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A GEOMETRIC PROGRAMMING ALGORITHM FOR SOLVING A CLASS OF NONLINEAR, SIGNOMIAL OPTIMIZATION PROBLEMS

by

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ABSTRACT

The ultimate objective in the area of research into optimization problems is to find algorithms that can easily solve all (or very large classes) of optimization problems, whether linear or nonlinear, unconstrained or constrained.

Geometric Programming is a good technique for problems that are functions of real variables, can be described in terms of posynomials, and have zero degrees of difficulty.

In order to generalize the technique, various methods have been developed to solve some signomial problems that have multiple degrees of difficulty. A generalized algorithm that can solve all such problems would be a significant breakthrough. There does not exist such an algorithm.

A condensation based dual algorithm that can solve a large class of nonlinear, signomial, multivariable, constrained optimization problems with multiple degrees of difficulty is presented. The algorithm incorporates significant advances in the sign table and in the methodology to transform the original problem to a suitable form.

The algorithm has been extensively tested and economic, engineering, and theoretical examples are presented to illustrate the use of the algorithm.

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soli Deo gloria

Chapter 1

INTRODUCTION

1.1 Operations Research and Optimization

There are many definitions of Operations Research (O.R.). Saaty (1959) gives the following definition:

> Operations Research is the art of giving bad answers to problems to which otherwise worse answers are given.

Churchman, Ackoff, and Arnoff (1957) provides the following "tentative working definition":

> O.R. is the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to the problems.

To be able to provide better or optimum solutions, the O.R. practitioner may be able to build a mathematical model of the problem. If the practitioner can then find the optimal solution to his mathematical model, it may help him to suggest better alternatives to address the problem.

Many optimization techniques have been developed to assist the O.R. practitioner in his trade, the most well known is perhaps linear programming. For example, a standard O.R. textbook such as Winston (1987) devotes 431 out of 986 pages to linear and integer programming.

The difficulty is that many problems encountered in other than academic environments involve nonlinear relationships between the components of the system. Such problems can sometimes be modeled as a system of nonlinear equations with a nonlinear objective function.

1.2 Nonlinear Programming

The field of nonlinear programming is much more difficult, and the methods more difficult to use than that of linear programming. A "global" method that would easily handle all the constrained nonlinear optimization problems of a certain degree of "smoothness" (continuous, Lipschitz continuous, or one or more times differentiable) does not seem to exist.

Many current methods seem to fall in three broad classes: search methods for unconstrained optimization (grid search, Fibonacci search, etc.), gradient methods for unconstrained optimization (steepest descent, secant methods, Newton-Raphson methods), and methods for constrained optimization (penalty and barrier methods, Lagrange-Newton methods). See for example Dennis and Schnabel (1983), Fletcher (1981), Luenberger (1984), and Walsh (1975).

Geometric programming is a technique to solve a certain class of nonlinear minimization problems. Because of the perceived limitations of the technique, it is generally not regarded as being in the mainstream of nonlinear optimization techniques (Peterson 1976; Dembo 1978).

1.3 Geometric Programming

Geometric programming developed as a technique to solve cost minimization problems in engineering design. Due to the physical considerations involved, many such problems can be formulated in posynomial form. (A posynomial is like a polynomial, except that all coefficients are positive and exponents can be real numbers.)

Geometric programming problems are also classified according to "degrees of difficulty" (d.d.), defined as the number of terms minus the number of variables minus one. Basic geometric programming techniques enable a person to solve zero d.d. posynomials easily.

There are three further assumptions that limit the usefulness of geometric programming:

(1) All the variables must be strictly positive.

(2) The objective function value must be strictly positive.

(3) All the constraints are satisfied as equalities, i. e. all the constraints are binding.

After some work in the early 1960's, Duffin, Peterson, and Zener published the first text in geometric programming (Duffin, Peterson, and Zener 1967). Since that time rapid development has taken place, see for example Duffin and Peterson (1973), Rijckaert and Martens (1978), and Ecker (1980). There has also been various attempts to develop software, which proved to be useful in certain special classes of problems (Dembo 1978; Ratliff 1986; Wall, Greening, and Woolsey 1986; Thome 1988; Jha, Kortanek, and No 1988).

Some texts have also been published describing the state of the art at the time, and showing many and varied successful applications of geometric programming, for example Beightler and Phillips (1976) and Wilde (1978).

Despite all the advances, a standard O.R. text like Winston does not mention geometric programming. It is also ignored in texts on nonlinear optimization, for example Walsh (1975), Dennis and Schnabel (1983), and Luenberger (1984). Fletcher (1981) devotes five out of 214 pages to geometric programming, although he concedes that:

> There could well be scope for even more applications . . . if the technique were more widely understood and appreciated.

1.4 Foundations and Development of Geometric Programming

The original idea leading to geometric programming was based on the arithmetic mean-geometric mean inequality, which can be proved using the Cauchy-Schwarz inequality (Royden 1968; Kreyszig 1978).

The posynomial optimization problem is transformed by substituting all the variables using a logarithmic transformation. Duffin, Peterson, and Zener (1967) proved that the problem in this form is a convex programming problem, thus any minimum found will be a global minimum. Fletcher (1981) provides an alternative and simpler proof.

By considering the dual of this transformed problem, the familiar form of the dual geometric programming problem is obtained. The attractive feature of this form is that it is now a nonlinear maximization problem subject to a set of linear constraints. In the case of a zero d.d. problem, there are as many constraints as there are variables, and the problem reduces to solving a set of simultaneous linear equations.

Problems with one d.d. are still fairly easy to solve, one approach being to solve for all variables but one using the set of linear equations. This leaves a nonlinear objective function in one variable, and the maximum can be found by a simple line search.

The factors that complicate the problem are:

(1) Multiple degrees of difficulty.

(2) Negative coefficients ("signomial" problems).

(3) Constraints (unconstrained problems are easier).

(4) Slack constraints.

(5) Objective function value not positive at optimality (this can occur in signomial problems).

(6) Allowing for variables that are not strictly positive.

Geometric programming researchers have concentrated on the first two problems above. Algorithms have been classified into "primal" and "dual" algorithms, depending on which form of the problem is addressed (Ecker 1980).

The unconstrained and constrained problems are sometimes addressed separately (Peterson 1976), although the difference in difficulty does not appear to be as great as in the case of "mainstream" optimization theory.

Various schemes have been proposed to handle slack constraints, but a successful general scheme has not yet appeared in the published literature.

The fifth and sixth problems above have not yet been addressed, although they are of crucial importance if geometric programming based algorithms are to compete with general nonlinear optimization algorithms.

Another important technique based on the arithmetic mean-geometric mean inequality is that of condensation. Using this technique, two or more terms in a problem can be combined into a single multiplicative term. By choosing the weights for the weighted arithmetic and geometric means correctly, the equality condition of the inequality can be satisfied. This leads to the useful result that two or more terms in the problem can be replaced by a single "condensed" term.

Condensation is extremely useful for two purposes: to reduce the degrees of difficulty, and to enable a signomial problem to be converted into a posynomial problem. For an extensive treatment of condensation, see Beightler and Phillips (1976).

1.5 Research at the Colorado School of Mines

Geometric programming research has been conducted, for the past 19 years, under the leadership of R. E. D. Woolsey in the Operations Research/Management Science program at the Colorado School of Mines.

An important step was the simplification of the representation of the dual of the geometric programming problem by R. E. D. Woolsey. The representation is in terms of four "rules", which makes it much easier to understand the issues involved and how to proceed with new developments (Woolsey and Swanson 1969; Hesse and Woolsey 1980; Woolsey 1985; Woolsey 1988).

Recent research have followed two paths. Some students have looked at specific applications and then developed algorithms to solve a class of problems relevant to the application. Problems with multiple degrees of difficulty and negative coefficients could be dealt with on a case by case basis, in part by exploiting the structure of the problem due to the physical nature of the problem (which would remain constant for the particular class of problems).

Examples of algorithms based on specific application areas are Wall, Greening, and Woolsey (1986) and Oatney (1987).

A second approach has been to examine simplified cases of the general problem, and to solve these cases in general. These algorithms are thus not bound to a specific application.

Examples of these kind of algorithms are Ratliff (1986) and Thome (1988). Ratliff's algorithm can solve any nonlinear, posynomial, multivariable, unconstrained optimization problem with multiple degrees of difficulty. Thome's algorithm can solve nonlinear, signomial, single variable, unconstrained optimization problems with multiple degrees of difficulty.

The algorithm presented in this dissertation is of the latter kind.

1.6 Objective

The ultimate objective in the area of research into optimization problems is to find algorithms that can easily solve all (or very large classes) of optimization problems, whether linear or nonlinear, unconstrained or constrained.

Geometric programming is, as indicated above, a good technique for problems that:

are functions of real variables,

can be described in terms of posynomials,

and have zero degrees of difficulty.

In order to generalize the technique, various methods have been developed to solve some signomial problems that have multiple degrees of difficulty.

A generalized algorithm that can solve all such problems would indeed be a significant breakthrough, since many problems of practical significance can be formulated as signomials with multiple degrees of difficulty. There does not exist such an algorithm.

The objective of this dissertation is to report on an algorithm that can solve a class of nonlinear, signomial, multivariable, constrained optimization problems with multiple degrees of difficulty.

Since this work represents a continuation of the ongoing research at the Colorado School of Mines, it will be assumed that the reader is familiar with the basic concepts of geometric programming, with condensation, and with Woolsey's four rules of geometric programming.

Referring to the list on page 6, factors (1) to (3) will be addressed in the algorithm, (4) and (5) will be indirectly dealt with, and (6) will be ignored (it will be assumed that all the variables are strictly positive at optimality).

Beale (1968) identifies five areas of research in mathematical programming:

1. Mathematical theory.
2. Development of algorithms. (The word algorithms is widely used to mean procedures for deriving numerical solutions, without specific reference to the precision to which the relevant numbers are to be computed or to the organization of the computations within the store of a computer.)
3. Development of efficient codes to solve practical problems on a computer.
4. Methods of organizing real problems so that they can be solved numerically. . . .
5. The assessment of actual and potential fields of application. . . .

This dissertation is primarily concerned with area no. 2 above. Area no. 4 will also be addressed, extending the work of Kirk (1988).

The algorithm will be presented in chapter 4, after some essential material is developed in chapters two and three. Twelve example problems will be discussed in chapter 5, although the detailed mathematical work will be presented in appendixes A to L. Areas for further research will be mentioned in chapter 6.

Chapter 2

THE SIGN TABLE

2.1 Monotonicity Analysis

Wilde (1978) presented monotonicity analysis as a method to determine which constraints have to be active or binding. He also used the insight gained to extract a bounded simplified problem from the original problem. He then used this simplified problem to construct bounds on the original problem, and then continued until the design and established bounds differed by an acceptable percentage.

