x,y) = \sum_{m=0}^{N+2}\sum_{n=0}^{N+2} u_{m,n}\mathcal{B}_m^{N\,\prime}(x)\mathcal{B}_n^N(y).$$

Hence,

$$D_x u_h^N(\tau_i^x,\tau_j^y) = \mathcal{B}_i^{N\,\prime}(\tau_i^x)\sum_{n=0}^{N+2} u_{i,n}\mathcal{B}_n^N(\tau_j^y), \quad i=0,N+2; \quad j=0,1,...,N+2. \tag{3.29}$$

For $i=0,N+2$, from (3.29), we obtain the following system of $N+3$ linear equations from (3.27) to solve for $u_{i,0},u_{i,1},...,u_{i,N+2}$:

$$M\mathbf{u}_i^x = \frac{-h^2/12}{\mathcal{B}_i^{N\,\prime}(\tau_i^x)}\mathbf{G}_i^x,$$

where $\mathcal{B}_i^{N\,\prime}(\tau_i^x),\ i=0,N+2$, are as in (3.16),

$$M=\frac{1}{8}\begin{bmatrix} 4 & 8 & & & & & \\ 1 & 7 & 1 & & & & \\ & 1 & 6 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 6 & 1 & \\ & & & & 1 & 7 & 1 \\ & & & & & 8 & 4 \end{bmatrix}_{(N+3)\times(N+3)}$$

$$\mathbf{u}_i^x = \left[u_{i,0}, u_{i,1}, ..., u_{i,N+2}\right]^T,$$

and

$$\mathbf{G}_i^x = \left[D_x g_{i,0}, D_x g_{i,1}, ..., D_x g_{i,N+2}\right]^T.$$

Similarly, from (3.25), we have

$$D_y u_h^N(x,y) = \sum_{m=0}^{N+2}\sum_{n=0}^{N+2} u_{m,n}\mathcal{B}_m^N(x)\mathcal{B}_n^{N\,\prime}(y).$$

Hence, for $i = 1, 2, ..., N+1;\ j = 0, N+2$,

$$\begin{aligned} D_y u_h^N(\tau_i^x, \tau_j^y) &= \mathcal{B}_j^{N\,\prime}(\tau_j^y)\sum_{m=0}^{N+2} u_{m,j}\mathcal{B}_m^N(\tau_i^x), \\ &= \mathcal{B}_j^{N\,\prime}(\tau_j^y)\left[u_{0,j}\mathcal{B}_0^N(\tau_i^x) + \sum_{m=1}^{N+1} u_{m,j}\mathcal{B}_m^N(\tau_i^x) + u_{N+2,j}\mathcal{B}_{N+2}^N(\tau_i^x)\right]. \end{aligned} \tag{3.30}$$

For $j = 0, N+2$, from (3.30), we then obtain the following system of linear equations from (3.28) to solve for $u_{1,j}, u_{2,j}, ..., u_{N+1,j}$:

$$B\mathbf{u}_j^y = \frac{-h^2/12}{\mathcal{B}_j^{N\,\prime}(\tau_j^y)}\mathbf{G}_j^y - \left[u_{0,j}\left(\boldsymbol{\mathcal{B}}_0^N\right)^x + u_{N+2,j}\left(\boldsymbol{\mathcal{B}}_{N+2}^N\right)^x\right], \quad j = 0, N+2,$$

where $\mathcal{B}_j^{N\,\prime}(\tau_j^y)$ and B are as in (3.16) and (3.6), respectively,

$$\mathbf{u}_j^y = \left[u_{1,j}, u_{2,j}, ..., u_{N+1,j}\right]^T,$$

$$\mathbf{G}_j^y = \left[D_y g_{1,j}, D_y g_{2,j}, ..., D_y g_{N+1,j}\right]^T,$$

and

$$\left(\boldsymbol{\mathcal{B}}_j^N\right)^x = \left[\mathcal{B}_j^N(\tau_1^x), \mathcal{B}_j^N(\tau_2^x), ..., \mathcal{B}_j^N(\tau_{N+1}^x)\right]^T.$$

Note that, from (3.3), we have

$$u_{0,j}\left(\boldsymbol{\mathcal{B}}_0^N\right)^x + u_{N+2,j}\left(\boldsymbol{\mathcal{B}}_{N+2}^N\right)^x = \left[u_{0,j}/8, 0, ..., 0, u_{N+2,j}/8\right]^T, \quad j = 0, N+2.$$

Collocating at the interior points (τ_i^x, τ_j^y), we seek $u_h^N \in S_2 \otimes S_2$ such that

$$(L_x + L_y + c)u_h^N(\tau_i^x, \tau_j^y) = g_{i,j}^N, \quad i, j = 1, ..., N+1, \tag{3.31}$$

where

$$g_{i,j}^N = \begin{cases} g_{i,j}, & i,j = 2,3,...,N, \\ g_{i,j} + \dfrac{h}{24} D_x g_{0,j}, & i = 1;\ j = 2,3,...,N, \\ g_{i,j} + \dfrac{h}{24} D_y g_{i,0}, & i = 2,3,...,N;\ j = 1, \\ g_{i,j} - \dfrac{h}{24} D_x g_{N+2,j}, & i = N+1;\ j = 2,3,...,N, \\ g_{i,j} - \dfrac{h}{24} D_y g_{i,N+2}, & i = 2,3,...,N;\ j = N+1, \\ g_{i,j} + \dfrac{h}{24} \left[D_x g_{0,j} + D_y g_{i,0} \right], & i,j = 1, \\ g_{i,j} + \dfrac{h}{24} \left[D_x g_{0,j} - D_y g_{i,N+2} \right], & i = 1;\ j = N+1, \\ g_{i,j} - \dfrac{h}{24} \left[D_x g_{N+2,j} - D_y g_{i,0} \right], & i = N+1;\ j = 1, \\ g_{i,j} - \dfrac{h}{24} \left[D_x g_{N+2,j} + D_y g_{i,N+2} \right], & i,j = N+1, \end{cases}$$

and, for each $j = 1,2,...,N+1$,

$$L_x u_h^N(\tau_i^x, \tau_j^y) = \begin{cases} \dfrac{1}{24} \left[23 D_x^2 u_h^N(\tau_i^x, \tau_j^y) + D_x^2 u_h^N(\tau_{i+1}^x, \tau_j^y) \right], & i = 1, \\ \dfrac{1}{24} \left[D_x^2 u_h^N(\tau_{i-1}^x, \tau_j^y) + 22 D_x^2 u_h^N(\tau_i^x, \tau_j^y) \right. \\ \qquad \left. + D_x^2 u_h^N(\tau_{i+1}^x, \tau_j^y) \right], & i = 2,3,...,N, \\ \dfrac{1}{24} \left[23 D_x^2 u_h^N(\tau_i^x, \tau_j^y) + D_x^2 u_h^N(\tau_{i-1}^x, \tau_j^y) \right], & i = N+1, \end{cases}$$

and similarly, for each $i = 1, 2, ..., N+1$,

$$L_y u_h^N(\tau_i^x, \tau_j^y) = \begin{cases} \frac{1}{24}\left[23D_y^2 u_h^N(\tau_i^x, \tau_j^y) + D_y^2 u_h^N(\tau_i^x, \tau_{j+1}^y)\right], & j = 1, \\ \frac{1}{24}\left[D_y^2 u_h^N(\tau_i^x, \tau_{j-1}^y) + 22D_y^2 u_h^N(\tau_i^x, \tau_j^y) \right. \\ \quad \left. + D_y^2 u_h^N(\tau_i^x, \tau_{j+1}^y)\right], & j = 2, 3, ..., N, \\ \frac{1}{24}\left[23D_y^2 u_h^N(\tau_i^x, \tau_j^y) + D_y^2 u_h^N(\tau_i^x, \tau_{j-1}^y)\right], & j = N+1. \end{cases}$$

From (3.25), for $i, j = 1, 2, ..., N+1$, we have

$$\begin{aligned} u_h^N(\tau_i^x, \tau_j^y) &= \sum_{m=0,N+2} \sum_{n=0}^{N+2} u_{m,n} \mathcal{B}_m^N(\tau_i^x) \mathcal{B}_n^N(\tau_j^y) + \sum_{m=1}^{N+1} \sum_{n=0,N+2} u_{m,n} \mathcal{B}_m^N(\tau_i^x) \mathcal{B}_n^N(\tau_j^y) \\ &+ \sum_{m=1}^{N+1} \sum_{n=1}^{N+1} u_{m,n} \mathcal{B}_m^N(\tau_i^x) \mathcal{B}_n^N(\tau_j^y), \end{aligned} \tag{3.32}$$

$$\begin{aligned} D_x^2 u_h^N(\tau_i^x, \tau_j^y) &= \sum_{m=0,N+2} \sum_{n=0}^{N+2} u_{m,n} \mathcal{B}_m^{N\,''}(\tau_i^x) \mathcal{B}_n^N(\tau_j^y) + \sum_{m=1}^{N+1} \sum_{n=0,N+2} u_{m,n} \mathcal{B}_m^{N\,''}(\tau_i^x) \mathcal{B}_n^N(\tau_j^y) \\ &+ \sum_{m=1}^{N+1} \sum_{n=1}^{N+1} u_{m,n} \mathcal{B}_m^{N\,''}(\tau_i^x) \mathcal{B}_n^N(\tau_j^y), \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} D_y^2 u_h^N(\tau_i^x, \tau_j^y) &= \sum_{m=0,N+2} \sum_{n=0}^{N+2} u_{m,n} \mathcal{B}_m^N(\tau_i^x) \mathcal{B}_n^{N\,''}(\tau_j^y) + \sum_{m=1}^{N+1} \sum_{n=0,N+2} u_{m,n} \mathcal{B}_m^N(\tau_i^x) \mathcal{B}_n^{N\,''}(\tau_j^y) \\ &+ \sum_{m=1}^{N+1} \sum_{n=1}^{N+1} u_{m,n} \mathcal{B}_m^N(\tau_i^x) \mathcal{B}''N_n(\tau_j^y). \end{aligned} \tag{3.34}$$

By substituting (3.32)–(3.34) into (3.31) and moving the terms with known coefficients $u_{k,0}, u_{k,1}, ..., u_{k,N+2}$ and $u_{1,k}, u_{2,k}, ..., u_{N+1,k}$, $k = 0, N+2$, to the right hand side of (3.31), we obtain the system of linear equations (2.34), with A and B given by (3.21) and (3.6), respectively.

Since the matrices A and B can be written in the form of (2.25), with the tridiagonal matrix T defined in (3.23), their eigenvalues are as in (2.38) where by [12], the eigenvalues of T are given by

$$\lambda_i^T = -4\sin^2\frac{(i-1)\pi}{2(N+1)}, \quad i = 1, 2, ..., N+1.$$

The columns of matrix Z used in the MDA are the corresponding eigenvectors of T, and Z is defined as follows:

$$Z_{i,j} = \begin{cases} \sqrt{\dfrac{1}{N+1}}, & i = 1, 2, ..., N+1;\ j = 1, \\[2ex] \sqrt{\dfrac{2}{N+1}}\cos\left[\dfrac{(2i-1)(j-1)\pi}{2(N+1)}\right], & i = 1, 2, ..., N+1;\ j = 2, 3, ..., N+1. \end{cases} \tag{3.35}$$

From (3.35), we have

$$Z^{-1} = Z^T.$$

3.2.2 Numerical Results

We consider four test problems for the Helmholtz equation (2.30) satisfying the Neumann BCs (3.24):

- Helmholtz N-1: $c = -1, \quad u(x,y) = e^{xy}(x^2-x)^2(y^2-y)^2$
- Helmholtz N-2: $c = -1, \quad u(x,y) = e^{2x+y}(x^2-x)^2(y^2-y)^2$
- Helmholtz N-3: $c = -1, \quad u(x,y) = \cos(\pi x)\cos(\pi y).$
- Helmholtz N-4: $c = -1, \quad u(x,y) = \cos(4\pi x)\cos(4\pi y).$

Numerical results are presented in Tables 3.10–3.13 in the same format as the Dirichlet Helmholtz case. The numerical results show optimal global accuracy and superonvergence as expected, very similar to the numerical results observed in the Helmholtz Dirichlet case.

Table 3.10. Helmholtz N-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	4.6912(-5)	1.1720(-5)	5.8959(-5)	6.0382(-5)				
10	4.2165(-6)	1.0352(-6)	9.8754(-6)	9.8283(-6)	3.9748	4.0036	2.9478	2.9951
15	9.4336(-7)	2.3145(-7)	3.2248(-6)	3.1808(-6)	3.9961	3.9980	2.9869	3.0108
20	3.1783(-7)	7.8112(-8)	1.4275(-6)	1.4007(-6)	4.0008	3.9944	2.9970	3.0159
25	1.3520(-7)	3.3279(-8)	7.5205(-7)	7.3548(-7)	4.0021	3.9949	3.0006	3.0164
$D_x u$								
5	3.5735(-3)	1.3092(-3)	4.3186(-4)	1.3092(-3)				
10	1.1154(-3)	4.6960(-4)	7.1840(-5)	4.6960(-4)	1.9209	1.6915	2.9592	1.6915
15	5.3195(-4)	2.3603(-4)	2.3445(-5)	2.3603(-4)	1.9761	1.8360	2.9885	1.8360
20	3.0963(-4)	1.4126(-4)	1.0383(-5)	1.4126(-4)	1.9901	1.8877	2.9954	1.8877
25	2.0219(-4)	9.3854(-5)	5.4731(-6)	9.3854(-5)	1.9955	1.9145	2.9979	1.9145
$D_y u$								
5	3.5735(-3)	1.3092(-3)	4.3186(-4)	1.3092(-3)				
10	1.1154(-3)	4.6960(-4)	7.1840(-5)	4.6960(-4)	1.9209	1.6915	2.9592	1.6915
15	5.3195(-4)	2.3603(-4)	2.3445(-5)	2.3603(-4)	1.9761	1.8360	2.9885	1.8360
20	3.0963(-4)	1.4126(-4)	1.0383(-5)	1.4126(-4)	1.9901	1.8877	2.9954	1.8877
25	2.0219(-4)	9.3854(-5)	5.4731(-6)	9.3854(-5)	1.9955	1.9145	2.9979	1.9145
$D_x^2 u$								
5		5.8135(-3)	6.7238(-2)	7.7268(-2)				
10		1.8725(-3)	3.9617(-2)	4.5677(-2)		1.8690	8.7273(-1)	8.6727(-1)
15		9.1219(-4)	2.7972(-2)	3.2334(-2)		1.9194	9.2885(-1)	9.2201(-1)
20		5.3802(-4)	2.1653(-2)	2.5036(-2)		1.9415	9.4159(-1)	9.4076(-1)
25		3.5550(-4)	1.7654(-2)	2.0402(-2)		1.9402	9.5625(-1)	9.5837(-1)
$D_y^2 u$								
5		5.8135(-3)	6.7238(-2)	7.7268(-2)				
10		1.8725(-3)	3.9617(-2)	4.5677(-2)		1.8690	8.7273(-1)	8.6727(-1)
15		9.1219(-4)	2.7972(-2)	3.2334(-2)		1.9194	9.2885(-1)	9.2201(-1)
20		5.3802(-4)	2.1653(-2)	2.5036(-2)		1.9415	9.4159(-1)	9.4076(-1)
25		3.5550(-4)	1.7654(-2)	2.0402(-2)		1.9402	9.5625(-1)	9.5837(-1)
$D_{xy}^2 u$								
5	1.6371(-2)	7.3687(-3)	2.3696(-3)	7.3687(-3)				
10	5.0848(-3)	2.2789(-3)	4.6657(-4)	2.2789(-3)	1.9291	1.9361	2.6810	1.9361
15	2.3209(-3)	1.1430(-3)	1.4358(-4)	1.1430(-3)	2.0932	1.8416	3.1453	1.8416
20	1.3635(-3)	6.6170(-4)	6.5542(-5)	6.6170(-4)	1.9559	2.0101	2.8837	2.0101
25	8.8052(-4)	4.3783(-4)	3.3914(-5)	4.3783(-4)	2.0475	1.9336	3.0850	1.9336

Table 3.11. Helmholtz N-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	1.2525(-3)	4.9855(-4)	8.5517(-4)	8.7129(-4)				
10	1.1518(-4)	4.4919(-5)	1.3261(-4)	1.3402(-4)	3.9371	3.9708	3.0751	3.0884
15	2.5884(-5)	1.0101(-5)	4.1644(-5)	4.1722(-5)	3.9842	3.9825	3.0912	3.1145
20	8.7331(-6)	3.4157(-6)	1.7993(-5)	1.7910(-5)	3.9955	3.9872	3.0860	3.1097
25	3.7170(-6)	1.4568(-6)	9.3388(-6)	9.2425(-6)	3.9995	3.9898	3.0705	3.0976
D_xu								
5	4.2227(-2)	8.5725(-3)	9.0039(-3)	1.4152(-2)				
10	1.2951(-2)	4.3183(-3)	1.5500(-3)	4.3183(-3)	1.9499	1.1313	2.9027	1.9583
15	6.1355(-3)	2.3589(-3)	5.1231(-4)	2.3589(-3)	1.9939	1.6138	2.9546	1.6138
20	3.5589(-3)	1.4744(-3)	2.2757(-4)	1.4744(-3)	2.0029	1.7281	2.9841	1.7281
25	2.3190(-3)	1.0020(-3)	1.2003(-4)	1.0020(-3)	2.0055	1.8084	2.9951	1.8084
D_yu								
5	2.1152(-2)	6.4214(-3)	3.0712(-3)	6.4214(-3)				
10	6.4412(-3)	2.5117(-3)	5.1664(-4)	2.5117(-3)	1.9617	1.5486	2.9407	1.5486
15	3.0536(-3)	1.2934(-3)	1.6930(-4)	1.2934(-3)	1.9920	1.7713	2.9776	1.7713
20	1.7735(-3)	7.8291(-4)	7.5128(-5)	7.8291(-4)	1.9982	1.8461	2.9877	1.8461
25	1.1569(-3)	5.2451(-4)	3.9654(-5)	5.2451(-4)	2.0002	1.8754	2.9919	1.8754
D_x^2u								
5		1.0105(-1)	6.5994(-1)	7.5961(-1)				
10		3.5812(-2)	4.2417(-1)	4.9230(-1)		1.7114	7.2923(-1)	7.1553(-1)
15		1.8156(-2)	3.0830(-1)	3.5681(-1)		1.8130	8.5155(-1)	8.5905(-1)
20		1.1008(-2)	2.4085(-1)	2.7846(-1)		1.8400	9.0786(-1)	9.1171(-1)
25		7.3615(-3)	1.9728(-1)	2.2797(-1)		1.8839	9.3449(-1)	9.3680(-1)
D_y^2u								
5		3.9212(-2)	3.7867(-1)	4.3768(-1)				
10		1.3063(-2)	2.2452(-1)	2.5963(-1)		1.8135	8.6238(-1)	8.6156(-1)
15		6.4476(-3)	1.5893(-1)	1.8375(-1)		1.8844	9.2207(-1)	9.2264(-1)
20		3.8296(-3)	1.2290(-1)	1.4206(-1)		1.9157	9.4535(-1)	9.4608(-1)
25		2.5388(-3)	1.0016(-1)	1.1576(-1)		1.9249	9.5787(-1)	9.5859(-1)
D_{xy}^2u								
5	1.7673(-1)	4.8236(-2)	3.8612(-2)	6.1210(-2)				
10	4.7740(-2)	1.8965(-2)	5.8641(-3)	1.8965(-2)	2.1594	1.5401	3.1094	1.9331
15	2.2070(-2)	9.7243(-3)	1.9203(-3)	9.7243(-3)	2.0592	1.7827	2.9794	1.7827
20	1.2773(-2)	5.8335(-3)	8.5091(-4)	5.8335(-3)	2.0111	1.8791	2.9932	1.8791
25	8.3774(-3)	3.8709(-3)	4.4623(-4)	3.8709(-3)	1.9750	1.9203	3.0222	1.9203