The table Wilde used was constructed from the orthogonality constraints on the dual variables, which consists of the exponents of the variables for each term. The table has a column for each term and a row for each variable in the problem. The problem is unbounded if there is not at least one positive and at least one negative element in each row.

As an example, consider Ratliff's degenerate problem (Ratliff 1986):

Minimize

$$TEC = N^{-\frac{7}{6}} \cdot D^{-1} \cdot L^{-\frac{4}{3}} + P^{\frac{8}{10}} \cdot N^{-\frac{2}{10}} \cdot L^{-1} + N \cdot D \cdot L + P^{-\frac{48}{10}} \cdot N^{-\frac{18}{10}} \cdot L$$
$$+ P^{-1} \cdot N^{-1} \cdot L^{-1}$$

Wilde's table for monotonicity analysis:

	T1	T2	T3	T4	T5
P:		$\frac{8}{10}$		$-\frac{48}{10}$	-1
N:	$-\frac{7}{6}$	$-\frac{2}{10}$	1	$-\frac{18}{10}$	-1
D:	-1		1		
L:	$-\frac{4}{3}$	-1	1	1	-1

Row N indicates that the third term is essential, row D indicates that the first term is essential, and row P indicates that the second term is essential. If it was necessary to extract a simplified problem to bound the answer (for example if L was a constant), it would have to include at least these three terms.

2.2 The Woolsey Sign Table

Woolsey (1985) simplified and generalized the concept by constructing a "sign table", which consists of the signs of the exponents of the variables, multiplied by the sign of the coefficient of the term in which the variable occurs.

Since the table consists only of + and - signs, it is easier to use than Wilde's table. It also makes provision for signomials.

As an example, consider Woolsey's sign table for the same problem as in the previous section:

	T1	T2	T3	T4	T5
P:		+		-	-
N:	-	-	+	-	-
D:	-		+		
L:	-	-	+	+	-

By looking at the pattern of the + and - signs, the same conclusions can be reached as with Wilde's table.

2.3 The Advanced Sign Table

The advanced sign table extends Wilde's and Woolsey's ideas by incorporating elementary row operations and column operations into the analysis. The format of the table is the same as that of Wilde (1978). Elementary row operations consist of adding a multiple of any row to another row. Column operations will be discussed in chapter 4.

Elementary row operations correspond to variable transformations in the original problem formulation. The objective is to minimize the density (or maximize the number of zero elements) of the table by a series of elementary row operations. One advantage is that computational efficiency is improved when it is used in calculations.

The major advantage is that the problem can be simplified to better reveal the structure and to make it easier to solve. Another important benefit is that at each intermediate step the structure can be analyzed to see which constraints must be binding. This information is cumulative; if a constraint must be binding at any step to prevent the problem from becoming unbounded, that constraint must be binding for any form of the problem.

This use of the sign table will be illustrated in the problems presented in chapter 5. As a first example, the same problem as in the previous two sections will be considered.

Referring to the table on page 12, add (-1) times row D to row N. This corresponds to the following transformation of variables: X = D•N. The sign table becomes:

	T1	T2	T3	T4	T5
P:		$\frac{8}{10}$		$-\frac{48}{10}$	-1
N:	$-\frac{1}{6}$	$-\frac{2}{10}$		$-\frac{18}{10}$	-1
X:	-1		1		
L:	$-\frac{4}{3}$	-1	1	1	-1

This clearly shows that the problem is unbounded, a result which Ratliff (1986) discovered by proving that there is no dual feasible solution.

By performing the advanced sign table analysis after the problem has been transformed to a posynomial in the all constraint form (see chapter 3), the problem of signomials is avoided. This analysis forms an integral part of the Wessels algorithm (see chapter 4).

2.4 The Sign Table and the Kuhn-Tucker Conditions

Kuhn and Tucker presented a set of necessary conditions for optimization problems (Walsh 1975; Luenberger 1984; Winston 1987). A number of points may satisfy these conditions, the problem is to find these points and then to find the best one of them.

To find these points, a generalized Lagrange multiplier is associated with each constraint. Each one of these multipliers must be zero or positive, corresponding to the associated constraint being nonbinding or binding, respectively. (In some cases a constraint can appear binding with the corresponding multiplier equal to zero. Such cases arise when the constraint is, in fact, redundant; the solution would be unchanged if the constraint was not there for that particular problem.)

An exhaustive strategy to find all the possible Kuhn-Tucker points would be to consider the following cases in sequence:

(1) No constraints (objective function only).

(2) All possible combinations of the objective function and one constraint.

(3) All possible combinations of the objective function and two constraints.

(n) All possible combinations of the objective function and $(n - 1)$ constraints $(n > 3)$.

Because more constraints can only reduce the feasible region, it is only necessary to continue to the case at which the first feasible solution to the problem is found.

The best point found after all the feasible combinations of that case are examined is the optimal solution to the problem.

This strategy can potentially lead to 2^n problems to be solved if there are n constraints, and each of these can be a difficult nonlinear optimization problem by itself. Use of the advanced sign table can greatly reduce the number of cases and problems to consider, since it can indicate some constraints which have to be tight. It can also indicate combinations of constraints which lead to unbounded problems, which of course can not be candidates for the optimal solution.

Thus although an easy way does not exist to incorporate slack constraints in the algorithm presented in chapter 4, the strategy outlined above (combining the Kuhn-Tucker necessary conditions and the advanced sign table) is very effective in reducing the number of combinations of constraints to consider.

The use of this strategy is demonstrated in some of the examples presented in chapter 5. See for example the two-phase sampling problem, Verma's grinding problem, and the helical spring design problem.

Chapter 3

TRANSFORMING TO THE ALL CONSTRAINT FORM

3.1 Signomials

Signomial problems are much more difficult to solve than posynomial problems. Various methods have been developed to address such problems. One method involves introducing a "signum" function, where $\sigma = \pm 1$, depending on the context (Beightler and Phillips 1976; Wilde 1978). Another method involves transforming the problem into a "reversed" geometric program (Duffin and Peterson 1973; Beightler and Phillips 1976).

With condensation it is in fact possible to transform any signomial problem into a posynomial (Beightler and Phillips 1976). By combining this fact and extending some of Thome's ideas (Thome 1988), all geometric programming problems will be transformed into a particularly useful form, which will be called the "all constraint form".

There are two steps to perform to transform the problem into the all constraint form. The first step is to replace the objective function by a single variable and a constraint. The second step is to transform the resulting problem into a posynomial with condensation.

3.2 Transforming to the All Constraint Form - Step 1

To replace the objective function with a single variable and a constraint, consider the following posynomial geometric programming problem:

Minimize $f(\underline{x})$

Subject to $\underline{x} > \underline{0}$

The following is an equivalent problem, but note that the objective function has been replaced by a single variable and a constraint:

Minimize y

Subject to $y \geq f(\underline{x})$

$\underline{x} > \underline{0}$

This problem can easily be converted into standard geometric programming format:

Minimize y

Subject to $y^{-1} \cdot f(\underline{x}) \leq 1$

$\underline{x} > \underline{0}$

This process can, with slight modification, be applied to maximization problems, problems where the optimal objective function value is negative, signomial problems, and to any combination of the above.

For maximization problems, the direction of the inequality of the generated constraint will be reversed. The maximization problem can easily be transformed into a minimization problem by minimizing the multiplicative inverse of the new variable.

If the optimal objective function value is negative, the new variable will be negative at optimality. This presents a problem, since all variables are assumed to be strictly positive in geometric programming. To solve this problem, multiply the objective function with -1 before the transformation. This will also change a minimization problem into a maximization problem (or a maximization problem into a minimization problem), thus proceed as for a maximization (or minimization) problem. Remember to multiply the optimal value found with -1 to find the optimal value for the original problem.

If the objective function contains terms with negative coefficients, the transformation will usually result in a constraint with terms with negative coefficients. The resulting problem is thus a signomial, which will be transformed into a posynomial as described in the next step.

In the special case where the objective function consists of a single term, a single term objective function can be achieved by a simple variable substitution, keeping the number of terms and variables constant.

3.3 Transforming to the All Constraint Form - Step 2

The problem is now in the all constraint form, but some constraints may still contain terms with negative coefficients. The purpose of this step is to transform the problem into an equivalent posynomial form using the technique of condensation.

By algebraic manipulation, arrange the terms in each inequality such that all the terms on either side of the inequality have positive coefficients. Condense all the terms on the greater side of the inequality into one term. Divide both sides of the inequality by the condensed term. Since all the variables are assumed to be positive and have been arranged to have positive signs, the condensed term has to be positive and the direction of the inequality does not change.

The problem has now been transformed into the all constraint form, and appears as a posynomial, possibly with condensed terms.

3.4 Advantages of the All Constraint Form

The all constraint form has several distinct advantages. Analysis using the advanced sign table is simplified. Since the problem is now in posynomial form, regular geometric programming techniques can be applied. Since the objective function consists of a single variable, the solution can be achieved by using only the second and fourth of Woolsey's four rules. A major benefit of this fact, as will be seen in the next chapter, is that during each iteration step needed to converge to the optimal solution and to resolve all the condensed terms, only one matrix (of size equal to the number of variables) has to be inverted.

The transformation process introduces one new variable and one new term, thus the degree of difficulty of the problem remains unchanged.

It should be noted that although a posynomial has a convex feasible region (Duffin, Peterson, and Zener 1967) and therefore a single global minimum, the feasible region of a signomial transformed into a posynomial is convex only in the area of the condensed point. The feasible region may thus not be totally convex, and any minimum found may in fact be only a local minimum. Thus, if the problem is a posynomial in its original formulation, the global minimum can be found. In all other cases a global minimum cannot be guaranteed.

Chapter 4

THE WESSELS ALGORITHM

4.1 Step 1 - Preprocessing

When presented with a nonlinear optimization problem, it can usually be preprocessed to simplify the problem that has to be addressed by a specific algorithm. Identifying constraints that have to be binding, that can not be binding (redundant, dominated by other constraints), and pairs of constraints representing upper and lower bounds (thus at most one of the pair can be binding) can greatly aid in simplifying the problem.

Many of these preprocessing techniques have been documented by Kirk (1988). These techniques are even more powerful when combined with the advanced sign table.

Thus the first step of the algorithm is to simplify the problem as far as possible, and to put it in the standard geometric programming form. If some of the constraints may be nonbinding, a strategy for examining a sequence of problems should be decided upon using the advanced sign table and Kuhn-Tucker theory as outlined in section 2.4. The problem is thus processed into standard geometric programming problem format, but may still be a signomial and may still have multiple degrees of difficulty.

4.2 Step 2 - Transforming into the All Constraint Form

The second step is to transform the problem into the all constraint form as described in chapter 3. The problem is thus transformed to posynomial form with one term in the objective function. The problem may have multiple degrees of difficulty and some condensed terms after this step.

4.3 Step 3 - Reducing the Degrees of Difficulty

The objective is to reduce the degrees of difficulty to zero by condensing some terms. Note that the degree of difficulty is equal to the number by which the columns exceed the rows in the advanced sign table. Condensation is a form of elementary column operation, except that there are some limitation on which columns can be combined, and the weight given to each column as they are combined must be chosen in the correct way. Another difference compared to regular column operations is that the two or more original columns that are condensed are replaced by the single weighted total of the original columns, thereby reducing the total number of columns in the matrix.

Each column in the advanced sign table correspond to a term in the problem. Condensation of terms (or combination of columns) is therefore restricted to terms in the same constraint: only two or more terms in the same constraint can be condensed. Expressed in a different way, one plus sign must be condensed away for each degree of difficulty.

One of the problems is to decide exactly which terms should be condensed. The following guidelines should be followed: columns that have entries of the same sign in the same rows of the respective columns can be condensed; each element in the combined column should have an unique sign regardless of the weights assigned to the terms that are condensed; the problem must still be balanced after condensation. Refer to appendixes B and E for simple examples.

The next step will indicate whether a different condensation pattern should be followed, if necessary.

4.4 Step 4 - Finding a Dual Feasible Solution

By using Woolsey's rule 2, values for the dual variables can now be found by inverting the exponent matrix. If condensed terms are present, an initial starting point should be chosen for each variable to have starting weights for the condensed terms. During subsequent iterations the previous current values of the primal variables constitute the starting point.

A typical choice is to start each variable at 1. Any positive value can be used, although computer overflow or underflow problems can arise due to very badly scaled problems, or very badly chosen starting points. (One of the attractive features of this algorithm is the relative insensitivity to the initial starting point.)

As described in chapter 1, this is a dual based problem and it is important that a dual feasible solution is found at each iteration. A dual feasible solution has been found if each dual variable is strictly positive.

If the current solution is dual infeasible, one or more of the following tactics can be tried:

(1) Try another transformation of the variables, or equivalently, change the exponent matrix by performing elementary row operations.

(2) Go back to step 3 and try another condensation pattern. A person familiar with matrix algebra may find it useful to examine the current condensation pattern and the current infeasible dual solution.

(3) As a last resort, different starting points may be tried.

If there does not exist any dual feasible solution, the primal problem is either unbounded or infeasible.

4.5 Step 5 - Finding the Values of the Primal Variables

By using the current feasible solution to the dual variables and Woolsey's rule 4, the corresponding values of the primal variables can be found. These values are usually infeasible in problems that were originally constrained.

Note that due to the structure of the all constraint form, the transpose of the inverse of the exponent matrix calculated in step 4 can be used in this step. An efficient feature of this algorithm is that only one matrix inverse has to be calculated during each iteration.

4.6 Step 6 - Deciding If Another Iteration Is Needed

Optimality is reached when the primal and dual objective function values are equal. More practical, however, is to recognize when optimality is achieved by noticing when the primal objective function value remains the same as in the previous iteration for the required number of significant figures.

The primal objective function value is contained in the variable introduced when the problem was transformed to the all constraint form (see chapter 3).

The dual objective function value can be calculated at each iteration using Woolsey's rule 1. This has been done in all the examples presented in chapter 5 for illustrative purposes, although it was not necessary to solve any of the problems.

At optimality the solution to the primal problem must also be feasible to the required number of significant figures, occasionally this may mean that some more iterations are required.

If another iteration is required, return to step 4 and use the current values of the primal variables as the new starting point. Repeat steps 4, 5, and 6 until accuracy to the required number of significant figures is achieved.

4.7 Step 7 - Finding the Optimal Solution

After the optimal solution to the transformed problem has been found, the optimal solution to the original formulation of the problem should be calculated.

It is always good practice to make the following three checks:

(1) Check that the objective function value of the transformed problem, as calculated in the previous steps and suitably transformed, is equal to the

objective function value of the problem as originally formulated.

(2) Check that the objective function value of the primal problem agrees with the objective function value of the dual problem.

(3) Check that all the constraints, if any, in the original formulation of the problem is satisfied. Any constraints that were active should be satisfied as equalities.

If the original problem was a posynomial, the solution is a global optimum. If condensation was used to transform the problem into a posynomial, global optimality cannot be guaranteed. One possible, but certainly not infallible, test would be to use many different starting points to see if convergence on any other local minima occurs.

Chapter 5

EXAMPLE PROBLEMS

5.1 Introduction

The algorithm presented in chapter 4 has been extensively tested. Each of the following example problems illustrates some features of the algorithm, including the advanced sign table and the transformation to the all constraint form.

A short description of each example problem is given in this chapter, and a complete printout of the solution is presented in the corresponding appendix.

The problems were solved by implementing the Wessels algorithm with the "MathCAD" software package on a Hewlett-Packard Vectra personal computer. The computer is built around an Intel 80286 microprocessor with an 80287 co-processor, both running at a clock speed of eight megahertz. The operating system used was MS/DOS, and 640 kilobytes of random access memory was available.

Due to the limitations imposed by memory, speed, and the software, problems with more than six variables become impractical to solve on this type of system without dedicated software.

The example problems are arranged in order of increasing degrees of difficulty, from 0 d.d. to 11 d.d.

5.2 The Shockum Electronics Problem

The objective in this problem is to maximize the net profit of a fictitious enterprise, given a sales function.

This is a signomial problem that illustrates the technique to transform a problem into the all constraint form. Due to the structure of the problem, the transformed problem is a 0 d.d. posynomial that can easily be solved.

5.2.1 Problem Characteristics. This is a 0 d.d. signomial problem without constraints. The problem has 4 terms and 3 variables.

5.2.2 Source. The problem was found in Woolsey (1985).

5.2.3 Number of Iterations. The number of iterations performed to achieve accuracy to four significant figures was 1; to eight significant figures was 1; to twelve significant figures was 1.

5.3 Woolsey's Problem (No. 1)

This is a small example that demonstrates what to do in the case of a negative objective function value.

5.3.1 <u>Problem Characteristics.</u> This is a 1 d.d. signomial problem without constraints. The problem has 3 terms and 1 variable.

5.3.2 <u>Source.</u> The problem was found in Woolsey (1988).

5.3.3 <u>Number of Iterations.</u> The number of iterations performed to achieve accuracy to four significant figures was 3; to eight significant figures was 5; to twelve significant figures was 8.

5.4 <u>Walsh's Problem (No. 1)</u>

This is a signomial problem with a negative objective function value at optimality. This problem also illustrates the concept of condensation on the right hand side of the inequality to transform the problem into a posynomial.

5.4.1 <u>Problem Characteristics.</u> This is a 1 d.d. signomial problem without constraints. The problem has 4 terms and 2 variables.

5.4.2 <u>Source.</u> The problem was found in Walsh (1975).

5.4.3 <u>Number of Iterations.</u> The number of iterations performed to achieve accuracy to four significant figures was 2; to eight significant figures was 5; to twelve significant figures was 7.

5.5 Woolsey's Problem (No. 2)

This is a very difficult problem with a deceptively simple appearance. Presented by Woolsey in 1985, it remained unsolved until now. Some valiant attempts using three-dimensional grid search methods have been made.

5.5.1 Problem Characteristics. This is a 1 d.d. signomial problem with 1 constraint. The problem has 5 terms and 3 variables.

5.5.2 Source. The problem was presented by Woolsey (1985).

5.5.3 Number of Iterations. The number of iterations performed to achieve accuracy to four significant figures was 6; to eight significant figures was 8; to twelve significant figures was 11.

5.6 Woolsey's Problem (No. 3)

This problem is representative of the class of problems addressed by Ratliff (1986). This illustrates that this algorithm can also solve such problems. The advanced sign table clearly suggests good candidates for condensation in this case.

5.6.1 Problem Characteristics. This is a 2 d.d. posynomial problem without constraints. The problem has 5 terms and 2 variables.

5.6.2 Source. The problem was found in Woolsey (1988).

5.6.3 Number of Iterations. The number of iterations performed to achieve accuracy to four significant figures was 5; to eight significant figures was 9; to twelve significant figures was 12.

5.7 Woolsey's Problem (No. 4)

This problem is representative of the class of problems addressed by Thome (1988). This illustrates that this algorithm can also solve such problems. Due to bad scaling care has to be taken with the selection of the starting point to avoid computer overflow.

5.7.1 Problem Characteristics. This is a 2 d.d. signomial problem without constraints. The problem has 4 terms and 1 variable.

5.7.2 Source. The problem was found in Woolsey (1988).

5.7.3 Number of Iterations. The number of iterations performed to achieve accuracy to four significant figures was 5; to eight significant figures was 7; to twelve significant figures was 11.

5.8 The Extended Rosenbrock Function

This is one of the set of test problems for unconstrained minimization provided by Moré, Garbow and Hillstrom in 1981 (Dennis and Schnabel 1983).

5.8.1 Problem Characteristics. This is a 2 d.d. signomial problem without constraints. The problem has 5 terms and 2 variables.

5.8.2 Source. The problem was found in Dennis and Schnabel (1983).

5.8.3 Number of Iterations. The number of iterations performed to achieve accuracy to four significant figures was 55; to eight significant figures was 95; to twelve significant figures was 125.

5.9 Fertilizer Plant Design Problem

The problem is to design a fertilizer plant consisting of a compressor and a reactor so that annual profit is maximized (Wilde 1978). Wilde did not solve the problem, he bounded it until he found a feasible solution that satisfied him.

5.9.1 Problem Characteristics. This is a 2 d.d. signomial problem without constraints. The problem has 5 terms and 2 variables.

5.9.2 Source. The problem was found in Wilde (1978).

5.9.3 Number of Iterations. The number of iterations performed to achieve accuracy to four significant figures was 4; to eight significant figures was 6; to twelve significant figures was 8.

5.10 Fleet Design Problem

This is one version of Folkers' fleet design problems as reported in Beightler and Phillips (1976). This problem illustrates that variable substitutions may be necessary to simplify the structure of the problem in order to find the solution.

5.10.1 Problem Characteristics. This is a 2 d.d. posynomial problem with 2 constraints. The problem has 7 terms and 4 variables.

5.10.2 Source. The problem was found in Beightler and Phillips (1976).

5.10.3 Number of Iterations. The number of iterations performed to achieve accuracy to four significant figures was 3; to eight significant figures was 4; to twelve significant figures was 5.

5.11 Walsh's Problem (No. 2)

This problem has a negative objective function value at optimality. It also has two constraints, the sign table indicates which one must always be binding. This solution is better than Walsh's solution by 2.4%, and it satisfies the constraints. Walsh's solution violates one of the constraints by 0.014%.

5.11.1 Problem Characteristics. This is a 3 d.d. signomial problem with 2 constraints. The problem has 6 terms and 2 variables.

5.11.2 Source. The problem was found in Walsh (1975).

5.11.3 Number of Iterations. The number of iterations performed to achieve accuracy to four significant figures was 3; to eight significant figures was 5; to twelve significant figures was 7.