Table 3.12. Helmholtz N-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	1.1452(-3)	5.2095(-5)	1.5413(-3)	1.6014(-3)				
10	1.0010(-4)	4.4659(-6)	2.1534(-4)	2.1682(-4)	4.0207	4.0529	3.2471	3.2988
15	2.2299(-5)	9.8975(-7)	6.7194(-5)	6.6622(-5)	4.0076	4.0213	3.1083	3.1494
20	7.5063(-6)	3.3250(-7)	2.8970(-5)	2.8514(-5)	4.0040	4.0114	3.0939	3.1207
25	3.1929(-6)	1.4129(-7)	1.5058(-5)	1.4765(-5)	4.0024	4.0070	3.0639	3.0816
$D_x u$								
5	7.0116(-2)	3.4947(-2)	7.4586(-3)	3.4947(-2)				
10	2.0997(-2)	1.0706(-2)	1.1847(-3)	1.0706(-2)	1.9892	1.9518	3.0354	1.9518
15	1.0062(-2)	5.0288(-3)	3.8309(-4)	5.0288(-3)	1.9633	2.0166	3.0131	2.0166
20	5.8322(-3)	2.9318(-3)	1.6912(-4)	2.9318(-3)	2.0055	1.9842	3.0068	1.9842
25	3.8178(-3)	1.9086(-3)	8.9032(-5)	1.9086(-3)	1.9840	2.0098	3.0042	2.0098
$D_y u$								
5	7.0116(-2)	3.4947(-2)	7.4586(-3)	3.4947(-2)				
10	2.0997(-2)	1.0706(-2)	1.1847(-3)	1.0706(-2)	1.9892	1.9518	3.0354	1.9518
15	1.0062(-2)	5.0288(-3)	3.8309(-4)	5.0288(-3)	1.9633	2.0166	3.0131	2.0166
20	5.8322(-3)	2.9318(-3)	1.6912(-4)	2.9318(-3)	2.0055	1.9842	3.0068	1.9842
25	3.8178(-3)	1.9086(-3)	8.9032(-5)	1.9086(-3)	1.9840	2.0098	3.0042	2.0098
$D_x^2 u$								
5		1.0394(-1)	1.4831	1.7152				
10		3.2749(-2)	8.1130(-1)	9.3710(-1)		1.9054	9.9524(-1)	9.9728(-1)
15		1.5676(-2)	5.5904(-1)	6.4566(-1)		1.9662	9.9391(-1)	9.9419(-1)
20		9.1433(-3)	4.2588(-1)	4.9181(-1)		1.9826	1.0005	1.0009
25		5.9784(-3)	3.4417(-1)	3.9745(-1)		1.9893	9.9737(-1)	9.9741(-1)
$D_y^2 u$								
5		1.0394(-1)	1.4831	1.7152				
10		3.2749(-2)	8.1130(-1)	9.3710(-1)		1.9054	9.9524(-1)	9.9728(-1)
15		1.5676(-2)	5.5904(-1)	6.4566(-1)		1.9662	9.9391(-1)	9.9419(-1)
20		9.1433(-3)	4.2588(-1)	4.9181(-1)		1.9826	1.0005	1.0009
25		5.9784(-3)	3.4417(-1)	3.9745(-1)		1.9893	9.9737(-1)	9.9741(-1)
$D_{xy}^2 u$								
5	4.5729(-1)	2.1877(-1)	2.4587(-2)	2.1877(-1)				
10	1.3201(-1)	6.7885(-2)	3.8627(-3)	6.7885(-2)	2.0498	1.9306	3.0535	1.9306
15	6.3544(-2)	3.1581(-2)	1.2380(-3)	3.1581(-2)	1.9513	2.0424	3.0367	2.0424
20	3.6651(-2)	1.8467(-2)	5.4331(-4)	1.8467(-2)	2.0237	1.9731	3.0287	1.9731
25	2.4034(-2)	1.1990(-2)	2.8483(-4)	1.1990(-2)	1.9757	2.0224	3.0237	2.0224

Table 3.13. Helmholtz N-4

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	3.3168(-1)	4.4256(-2)	2.3478(-1)	2.6923(-1)				
10	2.7446(-2)	1.9063(-3)	2.3299(-2)	2.5416(-2)	4.1112	5.1883	3.8114	3.8745
15	5.9065(-3)	2.8848(-4)	5.6181(-3)	5.9298(-3)	4.0998	5.0396	3.7963	3.9155
20	1.9594(-3)	1.0318(-4)	2.4008(-3)	2.5124(-3)	4.0577	3.7808	3.1265	3.1580
25	8.2736(-4)	4.1757(-5)	1.1745(-3)	1.2129(-3)	4.0367	4.2357	3.3476	3.4098
D_xu								
5	1.1470	3.3642	2.4915	3.3642				
10	1.1721	8.1705(-1)	3.6036(-1)	8.1705(-1)	-3.5632	2.3348	3.1900	2.3348
15	6.1017(-1)	3.0272(-1)	1.0439(-1)	3.0272(-1)	1.7422	2.6499	3.3065	2.6499
20	3.6251(-1)	1.9760(-1)	4.5629(-2)	1.9760(-1)	1.9147	2.5687	3.0434	1.5687
25	2.3813(-1)	1.2592(-1)	2.3493(-2)	1.2592(-1)	1.9678	2.1097	3.1083	2.1097
D_yu								
5	1.1470	3.3642	2.4915	3.3642				
10	1.1721	8.1705(-1)	3.6036(-1)	8.1705(-1)	-3.5632	2.3348	3.1900	2.3348
15	6.1017(-1)	3.0272(-1)	1.0439(-1)	3.0272(-1)	1.7422	2.6499	3.3065	2.6499
20	3.6251(-1)	1.9760(-1)	4.5629(-2)	1.9760(-1)	1.9147	2.5687	3.0434	1.5687
25	2.3813(-1)	1.2592(-1)	2.3493(-2)	1.2592(-1)	1.9678	2.1097	3.1083	2.1097
D_x^2u								
5		2.2534(+1)	8.2691(+1)	9.4939(+1)				
10		8.0858	5.2832(+1)	6.0551(+1)		1.6909	7.3909(-1)	7.4199(-1)
15		3.3720	3.5211(+1)	4.0811(+1)		2.3342	1.0830	1.0530
20		2.3202	2.7475(+1)	3.1657(+1)		1.3749	9.1219(-1)	9.3408(-1)
25		1.5218	2.2047(+1)	2.5486(+1)		1.9745	1.0305	1.0151
D_y^2u								
5		2.2534(+1)	8.2691(+1)	9.4939(+1)				
10		8.0858	5.2832(+1)	6.0551(+1)		1.6909	7.3909(-1)	7.4199(-1)
15		3.3720	3.5211(+1)	4.0811(+1)		2.3342	1.0830	1.0530
20		2.3202	2.7475(+1)	3.1657(+1)		1.3749	9.1219(-1)	9.3408(-1)
25		1.5218	2.2047(+1)	2.5486(+1)		1.9745	1.0305	1.0151
D_{xy}^2u								
5	9.8102(+1)	6.4301(+1)	3.1321(+1)	6.4301(+1)				
10	3.5759(+1)	1.9915(+1)	4.7216	1.9915(+1)	1.6650	1.9337	3.1216	1.9338
15	1.6739(+1)	7.5438	1.4042	7.5438	2.0258	2.5908	3.2365	2.5908
20	9.5433	4.9289	6.0574(-1)	4.9289	2.0664	1.5651	3.0917	1.5651
25	6.1316	3.1322	3.1256(-1)	3.1322	2.0713	2.1229	3.0980	2.1229

Chapter 4

MIXED BOUNDARY CONDITIONS

Since the second-order TPBVP for Dirichlet and Neumann BCs have been done, we only present the Helmholtz problem for the mixed (Neumann-Dirichlet and Dirichlet-Neumann) BCs.

4.1 Helmholtz Neumann-Dirichlet Boundary Conditions

4.1.1 New Modified Quadratic Spline Collocation

Consider the Helmholtz equation (2.30) subject to the homogeneous Neumann-Dirichlet BCs

$$\begin{cases} D_x u(0,y) = u(1,y) = 0, & y \in [0,1], \\ D_y u(x,0) = u(x,1) = 0, & x \in [0,1]. \end{cases} \tag{4.1}$$

Let S_2^{ND} be the space of quadratic splines satisfying the Dirichlet BC but not the Neumann BC, that is, $S_2^{ND} = S_2 \cap \{v : v(1) = 0\}$. Then a basis for S_2^{ND} is defined in terms of $\{B_m\}_{m=0}^{N+2}$ of (2.4) by

$$\{\mathcal{B}_m^{ND}\}_{m=0}^{N+1} = \{\mathcal{B}_0, \mathcal{B}_1 + \mathcal{B}_0, \mathcal{B}_2, ..., \mathcal{B}_N, \mathcal{B}_{N+1} - \mathcal{B}_{N+2}\}. \tag{4.2}$$

From (2.8) and (4.2), we have

$$\mathcal{B}_j^{ND}(\tau_i) = \begin{cases} 7/8, & i,j=1, \\ 1/8, & i=1,2,...,N+1;\ j=0,1,...,N+1;\ i=j\pm 1, \\ 3/4, & i,j=2,3,...,N;\ i=j, \\ 5/8, & i,j=N+1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{B}_j^{ND\,''}(\tau_i) = \begin{cases} -1/h^2, & i,j=1, \\ 1/h^2, & i=1,2,...,N+1;\ j=0,1,...,N+1;\ i=j\pm 1, \\ -2/h^2, & i,j=2,3,...,N;\ i=j, \\ -3/h^2, & i,j=N+1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

We seek an approximate solution $u_h^{ND} \in S_2^{ND} \otimes S_2^{ND}$ such that

$$u_h^{ND}(x,y) = \sum_{m=0}^{N+1}\sum_{n=0}^{N+1} u_{m,n}\mathcal{B}_m^{ND}(x)\mathcal{B}_n^{ND}(y). \tag{4.4}$$

According to the Helmholtz Neumann case, the BCs perturbations are

$$D_x u_h^{ND}(0,\tau_j^y) = -\frac{h^2}{12}D_x g_{0,j}, \quad j=0,1,...,N+1, \tag{4.5}$$

and

$$D_y u_h^{ND}(\tau_i^x,0) = -\frac{h^2}{12}D_y g_{i,0}, \quad i=1,2,...,N+1. \tag{4.6}$$

From (4.4), we have

$$D_x u_h^{ND}(x,y) = \sum_{m=0}^{N+1}\sum_{n=0}^{N+1} u_{m,n}\mathcal{B}_m^{ND\,\prime}(x)\mathcal{B}_n^{ND}(y).$$

Hence,

$$D_x u_h^{ND}(0,\tau_j^y) = \mathcal{B}_0^{P\,\prime}(0)\sum_{n=0}^{N+1} u_{0,n}\mathcal{B}_n^{P}(\tau_j^y), \quad j=0,...,N+1. \tag{4.7}$$

From (4.7), we obtain the following system of $N+2$ linear equations from (4.5) to solve for $u_{0,0}, u_{0,1}, ..., u_{0,N+1}$:

$$M\mathbf{u}_0^x = \frac{-h^2/12}{\mathcal{B}_0^{ND\,\prime}(0)}\mathbf{G}_0^x,$$

where $\mathcal{B}_0^{ND\,\prime}(0)$ is as in (3.16),

$$M = \frac{1}{8}\begin{bmatrix} 4 & 8 & & & & \\ 1 & 7 & 1 & & & \\ & 1 & 6 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 6 & 1 \\ & & & & 1 & 5 \end{bmatrix}_{(N+2)\times(N+2)}$$

$$\mathbf{u}_0^x = \left[u_{0,0}, u_{0,1}, ..., u_{0,N+1}\right]^T, \tag{4.8}$$

and

$$\mathbf{G}_0^x = \left[D_x g_{0,0}, D_x g_{0,1}, ..., D_x g_{0,N+1}\right]^T.$$

Similarly, from (4.4), we have

$$D_y u_h^{ND}(x,y) = \sum_{m=0}^{N+1}\sum_{n=0}^{N+1} u_{m,n}\mathcal{B}_m^{ND}(x)\mathcal{B}_n^{ND\,\prime}(y).$$

Hence, for $i = 1, 2, ..., N+1$,

$$D_y u_h^{ND}(\tau_i^x, 0) = \mathcal{B}_0^{ND\,\prime}(0) \sum_{m=0}^{N+1} u_{m,0} \mathcal{B}_m^{ND}(\tau_i^x), \tag{4.9}$$

$$= \mathcal{B}_0^{ND\,\prime}(0) \left[u_{0,0} \mathcal{B}_0^{ND}(\tau_i^x) + \sum_{m=1}^{N+1} u_{m,0} \mathcal{B}_m^{ND}(\tau_i^x) \right],. \tag{4.10}$$

From (4.10), we then obtain the following system of $N+1$ linear equations from (4.6) to solve for $u_{1,0}, u_{2,0}, ..., u_{N+1,0}$:

$$B\mathbf{u}_0^y = \frac{-h^2/12}{\mathcal{B}_0^{ND\,\prime}(0)} \mathbf{G}_0^y - u_{0,0} \left(\boldsymbol{\mathcal{B}}_0^{ND}\right)^x,$$

where $\mathcal{B}_0^{ND\,\prime}(0)$ is as in (3.16),

$$B = \frac{1}{8} \begin{bmatrix} 7 & 1 & & & & \\ 1 & 6 & 1 & & & \\ & 1 & 7 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 6 & 1 \\ & & & & 1 & 5 \end{bmatrix}_{(N+1)\times(N+1)} \tag{4.11}$$

$$\mathbf{u}_0^y = \left[u_{1,0}, u_{2,0}, ..., u_{N+1,0}\right]^T, \tag{4.12}$$

$$\mathbf{G}_0^y = \left[D_y g_{1,0}, D_y g_{2,0}, ..., D_y g_{N+1,0}\right]^T,$$

and

$$\left(\boldsymbol{\mathcal{B}}_0^{ND}\right)^x = \left[\mathcal{B}_0^{ND}(\tau_1^x), \mathcal{B}_0^{ND}(\tau_2^x), ..., \mathcal{B}_0^{ND}(\tau_{N+1}^x)\right]^T = [1/8, 0, ..., 0]^T.$$

Collocating at the interior points (τ_i^x, τ_j^y), we seek $u_h^{ND} \in S_2^{ND} \otimes S_2^{ND}$ such that

$$(L_x + L_y + c)u_h^{ND}(\tau_i^x, \tau_j^y) = g_{i,j}^{ND}, \quad i,j = 1, ..., N+1, \tag{4.13}$$

where

$$g_{i,j}^{ND} = \begin{cases} g_{i,j}, & i,j = 2,3,...,N, \\ g_{i,j} + \dfrac{h}{24} D_x g_{0,j}, & i = 1;\ j = 2,3,...,N, \\ g_{i,j} + \dfrac{h}{24} D_y g_{i,0}, & i = 2,3,...,N;\ j = 1, \\ g_{i,j} - \dfrac{1}{12} g_{N+2,j}, & i = N+1;\ j = 2,3,...,N, \\ g_{i,j} - \dfrac{1}{12} g_{i,N+2}, & i = 2,3,...,N;\ j = N+1, \\ g_{i,j} + \dfrac{h}{24}\left[D_x g_{0,j} + D_y g_{i,0}\right], & i,j = 1, \\ g_{i,j} + \dfrac{h}{24} D_x g_{0,j} - \dfrac{1}{12} g_{i,N+2}, & i = 1;\ j = N+1, \\ g_{i,j} - \dfrac{1}{12} g_{N+2,j} + \dfrac{h}{24} D_y g_{i,0}, & i = N+1;\ j = 1, \\ g_{i,j} - \dfrac{1}{12}\left[g_{N+2,j} + g_{i,N+2}\right], & i,j = N+1, \end{cases}$$

and, for each $j = 1, 2, ..., N+1$,

$$L_x u_h^{ND}(\tau_i^x,\tau_j^y) = \begin{cases} \frac{1}{24}\left[23D_x^2 u_h^{ND}(\tau_i^x,\tau_j^y) + D_x^2 u_h^{ND}(\tau_{i+1}^x,\tau_j^y)\right], & i=1, \\ \frac{1}{24}\left[D_x^2 u_h^{ND}(\tau_{i-1}^x,\tau_j^y) + 22D_x^2 u_h^{ND}(\tau_i^x,\tau_j^y) \right. \\ \quad \left. + D_x^2 u_h^{ND}(\tau_{i+1}^x,\tau_j^y)\right], & i=2,3,...,N, \\ \frac{1}{24}\left[21D_x^2 u_h^{ND}(\tau_i^x,\tau_j^y) + D_x^2 u_h^{ND}(\tau_{i-1}^x,\tau_j^y)\right], & i=N+1, \end{cases}$$

and similarly for each $i = 1,2,...,N+1$,

$$L_y u_h^{ND}(\tau_i^x,\tau_j^y) = \begin{cases} \frac{1}{24}\left[23D_y^2 u_h^{ND}(\tau_i^x,\tau_j^y) + D_y^2 u_h^{ND}(\tau_i^x,\tau_{j+1}^y)\right], & j=1, \\ \frac{1}{24}\left[D_y^2 u_h^{ND}(\tau_i^x,\tau_{j-1}^y) + 22D_y^2 u_h^{ND}(\tau_i^x,\tau_j^y) \right. \\ \quad \left. + D_y^2 u_h^{ND}(\tau_i^x,\tau_{j+1}^y)\right], & j=2,3,...,N, \\ \frac{1}{24}\left[21D_y^2 u_h^{ND}(\tau_i^x,\tau_j^y) + D_y^2 u_h^{ND}(\tau_i^x,\tau_{j-1}^y)\right], & j=N+1. \end{cases}$$

From (4.4), for $i,j = 1,2,...,N+1$, we have

$$\begin{aligned} u_h^{ND}(\tau_i^x,\tau_j^y) &= \mathcal{B}_0^{ND}(\tau_i^x)\sum_{n=0}^{N+1} u_{0,n}\mathcal{B}_n^{ND}(\tau_j^y) + \mathcal{B}_0^{ND}(\tau_j^y)\sum_{m=1}^{N+1} u_{m,0}\mathcal{B}_m^{ND}(\tau_i^x) \\ &+ \sum_{m=1}^{N+1}\sum_{n=1}^{N+1} u_{m,n}\mathcal{B}_m^{ND}(\tau_i^x)\mathcal{B}_n^{ND}(\tau_j^y), \end{aligned} \tag{4.14}$$

$$\begin{aligned} D_x^2 u_h^{ND}(\tau_i^x,\tau_j^y) &= \mathcal{B}_0^{ND\,''}(\tau_i^x)\sum_{n=0}^{N+1} u_{0,n}\mathcal{B}_n^{ND}(\tau_j^y) + \mathcal{B}_0^{ND}(\tau_j^y)\sum_{m=1}^{N+1} u_{m,0}\mathcal{B}_m^{ND\,''}(\tau_i^x) \\ &+ \sum_{m=1}^{N+1}\sum_{n=1}^{N+1} u_{m,n}\mathcal{B}_m^{ND\,''}(\tau_i^x)\mathcal{B}_n^{ND}(\tau_j^y), \end{aligned} \tag{4.15}$$

and

$$D_y^2 u_h^{ND}(\tau_i^x, \tau_j^y) = \mathcal{B}_0^{ND}(\tau_i^x) \sum_{n=0}^{N+1} u_{0,n} \mathcal{B}_n^{ND\,''}(\tau_j^y) + \mathcal{B}_0^{ND\,''}(\tau_j^y) \sum_{m=1}^{N+1} u_{m,0} \mathcal{B}_m^{ND}(\tau_i^x)$$
$$+ \sum_{m=1}^{N+1} \sum_{n=1}^{N+1} u_{m,n} \mathcal{B}_m^{ND}(\tau_i^x) \mathcal{B}_n^{ND}(\tau_j^y). \qquad (4.16)$$

Similar to the Neumann case, by substituting (4.14)–(4.16) into (4.13) and moving the terms with known coefficients $u_{0,0}, u_{0,1}, \ldots, u_{0,N+1}$ and $u_{1,0}, u_{2,0}, \ldots, u_{N+1,0}$ to the right hand side, (4.13) gives rise to a system of linear equations of the form (2.34) with the matrix B defined in (4.11) and

$$A = \frac{1}{24h^2} \begin{bmatrix} -22 & 21 & 1 & & & & \\ 21 & -42 & 20 & 1 & & & \\ 1 & 20 & -42 & 20 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 20 & -42 & 20 & 1 \\ & & & 1 & 20 & -42 & 19 \\ & & & & 1 & 19 & -62 \end{bmatrix}_{(N+1)\times(N+1)} \qquad (4.17)$$

The matrices A and B can also be written as in (2.25) with

$$T = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -3 \end{bmatrix}_{(N+1)\times(N+1)} \qquad (4.18)$$

Hence, their eigenvalues are as in (2.38) where by [14], the eigenvalues of the tridiagonal matrix T defined in (4.18) are given by

$$\lambda_i^T = -4\sin^2\frac{(2i-1)\pi}{4(N+1)}, \quad i = 1, 2, ..., N+1, \tag{4.19}$$

and the columns of matrix Z used in the MDA are the corresponding eigenvectors of the matrix T with Z defined as

$$Z_{i,j} = \sqrt{\frac{2}{N+1}}\cos\left[\frac{(2i-1)(2j-1)\pi}{4(N+1)}\right], \quad i, j = 1, 2, ..., N+1. \tag{4.20}$$

By (4.20), we have

$$Z^{-1} = Z^T.$$

4.1.2 Numerical Results

We consider four test problems for the Helmholtz equation (2.30) satisfying the Neumann-Dirichlet BCs (4.1):

- Helmholtz ND-1: $\quad c = -1, \quad u(x, y) = e^{xy}(x^2 - x)^2(y^2 - y)^2.$
- Helmholtz ND-2: $\quad c = -3, \quad u(x, y) = \cos\left(\frac{\pi x}{2}\right)\left(y^2 - y\right)^2 e^y.$
- Helmholtz ND-3: $\quad c = -1, \quad u(x, y) = \cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right).$
- Helmholtz ND-4: $\quad c = -1, \quad u(x, y) = \cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{3\pi y}{2}\right).$

Numerical results are presented in Tables 4.1–4.4 in the same format as in previous Helmholtz problems. The numerical results show optimal global accuracy and superconvergence as expected.

Table 4.1. Helmholtz ND-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	1.9131(-5)	1.7952(-5)	5.8351(-5)	5.4484(-5)				
10	1.6703(-6)	1.7859(-6)	9.7978(-6)	9.2743(-6)	4.0226	3.8073	2.9438	2.9212
15	3.7987(-7)	4.1672(-7)	3.2023(-6)	3.0466(-6)	3.9524	3.8839	2.9845	2.9710
20	1.2728(-7)	1.4375(-7)	1.4199(-6)	1.3573(-6)	4.0209	3.9141	2.9907	2.9733
25	5.4409(-8)	6.2204(-8)	7.4987(-7)	7.1746(-7)	3.9794	3.9220	2.9894	2.9850
$D_x u$								
5	3.3315(-3)	1.4568(-3)	3.9739(-4)	1.4568(-3)				
10	1.0769(-3)	4.9090(-4)	7.1674(-5)	4.9090(-4)	1.8632	1.7946	2.8257	1.7946
15	5.1975(-4)	2.4252(-4)	2.3999(-5)	2.4252(-4)	1.9442	1.8819	2.9201	1.8819
20	3.0433(-4)	1.4403(-4)	1.0768(-5)	1.4403(-4)	1.9683	1.9163	2.9470	1.9163
25	1.9943(-4)	9.5273(-5)	5.7227(-6)	9.5273(-5)	1.9789	1.9350	2.9599	1.9350
$D_y u$								
5	3.3315(-3)	1.4568(-3)	3.9739(-4)	1.4568(-3)				
10	1.0769(-3)	4.9090(-4)	7.1674(-5)	4.9090(-4)	1.8632	1.7946	2.8257	1.7946
15	5.1975(-4)	2.4252(-4)	2.3999(-5)	2.4252(-4)	1.9442	1.8819	2.9201	1.8819
20	3.0433(-4)	1.4403(-4)	1.0768(-5)	1.4403(-4)	1.9683	1.9163	2.9470	1.9163
25	1.9943(-4)	9.5273(-5)	5.7227(-6)	9.5273(-5)	1.9789	1.9350	2.9599	1.9350
$D_x^2 u$								
5		4.1816(-3)	6.5557(-2)	7.5585(-2)				
10		1.5390(-3)	3.9183(-2)	4.5242(-2)		1.6491	8.4912(-1)	8.4671(-1)
15		8.0142(-4)	2.7776(-2)	3.2148(-2)		1.7414	9.1822(-1)	9.1190(-1)
20		4.8913(-4)	2.1546(-2)	2.4928(-2)		1.8157	9.3403(-1)	9.3542(-1)
25		3.2899(-4)	1.7583(-2)	2.0331(-2)		1.8570	9.5158(-1)	9.5432(-1)
$D_y^2 u$								
5		4.1816(-3)	6.5557(-2)	7.5585(-2)				
10		1.5390(-3)	3.9183(-2)	4.5242(-2)		1.6491	8.4912(-1)	8.4671(-1)
15		8.0142(-4)	2.7776(-2)	3.2148(-2)		1.7414	9.1822(-1)	9.1190(-1)
20		4.8913(-4)	2.1546(-2)	2.4928(-2)		1.8157	9.3403(-1)	9.3542(-1)
25		3.2899(-4)	1.7583(-2)	2.0331(-2)		1.8570	9.5158(-1)	9.5432(-1)
$D_{xy}^2 u$								
5	1.4377(-2)	8.1852(-3)	3.0359(-3)	8.1852(-3)				
10	5.0729(-3)	2.3965(-3)	4.6463(-4)	2.3965(-3)	1.7186	2.0265	3.0967	2.0265
15	2.3142(-3)	1.1484(-3)	1.4563(-4)	1.1484(-3)	2.0946	1.9632	3.0964	1.9632
20	1.3607(-3)	6.6413(-4)	6.3219(-5)	6.6413(-4)	1.9528	2.0140	3.0686	2.0140
25	8.7929(-4)	4.3901(-4)	3.4022(-5)	4.3901(-4)	2.0446	1.9382	2.9011	1.9382

Table 4.2. Helmholtz ND-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	3.2695(-4)	7.0549(-4)	1.7595(-3)	1.6393(-3)				
10	4.0100(-5)	6.9729(-5)	2.9446(-4)	2.7812(-4)	3.4620	3.8181	2.9492	2.9267
15	1.0022(-5)	1.6173(-5)	9.6119(-5)	9.1287(-5)	3.7006	3.8999	2.9879	2.9732
20	3.5754(-6)	5.5535(-6)	4.2548(-5)	4.0530(-5)	3.7903	3.9309	2.9969	2.9859
25	1.5751(-6)	2.3903(-6)	2.2419(-5)	2.1396(-5)	3.8382	3.9472	3.0000	2.9912
D_xu								
5	9.9549(-4)	1.0393(-3)	2.8194(-3)	2.6403(-3)				
10	2.8525(-4)	1.0695(-4)	4.6527(-4)	4.4024(-4)	2.0620	3.7516	2.9724	2.9553
15	1.3420(-4)	5.9166(-5)	1.5140(-4)	1.4393(-4)	2.0124	1.5799	2.9962	2.9837
20	7.7746(-5)	3.6272(-5)	6.6943(-5)	6.3809(-4)	2.0074	1.7993	3.0012	2.9914
25	5.0650(-5)	2.4260(-5)	3.5253(-5)	3.3660(-5)	2.0064	1.8833	3.0026	2.9946
D_yu								
5	1.0296(-1)	3.9111(-2)	1.2529(-2)	3.9111(-2)				
10	3.2192(-2)	1.3945(-2)	2.4362(-3)	1.3945(-2)	1.9181	1.7014	2.7017	1.7014
15	1.5458(-2)	7.0141(-3)	8.4448(-4)	7.0141(-3)	1.9577	1.8340	2.8275	1.8340
20	9.0432(-3)	4.2017(-3)	3.8612(-4)	4.2017(-3)	1.9716	1.8844	2.8778	1.8844
25	5.9264(-3)	2.7936(-3)	2.0761(-4)	2.7936(-3)	1.9787	1.9112	2.9052	1.9112
D_x^2u								
5		1.8472(-3)	1.9234(-2)	2.2141(-2)				
10		2.7980(-4)	1.0743(-2)	1.2406(-2)		3.1137	9.6086(-1)	9.5566(-1)
15		1.1766(-4)	7.4233(-3)	8.5758(-3)		2.3118	9.8654(-1)	9.8538(-1)
20		6.5062(-5)	5.6630(-3)	6.5388(-3)		2.1788	9.9533(-1)	9.9726(-1)
25		4.1635(-5)	4.5720(-3)	5.2780(-3)		2.0902	1.0020	1.0030
D_y^2u								
5		1.4900(-1)	1.8723	2.1734				
10		5.2820(-2)	1.1459	1.3269		1.7109	8.1005(-1)	8.1411(-1)
15		2.7792(-2)	8.1857(-1)	9.4703(-1)		1.7137	8.9774(-1)	9.0009(-1)
20		1.7363(-2)	6.3568(-1)	7.3510(-1)		1.7299	9.2991(-1)	9.3160(-1)
25		1.1842(-2)	5.1931(-1)	6.0036(-1)		1.7917	9.4667(-1)	9.4799(-1)
D_{xy}^2u								
5	1.6265(-1)	6.2625(-2)	1.9317(-2)	6.2625(-2)				
10	5.0652(-2)	2.2153(-2)	3.8059(-3)	2.2153(-2)	1.9247	1.7145	2.6800	1.7145
15	2.4302(-2)	1.1106(-2)	1.3231(-3)	1.1106(-2)	1.9601	1.8427	2.8198	1.8427
20	1.4212(-2)	6.6411(-3)	6.0561(-4)	6.6411(-3)	1.9728	1.8910	2.8738	1.8910
25	9.3121(-3)	4.4103(-3)	3.2580(-4)	4.4103(-3)	1.9794	1.9165	2.9028	1.9165

Table 4.3. Helmholtz ND-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	7.1022(-5)	2.7482(-6)	1.6504(-4)	1.6516(-4)				
10	6.2660(-6)	2.4106(-7)	2.5168(-5)	2.4763(-5)	4.0055	4.0150	3.1027	3.1306
15	1.3988(-6)	5.3743(-8)	7.9825(-6)	7.7995(-6)	4.0020	4.0056	3.0647	3.0832
20	4.7123(-7)	1.8096(-8)	3.4879(-6)	3.3958(-6)	4.0010	4.0029	3.0447	3.0578
25	2.0052(-7)	7.6983(-9)	1.8239(-6)	1.7718(-6)	4.0006	4.0018	3.0356	3.0462
$D_x u$								
5	8.9214(-3)	4.4582(-3)	4.5562(-4)	4.4582(-3)				
10	2.6649(-3)	1.3322(-3)	7.3530(-5)	1.3322(-3)	1.9935	1.9928	3.0092	1.9928
15	1.2607(-3)	6.3028(-4)	2.3864(-5)	6.3028(-4)	1.9977	1.9974	3.0033	1.9974
20	7.3206(-4)	3.6601(-4)	1.0550(-5)	3.6601(-4)	1.9988	1.9987	3.0017	1.9987
25	4.7764(-4)	2.3881(-4)	5.5575(-6)	2.3881(-4)	1.9993	1.9992	3.0011	1.9992
$D_y u$								
5	8.9214(-3)	4.4582(-3)	4.5562(-4)	4.4582(-3)				
10	2.6649(-3)	1.3322(-3)	7.3530(-5)	1.3322(-3)	1.9935	1.9928	3.0092	1.9928
15	1.2607(-3)	6.3028(-4)	2.3864(-5)	6.3028(-4)	1.9977	1.9974	3.0033	1.9974
20	7.3206(-4)	3.6601(-4)	1.0550(-5)	3.6601(-4)	1.9988	1.9987	3.0017	1.9987
25	4.7764(-4)	2.3881(-4)	5.5575(-6)	2.3881(-4)	1.9993	1.9992	3.0011	1.9992
$D_x^2 u$								
5		6.9051(-3)	1.8260(-1)	2.1088(-1)				
10		2.0839(-3)	1.0108(-1)	1.1672(-1)		1.9765	9.7562(-1)	9.7579(-1)
15		9.8809(-4)	6.9722(-2)	8.0510(-2)		1.9915	9.9122(-1)	9.9128(-1)
20		5.7427(-4)	5.3187(-2)	6.1416(-2)		1.9956	9.9545(-1)	9.9549(-1)
25		3.7485(-4)	4.2984(-2)	4.9635(-2)		1.9973	9.9722(-1)	9.9724(-1)
$D_y^2 u$								
5		6.9051(-3)	1.8260(-1)	2.1088(-1)				
10		2.0839(-3)	1.0108(-1)	1.1672(-1)		1.9765	9.7562(-1)	9.7579(-1)
15		9.8809(-4)	6.9722(-2)	8.0510(-2)		1.9915	9.9122(-1)	9.9128(-1)
20		5.7427(-4)	5.3187(-2)	6.1416(-2)		1.9956	9.9545(-1)	9.9549(-1)
25		3.7485(-4)	4.2984(-2)	4.9635(-2)		1.9973	9.9722(-1)	9.9724(-1)
$D_{xy}^2 u$								
5	2.8284(-2)	1.3992(-2)	7.4242(-4)	1.3992(-2)				
10	8.3945(-3)	4.1841(-3)	1.1801(-4)	4.1841(-3)	2.0041	1.9917	3.0343	1.9917
15	3.9655(-3)	1.9798(-3)	3.8065(-5)	1.9798(-3)	2.0015	1.9970	3.0196	1.9970
20	2.3015(-3)	1.1498(-3)	1.6769(-5)	1.1498(-3)	2.0008	1.9985	3.0146	1.9985
25	1.5013(-3)	7.5022(-4)	8.8145(-6)	7.0522(-4)	2.0005	1.9991	3.0112	1.9991