5.12 Underwood's Problem

This is a problem presented by Underwood in 1987 to serve as a test problem for a nonlinear programming algorithm developed and implemented by a group of his students as a class project.

5.12.1 Problem Characteristics. This is a 4 d.d. signomial problem with 1 constraint. The problem has 7 terms and 2 variables.

5.12.2 Source. The problem was found in Underwood (1987).

5.12.3 Number of Iterations. The number of iterations performed to achieve accuracy to four significant figures was 7; to eight significant figures was 11; to twelve significant figure was 16.

5.13 Verma's Grinding Problem

This is a problem that was presented to Woolsey by Rao and Verma of the Indian Institute of Technology, Delhi, India. The problem is to find the optimal values for work speed, wheel speed, and depth of cut to minimize cost in a surface grinding operation. The problem presented here was slightly modified by Woolsey (1988).

This problem illustrates the use of the advanced sign table combined with the Kuhn-Tucker conditions as outlined in chapter 2. In the final step it is found that three constraints have to be binding for the three variables. The form of those three constraints is such that the final solution can be found by algebraic manipulation. The Wessels algorithm was used in an intermediate step.

5.13.1 Problem Characteristics. This is an 8 d.d. posynomial problem with 9 constraints. The problem has 12 terms and 3 variables.

5.13.2 Source. The original set of grinding problems on which this example is based was presented by Rao and Verma (1985).

5.13.3 Number of Iterations. The number of iterations performed to achieve accuracy (in an intermediate step) to four significant figures was 1; to eight significant figures was 1; to twelve significant figures was 2.

5.14 Bridge Design Problem

This is another difficult problem that was simplified by using a strategy based on the advanced sign table and the Kuhn-Tucker conditions. It was determined that three constraints have to be binding, and since there are three variables the algorithm serves as a simultaneous nonlinear equation solver for this problem.

5.14.1 Problem Characteristics. This is a 9 d.d. signomial problem with 4 constraints. The problem has 13 terms and 3 variables.

5.14.2 Source. The problem was found in Woolsey (1985).

5.14.3 Number of Iterations. The number of iterations performed to achieve accuracy to four significant figures was 3; to eight significant figures was 4; to twelve significant figures was 4.

5.15 Two-Phase Sampling Problem

The objective of this problem is to minimize sampling cost using direct and indirect measurements, subject to certain constraints. The solution strategy was again based on the advanced sign table and the Kuhn-Tucker conditions. An interesting feature of this problem is that it leads to cycling during the iteration process. A simple modification to the iteration step solved this problem.

5.15.1 Problem Characteristics. This is a 10 d.d. posynomial problem with 6 constraints. The problem has 13 terms and 2 variables.

5.15.2 Source. The problem was found in Woolsey (1985).

5.15.3 Number of Iterations. The number of iterations performed to achieve accuracy to four significant figures was 54; to eight significant figures was 82; to twelve significant figures was 102.

5.16 Helical Spring Design Problem

This is a solution to the optimal design of a helical spring for a cam driven system. It is presented as an exercise in Wilde (1978). Again using the strategy outlined in section 2.4, it is determined that this is a fully constrained design problem: there is one binding constraint for each variable. The final combination of active constraints was solved by algebraic manipulation. The Wessels algorithm was used in an intermediate step.

5.16.1 Problem Characteristics. This is an 11 d.d. signomial problem with 11 constraints. The problem has 17 terms and 5 variables.

5.16.2 Source. The problem was found in Wilde (1978).

5.16.3 Number of Iterations. The number of iterations performed to achieve accuracy (in an intermediate step) to four significant figures was 7; to eight significant figures was 15; to twelve significant figures was 20.

5.17 Conclusion

The above examples not only demonstrate how to implement the Wessels algorithm in a wide variety of cases, they also demonstrate that the algorithm actually works for all those cases. It was also shown that this algorithm extends the class of problems that can be solved to multivariable constrained signomials with multiple degrees of difficulty.

The computational experience as summarized above suggests very rapid convergence. A very rough approximation would be about one to two iterations for each significant figure required to be accurate in the final objective function.

Chapter 6

CONCLUSION AND AREAS FOR FURTHER RESEARCH

6.1 Conclusion

The objective of this dissertation was to expand the class of nonlinear optimization problems that could be solved by using geometric programming.

This objective has been achieved. An algorithm was presented that could solve some multivariable signomials with constraints and multiple degrees of difficulty. This represents a significant breakthrough. General algorithms (as opposed to algorithms designed for specific applications) developed recently in the research program at the Colorado School of Mines could solve only unconstrained posynomials (Ratliff 1986) or unconstrained single variable signomials of a certain class (Thome 1988).

An important side benefit of this algorithm was the development of the advanced sign table. The advanced sign table makes it possible to do a much more powerful analysis of the problem during the preprocessing stage than were previously possible.

The algorithm was tested and proved to be successful on a large variety of problems, some of which had never been solved before by a geometric programming based method. Convergence proved to be very rapid, and accuracy up to twelve significant figures was easily achieved for each problem.

This algorithm shows that geometric programming type methods can be used to solve problems that were previously thought to be the exclusive domain of Lagrange-Newton type methods. See, for example, the problems in appendixes C, G, J, and K.

The algorithm thus met and exceeded its initial goals, and represents a significant advance in the area of geometric programming based methods for nonlinear optimization.

6.2 Areas for Further Research

Although the algorithm presented constitutes a significant advance in the research program at the Colorado School of Mines, there remains a very great deal of research to be done.

As pointed out in section 1.6, the objective was to develop an algorithm for the solution of nonlinear optimization problems using geometric programming. The fact that the algorithm works successfully on a wide variety of problems was demonstrated in chapter 5. The following areas, as listed by Beale (1968) and quoted on page 10 above, still needs to be addressed:

(1) The mathematical theory on which this algorithm is based, specifically: a proof that the iteration procedure converges, the conditions under which convergence will not occur, and when a global optimum is achieved.

(2) Development of an efficient computer code.

The biggest obstacle still remaining before geometric programming can effectively compete with "mainstream" nonlinear optimization techniques is the assumption that all variables have to be strictly positive. This remains a virtually untouched area for research. This dissertation addresses the issue of a negative objective function value at optimality, but this is merely a start.

Another area of research that remains is how to identify (easily) all the constraints that will be nonbinding at optimality. A strategy based on the advanced sign table and the Kuhn-Tucker conditions has been proposed in this dissertation, but this approach can potentially be extremely time consuming if there are many constraints.

The class of problems that can be solved with the Wessels algorithm appears to be very large (multivariable unconstrained and constrained signomials with multiple degrees of difficulty). The algorithm successfully solved each test problem, but an important area of research that remains to be done is to determine if all problems of this class can be solved with this algorithm. If there exists a subclass of problems that can not be solved by this algorithm, it will be important to determine the extent of such a subclass.

Geometric programming methods are very effective and efficient to solve a large (and expanding) class of nonlinear optimization problems. Much remains to be done, however, on the path to the objective of a unified algorithm that can solve all optimization problems.

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Appendix A

THE SHOCKUM ELECTRONICS PROBLEM

THE SHOCKUM ELECTRONICS PROBLEM

MAXIMIZE: $\quad Profit(P,Q,A,S) := P \cdot Q - 10 \cdot Q - 38000 - A - S$

$$\text{Where} \quad Q(P,A,S) := 10^{5} \cdot P^{-2} \cdot A^{0.125} \cdot S^{0.25}$$

**STEP 1: Preprocessing.

Since the fixed cost does not vary if P, Q, A, or S changes, it can be ignored in the profit maximizaton problem. Let the profit without fixed cost (or "contribution") be C.

MAXIMIZE:

$$C(P,A,S) := 10^{5} \cdot P^{-1} \cdot A^{0.125} \cdot S^{0.25} - 10^{6} \cdot P^{-2} \cdot A^{0.125} \cdot S^{0.25} - A - S$$

A $4 - 3 - 1 = 0$ dd multi-variable signomial with no constraints.

**STEP 2: Transforming into the all constraint form.

If you maximize C, then

$$C(P,A,S) \le 10^{5} \cdot P^{-1} \cdot A^{0.125} \cdot S^{0.25} - 10^{6} \cdot P^{-2} \cdot A^{0.125} \cdot S^{0.25} - A - S$$

Put into standard GP form:

MINIMIZE: $\quad F(C) := C^{-1}$

SUBJECT TO: $\quad C + 10^{6} \cdot P^{-2} \cdot A^{0.125} \cdot S^{0.25} + A + S \le 10^{5} \cdot P^{-1} \cdot A^{0.125} \cdot S^{0.25}$

$$\begin{bmatrix} 10^{-5} \cdot C \cdot P \cdot A^{-0.125} \cdot S^{-0.25} + 10 \cdot P^{-1} \ldots \\ + 10^{-5} \cdot P \cdot A^{0.875} \cdot S^{-0.25} + 10^{-5} \cdot P \cdot A^{-0.125} \cdot S^{0.75} \end{bmatrix} \le 1$$

A $5 - 4 - 1 = 0$ dd multi-variable posynomial with a constraint.

**STEP 3: Reducing the d.d. - Not necessary for this problem.

**STEP 4: Finding a dual feasible solution.

RULE 1:

$$F'(\delta) := 10^{-5} \cdot \left[\frac{10}{\delta_3}\right]^{\delta_3} \cdot \left[\frac{10^{-5}}{\delta_4}\right]^{\delta_4} \cdot \left[\frac{10^{-5}}{\delta_5}\right]^{\delta_5} \cdot \left[1 + \delta_3 + \delta_4 + \delta_5\right]^{1+\delta_3+\delta_4+\delta_5}$$

NOTE: Rule 1 is not necessary for the algorithm, it is provided for illustrative and comparative purposes only.

RULE 2A: $\delta_1 := 1$

RULE 2B: (C) $-\delta_1 + \delta_2 := 0$ (P) $\delta_2 - \delta_3 + \delta_4 + \delta_5 := 0$

(A) $-0.125 \cdot \delta_2 + 0.875 \cdot \delta_4 - 0.125 \cdot \delta_5 := 0$

(S) $-0.25 \cdot \delta_2 - 0.25 \cdot \delta_4 + 0.75 \cdot \delta_5 := 0$

Advanced sign table / exponent matrix:

$$\delta := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & -0.125 & 0 & 0.875 & -0.125 \\ 0 & -0.25 & 0 & -0.25 & 0.75 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{thus} \quad \delta = \begin{bmatrix} 1 \\ 1 \\ 1.6 \\ 0.2 \\ 0.4 \end{bmatrix}$$

All the δ's are positive, thus the solution is dual feasible. In the examples that follow the Greek letter δ will be used for weights in condensed terms. Dual variables will be represented by the Greek letter ω.