Table 4.4. Helmholtz ND-4

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	2.8702(-3)	2.6890(-4)	4.3160(-3)	4.3880(-3)				
10	2.4850(-4)	2.1819(-5)	6.6040(-4)	6.4337(-4)	4.0366	4.1436	3.0970	3.1675
15	5.5236(-5)	4.6903(-6)	2.0880(-4)	2.0252(-4)	4.0134	4.1027	3.0731	3.0849
20	1.8578(-5)	1.5486(-6)	9.1563(-5)	8.8581(-5)	4.0070	4.0752	3.0314	3.0408
25	7.8993(-6)	6.5799(-7)	4.8151(-5)	4.6441(-5)	4.0043	4.0075	3.0092	3.0235
D_xu								
5	4.4992(-3)	3.7372(-3)	6.7518(-3)	6.2549(-3)				
10	2.2837(-3)	1.2983(-3)	1.0325(-3)	1.2983(-3)	1.1187	1.7442	3.0981	2.5939
15	1.1760(-3)	6.2300(-4)	3.2893(-4)	6.2300(-4)	1.7712	1.9597	3.0528	1.9597
20	7.0360(-4)	3.6156(-4)	1.4377(-4)	3.6156(-4)	1.8890	2.0009	3.0434	2.0009
25	4.6555(-4)	2.3779(-4)	7.5572(-5)	2.3779(-4)	1.9337	1.9620	3.0113	1.9620
D_yu								
5	2.4391(-1)	1.2201(-1)	3.7869(-2)	1.2201(-1)				
10	7.2229(-2)	3.6911(-2)	6.0037(-3)	3.6911(-2)	2.0077	1.9725	3.0385	1.9725
15	3.4100(-2)	1.7228(-2)	1.9404(-3)	1.7228(-2)	2.0030	2.0335	3.0145	2.0335
20	1.9787(-2)	9.8976(-3)	8.5642(-4)	9.8976(-3)	2.0016	2.0382	3.0076	2.0382
25	1.2905(-2)	6.4782(-3)	4.5081(-4)	6.4782(-3)	2.0010	1.9846	3.0047	1.9846
D_x^2u								
5		7.0936(-3)	1.8329(-1)	2.1274(-1)				
10		2.1372(-3)	1.0100(-1)	1.1649(-1)		1.9793	9.8308(-1)	9.9351(-1)
15		9.9953(-4)	6.9884(-2)	8.0647(-2)		2.0282	9.8302(-1)	9.8151(-1)
20		5.7483(-4)	5.3219(-2)	6.1477(-2)		2.0343	1.0018(-1)	9.9811(-1)
25		3.7645(-4)	4.2978(-2)	4.9615(-2)		1.9819	1.0007(-1)	1.0037
D_y^2u								
5		5.0794(-1)	5.0335	5.8060				
10		1.6757(-1)	2.7389	3.1579		1.8296	1.0040	1.0047
15		7.9762(-2)	1.8910	2.1829		1.9812	9.8877(-1)	9.8552(-1)
20		4.6164(-2)	1.4387	1.6611		2.0109	1.0052	1.0045
25		3.0324(-2)	1.1613	1.3406		1.9678	1.0028	1.0036
D_{xy}^2u								
5	4.2790(-1)	2.1069(-1)	5.9449(-2)	2.1069(-1)				
10	1.2625(-1)	6.4209(-2)	9.3777(-3)	6.4209(-2)	2.0137	1.9604	3.0468	1.9604
15	5.9559(-2)	3.0022(-2)	3.0398(-3)	3.0022(-2)	2.0052	2.0289	3.0066	2.0289
20	3.4548(-2)	1.7259(-2)	1.3452(-3)	1.7259(-2)	2.0027	2.0358	2.9980	2.0358
25	2.2530(-2)	1.1300(-2)	7.0742(-4)	1.1300(-2)	2.0017	1.9831	3.0091	1.9831

4.2 Helmholtz Dirichlet-Neumann BCs

4.2.1 New Modified Quadratic Spline Collocation

Consider the Helmholtz equation (2.30) subject to the homogeneous Dirichlet-Neumann BCs

$$\begin{cases} D_x u(1,y) = u(0,y) = 0, & y \in [0,1], \\ D_y u(x,1) = u(x,0) = 0, & x \in [0,1]. \end{cases} \tag{4.21}$$

Let S_2^{DN} be the space of quadratic splines satisfying the Dirichlet BC but not the Neumann BC, that is, $S_2^{DN} = S_2 \cap \{v : v(0) = 0\}$. Then a basis for S_2^{DN} is defined in terms of $\{\mathcal{B}_m\}_{m=1}^{N+2}$ of (2.4) by

$$\{\mathcal{B}_m^{DN}\}_{m=1}^{N+2} = \{\mathcal{B}_1 - \mathcal{B}_0, \mathcal{B}_2, ..., \mathcal{B}_N, \mathcal{B}_{N+1} + \mathcal{B}_{N+2}, \mathcal{B}_{N+2}\}\,. \tag{4.22}$$

From (2.8) and (4.22), we have

$$\mathcal{B}_j^{DN}(\tau_i) = \begin{cases} 5/8, & i,j=1, \\ 1/8, & i=1,2,...,N+1;\ j=1,2,...,N+2;\ i=j\pm 1, \\ 3/4, & i,j=2,3,...,N;\ i=j, \\ 7/8, & i,j=N+1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{B}_j^{DN\,''}(\tau_i) = \begin{cases} -3/h^2, & i,j=1, \\ 1/h^2, & i=1,2,...,N+1;\ j=1,2,...,N+2;\ i=j\pm 1, \\ -2/h^2, & i,j=2,3,...,N;\ i=j, \\ -1/h^2, & i,j=N+1, \\ 0, & \text{otherwise}. \end{cases} \tag{4.23}$$

We seek an approximate solution $u_h^{DN} \in S_2^{DN} \otimes S_2^{DN}$ such that

$$u_h^{DN}(x,y) = \sum_{m=1}^{N+2}\sum_{n=1}^{N+2} u_{m,n}\mathcal{B}_m^{DN}(x)\mathcal{B}_n^{DN}(y). \tag{4.24}$$

According to the Helmholtz Neumann case, the BCs perturbations are

$$D_x u_h^{DN}(1,\tau_j^y) = -\frac{h^2}{12}D_x g_{N+2,j}, \quad j=1,2,...,N+2, \tag{4.25}$$

and

$$D_y u_h^{DN}(\tau_i^x,1) = -\frac{h^2}{12}D_y g_{i,N+2}, \quad i=1,2,...,N+1. \tag{4.26}$$

From (4.24), we have

$$D_x u_h^{DN}(x,y) = \sum_{m=1}^{N+2}\sum_{n=1}^{N+2} u_{m,n}\mathcal{B}_m^{DN\,\prime}(x)\mathcal{B}_n^{DN}(y).$$

Hence,

$$D_x u_h^{DN}(1,\tau_j^y) = \mathcal{B}_{N+2}^{DN\,\prime}(1)\sum_{n=1}^{N+2} u_{N+2,n}\mathcal{B}_n^{DN}(\tau_j^y), \quad j=1,2,...,N+2. \tag{4.27}$$

From (4.27), we obtain the following system of $N+2$ linear equations from (4.25) to solve for $u_{N+2,1}, u_{N+2,2}, ..., u_{N+2,N+2}$:

$$M\mathbf{u}_{N+2}^x = \frac{-h^2/12}{\mathcal{B}_{N+2}^{DN\,\prime}(1)}\mathbf{G}_{N+2}^x,$$

where $\mathcal{B}_{N+2}^{DN\,\prime}(1)$ is as in (3.16),

$$M = \frac{1}{8}\begin{bmatrix} 5 & 1 & & & & \\ 1 & 6 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & 6 & 1 & \\ & & & 1 & 7 & 1 \\ & & & & 8 & 4 \end{bmatrix}_{(N+2)\times(N+2)}$$

$$\mathbf{u}_{N+2}^x = \left[u_{N+2,1}, u_{N+2,2}, ..., u_{N+2,N+2}\right]^T,$$

and

$$\mathbf{G}_{N+2}^x = \left[D_x g_{N+2,1}, D_x g_{N+2,2}, ..., D_x g_{N+2,N+2})\right]^T.$$

Similarly, from (4.24), we have

$$D_y u_h^{DN}(x,y) = \sum_{m=1}^{N+2}\sum_{n=1}^{N+2} u_{m,n}\mathcal{B}_m^{DN}(x)\mathcal{B}_n^{DN\,\prime}(y).$$

Hence, for $i = 1, ..., N+1$,

$$\begin{aligned} D_y u_h^{DN}(\tau_i^x, 1) &= \mathcal{B}_{N+2}^{DN\,\prime}(1)\sum_{m=1}^{N+2} u_{m,N+2}\mathcal{B}_m^{DN}(\tau_i^x), \\ &= \mathcal{B}_{N+2}^{DN\,\prime}(1)\left[\sum_{m=1}^{N+1} u_{m,N+2}\mathcal{B}_m^{DN}(\tau_i^x) + u_{N+2,N+2}\mathcal{B}_{N+2}^{DN}(\tau_i^x)\right],. \end{aligned} \tag{4.28}$$

From (4.28, we then obtain the following system of linear equations from (4.26) to solve for $u_{1,N+2}, u_{2,N+2}, ..., u_{N+1,N+2}$:

$$B\mathbf{u}_{N+2}^y = \frac{-h^2/12}{\mathcal{B}_{N+2}^{DN\,\prime}(1)}\mathbf{G}_{N+2}^y - u_{N+2,N+2}\left(\boldsymbol{\mathcal{B}}_{N+2}^{DN}\right)^x,$$

where $\mathcal{B}_{N+2}^{DN\,\prime}(1)$ is as in (3.16),

$$B = \frac{1}{8}\begin{bmatrix} 5 & 1 & & & & \\ 1 & 6 & 1 & & & \\ & 1 & 7 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 6 & 1 \\ & & & & 1 & 7 \end{bmatrix}_{(N+1)\times(N+1)}. \tag{4.29}$$

$$\mathbf{u}_{N+2}^y = \left[u_{1,N+2}, u_{2,N+2}, ..., u_{N+1,N+2}\right]^T,$$

$$\mathbf{G}_{N+2}^y = \left[D_y g_{1,N+2}, D_y g_{2,N+2} ..., D_y g_{N+1,N+2}\right]^T,$$

and

$$\left(\mathcal{B}_{N+2}^{DN}\right)^x = \left[\mathcal{B}_{N+2}^{DN}(\tau_1^x), ..., \mathcal{B}_{N+2}^{DN}(\tau_{N+1}^x)\right]^T = [0, ..., 0, 1/8]^T .$$

Collocating at the interior points (τ_i^x, τ_j^y), we seek $u_h^{DN} \in S_2^{DN} \otimes S_2^{DN}$ such that

$$(L_x + L_y + c)u_h^{DN}(\tau_i^x, \tau_j^y) = g_{i,j}^{DN}, \quad i, j = 1, 2, ..., N+1, \tag{4.30}$$

where

$$g_{i,j}^{DN} = \begin{cases} g_{i,j}, & i, j = 2, 3, ..., N, \\ g_{i,j} - \dfrac{1}{12} g_{0,j}, & i = 1;\ j = 2, 3, ..., N, \\ g_{i,j} - \dfrac{1}{12} g_{i,0}, & i = 2, 3, ..., N;\ j = 1, \\ g_{i,j} - \dfrac{h}{24} D_x g_{N+2,j}, & i = N+1;\ j = 2, 3, ..., N, \\ g_{i,j} - \dfrac{h}{24} D_y g_{i,N+2}, & i = 2, 3, ..., N;\ j = N+1, \\ g_{i,j} - \dfrac{1}{12}\left[g_{0,j} + g_{i,0}\right], & i, j = 1, \\ g_{i,j} - \dfrac{1}{12} g_{0,j} - \dfrac{h}{24} D_y g_{i,N+2}, & i = 1;\ j = N+1, \\ g_{i,j} - \dfrac{h}{24} D_x g_{N+2,j} - \dfrac{1}{12} g_{i,0}, & i = N+1;\ j = 1, \\ g_{i,j} - \dfrac{h}{24}\left[D_x g_{N+2,j} + D_y g_{i,N+2}\right], & i, j = N+1, \end{cases}$$

and, for each $j = 1, 2, ..., N+1$,

$$L_x u_h^{DN}(\tau_i^x, \tau_j^y) = \begin{cases} \frac{1}{24}\left[21 D_x^2 u_h^{DN}(\tau_i^x, \tau_j^y) + D_x^2 u_h^{DN}(\tau_{i+1}^x, \tau_j^y)\right], & i = 1, \\ \frac{1}{24}\left[D_x^2 u_h^{DN}(\tau_{i-1}^x, \tau_j^y) + 22 D_x^2 u_h^{DN}(\tau_i^x, \tau_j^y) \right. \\ \qquad \left. + D_x^2 u_h^{DN}(\tau_{i+1}^x, \tau_j^y)\right], & i = 2, ..., N, \\ \frac{1}{24}\left[23 D_x^2 u_h^{DN}(\tau_i^x, \tau_j^y) + D_x^2 u_h^{DN}(\tau_{i-1}^x, \tau_j^y)\right], & i = N+1, \end{cases}$$

and similarly for each $i = 1, ..., N+1$,

$$L_y u_h^{DN}(\tau_i^x, \tau_j^y) = \begin{cases} \frac{1}{24}\left[21 D_y^2 u_h^{DN}(\tau_i^x, \tau_j^y) + D_y^2 u_h^{DN}(\tau_i^x, \tau_{j+1}^y)\right], & j = 1, \\ \frac{1}{24}\left[D_y^2 u_h^{DN}(\tau_i^x, \tau_{j-1}^y) + 22 D_y^2 u_h^{DN}(\tau_i^x, \tau_j^y) \right. \\ \qquad \left. + D_y^2 u_h^{DN}(\tau_i^x, \tau_{j+1}^y)\right], & j = 2, ..., N, \\ \frac{1}{24}\left[23 D_y^2 u_h^{DN}(\tau_i^x, \tau_j^y) + D_y^2 u_h^{DN}(\tau_i^x, \tau_{j-1}^y)\right], & j = N+1. \end{cases}$$

From (4.24), for $i, j = 1, 2, ..., N+1$, we have

$$\begin{aligned} u_h^{DN}(\tau_i^x, \tau_j^y) &= \sum_{m=1}^{N+1}\sum_{n=1}^{N+1} u_{m,n} \mathcal{B}_m^{DN}(\tau_i^x) \mathcal{B}_n^{DN}(\tau_j^y) \\ &+ \mathcal{B}_{N+2}^{DN}(\tau_i^x) \sum_{n=1}^{N+2} u_{N+2,n} \mathcal{B}_n^{DN}(\tau_j^y) + \mathcal{B}_{N+2}^{DN}(\tau_j^y) \sum_{m=1}^{N+1} u_{m,N+2} \mathcal{B}_m^{DN}(\tau_i^x), \end{aligned} \tag{4.31}$$

$$D_x^2 u_h^{DN}(\tau_i^x, \tau_j^y) = \sum_{m=1}^{N+1}\sum_{n=1}^{N+1} u_{m,n}\mathcal{B}_m^{DN\,''}(\tau_i^x)\mathcal{B}_n^{DN}(\tau_j^y)$$
$$+ \mathcal{B}_{N+2}^{DN\,''}(\tau_i^x)\sum_{n=1}^{N+2} u_{N+2,n}\mathcal{B}_n^{DN}(\tau_j^y) + \mathcal{B}_{N+2}^{DN}(\tau_j^y)\sum_{m=1}^{N+1} u_{m,N+2}\mathcal{B}_m^{DN\,''}(\tau_i^x), \tag{4.32}$$

$$D_y^2 u_h^{DN}(\tau_i^x, \tau_j^y) = \sum_{m=1}^{N+1}\sum_{n=1}^{N+1} u_{m,n}\mathcal{B}_m^{DN}(\tau_i^x)\mathcal{B}_n^{DN\,''}(\tau_j^y)$$
$$+ \mathcal{B}_{N+2}^{DN}(\tau_i^x)\sum_{n=1}^{N+2} u_{N+2,n}\mathcal{B}_n^{DN\,''}(\tau_j^y) + \mathcal{B}_{N+2}^{DN\,''}(\tau_j^y)\sum_{m=1}^{N+1} u_{m,N+2}\mathcal{B}_m^{DN}(\tau_i^x). \tag{4.33}$$

On substituting (4.31)–(4.33) into (4.13) and moving the terms with known coefficients $u_{N+2,1}, u_{N+2,2}, \ldots, u_{N+2,N+2}$ and $u_{1,N+2}, u_{2,N+2}, \ldots, u_{N+1,N+2}$ to the right hand side, (4.30) gives rise to a system of linear equations of the form (2.34) with

$$A = \frac{1}{24h^2}\begin{bmatrix} -62 & 19 & 1 & & & & \\ 19 & -42 & 20 & 1 & & & \\ 1 & 20 & -42 & 20 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 20 & -42 & 20 & 1 \\ & & & 1 & 20 & -42 & 21 \\ & & & & 1 & 21 & -22 \end{bmatrix}_{(N+1)\times(N+1)} \tag{4.34}$$

and the matrix B defined in (4.29).

The matrices A and B can also be written as in (2.25) with

$$T = \begin{bmatrix} -3 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}_{(N+1)\times(N+1)} \tag{4.35}$$

Hence, their eigenvalues are as in (2.38), respectively, where by [14], the eigenvalues of the tridiagonal matrix T defined in (4.35) are given in (4.19). The columns of matrix Z are the corresponding eigenvectors of the matrix T where these eigenvectors are in reverse order of those in the Neumann-Dirichlet case given in (4.20), that is,

$$Z = KZ_{ND}, \tag{4.36}$$

where Z_{ND} is as in (4.20) and

$$K = \begin{bmatrix} & & 1 \\ & \cdot & \\ 1 & & \end{bmatrix}_{(N+1)\times(N+1)}$$

From (4.36), we have

$$Z^{-1} = Z^T.$$

4.2.2 Numerical Results

We consider four test problems for the Helmholtz equation (2.30) satisfying the Dirichlet-Neumann BCs (4.21):

- Helmholtz DN-1: $c = -1, \quad u(x, y) = e^{xy}(x^2 - x)^2(y^2 - y)^2.$

- Helmholtz DN-2: $c = 0, \quad u(x,y) = \sin\left(\frac{\pi x}{2}\right)\left(y^2 - \frac{3y}{2}\right)e^y.$
- Helmholtz DN-3: $c = -1, \quad u(x,y) = \sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi y}{2}\right).$
- Helmholtz DN-4: $c = -1, \quad u(x,y) = \sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{3\pi y}{2}\right).$

Numerical results are presented in Tables 4.5–4.8 in the same format as previous Helmholtz problems. The numerical results again show the expected optimal global accuracy and superconvergence.