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$\delta_2 := 10^{-5} \cdot C \cdot P \cdot A^{-0.125} \cdot S^{-0.25} \cdot \left[\delta_2 + \delta_3 + \delta_4 + \delta_5\right]$$

$$\delta_3 := 10 \cdot P^{-1} \cdot \left[\delta_2 + \delta_3 + \delta_4 + \delta_5\right]$$

$$\delta_4 := 10^{-5} \cdot P \cdot A^{0.875} \cdot S^{-0.25} \cdot \left[\delta_2 + \delta_3 + \delta_4 + \delta_5\right]$$

$$\delta_5 := 10^{-5} \cdot P \cdot A^{-0.125} \cdot S^{0.75} \cdot \left[\delta_2 + \delta_3 + \delta_4 + \delta_5\right]$$

Let $x_1 := \ln(C)$ $x_2 := \ln(P)$ $x_3 := \ln(A)$ $x_4 := \ln(S)$

Then

$$x := \begin{bmatrix} 1 & 1 & -0.125 & -0.25 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0.875 & -0.25 \\ 0 & 1 & -0.125 & 0.75 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \ln\left[5 \cdot 16^{-1} \cdot 10^{5}\right] \\ \ln\left[8 \cdot 16^{-1} \cdot 10^{-1}\right] \\ \ln\left[1 \cdot 16^{-1} \cdot 10^{5}\right] \\ \ln\left[2 \cdot 16^{-1} \cdot 10^{5}\right] \end{bmatrix} \quad x = \begin{bmatrix} 11.0780639353 \\ 2.9957322736 \\ 9.4686260229 \\ 10.1617732035 \end{bmatrix}$$

NOTE: The matrix in the previous step is the transpose of the matrix constructed with rule 2 above, if we remove the first row and the first column of that matrix. This observation will be exploited for computational efficiency in the examples that follow.

**STEP 6: Iterations - Not necessary for this problem.

**STEP 7: Finding the optimal solution.

$$\begin{bmatrix} C' \\ P' \\ A' \\ S' \end{bmatrix} := \overrightarrow{\exp(x)} \qquad \text{thus} \qquad \text{SOLUTION:} \quad \begin{bmatrix} C' \\ P' \\ A' \\ S' \end{bmatrix} = \begin{bmatrix} 64735.4300261715 \\ 20 \\ 12947.0860052343 \\ 25894.1720104686 \end{bmatrix}$$

Let $Q' := Q(P',A',S')$ $\qquad Q' = 10357.6688041874$

CHECK:

$$F(C')^{-1} = 64735.4300261715 \qquad \text{(Objective function)}$$

$$F'(\delta)^{-1} = 64735.4300261716 \qquad \text{(Rule 1)}$$

$$C' = 64735.4300261715 \qquad \text{(Rule 4)}$$

$$\text{Profit}(P',Q',A',S') + 38000 = 64735.4300261716 \qquad \text{(Original Objective function)}$$

Appendix B

WOOLSEY'S PROBLEM (NO. 1)

WOOLSEY'S PROBLEM (NO. 1)

MINIMIZE: $Y(Q) := Q^{3} + Q^{-2} - 3\cdot Q$

A 3 - 1 - 1 = 1 dd single-variable signomial without constraints.

**STEP 1: Preprocessing.

Since $Y(1) = -1$ we know that optimal Y must be negative.

Let $F(Q) := -Y(Q)$ ▫ and let $Z := F(Q)$ ▫

The problem becomes:

MAXIMIZE: $F(Q) := -Q^{3} - Q^{-2} + 3\cdot Q$

**STEP 2: Transforming into the all constraint form.

MINIMIZE: Z^{-1} ▫ Subject to: $\left[\frac{1}{3}\cdot Z\cdot Q^{-1} + \frac{1}{3}\cdot Q^{-3}\right] + \frac{1}{3}\cdot Q^{2} \le 1$ ▫

A 4 - 2 - 1 = 1 dd multi-variable posynomial with a constraint.

RULE 2A:
RULE 2B:

$$\begin{matrix} \\ Z: \\ Q: \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & -3 & 2 \end{bmatrix} \cdot \S := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ ▫

**STEP 3: Reducing the d.d.

Note that for the second and third columns, the elements in the corresponding rows either have the same sign or are equal to zero. Condense the second and third term:

$$\S 1(Z,Q) := \frac{\frac{1}{3}\cdot Z\cdot Q^{-1}}{\frac{1}{3}\cdot Z\cdot Q^{-1} + \frac{1}{3}\cdot Q^{-3}} \qquad \S 2(Z,Q) := \frac{\frac{1}{3}\cdot Q^{-3}}{\frac{1}{3}\cdot Z\cdot Q^{-1} + \frac{1}{3}\cdot Q^{-3}}$$

$$k(Z,Q) := \frac{1}{3}\cdot \S 1(Z,Q)^{-\S 1(Z,Q)}\cdot \S 2(Z,Q)^{-\S 2(Z,Q)}$$

Thus the problem is now:

MINIMIZE: Z^{-1} ▫

Subject to: $k(Z,Q)\cdot Z^{\S 1(Z,Q)}\cdot Q^{-1-2\cdot \S 2(Z,Q)} + \frac{1}{3}\cdot Q^{2} \le 1$ ▫

Note that the problem is still balanced in each term after the condensation.

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(Z,Q) := \begin{bmatrix} \delta 1(Z,Q) & 0 \\ -1 - 2\cdot\delta 2(Z,Q) & 2 \end{bmatrix}^{-1}$$

$$\omega(Z,Q) := \left[\text{augment}\left[\begin{bmatrix}1\\0\end{bmatrix}, W(Z,Q)^T\right]\right]^T \cdot \begin{bmatrix}1\\0\end{bmatrix} \qquad \omega(1,1) = \begin{bmatrix}1\\2\\2\end{bmatrix}$$

Note that a starting point of Z = 1, Q = 1 yields a feasible dual solution.

RULE 1:

$$G(Z,Q) := \left[\frac{k(Z,Q)}{\omega(Z,Q)_1}\right]^{\omega(Z,Q)_1} \cdot \left[\frac{1}{3\cdot\omega(Z,Q)_2}\right]^{\omega(Z,Q)_2}$$

$$F'(Z,Q) := G(Z,Q)\cdot\left[\omega(Z,Q)_1 + \omega(Z,Q)_2\right]^{\omega(Z,Q)_1 + \omega(Z,Q)_2}$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(Z,Q) := W(Z,Q)^T\cdot\begin{bmatrix} \ln\left[\omega(Z,Q)_1\cdot\left[\omega(Z,Q)_1 + \omega(Z,Q)_2\right]^{-1}\cdot k(Z,Q)^{-1}\right] \\ \ln\left[\omega(Z,Q)_2\cdot\left[\omega(Z,Q)_1 + \omega(Z,Q)_2\right]^{-1}\cdot 3\right] \end{bmatrix}$$

$$A(Z,Q) := \overrightarrow{\exp(B(Z,Q))}$$

SOLUTION: $N := 9 \quad i := 0..N$

$$\begin{bmatrix} Z_0 \\ Q_0 \end{bmatrix} := \begin{bmatrix}1\\1\end{bmatrix} \qquad \begin{bmatrix} Z_{i+1} \\ Q_{i+1} \end{bmatrix} := A\left[Z_i, Q_i\right]$$

$i := 0..N+1$

$$Y_i := Y\left[Q_i\right]$$

$$F'_i := -F'\left[Z_i, Q_i\right]^{-1}$$

**STEP 6: Iterations.

i	F'_i	$-Z_i$	Y_i	Q_i
0	-1.265625	-1	-1	1
1	-1.1834120138	-1.265625	-1.17045064042	1.2247448714
2	-1.1789290606	-1.1834120138	-1.17815828971	1.1721755915
3	-1.17886389	-1.1789290606	-1.17885308106	1.1853016123
4	-1.1788625131	-1.17886389	-1.1788622838	1.183741916
5	-1.1788624852	-1.1788625131	-1.17886248055	1.1839690302
6	-1.1788624846	-1.1788624852	-1.17886248452	1.1839367223
7	-1.1788624846	-1.1788624846	-1.1788624846	1.1839413339
8	-1.1788624846	-1.1788624846	-1.17886248461	1.183940676
9	-1.1788624846	-1.1788624846	-1.17886248461	1.1839407699
10	-1.1788624846	-1.1788624846	-1.17886248461	1.1839407565

**STEP 7: Finding the optimal solution.

Objective function value: $Y_{N+1} = -1.1788624846$

Variable value: $Q_{N+1} = 1.1839407565$

Note that the solution is on the last row of the columns showing the values of the objective function and variable for each iteration. These values will not always be repeated in the following examples.

Check the first order condition:

$$3 \cdot Q_{N+1}^{2} - 2 \cdot Q_{N+1}^{-3} - 3 = 0$$

Check the second order condition:

$$6 \cdot Q_{N+1} + 6 \cdot Q_{N+1}^{-4} = 10.1573796728$$

Appendix C

WALSH'S PROBLEM (NO. 1)

WALSH'S PROBLEM (NO. 1)

MINIMIZE: $f(x,y) := 2\cdot x^{4} + y^{2} - 4\cdot x\cdot y - 5\cdot y$

A 4 - 2 - 1 = 1 dd multivariable signomial without constraints.

**STEP 1: Preprocessing.

Since f(1,1) = -6 it is clear that the optimal value must be negative. Multiply with -1 and change the problem into a maximization problem.