Table 4.5. Helmholtz DN-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	4.0139(-5)	5.1100(-6)	5.1979(-5)	5.3608(-5)				
10	3.6129(-6)	4.5060(-7)	9.2624(-6)	9.2146(-6)	3.9724	4.0063	2.8457	2.9051
15	8.0813(-7)	1.0064(-7)	3.0882(-6)	3.0441(-6)	3.9967	4.0006	2.9314	2.9560
20	2.7219(-7)	3.3914(-8)	1.3815(-6)	1.3547(-6)	4.0018	4.0002	2.9582	2.9772
25	1.1576(-7)	1.4443(-8)	7.3248(-7)	7.1591(-7)	4.0031	3.9968	2.9707	2.9863
$D_x u$								
5	3.5735(-3)	1.3089(-3)	4.3197(-4)	1.3089(-3)				
10	1.1154(-3)	4.6959(-4)	7.1845(-5)	4.6959(-4)	1.9209	1.6912	2.9595	1.6912
15	5.3195(-4)	2.3602(-4)	2.3446(-5)	2.3602(-4)	1.9761	1.8360	2.9886	1.8360
20	3.0963(-4)	1.4126(-4)	1.0383(-5)	1.4126(-4)	1.9901	1.8877	2.9955	1.8877
25	2.0219(-4)	9.3854(-5)	5.4732(-6)	9.3854(-5)	1.9955	1.9145	2.9979	1.9145
$D_y u$								
5	3.5735(-3)	1.3089(-3)	4.3197(-4)	1.3089(-3)				
10	1.1154(-3)	4.6959(-4)	7.1845(-5)	4.6959(-4)	1.9209	1.6912	2.9595	1.6912
15	5.3195(-4)	2.3602(-4)	2.3446(-5)	2.3602(-4)	1.9761	1.8360	2.9886	1.8360
20	3.0963(-4)	1.4126(-4)	1.0383(-5)	1.4126(-4)	1.9901	1.8877	2.9955	1.8877
25	2.0219(-4)	9.3854(-5)	5.4732(-6)	9.3854(-5)	1.9955	1.9145	2.9979	1.9145
$D_x^2 u$								
5		5.8104(-3)	6.7235(-2)	7.7264(-2)				
10		1.8723(-3)	3.9616(-2)	4.5677(-2)		1.8684	8.7266(-1)	8.6721(-1)
15		9.1212(-4)	2.7972(-2)	3.2334(-2)		1.9192	9.2884(-1)	9.2200(-1)
20		5.3800(-4)	2.1653(-2)	2.5036(-2)		1.9414	9.4158(-1)	9.4076(-1)
25		3.5549(-4)	1.7653(-2)	2.0402(-2)		1.9401	9.5625(-1)	9.5837(-1)
$D_y^2 u$								
5		5.8104(-3)	6.7235(-2)	7.7264(-2)				
10		1.8723(-3)	3.9616(-2)	4.5677(-2)		1.8684	8.7266(-1)	8.6721(-1)
15		9.1212(-4)	2.7972(-2)	3.2334(-2)		1.9192	9.2884(-1)	9.2200(-1)
20		5.3800(-4)	2.1653(-2)	2.5036(-2)		1.9414	9.4158(-1)	9.4076(-1)
25		3.5549(-4)	1.7653(-2)	2.0402(-2)		1.9401	9.5625(-1)	9.5837(-1)
$D_{xy}^2 u$								
5	1.6371(-2)	7.3687(-3)	2.3696(-3)	7.3687(-3)				
10	5.0848(-3)	2.2789(-3)	4.6657(-4)	2.2789(-3)	1.9291	1.9361	2.6810	1.9361
15	2.3209(-3)	1.1430(-3)	1.4358(-4)	1.1430(-3)	2.0932	1.8416	3.1453	1.8416
20	1.3635(-3)	6.6170(-4)	6.5542(-5)	6.6170(-4)	1.9559	2.0101	2.8837	2.0101
25	8.8052(-4)	4.3783(-4)	3.3914(-5)	4.3783(-4)	2.0475	1.9336	3.0850	1.9336

Table 4.6. Helmholtz DN-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	1.8629(-4)	1.8317(-5)	6.4805(-4)	6.3553(-4)				
10	1.6553(-5)	1.8217(-6)	1.1222(-4)	1.0921(-4)	3.9938	3.8078	2.8929	2.9056
15	3.7010(-6)	4.2329(-7)	3.7217(-5)	3.6100(-5)	3.9977	3.8951	2.9457	2.9543
20	1.2476(-6)	1.4548(-7)	1.6622(-5)	1.6095(-5)	3.9988	3.9275	2.9640	2.9706
25	5.3103(-7)	6.2651(-8)	8.8085(-6)	8.5194(-6)	3.9993	3.9446	2.9733	2.9786
$D_x u$								
5	1.2572(-2)	5.9970(-3)	1.1540(-3)	5.9970(-3)				
10	3.6614(-3)	1.8047(-3)	1.9654(-4)	1.8047(-3)	2.0353	1.9811	2.9204	1.9811
15	1.7222(-3)	8.5528(-4)	6.5013(-5)	8.5528(-4)	2.0129	1.9929	2.9525	1.9929
20	9.9793(-4)	4.9700(-4)	2.8964(-5)	4.9700(-4)	2.0067	1.9963	2.9733	1.9963
25	6.5045(-4)	3.2438(-4)	1.5331(-5)	3.2438(-4)	2.0041	1.9977	2.9788	1.9977
$D_y u$								
5	4.4045(-2)	1.8234(-2)	2.7071(-3)	1.8234(-2)				
10	1.3105(-2)	5.9535(-3)	4.4145(-4)	5.9535(-3)	1.9999	1.8466	2.9920	1.8466
15	6.1940(-3)	2.9047(-3)	1.4360(-4)	2.9047(-3)	2.0000	1.9153	2.9973	1.9153
20	3.5956(-3)	1.7133(-3)	6.3532(-5)	1.7133(-3)	2.0000	1.9413	2.9987	1.9413
25	2.3457(-3)	1.1285(-3)	3.3481(-5)	1.1285(-3)	2.0000	1.9550	2.9993	1.9550
$D_x^2 u$								
5		9.3399(-3)	2.4796(-1)	2.8648(-1)				
10		2.8379(-3)	1.3734(-1)	1.5861(-1)		1.9711	9.7477(-1)	9.7537(-1)
15		1.3419(-3)	9.4745(-2)	1.0941(-2)		1.9894	9.9080(-1)	9.9106(-1)
20		7.8017(-4)	7.2281(-2)	8.3468(-2)		1.9945	9.9521(-1)	9.9535(-1)
25		5.0932(-4)	5.8418(-2)	6.7457(-2)		1.9966	9.9705(-1)	9.9714(-1)
$D_y^2 u$								
5		3.6861(-2)	8.5201(-1)	9.8564(-1)				
10		1.1714(-2)	4.8061(-1)	5.5540(-1)		1.8913	9.4460(-1)	9.4633(-1)
15		5.6715(-3)	3.3448(-1)	3.8641(-1)		1.9357	9.6741(-1)	9.6822(-1)
20		3.3336(-3)	2.5645(-1)	2.9623(-1)		1.9541	9.7681(-1)	9.7732(-1)
25		2.1914(-3)	2.0793(-1)	2.4017(-1)		1.9643	9.8197(-1)	9.8234(-1)
$D_{xy}^2 u$								
5	6.9583(-2)	3.0826(-2)	4.1522(-3)	3.0826(-2)				
10	2.0620(-2)	9.7309(-3)	6.8856(-4)	9.7309(-3)	2.0066	1.9023	2.9643	1.9023
15	9.7373(-3)	4.6891(-3)	2.2481(-4)	4.6891(-3)	2.0024	1.9484	2.9874	1.9484
20	5.6506(-3)	2.7480(-3)	9.9603(-5)	2.7480(-3)	2.0012	1.9651	2.9936	1.9651
25	3.6857(-3)	1.8028(-3)	5.2525(-5)	1.8028(-3)	2.0008	1.9737	2.9961	1.9737

Table 4.7. Helmholtz DN-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	7.1022(-5)	2.7482(-6)	1.6504(-4)	1.6516(-4)				
10	6.2660(-6)	2.4106(-7)	2.5168(-5)	2.4763(-5)	4.0055	4.0150	3.1027	3.1306
15	1.3988(-6)	5.3743(-8)	7.9825(-6)	7.7995(-6)	4.0020	4.0056	3.0647	3.0832
20	4.7123(-7)	1.8096(-8)	3.4879(-6)	3.3958(-6)	4.0010	4.0029	3.0447	3.0578
25	2.0052(-7)	7.6983(-9)	1.8239(-6)	1.7718(-6)	4.0006	4.0018	3.0356	3.0462
D_xu								
5	8.9214(-3)	4.4582(-3)	4.5562(-4)	4.4582(-3)				
10	2.6649(-3)	1.3322(-3)	7.3530(-5)	1.3322(-3)	1.9935	1.9928	3.0092	1.9928
15	1.2607(-3)	6.3028(-4)	2.3864(-5)	6.3028(-4)	1.9977	1.9974	3.0033	1.9974
20	7.3206(-4)	3.6601(-4)	1.0550(-5)	3.6601(-4)	1.9988	1.9987	3.0017	1.9987
25	4.7764(-4)	2.3881(-4)	5.5575(-6)	2.3881(-4)	1.9993	1.9992	3.0011	1.9992
D_yu								
5	8.9214(-3)	4.4582(-3)	4.5562(-4)	4.4582(-3)				
10	2.6649(-3)	1.3322(-3)	7.3530(-5)	1.3322(-3)	1.9935	1.9928	3.0092	1.9928
15	1.2607(-3)	6.3028(-4)	2.3864(-5)	6.3028(-4)	1.9977	1.9974	3.0033	1.9974
20	7.3206(-4)	3.6601(-4)	1.0550(-5)	3.6601(-4)	1.9988	1.9987	3.0017	1.9987
25	4.7764(-4)	2.3881(-4)	5.5575(-6)	2.3881(-4)	1.9993	1.9992	3.0011	1.9992
D_x^2u								
5		6.9051(-3)	1.8260(-1)	2.1088(-1)				
10		2.0839(-3)	1.0108(-1)	1.1672(-1)		1.9765	9.7562(-1)	9.7579(-1)
15		9.8809(-4)	6.9722(-2)	8.0510(-2)		1.9915	9.9122(-1)	9.9128(-1)
20		5.7427(-4)	5.3187(-2)	6.1416(-2)		1.9956	9.9545(-1)	9.9549(-1)
25		3.7485(-4)	4.2984(-2)	4.9635(-2)		1.9973	9.9722(-1)	9.9724(-1)
D_y^2u								
5		6.9051(-3)	1.8260(-1)	2.1088(-1)				
10		2.0839(-3)	1.0108(-1)	1.1672(-1)		1.9765	9.7562(-1)	9.7579(-1)
15		9.8809(-4)	6.9722(-2)	8.0510(-2)		1.9915	9.9122(-1)	9.9128(-1)
20		5.7427(-4)	5.3187(-2)	6.1416(-2)		1.9956	9.9545(-1)	9.9549(-1)
25		3.7485(-4)	4.2984(-2)	4.9635(-2)		1.9973	9.9722(-1)	9.9724(-1)
D_{xy}^2u								
5	2.8284(-2)	1.3992(-2)	7.4242(-4)	1.3992(-2)				
10	8.3945(-3)	4.1841(-3)	1.1801(-4)	4.1841(-3)	2.0041	1.9917	3.0343	1.9917
15	3.9655(-3)	1.9798(-3)	3.8065(-5)	1.9798(-3)	2.0015	1.9970	3.0196	1.9970
20	2.3015(-3)	1.1498(-3)	1.6769(-5)	1.1498(-3)	2.0008	1.9985	3.0146	1.9985
25	1.5013(-3)	7.5022(-4)	8.8145(-6)	7.5022(-4)	2.0005	1.9991	3.0112	1.9991

Table 4.8. Helmholtz DN-4

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	2.8702(-3)	2.6890(-4)	4.3160(-3)	4.3880(-3)				
10	2.4850(-4)	2.1819(-5)	6.6040(-4)	6.4337(-4)	4.0366	4.1436	3.0970	3.1675
15	5.5236(-5)	4.6903(-6)	2.0880(-4)	2.0252(-4)	4.0134	4.1027	3.0731	3.0849
20	1.8578(-5)	1.5486(-6)	9.1563(-5)	8.8581(-5)	4.0070	4.0752	3.0314	3.0408
25	7.8993(-6)	6.5799(-7)	4.8151(-5)	4.6441(-5)	4.0043	4.0075	3.0092	3.0235
D_xu								
5	4.4992(-3)	3.7372(-3)	6.7518(-3)	6.2549(-3)				
10	2.2837(-3)	1.2983(-3)	1.0325(-3)	1.2983(-3)	1.1187	1.7442	3.0981	2.5939
15	1.1760(-3)	6.2300(-4)	3.2893(-4)	6.2300(-4)	1.7712	1.9597	3.0528	1.9597
20	7.0360(-4)	3.6156(-4)	1.4377(-4)	3.6156(-4)	1.8890	2.0009	3.0434	2.0009
25	4.6555(-4)	2.3779(-4)	7.5572(-5)	2.3779(-4)	1.9337	1.9620	3.0113	1.9620
D_yu								
5	2.4391(-1)	1.2201(-1)	3.7869(-2)	1.2201(-1)				
10	7.2229(-2)	3.6911(-2)	6.0037(-3)	3.6911(-2)	2.0077	1.9725	3.0385	1.9725
15	3.4100(-2)	1.7228(-2)	1.9404(-3)	1.7228(-2)	2.0030	2.0335	3.0145	2.0335
20	1.9787(-2)	9.8976(-3)	8.5642(-4)	9.8976(-3)	2.0016	2.0382	3.0076	2.0382
25	1.2905(-2)	6.4782(-3)	4.5081(-4)	6.4782(-3)	2.0010	1.9846	3.0047	1.9846
D_x^2u								
5		7.0936(-3)	1.8229(-1)	2.1274(-1)				
10		2.1372(-3)	1.0100(-1)	1.1649(-1)		1.9793	9.8308(-1)	9.9351(-1)
15		9.9953(-4)	6.9884(-2)	8.0647(-2)		2.0282	9.8302(-1)	9.8151(-1)
20		5.7483(-4)	5.3219(-2)	6.1477(-2)		2.0343	1.0018	9.9811(-1)
25		3.7645(-4)	4.2978(-2)	4.9615(-2)		1.9819	1.0007	1.0037
D_y^2u								
5		5.0794(-1)	5.0335	5.8060				
10		1.6757(-1)	2.7389	3.1579		1.8296	1.0040	1.0047
15		7.9762(-2)	1.8910	2.1829		1.9812	9.8877(-1)	9.8552(-1)
20		4.6164(-2)	1.4387	1.6611		2.0109	1.0052	1.0045
25		3.0324(-2)	1.1613	1.3406		1.9678	1.0028	1.0036
D_{xy}^2u								
5	4.2790(-1)	2.1069(-1)	5.9449(-2)	2.1069(-1)				
10	1.2625(-1)	6.4209(-2)	9.3777(-3)	6.4209(-2)	2.0137	1.9604	3.0468	1.9604
15	5.9559(-2)	3.0022(-2)	3.0398(-3)	3.0022(-2)	2.0052	2.0289	3.0066	2.0289
20	3.4548(-2)	1.7259(-2)	1.3452(-3)	1.7259(-2)	2.0027	2.0358	2.9980	2.0358
25	2.2530(-2)	1.1300(-2)	7.0742(-4)	1.1300(-2)	2.0017	1.9831	3.0091	1.9831

Chapter 5

PERIODIC BOUNDARY CONDITIONS

Note that the periodic case is not considered in [17] and only the TSM for periodic BCs is considered in [12].

5.1 Two-Point Boundary Value Problem

Consider the TPBVP comprising (2.1) subject to the periodic BCs,

$$u(0) = u(1), \qquad u'(0) = u'(1). \tag{5.1}$$

Let $S_2^P = S_2 \cap \{v : v(0) = v(1)\}$. Then a basis for S_2^P is defined by

$$\{\mathcal{B}_m^P\}_{m=0}^{N+1} = \{\mathcal{B}_0 + \mathcal{B}_{N+2}, \mathcal{B}_1 + \mathcal{B}_{N+2}, \mathcal{B}_2, ..., \mathcal{B}_N, \mathcal{B}_{N+1} + \mathcal{B}_0\}, \tag{5.2}$$

where $\mathcal{B}_m,\ m = 0, 1, ..., N + 2$, is defined in (2.4). From (2.8) and (5.2), we have

$$\mathcal{B}_j^P(\tau_i) = \begin{cases} 1/8, & i = 1, 2, ..., N+1;\ j = 0, 1, ..., N+2;\ i = j \pm 1, \\ & \text{or } i = 1, j = N+1, \text{ or } i = N+1, j = 0, 1, \\ 3/4, & i, j = 2, 3, ..., N;\ i = j, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{B}_j^{P\,\prime\prime}(\tau_i) = \begin{cases} 1/h^2, & i = 1, 2, ..., N+1;\ j = 0, 1, ..., N+2;\ i = j \pm 1, \\ & \text{or } i = 1, j = N+1, \text{ or } i = N+1, j = 0, 1, \\ -2/h^2, & i, j = 2, 3, ..., N;\ i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{5.3}$$

5.1.1 New Modified Quadratic Spline Collocation Method

Similar to the Neumann case, in this method, we perturb the periodic BCs based on the work of [17]. According to Houstis et al., [17, Eqn (30b)], from (3.13), we have

$$-w'(0) + w'(1) = \frac{h^2}{12}\left[u^{(3)}(0) - u^{(3)}(1)\right] + O(h^4), \tag{5.4}$$

From (2.1) and the periodic BCs (5.1), we have

$$u^{(3)}(0) = f'(0) - cu'(1), \qquad u^{(3)}(1) = f'(1) - cu'(1),$$

thus, (5.4) becomes

$$-w'(0) + w'(1) = \frac{h^2}{12}\left[f'(0) - f'(1)\right] + O(h^4).$$

As before, we replace w by u_h^P and drop the $O(h^4)$ term.