**STEP 2: Transforming into the all constraint form.

MINIMIZE: f'^{-1} ▫

SUBJECT TO: $f' + 2\cdot x^{4} + y^{2} \le (4\cdot x\cdot y + 5\cdot y)$ ▫

Condense the right hand side:

$$\delta 1(x,y) := \frac{4\cdot x\cdot y}{4\cdot x\cdot y + 5\cdot y} \qquad \delta 2(x,y) := \frac{5\cdot y}{4\cdot x\cdot y + 5\cdot y}$$

$$k(x,y) := \left[\frac{\delta 1(x,y)}{4}\right]^{\delta 1(x,y)} \cdot \left[\frac{\delta 2(x,y)}{5}\right]^{\delta 2(x,y)}$$

The problem becomes

MINIMIZE: f'^{-1} ▫

SUBJECT TO:

$$\left[\begin{matrix} k(x,y)\cdot f'\cdot x^{-\delta 1(x,y)}\cdot y^{-1} \;\ldots \\ + 2\cdot k(x,y)\cdot x^{4-\delta 1(x,y)}\cdot y^{-1} + k(x,y)\cdot x^{-\delta 1(x,y)}\cdot y \end{matrix}\right] \le 1 \;▫$$

Degree of difficulty: 4 - 3 - 1 = 0

**STEP 3: Reducing the d.d. - No further reduction necessary.

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(x,y) := \begin{bmatrix} 1 & 0 & 0 \\ -\delta 1(x,y) & 4 - \delta 1(x,y) & -\delta 1(x,y) \\ -1 & -1 & 1 \end{bmatrix}^{-1}$$

$$\omega(x,y) := \left[\text{augment}\left[\begin{bmatrix}1\\0\\0\end{bmatrix}, W(x,y)^{T}\right]\right]^{T}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \omega(1,1) = \begin{bmatrix}1\\1\\0.2857142857\\1.2857142857\end{bmatrix}$$

RULE 1:

$$g(x,y) := \left[\frac{k(x,y)}{\omega(x,y)_1}\right]^{\omega(x,y)_1}\cdot\left[\frac{2\cdot k(x,y)}{\omega(x,y)_2}\right]^{\omega(x,y)_2}\cdot\left[\frac{k(x,y)}{\omega(x,y)_3}\right]^{\omega(x,y)_3}$$

$$f''(x,y) := g(x,y)\cdot\left[\omega(x,y)_1 + \omega(x,y)_2 + \omega(x,y)_3\right]^{\omega(x,y)_1+\omega(x,y)_2+\omega(x,y)_3}$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(x,y) := W(x,y)^{T}\cdot\begin{bmatrix}\ln\left[\omega(x,y)_1\cdot\left[\omega(x,y)_1+\omega(x,y)_2+\omega(x,y)_3\right]^{-1}\cdot k(x,y)^{-1}\right]\\ \ln\left[\omega(x,y)_2\cdot\left[\omega(x,y)_1+\omega(x,y)_2+\omega(x,y)_3\right]^{-1}\cdot\frac{1}{2}\cdot k(x,y)^{-1}\right]\\ \ln\left[\omega(x,y)_3\cdot\left[\omega(x,y)_1+\omega(x,y)_2+\omega(x,y)_3\right]^{-1}\cdot k(x,y)^{-1}\right]\end{bmatrix}$$

$$A(x,y) := \overrightarrow{\exp(B(x,y))}$$

SOLUTION: $N := 7$ $\quad i := 0\,..N$

$$\begin{bmatrix}f'_0\\x_0\\y_0\end{bmatrix} := \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \begin{bmatrix}f'_{i+1}\\x_{i+1}\\y_{i+1}\end{bmatrix} := A\left[x_i, y_i\right]$$

$i := 0\,..N + 1$

$$f''_i := f''\left[x_i, y_i\right]^{-1}$$

$$f_i := f\left[x_i, y_i\right]$$

**STEP 6: Iterations.

i	$-f''_i$	$-f'_i$	f_i
0	-19.8565655923	-1	-6
1	-20.3923004282	-19.8565655923	-20.2896180435
2	-20.4135090776	-20.3923004282	-20.4103651311
3	-20.4141082437	-20.4135090776	-20.4140241887
4	-20.414124013	-20.4141082437	-20.4141218214
5	-20.414124423	-20.414124013	-20.4141243661
6	-20.4141244337	-20.414124423	-20.4141244322
7	-20.4141244339	-20.4141244337	-20.4141244339
8	-20.4141244339	-20.4141244339	-20.4141244339

**STEP 7: Finding the optimal solution.

f_{N+1} = -20.4141244339

x_{N+1} = 1.3804086665

y_{N+1} = 5.2608173329

Appendix D

WOOLSEY'S PROBLEM (NO. 2)

WOOLSEY'S PROBLEM (NO. 2)

MINIMIZE:
$$f(x,y,z) := -x\cdot z + 0.5\cdot y^{2}\cdot z^{2}$$

SUBJECT TO:
$$\left[\frac{4}{3}\cdot 10^{-3}\right]\cdot x^{2.1} + \left[\frac{4}{3}\right]\cdot y^{-1.1} + \left[\frac{2}{3}\right]\cdot z^{0.6} \le 1$$

**STEP 1: Preprocessing.

Since $f(5,4,0.5) = -0.5$

and $$\left[\frac{4}{3}\cdot 10^{-3}\right]\cdot 5^{2.1} + \left[\frac{4}{3}\right]\cdot 4^{-1.1} + \left[\frac{2}{3}\right]\cdot 0.5^{0.6} = 0.7691734561$$

it is clear that the minimum is negative.

Multiply with -1 and change it into a maximization problem.

MAXIMIZE:
$$f' := x\cdot z - 0.5\cdot y^{2}\cdot z^{2}$$

(subject to the same constraint)

**STEP 2: Transforming into the all constraint form.

The problem becomes:

MINIMIZE:
$$f'^{-1}$$

SUBJECT TO:
$$f'\cdot x^{-1}\cdot z^{-1} + 0.5\cdot x^{-1}\cdot y^{2}\cdot z \le 1$$

$$\left[\frac{4}{3}\cdot 10^{-3}\right]\cdot x^{2.1} + \left[\frac{4}{3}\right]\cdot y^{-1.1} + \left[\frac{2}{3}\right]\cdot z^{0.6} \le 1$$

Construct the advanced sign table:

$$P := \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2.1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1.1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0.6 \end{bmatrix} \quad \begin{bmatrix} f' \\ x \\ y \\ z \end{bmatrix}$$

Subtract row z from row x, this is the same as the following variable substitution:

Let $z := s\cdot x^{-1}$ or $s := z\cdot x$

$$Q1 := \text{augment}\left[\left(P^{T}\right)^{<0>}, \left[\left(P^{T}\right)^{<1>} - \left(P^{T}\right)^{<3>}\right]\right]$$

$$Q2 := \text{augment}\left[\left(P^T\right)^{<2>}, \left(P^T\right)^{<3>}\right]$$

$$Q := \left(\text{augment}(Q1, Q2)\right)^T$$

$$Q = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2.1 & 0 & -0.6 \\ 0 & 0 & 2 & 0 & -1.1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0.6 \end{bmatrix} \quad \begin{bmatrix} f' \\ x \\ y \\ s \end{bmatrix}$$

Add row y to row x and subtract row y from row s, this is the same as the following variable substitution:

Let $\quad y := t \cdot x \cdot s^{-1} \quad$ or $\quad t := y \cdot x^{-1} \cdot s$

$$R1 := \text{augment}\left[\left(Q^T\right)^{<0>}, \left[\left(Q^T\right)^{<1>} + \left(Q^T\right)^{<2>}\right]\right]$$

$$R2 := \text{augment}\left[\left(Q^T\right)^{<2>}, \left[\left(Q^T\right)^{<3>} - \left(Q^T\right)^{<2>}\right]\right]$$

$$R := \left(\text{augment}(R1, R2)\right)^T$$

$$R = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.1 & -1.1 & -0.6 \\ 0 & 0 & 2 & 0 & -1.1 & 0 \\ 0 & -1 & -1 & 0 & 1.1 & 0.6 \end{bmatrix} \quad \begin{bmatrix} f' \\ x \\ t \\ s \end{bmatrix}$$

Make row s the second row and row x the last row:

$$R3 := \text{augment}\left[\left(R^T\right)^{<0>}, \left(R^T\right)^{<3>}\right]$$

$$R4 := \text{augment}\left[\left(R^T\right)^{<2>}, \left(R^T\right)^{<1>}\right]$$

$$R' := \left(\text{augment}(R3, R4)\right)^T$$

$$R' = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1.1 & 0.6 \\ 0 & 0 & 2 & 0 & -1.1 & 0 \\ 0 & 0 & 0 & 2.1 & -1.1 & -0.6 \end{bmatrix} \quad \begin{bmatrix} f' \\ s \\ t \\ x \end{bmatrix}$$

Note the pattern achieved in the first four columns. It is also now clear that the last two terms should be condensed. Although the density of the matrix was not reduced in this case, the structure of the problem became clearer.

The problem becomes:

MINIMIZE: $\quad f'^{-1}$

SUBJECT TO: $\quad f' \cdot s^{-1} + 0.5 \cdot s^{-1} \cdot t^2 \le 1$

AND TO: $$\left[\frac{4}{3}\cdot 10^{-3}\right]\cdot x^{2.1} + \left[\frac{4}{3}\right]\cdot s^{1.1}\cdot t^{-1.1}\cdot x^{-1.1} + \left[\frac{2}{3}\right]\cdot s^{0.6}\cdot x^{-0.6} \le 1$$

**STEP 3: Reducing the d.d.

Extract common variables in the last two terms and condense the result:

$$\left[\frac{2}{3}\cdot s^{0.6}\cdot x^{-0.6}\right]\cdot\left[2\cdot s^{0.5}\cdot t^{-1.1}\cdot x^{-0.5} + 1\right]$$

$$\delta 1(s,t,x) := \frac{2\cdot s^{0.5}\cdot t^{-1.1}\cdot x^{-0.5}}{2\cdot s^{0.5}\cdot t^{-1.1}\cdot x^{-0.5} + 1}$$

$$\delta 2(s,t,x) := \frac{1}{2\cdot s^{0.5}\cdot t^{-1.1}\cdot x^{-0.5} + 1}$$

$$k(s,t,x) := \frac{2}{3}\cdot\left[\frac{2}{\delta 1(s,t,x)}\right]^{\delta 1(s,t,x)}\cdot\delta 2(s,t,x)^{-\delta 2(s,t,x)}$$

The problem is now:

MINIMIZE: $$f'^{-1}$$

SUBJECT TO: $$f'\cdot s^{-1} + 0.5\cdot s^{-1}\cdot t^{2} \le 1$$

$$\left[\left[\frac{4}{3}\cdot 10^{-3}\right]\cdot x^{2.1} \;\ldots\; + k(s,t,x)\cdot s^{0.6+0.5\cdot\delta 1(s,t,x)}\cdot t^{-1.1\cdot\delta 1(s,t,x)}\cdot x^{-0.6-0.5\cdot\delta 1(s,t,x)}\right] \le 1$$

**STEP 4: Finding a dual feasible solution.

RUlE 2:

$$W(s,t,x) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0.6 + 0.5\cdot\delta 1(s,t,x) \\ 0 & 2 & 0 & -1.1\cdot\delta 1(s,t,x) \\ 0 & 0 & 2.1 & -0.6 - 0.5\cdot\delta 1(s,t,x) \end{bmatrix}^{-1}$$

$$\omega(s,t,x) := \left[\mathrm{augment}\left[\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, W(s,t,x)^{T}\right]\right]^{T}\cdot\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad \omega(1,1,1) = \begin{bmatrix}1\\1\\0.6470588235\\0.7843137255\\1.7647058824\end{bmatrix}$$