In this method, we seek $u_h^P \in S_2^P$ such that

$$u_h^P(x) = \sum_{j=0}^{N+1} u_j \mathcal{B}_j^P(x), \tag{5.5}$$

and

$$\begin{cases} -u_h^{P\prime}(0) + u_h^{P\prime}(1) = \dfrac{h^2}{12}\left[f'(0) - f'(1)\right], \\ \\ (L_h + c)\, u_h^P(\tau_i) = f_h^P(\tau_i), \qquad i = 0, ..., N+1, \end{cases} \tag{5.6}$$

where

$$L_h u_h^P(\tau_i) = \begin{cases} \dfrac{1}{24}\left[22u_h^{P\prime\prime}(\tau_i) + u_h^{P\prime\prime}(\tau_{i+1}) + u_h^{P\prime\prime}(\tau_{N+1})\right], & i = 1, \\ \\ \dfrac{1}{24}\left[u_h^{P\prime\prime}(\tau_{i-1}) + 22u_h^{P\prime\prime}(\tau_i) + u_h^{P\prime\prime}(\tau_{i+1})\right], & i = 2, 3, ..., N, \\ \\ \dfrac{1}{24}\left[22u_h^{P\prime\prime}(\tau_i) + u_h^{P\prime\prime}(\tau_{i-1}) + u_h^{P\prime\prime}(\tau_1)\right], & i = N+1, \end{cases}$$

and

$$f_h^P(\tau_i) = \begin{cases} f_i + \dfrac{1}{24}\left\{[f(0) - f(1)]\left(\dfrac{ch^2}{12} - 1\right) \right. \\ \quad \left. + \dfrac{h}{2}[f'(0) - f'(1)] - \dfrac{h^2}{12}[f''(0) - f''(1)]\right\}, & i = 1, \\ \\ f_i, & i = 2, 3, ..., N, \\ \\ f_i + \dfrac{1}{24}\left\{[f(1) - f(0)]\left(\dfrac{ch^2}{12} - 1\right) \right. \\ \quad \left. - \dfrac{h}{2}[f'(1) - f'(0)] - \dfrac{h^2}{12}[f''(1) - f''(0)]\right\}, & i = N+1, \end{cases}$$

with $\tau_0 = 0$. Note that the first equation of (5.6) is due to the perturbed periodic BCs.

From (2.4) and (5.2), we have

$$\mathcal{B}_j^P(0) = \begin{cases} \frac{1}{2}, & j = 0, 1, N+1 \\ 0, & \text{otherwise,} \end{cases} \qquad \mathcal{B}_j^P(1) = \begin{cases} \frac{1}{2}, & j = 0, 1, N+1, \\ 0, & \text{otherwise.} \end{cases} \tag{5.7}$$

Hence, in this case, we obtain the system of linear equations (3.17) from (5.6) with

$$\tilde{\mathbf{u}} = [u_0,\ u_1, ..., u_{N+1}]^T,$$

$$\tilde{\mathbf{f}} = \left[\frac{f_h^P(\tau_0)}{2}, f_h^P(\tau_1), ..., f_h^P(\tau_{N+1})\right]^T,$$

$$\tilde{A} = \frac{1}{24h^2}\begin{bmatrix} 24h & 0 & & & & & & 0 \\ 23 & -42 & 20 & 1 & & & 1 & 20 \\ 1 & 20 & -42 & 20 & 1 & & & 1 \\ & 1 & 20 & -42 & 20 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 20 & -42 & 20 & 1 \\ 1 & 1 & & & 1 & 20 & -42 & 20 \\ 23 & 20 & 1 & & & 1 & 20 & -42 \end{bmatrix}_{(N+2)\times(N+2)}$$

and

$$\tilde{B} = \frac{1}{8}\begin{bmatrix} 0 & & & & & & 0 \\ 1 & 6 & 1 & & & & 1 \\ & 1 & 6 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & & 6 & 1 \\ 1 & 1 & & & & 1 & 6 \end{bmatrix}_{(N+2)\times(N+2)}$$

We now show how the second and last equations in (5.6) are derived. From (2.12) with $i = 1$, it follows that

$$\begin{aligned} \frac{1}{24}(22w_1'' + w_2'' + w_{N+1}'') + cw_1 &= \frac{1}{24}(-2w_1'' + w_2'' + w_{N+1}'') + w_1'' + cw_1 \\ &= \frac{1}{24}(-2w_1'' + w_2'' + w_{N+1}'') + Lw_1 \\ &= \frac{1}{24}(-2w_1'' + w_2'' + w_{N+1}'') + f_1 - \frac{h^2}{24}u_1^{(4)} + O(h^4). \end{aligned} \tag{5.8}$$

From (2.11), we have

$$-2w_1'' + w_2'' + w_{N+1}'' = (-2u_1'' + u_2'' + u_{N+1}'') - \frac{h^2}{24}\left[-2u_1^{(4)} + u_2^{(4)} + u_{N+1}^{(4)}\right] + O(h^4). \tag{5.9}$$

Similar to (2.28), using Taylor series expansions about $x_{N+2} = 1$, we have

$$\begin{aligned} &\text{(i)} \quad & u_{N+1}'' &= u_{N+2}'' - \frac{h}{2}u_{N+2}^{(3)} + \frac{h^2}{8}u_{N+2}^{(4)} + O(h^3), \\ &\text{(ii)} \quad & u_{N+1}^{(4)} &= u_{N+2}^{(4)} + O(h). \end{aligned} \tag{5.10}$$

Then (2.28(a)–(b)) and (5.10(i)) imply

$$-2u_1'' + u_2'' + u_{N+1}'' = -u_0'' + u_{N+2}'' + \frac{h}{2}\left(u_0^{(3)} - u_{N+2}^{(3)}\right) + \frac{h^2}{8}\left(7u_0^{(4)} + u_{N+2}^{(4)}\right) + O(h^3),$$

and on using (2.28(c)–(d)) and (5.10(ii)), the right hand side of (5.8) becomes

$$\begin{aligned}
& f_1 + \frac{1}{24}\left[-u_0'' + u_{N+2}'' + \frac{h}{2}\left(u_0^{(3)} - u_{N+2}^{(3)}\right) - \frac{h^2}{12}\left(u_0^{(4)} - u_{N+2}^{(4)}\right)\right] + O(h^3) \\
= \; & f_1 + \frac{1}{24}\left\{[f(0) - f(1)]\left(\frac{ch^2}{12} - 1\right) + \frac{h}{2}[f'(0) - f'(1)] - \frac{h^2}{12}[f''(0) - f''(1)]\right\} + O(h^3),
\end{aligned}$$

with

$$\begin{aligned}
u_{N+2}'' &= f(1) - cu_{N+2} = f(1) - cu_0, \\
u_{N+2}^{(3)} &= f'(1) - cu_{N+2}' = f'(1) - cu_0', \\
u_{N+2}^{(4)} &= f''(1) - cu_{N+2}'' = f''(1) - cf(1) - c^2 u_0,
\end{aligned}$$

on using (2.1) and the periodic BCs in (5.1).

Similarly, we have

$$\begin{aligned}
\frac{1}{24}(22w_{N+1}'' + w_N'' + w_1'') + cw_{N+1} \; &= f_{N+1} + \frac{1}{24}\left\{[f(1) - f(0)]\left(\frac{ch^2}{12} - 1\right)\right. \\
& \left. -\frac{h}{2}[f'(1) - f'(0)] - \frac{h^2}{12}[f''(1) - f''(0)]\right\} + O(h^3).
\end{aligned}$$

Replacing w by u_h^P and dropping the $O(h^3)$ terms, we obtain (5.6) with $i = 1, N+1$.

From the system of linear equations (5.6), we obtain

$$u_0 = \frac{h^3}{24}\left[f'(0) - f'(1)\right]. \tag{5.11}$$

Moving the term with u_0 to the right hand side of the system, we obtain an equivalent system of the form (2.16) where $\mathbf{u}$ is as in (2.17),

$$\begin{aligned}
\mathbf{f} = \; & \left[f_h^P(\tau_1) - \left(\frac{23}{24h^2} + \frac{c}{8}\right)u_0,\; f_h^P(\tau_2) - \frac{u_0}{24h^2}, \ldots,\right. \\
& \left. f_h^P(\tau_N) - \frac{u_0}{24h^2},\; f_h^P(\tau_{N+1}) - \left(\frac{23}{24h^2} + \frac{c}{8}\right)u_0\right]^T,
\end{aligned} \tag{5.12}$$

$$A = \frac{1}{24h^2} \begin{bmatrix} -42 & 20 & 1 & & & 1 & 20 \\ 20 & -42 & 20 & 1 & & & 1 \\ 1 & 20 & -42 & 20 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 20 & -42 & 20 & 1 \\ 1 & & & 1 & 20 & -42 & 20 \\ 20 & 1 & & & 1 & 20 & -42 \end{bmatrix}_{(N+1)\times(N+1)} \tag{5.13}$$

and

$$B = \frac{1}{8} \begin{bmatrix} 6 & 1 & & & 1 \\ 1 & 6 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 6 & 1 \\ 1 & & & 1 & 6 \end{bmatrix}_{(N+1)\times(N+1)} \tag{5.14}$$

Note that A and B can be written in (2.25) where T is now the circulant matrix given by

$$T = \begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}_{(N+1)\times(N+1)} \tag{5.15}$$

5.1.2 Numerical Results

We consider three test problems for the second-order TPBVP (2.1) satisfying the Periodic BCs (5.1):

- Problem P-1: $c = -1, \quad u(x) = e^x(x^2 - x)^2.$
- Problem P-2: $c = -1, \quad u(x) = e^{2x} + (1 - e^2)x^2.$

- Problem P-3: $c = -1, \quad u(x) = \left(x^2 - \frac{1}{2}\right)\sin(\pi x).$

Numerical results are presented in Tables 5.1–5.6 for (5.6) and for the simplified version of (5.6) in which the term

$$\frac{h^2}{12}\left[f''(0) - f''(1)\right] \tag{5.16}$$

is omitted from (5.6) when $i = 1, N+1$. The tables are constructed in the same format as previous second-order TPBVPs. The test problems exhibit the expected optimal global accuracy and superconvergence.

Table 5.1. New Modified QSC - Periodic BCs, Problem P-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	2.0469(-5)	2.0800(-5)	7.3671(-5)	7.3756(-5)				
32	1.4429(-6)	1.4929(-6)	1.0477(-5)	1.0404(-5)	3.9986	3.9714	2.9405	2.9528
64	9.5881(-8)	1.0015(-7)	1.3993(-6)	1.3833(-6)	3.9996	3.9857	2.9699	2.9765
128	6.1810(-9)	6.4875(-9)	1.8087(-7)	1.7838(-7)	3.9999	3.9928	2.9849	2.9883
256	3.9205(-10)	4.1252(-10)	2.2993(-8)	2.2649(-8)	4.0011	3.9975	2.9925	2.9942
u'								
16	1.4081(-2)	6.0944(-3)	7.1754(-4)	7.4311(-3)				
32	3.7423(-3)	1.7402(-3)	9.9970(-5)	1.9330(-3)	1.9978	1.8896	2.9715	2.0302
64	9.6494(-4)	4.6524(-4)	1.3207(-5)	4.9259(-4)	1.9994	1.9461	2.9859	2.0168
128	2.4501(-4)	1.2030(-4)	1.6977(-6)	1.2431(-4)	1.9999	1.9734	2.9930	2.0088
256	6.1733(-5)	3.0587(-5)	2.1521(-7)	3.1221(-5)	2.0000	1.9868	2.9965	2.0045
u''								
16		3.0064(-2)	7.8252(-1)	1.1149				
32		8.3368(-3)	4.1520(-1)	5.8996(-1)		1.9338	9.5547(-1)	9.5960(-1)
64		2.1989(-3)	2.1400(-1)	3.0366(-1)		1.9660	9.7773(-1)	9.7971(-1)
128		5.6490(-4)	1.0866(-1)	1.5408(-1)		1.9828	9.8887(-1)	9.8983(-1)
256		1.4318(-4)	5.4749(-2)	7.7611(-2)		1.9913	9.9444(-1)	9.9491(-1)

Table 5.2. New Modified QSC (simplififed version) - Periodic BCs, Problem P-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	2.2714(-5)	1.7126(-5)	7.8183(-5)	7.8301(-5)				
32	1.6958(-6)	1.2236(-6)	1.0811(-5)	1.0741(-5)	3.9120	3.9783	2.9828	2.9949
64	1.1620(-7)	8.1890(-8)	1.4221(-6)	1.4063(-6)	3.9545	3.9892	2.9924	2.9992
128	7.6099(-9)	5.2983(-9)	1.8236(-7)	1.7988(-7)	3.9768	3.9946	2.9965	3.0002
256	4.8665(-10)	3.3664(-10)	2.3088(-8)	2.2745(-8)	3.9893	3.9987	2.9984	3.0003
u'								
16	1.3909(-2)	6.1714(-3)	6.6521(-4)	7.2757(-3)				
32	3.7179(-3)	1.7514(-3)	9.8146(-5)	1.9110(-3)	1.9891	1.8988	2.8851	2.0156
64	9.6171(-4)	4.6676(-4)	1.3360(-5)	4.8967(-4)	1.9948	1.9508	2.9418	2.0087
128	2.4460(-4)	1.2049(-4)	1.7437(-6)	1.2393(-4)	1.9974	1.9757	2.9707	2.0046
256	6.1680(-5)	3.0612(-5)	2.2276(-7)	3.1174(-5)	1.9987	1.9879	2.9853	2.0024
u''								
16		2.6808(-2)	7.7927(-1)	1.1117				
32		7.6149(-3)	4.1434(-1)	5.8909(-1)		1.8975	9.5233(-1)	9.5741(-1)
64		2.1057(-3)	2.1378(-1)	3.0344(-1)		1.8963	9.7620(-1)	9.7863(-1)
128		5.5402(-4)	1.0860(-1)	1.5402(-1)		1.9480	9.8811(-1)	9.8930(-1)
256		1.4211(-4)	5.4734(-2)	7.7597(-2)		1.9740	9.9406(-1)	9.9465(-1)

Table 5.3. New Modified QSC - Periodic BCs, Problem P-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	9.3581(-6)	7.5364(-6)	9.2108(-5)	9.1298(-5)				
32	6.5944(-7)	5.4215(-7)	1.2877(-5)	1.2719(-5)	3.9991	3.9680	2.9663	2.9716
64	4.3818(-8)	3.6412(-8)	1.7051(-6)	1.6809(-6)	3.9998	3.9839	2.9826	2.9854
128	2.8249(-9)	2.3604(-9)	2.1946(-7)	2.1612(-7)	3.9998	3.9918	2.9912	2.9926
256	1.7332(-10)	1.4425(-10)	2.7833(-8)	2.7396(-8)	4.0494	4.0552	2.9959	2.9966
u'								
16	1.7034(-2)	8.0260(-3)	3.7446(-4)	8.7608(-3)				
32	4.5226(-3)	2.1935(-3)	5.1961(-5)	2.3041(-3)	1.9993	1.9557	2.9776	2.0136
64	1.1659(-3)	5.7401(-4)	6.8513(-6)	5.9093(-4)	1.9998	1.9776	2.9888	2.0074
128	2.9601(-4)	1.4686(-4)	8.7984(-7)	1.4964(-4)	2.0000	1.9888	2.9944	2.0038
256	7.4581(-5)	3.7146(-5)	1.1148(-7)	3.7651(-5)	2.0000	1.9944	2.9972	2.0020
u''								
16		1.6080(-2)	9.7877(-1)	1.3901				
32		4.3890(-3)	5.1047(-1)	7.2420(-1)		1.9756	9.8142(-1)	9.8307(-1)
64		1.1482(-3)	2.6082(-1)	3.6982(-1)		1.9781	9.9059(-1)	9.9140(-1)
128		2.9373(-4)	1.3185(-1)	1.8690(-1)		1.9889	9.9527(-1)	9.9567(-1)
256		7.4292(-5)	6.6289(-2)	9.3953(-2)		1.9944	9.9763(-1)	9.9783(-1)

Table 5.4. New Modified QSC (simplified version) - Periodic BCs, Problem P-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	1.0174(-5)	6.3475(-6)	9.3569(-5)	9.2769(-5)				
32	7.4478(-7)	4.5500(-7)	1.2985(-5)	1.2828(-5)	3.9417	3.9734	2.9774	2.9828
64	5.0513(-8)	3.0503(-8)	1.7125(-6)	1.6883(-6)	3.9695	3.9866	2.9886	2.9915
128	3.2914(-9)	1.9756(-9)	2.1994(-7)	2.1661(-7)	3.9843	3.9931	2.9943	2.9958
256	2.0406(-10)	1.1969(-10)	2.7863(-8)	2.7427(-8)	4.0342	4.0677	2.9975	2.9982
u'								
16	1.6978(-2)	8.0509(-3)	3.6062(-4)	8.7105(-3)				
32	4.5148(-3)	2.1971(-3)	5.1603(-5)	2.2970(-3)	1.9969	1.9579	2.9312	2.0095
64	1.1648(-3)	5.7450(-4)	6.9166(-6)	5.8999(-4)	1.9986	1.9788	2.9646	2.0052
128	2.9588(-4)	1.4693(-4)	8.9579(-7)	1.4952(-4)	1.9993	1.9894	2.9821	2.0027
256	7.4564(-5)	3.7154(-5)	1.1399(-7)	3.7635(-5)	1.9997	1.9947	2.9910	2.0014
u''								
16		1.5026(-2)	9.7772(-1)	1.3891				
32		4.1438(-3)	5.1019(-1)	7.2392(-1)		1.9421	9.8063(-1)	9.8251(-1)
64		1.1167(-3)	2.6075(-1)	3.6975(-1)		1.9343	9.9019(-1)	9.9112(-1)
128		2.9005(-4)	1.3183(-1)	1.8688(-1)		1.9667	9.9506(-1)	9.9553(-1)
256		7.3926(-5)	6.6284(-2)	9.3948(-2)		1.9833	9.9752(-1)	9.9776(-1)

Table 5.5. New Modified QSC - Periodic BCs, Problem P-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	2.9855(-5)	2.9104(-5)	8.7297(-5)	8.6118(-5)				
32	2.1002(-6)	2.0728(-6)	1.0963(-5)	1.0800(-5)	4.0017	3.9831	3.1280	3.1300
64	1.3948(-7)	1.3856(-7)	1.3718(-6)	1.3508(-6)	4.0005	3.9909	3.0661	3.0667
128	8.9904(-9)	8.9603(-9)	1.7147(-7)	1.6876(-7)	4.0001	3.9953	3.0338	3.0346
256	5.7141(-10)	5.7045(-10)	2.1428(-8)	2.1085(-8)	3.9982	3.9958	3.0173	3.0176
u'								
16	1.2934(-2)	6.4941(-3)	7.9098(-4)	6.5516(-3)				
32	3.4311(-3)	1.7158(-3)	1.0932(-4)	1.7313(-3)	2.0006	2.0067	2.9835	2.0064
64	8.8401(-4)	4.4205(-4)	1.4392(-5)	4.4577(-4)	2.0006	2.0007	2.9911	2.0016
128	2.2439(-4)	1.1222(-4)	1.8471(-6)	1.1314(-4)	2.0003	2.0001	2.9954	2.0005
256	5.6540(-5)	2.8271(-5)	2.3397(-7)	2.8505(-5)	1.9999	2.0002	2.9976	2.0000
u''								
16		3.3487(-2)	7.6240(-1)	1.0798				
32		9.1987(-3)	3.9217(-1)	5.5569(-1)		1.9480	1.0022	1.0015
64		2.4094(-3)	1.9905(-1)	2.8207(-1)		1.9763	1.0004	1.0003
128		6.1646(-4)	1.0028(-1)	1.4211(-1)		1.9888	1.0002	1.0001
256		1.5590(-4)	5.0337(-2)	7.1334(-2)		1.9945	9.9998(-1)	9.9996(-1)

Table 5.6. New Modified QSC (simplified version) - Periodic BCs, Problem P-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
16	3.2839(-5)	2.5063(-5)	8.9321(-5)	8.6161(-5)				
32	2.4100(-6)	1.7766(-6)	1.1104(-5)	1.0954(-5)	3.9379	3.9903	3.1433	3.1440
64	1.6313(-7)	1.1847(-7)	1.3818(-6)	1.3609(-6)	3.9724	3.9945	3.0742	3.0767
128	1.0608(-8)	7.6520(-9)	1.7209(-7)	1.6938(-7)	3.9871	3.9971	3.0391	3.0401
256	6.7701(-10)	4.8697(-10)	2.1467(-8)	2.1124(-8)	3.9923	3.9963	3.0199	3.0202
u'								
16	1.2923(-2)	6.4947(-3)	7.5004(-4)	6.5410(-3)				
32	3.4304(-3)	1.7166(-3)	1.0931(-4)	1.7306(-3)	1.9997	2.0084	2.9036	2.0046
64	8.8396(-4)	4.4210(-4)	1.4722(-5)	4.4572(-4)	2.0004	2.0012	2.9575	2.0011
128	2.2439(-4)	1.1223(-4)	1.9091(-6)	1.1314(-4)	2.0003	2.0002	2.9802	2.0004
256	5.6540(-5)	2.8272(-5)	2.4303(-7)	2.8504(-5)	1.9999	2.0002	2.9905	2.0000
u''								
16		2.9905(-2)	7.6240(-1)	1.0798				
32		8.5800(-3)	3.9217(-1)	5.5569(-1)		1.8824	1.0022	1.0015
64		2.3414(-3)	1.9905(-1)	2.8207(-1)		1.9158	1.0004	1.0003
128		6.0969(-4)	1.0028(-1)	1.4211(-1)		1.9631	1.0002	1.0001
256		1.5544(-4)	5.0337(-2)	7.1334(-2)		1.9829	9.9998(-1)	9.9996(-1)

5.2 Helmholtz Problem

Consider the Helmholtz equation (2.30) subject to the periodic BCs

$$\begin{cases} u(0,y) = u(1,y), \quad D_x u(0,y) = D_x u(1,y), & y \in [0,1], \\ u(x,0) = u(x,y), \quad D_y u(x,0) = D_y u(x,1), & x \in [0,1]. \end{cases} \tag{5.17}$$

We seek an approximate solution $u_h^P \in S_2^P \otimes S_2^P$ such that

$$u_h^P(x,y) = \sum_{m=0}^{N+1}\sum_{n=0}^{N+1} u_{m,n}\mathcal{B}_m^P(x)\mathcal{B}_n^P(y). \tag{5.18}$$

Note that from (2.4) and (5.2), we have

$$\mathcal{B}_j^{P\,\prime}(0) = \begin{cases} -\frac{1}{h}, & j = 0, N+1, \\ \frac{1}{h}, & j = 1, \quad j = 1, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{B}_j^{P\,\prime}(1) = \begin{cases} \frac{1}{h}, & j = 0, 1, \\ -\frac{1}{h}, & j = N+1, \\ 0, & \text{otherwise}. \end{cases} \tag{5.19}$$

5.2.1 New Modified Quadratic Spline Collocation

Similar to the second-order TPBVPs, the BCs perturbations are

$$-D_x u_h^P(0,\tau_j^y) + D_x u_h^P(1,\tau_j^y) = \frac{h^2}{12}\left[D_x g_{0,j} - D_x g_{N+2,j}\right], \quad j = 0, 1, \ldots, N+1, \tag{5.20}$$

and

$$-D_y u_h^P(\tau_i^x, 0) + D_y u_h^P(\tau_i^x, 1) = \frac{h^2}{12}\left[D_y g_{i,0} - D_y g_{i,N+2}\right], \quad i = 1, 2, \ldots, N+1. \tag{5.21}$$

From (5.18), we have

$$D_x u_h^P(x,y) = \sum_{m=0}^{N+1}\sum_{n=0}^{N+1} u_{m,n}\mathcal{B}_m^{P\,\prime}(x)\mathcal{B}_n^P(y).$$

Hence, for $j = 0, 1, ..., N+1$,

$$\begin{aligned} D_x u_h^P(0,\tau_j^y) &= \mathcal{B}_0^{P\,\prime}(0)\sum_{n=0}^{N+1} u_{0,n}\mathcal{B}_n^P(\tau_j^y) + \mathcal{B}_1^{P\,\prime}(0)\sum_{n=0}^{N+1} u_{1,n}\mathcal{B}_n^P(\tau_j^y) \\ &+ \mathcal{B}_{N+1}^{P\,\prime}(0)\sum_{n=0}^{N+1} u_{N+1,n}\mathcal{B}_n^P(\tau_j^y) \\ &= -\frac{1}{h}\sum_{n=0}^{N+1}\left[\left(u_{0,n} - u_{1,n} + u_{N+1,n}\right)\mathcal{B}_n^P(\tau_j^y)\right], \end{aligned} \tag{5.22}$$

and similarly,

$$\begin{aligned} D_x u_h^P(1,\tau_j^y) &= \mathcal{B}_0^{P\,\prime}(1)\sum_{n=0}^{N+1} u_{0,n}\mathcal{B}_n^P(\tau_j^y) + \mathcal{B}_1^{P\,\prime}(1)\sum_{n=0}^{N+1} u_{1,n}\mathcal{B}_n^P(\tau_j^y) \\ &+ \mathcal{B}_{N+1}^{P\,\prime}(1)\sum_{n=0}^{N+1} u_{N+1,n}\mathcal{B}_n^P(\tau_j^y) \\ &= \frac{1}{h}\sum_{n=0}^{N+1}\left[\left(u_{0,n} + u_{1,n} - u_{N+1,n}\right)\mathcal{B}_n^P(\tau_j^y)\right]. \end{aligned} \tag{5.23}$$

From (5.22) and (5.23), we obtain the following system of $N+2$ linear equations from (5.20) to solve for $u_{0,0}, u_{0,1}, ..., u_{0,N+1}$:

$$M\mathbf{u}_0^x = \frac{h^3}{24}\mathbf{G}_0^x,$$

where $\mathbf{u}_0^x$ is as in (4.8),

$$M = \frac{1}{8}\begin{bmatrix} 4 & 4 & & & & 4 \\ 8 & 6 & 1 & & & 1 \\ & 1 & 6 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 6 & 1 \\ 8 & 1 & & & 1 & 6 \end{bmatrix}_{(N+2)\times(N+2)}$$

and

$$\mathbf{G}_0^x = \left[D_x g_{0,0} - D_x g_{N+2,0}, D_x g_{0,1} - D_x g_{N+2,1}, ..., D_x g_{0,N+1} - D_x g_{N+2,N+1}\right]^T.$$

Similarly, for $i = 1, 2, ..., N+1$, we have

$$\begin{aligned} -D_y u_h^P(\tau_i^x, 0) + D_y u_h^P(\tau_i^x, 1) &= \frac{2}{h}\sum_{m=0}^{N+1} \mathcal{B}_m^P(\tau_i^x) \\ &= \frac{2}{h}\left[u_{0,0}\mathcal{B}_0^P(\tau_i^x) + \sum_{m=1}^{N+1} u_{m,0}\mathcal{B}_m^P(\tau_i^x)\right]. \end{aligned} \tag{5.24}$$

From (5.24), we obtain the following system of $N+1$ linear equations from (5.21) to solve for $u_{1,0}, u_{2,0}, ..., u_{N+1,0}$:

$$B\mathbf{u}_0^y = \frac{h^3}{24}\mathbf{G}_0^y - u_{0,0}\left(\boldsymbol{\mathcal{B}}_0^P\right)^x,$$

where B is given in (5.14), $\mathbf{u}_0^y$ is as in (4.12),

$$\mathbf{G}_0^y = \left[D_y g_{1,0} - D_y g_{1,N+2}, D_y g_{2,0} - D_y g_{2,N+2}, ..., D_y g_{N+1,0} - D_y g_{N+1,N+2}\right]^T,$$

and

$$\left(\boldsymbol{\mathcal{B}}_0^P\right)^x = \left[\mathcal{B}_0^P(\tau_1^x), \mathcal{B}_0^P(\tau_2^x), ..., \mathcal{B}_0^P(\tau_{N+1}^x)\right]^T = [1/8, 0, ..., 0, 1/8]^T.$$

Collocating at the interior points (τ_i^x, τ_j^y), we seek $u_h^P \in S_2^P \otimes S_2^P$ such that

$$(L_x + L_y + c)u_h^P(\tau_i^x, \tau_j^y) = g_{i,j}^P, \quad i, j = 1, 2, ..., N+1. \tag{5.25}$$

For $i = 1,\ j = 2, 3, ..., N$,

$$g_{i,j}^P = g_{i,j} + \frac{1}{24}\left\{[g_{0,j} - g_{N+2,j}]\left(\frac{ch^2}{12} - 1\right) + \frac{h}{2}[D_x g_{0,j} - D_x g_{N+2,j}]\right\}.$$

For $i = 2, 3, ..., N,\ j = 1$,

$$g_{i,j}^P = g_{i,j} + \frac{1}{24}\left\{[g_{i,0} - g_{i,N+2}]\left(\frac{ch^2}{12} - 1\right) + \frac{h}{2}[D_y g_{i,0} - D_y g_{i,N+2}]\right\}.$$

For $i = N + 1,\ j = 2, 3, ..., N$,

$$g_{i,j}^P = g_{i,j} + \frac{1}{24}\left\{[gN + 2, j - g_{0,j}]\left(\frac{ch^2}{12} - 1\right) - \frac{h}{2}[D_x g_{N+2,j} - D_x g_{0,j}]\right\}.$$

For $i = 2, 3, ..., N,\ j = N + 1$,

$$g_{i,j}^P = g_{i,j} + \frac{1}{24}\left\{[g_{i,N+2} - g_{i,0}]\left(\frac{ch^2}{12} - 1\right) - \frac{h}{2}[D_y g_{i,N+2} - D_y g_{i,0}]\right\}.$$

For $i, j = 1$,

$$\begin{aligned} g_{i,j}^P &= g_{i,j} + \frac{1}{24}\left\{[g_{0,j} - g_{N+2,j} + g_{i,0} - g_{i,N+2}]\left(\frac{ch^2}{12} - 1\right)\right. \\ &+ \left.\frac{h}{2}[D_x g_{0,j} - D_x g_{N+2,j} + D_y g_{i,0} - D_y g_{i,N+2}]\right\}. \end{aligned}$$

For $i = 1,\ j = N + 1$,

$$\begin{aligned} g_{i,j}^P &= g_{i,j} + \frac{1}{24}\left\{[g_{0,j} - g_{N+2,j} + g_{i,N+2} - g_{i,0}]\left(\frac{ch^2}{12} - 1\right)\right. \\ &+ \left.\frac{h}{2}[D_x g_{0,j} - D_x g_{N+2,j} + D_y g_{i,0} - D_y g_{i,N+2}]\right\}. \end{aligned}$$

For $i = N+1,\ j = 1$,

$$\begin{aligned} g^P_{i,j} &= g_{i,j} + \frac{1}{24}\left\{ [g_{N+2,j} - g_{0,j} + g_{i,0} - g_{i,N+2}] \left(\frac{ch^2}{12} - 1\right) \right. \\ &+ \left. \frac{h}{2}[D_x g_{0,j} - D_x g_{N+2,j} + D_y g_{i,0} - D_y g_{i,N+2}] \right\}. \end{aligned}$$

For $i, j = N+1$,

$$\begin{aligned} g^P_{i,j} &= g_{i,j} + \frac{1}{24}\left\{ [gN+2,j - g_{0,j} + g_{i,N+2} - g_{i,0}] \left(\frac{ch^2}{12} - 1\right) \right. \\ &- \left. \frac{h}{2}[D_x g_{N+2,j} - D_x g_{0,j} + D_y g_{i,N+2} - D_y g_{i,0}] \right\}. \end{aligned}$$

For each $j = 1, 2, ..., N+1$,

$$L_x u^P_h(\tau^x_i, \tau^y_j) = \begin{cases} \frac{1}{24}\left[22 D^2_x u^P_h(\tau^x_i, \tau^y_j) + D^2_x u^P_h(\tau^x_{i+1}, \tau^y_j) + D^2_x u^P_h(\tau^x_{N+1}, \tau^y_j)\right], & i = 1, \\ \frac{1}{24}\left[D^2_x u^P_h(\tau^x_{i-1}, \tau^y_j) + 22 D^2_x u^P_h(\tau^x_i, \tau^y_j) \right. \\ \quad \left. + D^2_x u^P_h(\tau^x_{i+1}, \tau^y_j)\right], & i = 2, 3, ..., N, \\ \frac{1}{24}\left[22 D^2_x u^P_h(\tau^x_i, \tau^y_j) + D^2_x u^P_h(\tau^x_{i-1}, \tau^y_j) + D^2_x u^P_h(\tau^x_1, \tau^y_j)\right], & i = N+1, \end{cases}$$

and, for each $i = 1, 2, ..., N+1$,

$$L_y u_h^P(\tau_i^x,\tau_j^y) = \begin{cases} \dfrac{1}{24}\left[22D_y^2 u_h^P(\tau_i^x,\tau_j^y) + D_y^2 u_h^P(\tau_i^x,\tau_{j+1}^y) + D_y^2 u_h^P(\tau_i^x,\tau_{N+1}^y)\right], & j=1,\ . \\ \dfrac{1}{24}\left[D_y^2 u_h^P(\tau_i^x,\tau_{j-1}^y) + 22D_y^2 u_h^P(\tau_i^x,\tau_j^y) \right. \\ \quad \left. + D_y^2 u_h^P(\tau_i^x,\tau_{j+1}^y)\right], & j=2,3,\dots,N, \\ \dfrac{1}{24}\left[22D_y^2 u_h^P(\tau_i^x,\tau_j^y) + D_y^2 u_h^P(\tau_i^x,\tau_{j-1}^y) + D_y^2 u_h^P(\tau_i^x,\tau_1^y)\right], & j=N+1. \end{cases}$$

From (5.18), for $i,j=1,2,\dots,N+1$, we have

$$\begin{aligned} u_h^P(\tau_i^x,\tau_j^y) &= \mathcal{B}_0^P(\tau_i^x)\sum_{n=0}^{N+1} u_{0,n}\mathcal{B}_n^P(\tau_j^y) + \mathcal{B}_0^P(\tau_j^y)\sum_{m=1}^{N+1} u_{m,0}\mathcal{B}_m^P(\tau_i^x) \\ &+ \sum_{m=1}^{N+1}\sum_{n=1}^{N+1} u_{m,n}\mathcal{B}_m^P(\tau_i^x)\mathcal{B}_n^P(\tau_j^y), \end{aligned} \tag{5.26}$$

$$\begin{aligned} D_x^2 u_h^P(\tau_i^x,\tau_j^y) &= \mathcal{B}_0^{P\,''}(\tau_i^x)\sum_{n=0}^{N+1} u_{0,n}\mathcal{B}_n^P(\tau_j^y) + \mathcal{B}_0^P(\tau_j^y)\sum_{m=1}^{N+1} u_{m,0}\mathcal{B}_m^{P\,''}(\tau_i^x) \\ &+ \sum_{m=1}^{N+1}\sum_{n=1}^{N+1} u_{m,n}\mathcal{B}_m^{P\,''}(\tau_i^x)\mathcal{B}_n^P(\tau_j^y), \end{aligned} \tag{5.27}$$

and

$$\begin{aligned} D_y^2 u_h^P(\tau_i^x,\tau_j^y) &= \mathcal{B}_0^P(\tau_i^x)\sum_{n=0}^{N+1} u_{0,n}\mathcal{B}_n^{P\,''}(\tau_j^y) + \mathcal{B}_0^{P\,''}(\tau_j^y)\sum_{m=1}^{N+1} u_{m,0}\mathcal{B}_m^P(\tau_i^x) \\ &+ \sum_{m=1}^{N+1}\sum_{n=1}^{N+1} u_{m,n}\mathcal{B}_m^P(\tau_i^x)\mathcal{B}_n^P(\tau_j^y). \end{aligned} \tag{5.28}$$

On substituting (5.26)–(5.28) into (5.25) and moving the terms with known coefficients $u_{0,0}, u_{0,1}, \dots, u_{0,N+1}$ and $u_{1,0}, u_{2,0}, \dots, u_{N+1,0}$ to the right hand side, we obtain the system of linear equations (2.34) where the matrices A and B are defined in (5.13) and (5.14), respectively.