RULE 1:

$$G(s,t,x) := \left[\frac{1}{\omega(s,t,x)_1}\right]^{\omega(s,t,x)_1} \cdot \left[\frac{0.5}{\omega(s,t,x)_2}\right]^{\omega(s,t,x)_2}$$

$$H(s,t,x) := \left[\omega(s,t,x)_1 + \omega(s,t,x)_2\right]^{\omega(s,t,x)_1 + \omega(s,t,x)_2}$$

$$I(s,t,x) := \left[\frac{4}{3000 \cdot \omega(s,t,x)_3}\right]^{\omega(s,t,x)_3} \cdot \left[\frac{k(s,t,x)}{\omega(s,t,x)_4}\right]^{\omega(s,t,x)_4}$$

$$J(s,t,x) := \left[\omega(s,t,x)_3 + \omega(s,t,x)_4\right]^{\omega(s,t,x)_3 + \omega(s,t,x)_4}$$

$$F(s,t,x) := G(s,t,x) \cdot H(s,t,x) \cdot I(s,t,x) \cdot J(s,t,x)$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(s,t,x) := W(s,t,x)^{T} \cdot \begin{bmatrix} \ln\left[\omega(s,t,x)_1 \cdot \left[\omega(s,t,x)_1 + \omega(s,t,x)_2\right]^{-1}\right] \\ \ln\left[\omega(s,t,x)_2 \cdot \left[\omega(s,t,x)_1 + \omega(s,t,x)_2\right]^{-1} \cdot 2\right] \\ \ln\left[\omega(s,t,x)_3 \cdot \left[\omega(s,t,x)_3 + \omega(s,t,x)_4\right]^{-1} \cdot 750\right] \\ \ln\left[\omega(s,t,x)_4 \cdot \left[\omega(s,t,x)_3 + \omega(s,t,x)_4\right]^{-1} \cdot k(s,t,x)^{-1}\right] \end{bmatrix}$$

$$A(s,t,x) := \overrightarrow{\exp(B(s,t,x))}$$

SOLUTION: $N := 14$ $i := 0 \,..N$

$$\begin{bmatrix} f'_0 \\ s_0 \\ t_0 \\ x_0 \end{bmatrix} := \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} f'_{i+1} \\ s_{i+1} \\ t_{i+1} \\ x_{i+1} \end{bmatrix} := A\left[s_i, t_i, x_i\right]$$

$$i := 0 \,..N + 1$$

$$F_i := F\left[s_i, t_i, x_i\right]$$

$$y_i := t_i \cdot x_i \cdot s_i^{-1}$$

$z_i := s_i \cdot x_i^{-1}$ $f_i := f\left[x_i, y_i, z_i\right]$

**STEP 6: Iterations.

i	$-f'_i$	$-F_i^{-1}$	f_i
0	-1	-5.70084720388	-0.5
1	-5.70084720388	-4.35382176634	-5.70084720388
2	-4.35382176634	-4.29953077362	-4.35382176634
3	-4.29953077362	-4.29485844767	-4.29953077362
4	-4.29485844767	-4.2945194173	-4.29485844767
5	-4.2945194173	-4.29449358578	-4.2945194173
6	-4.29449358578	-4.29449164351	-4.29449358578
7	-4.29449164351	-4.29449149694	-4.29449164351
8	-4.29449149694	-4.29449148589	-4.29449149694
9	-4.29449148589	-4.29449148506	-4.29449148589
10	-4.29449148506	-4.29449148499	-4.29449148506
11	-4.29449148499	-4.29449148499	-4.29449148499
12	-4.29449148499	-4.29449148499	-4.29449148499
13	-4.29449148499	-4.29449148499	-4.29449148499
14	-4.29449148499	-4.29449148499	-4.29449148499
15	-4.29449148499	-4.29449148499	-4.29449148499

**STEP 7: Finding the optimal solution.

$f_{N+1} = -4.294491485$

$x_{N+1} = 12.7042922615$

$y_{N+1} = 3.8997157399$

$z_{N+1} = 0.4705716091$

Check the original constraint:

$$\left[\frac{4}{3} \cdot 10^{-3}\right] \cdot x_{N+1}^{2.1} + \left[\frac{4}{3}\right] \cdot y_{N+1}^{-1.1} + \left[\frac{2}{3}\right] \cdot z_{N+1}^{0.6} = 1$$

Appendix E

WOOLSEY'S PROBLEM (NO. 3)

WOOLSEY'S PROBLEM (NO. 3)

MINIMIZE:

$$T(X,Y) := 5\cdot X\cdot Y + 7\cdot X + 8\cdot Y + 4\cdot X^{-2} + 8\cdot Y^{-2}$$

A 5 - 2 - 1 = 2 dd multi-variable posynomial without constraints.

**STEP 1: Preprocessing - Not necessary for this problem.

**STEP 2: Transforming into the all constraint form.

The problem becomes:

MINIMIZE: Z ▫

Subject to:

$$\left[5\cdot Z^{-1}\cdot X\cdot Y + 7\cdot Z^{-1}\cdot X + 8\cdot Z^{-1}\cdot Y\right] + 4\cdot Z^{-1}\cdot X^{-2} + 8\cdot Z^{-1}\cdot Y^{-2} \le 1 \;▫$$

A 6 - 3 - 1 = 2 dd multi-variable posynomial with a constraint.

RULE 2A:
RULE 2B:

$$\begin{matrix} \\ Z: \\ X: \\ Y: \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 & -2 \end{bmatrix} \cdot \delta := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \;▫$$

The second, third, and fourth columns can be combined to make the matrix square, this corresponds to condensing the terms in brackets above.

**STEP 3: Reducing the d.d.

Condense the terms in brackets above:

$$\delta 1(X,Y) := \frac{5\cdot X\cdot Y}{5\cdot X\cdot Y + 7\cdot X + 8\cdot Y} \qquad \delta 2(X,Y) := \frac{7\cdot X}{5\cdot X\cdot Y + 7\cdot X + 8\cdot Y}$$

$$\delta 3(X,Y) := \frac{8\cdot Y}{5\cdot X\cdot Y + 7\cdot X + 8\cdot Y}$$

$$k(X,Y) := \left[\frac{5}{\delta 1(X,Y)}\right]^{\delta 1(X,Y)} \cdot \left[\frac{7}{\delta 2(X,Y)}\right]^{\delta 2(X,Y)} \cdot \left[\frac{8}{\delta 3(X,Y)}\right]^{\delta 3(X,Y)}$$

Thus the problem is now:

MINIMIZE: Z ▫

Subject to:

$$k(X,Y)\cdot Z^{-1}\cdot X^{\delta 1(X,Y)+\delta 2(X,Y)}\cdot Y^{\delta 1(X,Y)+\delta 3(X,Y)} + 4\cdot Z^{-1}\cdot X^{-2} + 8\cdot Z^{-1}\cdot Y^{-2} \le 1 \;▫$$

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(X,Y) := \begin{bmatrix} -1 & -1 & -1 \\ \delta 1(X,Y) + \delta 2(X,Y) & -2 & 0 \\ \delta 1(X,Y) + \delta 3(X,Y) & 0 & -2 \end{bmatrix}^{-1}$$

$$\omega(X,Y) := \left[\text{augment}\left[\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, W(X,Y)^{T}\right]\right]^{T} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

RULE 1:

$$G(X,Y) := \left[\frac{k(X,Y)}{\omega(X,Y)_1}\right]^{\omega(X,Y)_1} \cdot \left[\frac{4}{\omega(X,Y)_2}\right]^{\omega(X,Y)_2} \cdot \left[\frac{8}{\omega(X,Y)_3}\right]^{\omega(X,Y)_3}$$

$$F(X,Y) := G(X,Y) \cdot \left[\omega(X,Y)_1 + \omega(X,Y)_2 + \omega(X,Y)_3\right]^{\omega(X,Y)_1 + \omega(X,Y)_2 + \omega(X,Y)_3}$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(X,Y) := W(X,Y)^{T} \cdot \begin{bmatrix} \ln\left[\omega(X,Y)_1 \cdot \left[\omega(X,Y)_1 + \omega(X,Y)_2 + \omega(X,Y)_3\right]^{-1} \cdot k(X,Y)^{-1}\right] \\ \ln\left[\omega(X,Y)_2 \cdot \left[\omega(X,Y)_1 + \omega(X,Y)_2 + \omega(X,Y)_3\right]^{-1} \cdot 0.25\right] \\ \ln\left[\omega(X,Y)_3 \cdot \left[\omega(X,Y)_1 + \omega(X,Y)_2 + \omega(X,Y)_3\right]^{-1} \cdot 0.125\right] \end{bmatrix}$$

$$A(X,Y) := \overrightarrow{\exp(B(X,Y))}$$

SOLUTION:

$N := 14 \qquad i := 0 \,..N$

$$\begin{bmatrix} Z_0 \\ X_0 \\ Y_0 \end{bmatrix} := \begin{bmatrix} 1 & 6 \\ 10 & -6 \\ 10 & \end{bmatrix} \qquad \begin{bmatrix} Z_{i+1} \\ X_{i+1} \\ Y_{i+1} \end{bmatrix} := A\left[X_i, Y_i\right]$$

$i := 0 \,..N + 1$

$T_i := T\left[X_i, Y_i\right]$

$F_i := F\left[X_i, Y_i\right]$

**STEP 6: Iterations.

i	F_i	Z_i	T_i
0	10.9780362287	1	$8.000007 \cdot 10^{12}$
1	24.737310624	10.9780362287	23152.2186863937
2	31.2381759055	24.737310624	32.9715464123
3	31.541510589	31.2381759055	31.6582497479
4	31.5659770696	31.541510589	31.5769617803
5	31.568317174	31.5659770696	31.5693460332
6	31.5685356296	31.568317174	31.5686323082
7	31.5685561843	31.5685356296	31.5685652635
8	31.5685581139	31.5685561843	31.5685589667
9	31.5685582952	31.5685581139	31.5685583753
10	31.5685583122	31.5685582952	31.5685583197
11	31.5685583138	31.5685583122	31.5685583145
12	31.5685583139	31.5685583138	31.568558314
13	31.568558314	31.5685583139	31.568558314
14	31.568558314	31.568558314	31.568558314
15	31.568558314	31.568558314	31.568558314

**STEP 7: Finding the optimal solution.

$T_{N+1} = 31.568558314$

$X_{N+1} = 0.8627863847$

$Y_{N+1} = 1.0912085394$

Appendix F

WOOLSEY'S PROBLEM (NO. 4)

WOOLSEY'S PROBLEM (NO. 4)

MINIMIZE: $F(S) := 100000 \cdot S^{3} - 27900 \cdot S^{2} - 1200 \cdot S + 1095 \cdot S^{-1} + 2875$

A 4 - 1 - 1 = 2 dd single-variable signomial without constraints.

**STEP 1: Preprocessing.

Ignore the constant term while minimizing.