With the circulant matrix T defined in (5.15), since the matrices A and B can be written as in (2.25) so that their eigenvalues are as in (2.38) where by [6], the eigenvalues of T are given by

$$\lambda_i^T = -4\sin^2\frac{(i-1)\pi}{(N+1)}, \quad i = 1, 2, ..., N+1,$$

and the columns of matrix Z used in the MDA are the corresponding eigenvectors of T, with Z is defined as: for $i = 1, 2, ..., N+1$,

$$Z_{i,1} = \frac{1}{\sqrt{N+1}},$$

for even N,

$$\begin{aligned}
(Z_{i,j+1})_{j=1}^{N/2} &= \sqrt{\frac{2}{N+1}}\cos\left[\frac{2(i-1)j\pi}{N+1}\right], \\
(Z_{i,N+2-j})_{j=1}^{N/2} &= \sqrt{\frac{2}{N+1}}\sin\left[\frac{2(i-1)j\pi}{N+1}\right],
\end{aligned}$$

and for odd N,

$$\begin{aligned}
(Z_{i,j+1})_{j=1}^{(N-1)/2} &= \sqrt{\frac{2}{N+1}}\cos\left[\frac{2(i-1)j\pi}{N+1}\right], \\
(Z_{i,N+2-j})_{j=1}^{(N-1)/2} &= \sqrt{\frac{2}{N+1}}\sin\left[\frac{2(i-1)j\pi}{N+1}\right], \\
Z_{i,(N+3)/2} &= \frac{(-1)^{i-1}}{\sqrt{N+1}}.
\end{aligned}$$

We still have

$$Z^{-1} = Z^T.$$

5.2.2 Numerical Results

We consider three test problems for the Helmholtz equation (2.30) satisfying the periodic BCs (5.17):

- Helmholtz P-1: $c = -1, \quad u(x,y) = e^{xy}(x^2 - x)^2(y^2 - y)^2.$
- Helmholtz P-2: $c = -1, \quad u(x,y) = \left[e^{2x} + (1-e^2)x^2\right]\left[e^{2y} + (1-e^2)y^2\right].$
- Helmholtz P-3: $c = -1, \quad u(x,y) = \left(x^2 - \frac{1}{2}\right)\sin(\pi x)\left(y^2 - \frac{1}{2}\right)\sin(\pi y).$

Numerical results are presented in Tables 5.7–5.9 in the same format as previous Helmholtz problems. The numerical results show expected optimal global accuracy and supercovergence.

Table 5.7. Helmholtz P-1

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	3.5486(-5)	1.7191(-5)	5.2539(-5)	5.3852(-5)				
10	3.1340(-6)	1.6540(-6)	9.0949(-6)	9.0759(-6)	4.0038	3.8625	2.8935	2.9377
15	7.0622(-7)	3.8173(-7)	3.0301(-6)	2.9914(-6)	3.9770	3.9131	2.9334	2.9622
20	2.3831(-7)	1.3086(-7)	1.3580(-6)	1.3329(-6)	3.9949	3.9370	2.9512	2.9727
25	1.0131(-7)	5.6282(-8)	7.2147(-7)	7.0552(-7)	4.0053	3.9506	2.9615	2.9787
D_xu								
5	3.4486(-3)	1.3924(-3)	3.7765(-4)	1.3924(-3)				
10	1.0957(-3)	4.8110(-4)	6.7695(-5)	4.8110(-4)	1.8916	1.7532	2.8359	1.7532
15	5.2576(-4)	2.3946(-4)	2.2720(-5)	2.3946(-4)	1.9598	1.8620	2.9137	1.8620
20	3.0695(-4)	1.4271(-4)	1.0219(-5)	1.4271(-4)	1.9790	1.9034	2.9383	1.9034
25	2.0080(-4)	9.4588(-5)	5.4409(-6)	9.4588(-5)	1.9871	1.9256	2.9512	1.9256
D_yu								
5	3.4486(-3)	1.3924(-3)	3.7765(-4)	1.3924(-3)				
10	1.0957(-3)	4.8110(-4)	6.7695(-5)	4.8110(-4)	1.8916	1.7532	2.8359	1.7532
15	5.2576(-4)	2.3946(-4)	2.2720(-5)	2.3946(-4)	1.9598	1.8620	2.9137	1.8620
20	3.0695(-4)	1.4271(-4)	1.0219(-5)	1.4271(-4)	1.9790	1.9034	2.9383	1.9034
25	2.0080(-4)	9.4588(-5)	5.4409(-6)	9.4588(-5)	1.9871	1.9256	2.9512	1.9256
D_x^2u								
5		4.9962(-3)	6.6380(-2)	7.6407(-2)				
10		1.6596(-3)	3.9399(-2)	4.5458(-2)		1.8183	8.6064(-1)	8.5669(-1)
15		8.1578(-4)	2.7874(-2)	3.2243(-2)		1.8953	9.2348(-1)	9.1671(-1)
20		4.8744(-4)	2.1601(-2)	2.4983(-2)		1.8938	9.3765(-1)	9.3819(-1)
25		3.2774(-4)	1.7619(-2)	2.0367(-2)		1.8586	9.5399(-1)	9.5641(-1)
D_y^2u								
5		4.9962(-3)	6.6380(-2)	7.6407(-2)				
10		1.6596(-3)	3.9399(-2)	4.5458(-2)		1.8183	8.6064(-1)	8.5669(-1)
15		8.1578(-4)	2.7874(-2)	3.2243(-2)		1.8953	9.2348(-1)	9.1671(-1)
20		4.8744(-4)	2.1601(-2)	2.4983(-2)		1.8938	9.3765(-1)	9.3819(-1)
25		3.2774(-4)	1.7619(-2)	2.0367(-2)		1.8586	9.5399(-1)	9.5641(-1)
D_{xy}^2u								
5	1.5325(-2)	7.8860(-3)	2.7461(-3)	7.8860(-3)				
10	5.0712(-3)	2.3444(-3)	4.5207(-4)	2.3444(-3)	1.8245	2.0013	2.9764	2.0013
15	2.3161(-3)	1.1474(-3)	1.4346(-4)	1.1474(-3)	2.0916	1.9069	3.0632	1.9069
20	1.3615(-3)	6.6344(-4)	6.3673(-5)	6.6344(-4)	1.9536	2.0146	2.9872	2.0146
25	8.7967(-4)	4.3867(-4)	3.3651(-5)	4.3867(-4)	2.0453	1.9370	2.9859	1.9370

Table 5.8. Helmholtz P-2

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	1.2255(-3)	8.2763(-4)	3.5041(-3)	3.4767(-3)				
10	1.0885(-4)	7.5817(-5)	6.1232(-4)	5.9821(-4)	3.9943	3.9434	2.8780	2.9035
15	2.4483(-5)	1.7094(-5)	2.0704(-4)	2.0122(-4)	3.9819	3.9755	2.8940	2.9078
20	8.4573(-6)	5.7821(-6)	9.3809(-5)	9.0883(-5)	3.9089	3.9861	2.9112	2.9229
25	3.6843(-6)	2.4655(-6)	5.0286(-5)	4.8557(-5)	3.8907	3.9910	2.9195	2.9350
$D_x u$								
5	1.6732(-1)	7.2300(-2)	8.4273(-3)	7.2300(-2)				
10	5.0584(-2)	2.3195(-2)	1.4939(-3)	2.3195(-2)	1.9736	1.8756	2.8543	1.8756
15	2.3944(-2)	1.1318(-2)	5.0358(-4)	1.1318(-2)	1.9961	1.9150	2.9021	1.9150
20	1.3912(-2)	6.6699(-3)	2.2684(-4)	6.6699(-3)	1.9966	1.9446	2.9327	1.9446
25	9.0887(-3)	4.3875(-3)	1.2083(-4)	4.3875(-3)	1.9933	1.9611	2.9489	1.9611
$D_y u$								
5	1.6732(-1)	7.2300(-2)	8.4273(-3)	7.2300(-2)				
10	5.0584(-2)	2.3195(-2)	1.4939(-3)	2.3195(-2)	1.9736	1.8756	2.8543	1.8756
15	2.3944(-2)	1.1318(-2)	5.0358(-4)	1.1318(-2)	1.9961	1.9150	2.9021	1.9150
20	1.3912(-2)	6.6699(-3)	2.2684(-4)	6.6699(-3)	1.9966	1.9446	2.9327	1.9446
25	9.0887(-3)	4.3875(-3)	1.2083(-4)	4.3875(-3)	1.9933	1.9611	2.9489	1.9611
$D_x^2 u$								
5		1.3413(-1)	3.2811	3.7924				
10		4.3137(-2)	1.8598	2.1501		1.8716	9.3661(-1)	9.3620(-1)
15		2.1095(-2)	1.2934	1.4950		1.9091	9.6927(-1)	9.6978(-1)
20		1.2449(-2)	9.9424(-1)	1.1485		1.9396	9.6734(-1)	9.6977(-1)
25		8.1962(-3)	8.0649(-1)	9.3164(-1)		1.9569	9.7993(-1)	9.7975(-1)
$D_y^2 u$								
5		1.3413(-1)	3.2811	3.7924				
10		4.3137(-2)	1.8598	2.1501		1.8716	9.3661(-1)	9.3620(-1)
15		2.1095(-2)	1.2934	1.4950		1.9091	9.6927(-1)	9.6978(-1)
20		1.2449(-2)	9.9424(-1)	1.1485		1.9396	9.6734(-1)	9.6977(-1)
25		8.1962(-3)	8.0649(-1)	9.3164(-1)		1.9569	9.7993(-1)	9.7975(-1)
$D_{xy}^2 u$								
5	5.1770(-1)	9.6115(-2)	1.8462(-2)	9.6115(-2)				
10	1.5958(-1)	4.8807(-2)	3.7040(-3)	4.8807(-2)	1.9416	1.1180	2.6500	1.1180
15	7.6120(-2)	2.7379(-2)	1.3090(-3)	2.7379(-2)	1.9755	1.5428	2.7759	1.5428
20	4.4357(-2)	1.7306(-2)	6.0611(-4)	1.7306(-2)	1.9859	1.6869	2.8315	1.6869
25	2.8995(-2)	1.1880(-2)	3.2881(-4)	1.1880(-2)	1.9906	1.7612	2.8636	1.7612

Table 5.9. Helmholtz P-3

N	Maximum Absolute Error				Rate of Convergence			
	$E_n(N)$	$E_m(N)$	$E_G(N)$	$E_g(N)$	$R_n(N)$	$R_m(N)$	$R_G(N)$	$R_g(N)$
u								
5	6.2586(-4)	4.4303(-4)	9.7058(-4)	9.8807(-4)				
10	6.3012(-5)	4.0333(-5)	1.1885(-4)	1.1831(-4)	3.7876	3.9537	3.4646	3.5016
15	1.4272(-5)	9.0910(-6)	3.5725(-5)	3.4982(-5)	3.9632	3.9763	3.2080	3.2518
20	4.8235(-6)	3.0913(-6)	1.5029(-5)	1.4672(-5)	3.9893	3.9668	3.1841	3.1954
25	2.0444(-6)	1.3216(-6)	7.7246(-6)	7.4891(-6)	4.0191	3.9787	3.1163	3.1486
$D_x u$								
5	3.3583(-2)	1.7277(-2)	4.8355(-3)	1.7277(-2)				
10	1.0141(-2)	5.2919(-3)	9.2799(-4)	5.2919(-3)	1.9755	1.9521	2.7233	1.9521
15	4.8614(-3)	2.4698(-3)	3.2822(-4)	2.4698(-3)	1.9623	2.0338	2.7738	2.0338
20	2.8350(-3)	1.4301(-3)	1.5049(-4)	1.4301(-3)	1.9832	2.0093	2.8676	2.0093
25	1.8572(-3)	9.3045(-4)	8.0860(-5)	9.3045(-4)	1.9804	2.0125	2.9085	2.0125
$D_y u$								
5	3.3583(-2)	1.7277(-2)	4.8355(-3)	1.7277(-2)				
10	1.0141(-2)	5.2919(-3)	9.2799(-4)	5.2919(-3)	1.9755	1.9521	2.7233	1.9521
15	4.8614(-3)	2.4698(-3)	3.2822(-4)	2.4698(-3)	1.9623	2.0338	2.7738	2.0338
20	2.8350(-3)	1.4301(-3)	1.5049(-4)	1.4301(-3)	1.9832	2.0093	2.8676	2.0093
25	1.8572(-3)	9.3045(-4)	8.0860(-5)	9.3045(-4)	1.9804	2.0125	2.9085	2.0125
$D_x^2 u$								
5		6.1002(-2)	7.2577(-1)	8.4006(-1)				
10		2.2573(-2)	3.9707(-1)	4.5786(-1)		1.6401	9.9502(-1)	1.0013
15		1.1255(-2)	2.7237(-1)	3.1417(-1)		1.8574	1.0060	1.0052
20		6.6623(-3)	2.0709(-1)	2.3923(-1)		1.9282	1.0070	1.0020
25		4.4826(-3)	1.6758(-1)	1.9349(-1)		1.8554	9.9121(-1)	9.9371(-1)
$D_y^2 u$								
5		6.1002(-2)	7.2577(-1)	8.4006(-1)				
10		2.2573(-2)	3.9707(-1)	4.5786(-1)		1.6401	9.9502(-1)	1.0013
15		1.1255(-2)	2.7237(-1)	3.1417(-1)		1.8574	1.0060	1.0052
20		6.6623(-3)	2.0709(-1)	2.3923(-1)		1.9282	1.0070	1.0020
25		4.4826(-3)	1.6758(-1)	1.9349(-1)		1.8554	9.9121(-1)	9.9371(-1)
$D_{xy}^2 u$								
5	2.6803(-1)	1.2835(-1)	2.9208(-2)	1.2835(-1)				
10	7.7554(-2)	3.9160(-2)	5.7677(-3)	3.9160(-2)	2.0459	1.9584	2.6763	1.9584
15	3.6054(-2)	1.8548(-2)	2.0251(-3)	1.8548(-2)	2.0442	1.9944	2.7933	1.9944
20	2.1171(-2)	1.0708(-2)	9.3405(-4)	1.0708(-2)	1.9579	2.0202	2.8458	2.0202
25	1.3834(-2)	6.9634(-3)	5.3535(-4)	6.9634(-3)	1.9920	2.0150	2.6062	2.0150

Chapter 6

CONCLUDING REMARKS AND FUTURE RESEARCH

6.1 Concluding Remarks

In this thesis, we develop new one-step modified QSC methods for the Helmholtz problem in the unit square subject to four BCs: Dirichlet, Neumann, mixed (Neumann-Dirichlet and Dirichlet-Neumann), and periodic. All of these methods involve a perturbation of the right hand side of the differential equation as well as the differential operator. The methods are of optimal global accuracy, that is, for the k^{th} derivative, $k = 0, 1, 2$, we obtain a rate of convergence of order $3-k$ globally. The methods also exhibit superconvergence: fourth-order at the nodal points and the collocation points (the mid-points in each uniform partition in x and y directions), third-order at the Gauss points for the first and mixed derivatives of the solution, and second-order at collocation points for the second derivatives.

There are several differences between these one-step modified QSC methods and the corresponding methods for CSC. In the Dirichlet problem, the QSC method is more straightforward as it involves no boundary collocation equations. Moreover, the perturbation of the right hand side of the differential equation is simpler than one might first expect since experiments on TPBVPs show that an $O(h^2)$ term can be omitted without any degradation of the accuracy. Interestingly, in the Neumann problem, the QSC approximation does not satisfy the BCs, in contrast to the cubic case. Mixed BCs are easily handled from knowledge of the Dirichlet and Neumann cases. In the periodic case, the term (5.16) can be dropped from right hand side perturbations as in the Dirichlet case.

6.2 Future Research

Future research will include the following topics:

- *FFTs in the MDA.* Since the main purpose of this thesis is to show the viability of the new one-step modified QSC methods, we compute Steps 1 and 3 of the MDA in the usual way by finding matrix-vector products. In the future, FFTs should be used to increase the efficiency of the MDA since these steps involve linear combinations of sines and cosines for any choice of the four BCs.

- *Generalization of the methods to separable elliptic equations of the form*

$$D_x^2 u + a(y) D_y^2 u + b(y) D_y u + c(y) u = f(x,y), \qquad (x,y) \in \Omega = [0,1] \times [0,1],$$

  cf. [7] for the bicubic case. The new one-step modified QSC approach to solving TPBVPs formulated in this thesis would be used to approximate the term u_{xx} and for the discretization in the y-direction, the approach of [17] will be investigated. It is also possible to use a non-uniform partition in the y-direction based on the approach of [20].

- *Extension of the methods to Dirichlet biharmonic problems of the form*

$$\begin{cases} \Delta^2 u(x,y) = f(x,y), & (x,y) \in \Omega = [0,1] \times [0,1], \\ u(x,y) = 0, \quad \dfrac{\partial u}{\partial n} = 0, & (x,y) \in \partial\Omega, \end{cases}$$

  where $\partial/\partial n$ denotes the outward normal derivative on the boundary $\partial\Omega$.

- *Extension to non-rectangular regions using domain decomposition.*

- *A rigorous theoretical analysis of the methods has yet to be obtained.*

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