**STEP 2: Transforming into the all constraint form.

The problem becomes:

MINIMIZE: Z ▫

Subject to: $100000 \cdot S^{3} + 1095 \cdot S^{-1} \le \left[Z + 27900 \cdot S^{2} + 1200 \cdot S \right]$ ▫

Condense the terms in brackets:

$$\delta 1(Z,S) := \frac{Z}{Z + 27900 \cdot S^{2} + 1200 \cdot S} \qquad \delta 2(Z,S) := \frac{27900 \cdot S^{2}}{Z + 27900 \cdot S^{2} + 1200 \cdot S}$$

$$\delta 3(Z,S) := \frac{1200 \cdot S}{Z + 27900 \cdot S^{2} + 1200 \cdot S}$$

$$k(Z,S) := \delta 1(Z,S)^{\delta 1(Z,S)} \cdot \left[\frac{\delta 2(Z,S)}{27900} \right]^{\delta 2(Z,S)} \cdot \left[\frac{\delta 3(Z,S)}{1200} \right]^{\delta 3(Z,S)}$$

The problem becomes:

MINIMIZE: Z ▫

Subject to: $$\left[\begin{array}{l} 100000 \cdot k(Z,S) \cdot Z^{-\delta 1(Z,S)} \cdot S^{3-2\cdot\delta 2(Z,S)-\delta 3(Z,S)} \ldots \\ + 1095 \cdot k(Z,S) \cdot Z^{-\delta 1(Z,S)} \cdot S^{-1-2\cdot\delta 2(Z,S)-\delta 3(Z,S)} \end{array} \right] \le 1 \quad ▫$$

A 3 - 2 - 1 = 0 dd multi-variable posynomial with a constraint.

**STEP 3: Reducing the d.d. - The d.d. was reduced in the previous step in order to formulate the problem as a posynomial in the all constraint form.

**STEP 4: Finding a dual feasible solution.

RULE 2A:
RULE 2B:

$$\begin{matrix} \\ Z: \\ S: \end{matrix}\begin{bmatrix}1 & 0 & 0\\ 1 & -\S 1(Z,S) & -\S 1(Z,S)\\ 0 & 3-2\cdot\S 2(Z,S)-\S 3(Z,S) & -1-2\cdot\S 2(Z,S)-\S 3(Z,S)\end{bmatrix}\cdot\omega := \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Solving for ω:

$$W(Z,S) := \begin{bmatrix} -\S 1(Z,S) & -\S 1(Z,S)\\ 3-2\cdot\S 2(Z,S)-\S 3(Z,S) & -1-2\cdot\S 2(Z,S)-\S 3(Z,S)\end{bmatrix}^{-1}$$

$$\omega(Z,S) := \left[\mathrm{augment}\left[\begin{bmatrix}-1\\0\end{bmatrix}, W(Z,S)^{T}\right]\right]^{T}\cdot\begin{bmatrix}-1\\0\end{bmatrix}$$

$$\omega(1,1) = \begin{bmatrix}1\\21525.25\\7575.75\end{bmatrix}$$

Note: Although this is a dual feasible solution, the difference in size of the dual variables indicates scaling problems which may lead to computer overflow.

RULE 1:

$$G(Z,S) := \left[\frac{100000\cdot k(Z,S)}{\omega(Z,S)_1}\right]^{\omega(Z,S)_1}\cdot\left[\frac{1095\cdot k(Z,S)}{\omega(Z,S)_2}\right]^{\omega(Z,S)_2}$$

$$F'(Z,S) := G(Z,S)\cdot\left[\omega(Z,S)_1+\omega(Z,S)_2\right]^{\omega(Z,S)_1+\omega(Z,S)_2}$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(Z,S) := W(Z,S)^{T}\cdot\begin{bmatrix}\ln\left[\omega(Z,S)_1\cdot\left[\omega(Z,S)_1+\omega(Z,S)_2\right]^{-1}\cdot 10^{-5}\cdot k(Z,S)^{-1}\right]\\ \ln\left[\omega(Z,S)_2\cdot\left[\omega(Z,S)_1+\omega(Z,S)_2\right]^{-1}\cdot 1095^{-1}\cdot k(Z,S)^{-1}\right]\end{bmatrix}$$

$$A(Z,S) := \overrightarrow{\exp(B(Z,S))}$$

SOLUTION: $N := 19$ $\quad i := 0 ..N$

$$\begin{bmatrix}Z_0\\S_0\end{bmatrix} := \begin{bmatrix}1000000\\10\end{bmatrix} \qquad \begin{bmatrix}Z_{i+1}\\S_{i+1}\end{bmatrix} := A\left[Z_i, S_i\right]$$

$i := 0 ..N + 1$

$F_i := F\left[S_i\right]$

$F'_i := F'\left[Z_i, S_i\right]$

**STEP 6: Iterations.

i	$F'_i + 2875$	$Z_i + 2875$	F_i	S_i
0	9689.2258937179	1002875	97200984.5	10
1	6420.6401135645	9689.2258937179	6579.788966322	0.364709782062
2	6338.0072056514	6420.6401135645	6350.3045443262	0.301240338743
3	6330.8964531941	6338.0072056514	6332.2307313158	0.31097305781
4	6330.4242583714	6330.8964531941	6330.5135677505	0.314143670449
5	6330.3970792585	6330.4242583714	6330.4022255357	0.314964062743
6	6330.3955763664	6330.3970792585	6330.3958609836	0.315161042555
7	6330.3954940792	6330.3955763664	6330.3955096632	0.315207369743
8	6330.3954895843	6330.3954940792	6330.3954904355	0.315218210336
9	6330.3954893389	6330.3954895843	6330.3954893853	0.315220744012
10	6330.3954893255	6330.3954893389	6330.395489328	0.31522133602
11	6330.3954893247	6330.3954893255	6330.3954893249	0.315221474337
12	6330.3954893247	6330.3954893247	6330.3954893247	0.315221506653
13	6330.3954893247	6330.3954893247	6330.3954893247	0.315221514203
14	6330.3954893247	6330.3954893247	6330.3954893247	0.315221515967
15	6330.3954893247	6330.3954893247	6330.3954893247	0.315221516379
16	6330.3954893247	6330.3954893247	6330.3954893247	0.315221516476
17	6330.3954893247	6330.3954893247	6330.3954893247	0.315221516498
18	6330.3954893247	6330.3954893247	6330.3954893247	0.315221516503
19	6330.3954893247	6330.3954893247	6330.3954893247	0.315221516504
20	6330.3954893247	6330.3954893247	6330.3954893247	0.315221516505

**STEP 7: Finding the optimal solution.

$$F_{N+1} = 6330.3954893247$$

$$S_{N+1} = 0.3152215165$$

Check the first order condition:

$$300000 \cdot S_{N+1}^{2} - 55800 \cdot S_{N+1} - 1200 - 1095 \cdot S_{N+1}^{-2} = 0$$

Check the second order condition:

$$600000 \cdot S_{N+1} - 55800 + 2190 \cdot S_{N+1}^{-3} = 203252.1263416098$$

Appendix G

THE EXTENDED ROSENBROCK FUNCTION

THE EXTENDED ROSENBROCK FUNCTION

MINIMIZE:

$$f(X,Y) := \left[10\cdot\left[Y - X^2\right]\right]^2 + (1 - X)^2$$

**STEP 1: Preprocessing.

Multiply to remove the squared brackets. The problem becomes:

MINIMIZE:

$$f(X,Y) := 100\cdot Y^2 - 200\cdot X^2\cdot Y + 100\cdot X^4 + 1 - 2\cdot X + X^2 \quad \square$$

A $5 - 2 - 1 = 2$ dd multi-variable signomial with no constraints.

If we ignore the constant, it is clear that the objective function value will be negative. Multiply with -1.

MAXIMIZE:

$$f'(X,Y) := -100\cdot Y^2 + 200\cdot X^2\cdot Y - 100\cdot X^4 + 2\cdot X - X^2 \quad \square$$

**STEP 2: Transforming into the all constraint form.

The problem is now:

MINIMIZE: $g(Z) := Z^{-1}$

SUBJECT TO:

$$Z + 100\cdot Y^2 + \left[100\cdot X^4 + X^2\right] \le \left[200\cdot X^2\cdot Y + 2\cdot X\right] \quad \square$$

**STEP 3: Reducing the d.d.

Note that to simplify the algebra, the condensation of the terms on the right hand side of the inequality and the condensation needed to reduce the d.d. to zero will be done simultaneously.

Condense the terms in brackets:

$$\delta 1(X) := \frac{100\cdot X^2}{100\cdot X^2 + 1} \qquad \delta 2(X) := \frac{1}{100\cdot X^2 + 1}$$

$$k1(X) := \left[\frac{100}{\delta 1(X)}\right]^{\delta 1(X)} \cdot \left[\frac{1}{\delta 2(X)}\right]^{\delta 2(X)}$$

$$\delta 3(X,Y) := \frac{100\cdot X\cdot Y}{100\cdot X\cdot Y + 1} \qquad \delta 4(X,Y) := \frac{1}{100\cdot X\cdot Y + 1}$$

$$k2(X,Y) := \left[\frac{\delta 3(X,Y)}{100}\right]^{\delta 3(X,Y)} \cdot \delta 4(X,Y)^{\delta 4(X,Y)}$$

Thus the problem becomes:

MINIMIZE:

$$g(Z) := Z^{-1}$$

SUBJECT TO:

$$\begin{bmatrix} 0.5\cdot k2(X,Y)\cdot X^{-1-\delta 3(X,Y)}\cdot Y^{-\delta 3(X,Y)}\cdot Z \ldots \\ + 50\cdot k2(X,Y)\cdot X^{-1-\delta 3(X,Y)}\cdot Y^{2-\delta 3(X,Y)} \ldots \\ + 0.5\cdot k1(X)\cdot k2(X,Y)\cdot X^{1+2\cdot\delta 1(X)-\delta 3(X,Y)}\cdot Y^{-\delta 3(X,Y)} \end{bmatrix} \le 1 \;\square$$

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(X,Y) := \begin{bmatrix} 1 & 0 & 0 \\ -1-\delta 3(X,Y) & -1-\delta 3(X,Y) & 1+2\cdot\delta 1(X)-\delta 3(X,Y) \\ -\delta 3(X,Y) & 2-\delta 3(X,Y) & -\delta 3(X,Y) \end{bmatrix}^{-1}$$

$$\omega(X,Y) := \left[\mathrm{augment}\left[\begin{bmatrix}1\\0\\0\end{bmatrix}, W(X,Y)^{T}\right]\right]^{T}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\omega(1,1) = \begin{bmatrix}1\\1\\100\\101\end{bmatrix}$$

RULE 1:

$$G(X,Y) := 0.5\cdot k2(X,Y)\cdot\left[\left[\frac{50\cdot k2(X,Y)}{\omega(X,Y)_2}\right]\cdot\left[1+\omega(X,Y)_2+\omega(X,Y)_3\right]\right]^{\omega(X,Y)_2}$$

$$H(X,Y) := \left[\left[\frac{0.5\cdot k1(X)\cdot k2(X,Y)}{\omega(X,Y)_3}\right]\cdot\left[1+\omega(X,Y)_2+\omega(X,Y)_3\right]\right]^{\omega(X,Y)_3}$$

$$F(X,Y) := G(X,Y)\cdot H(X,Y)\cdot\left[1+\omega(X,Y)_2+\omega(X,Y)_3\right]$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(X,Y) := W(X,Y)^{T}\cdot\begin{bmatrix} \ln\left[\begin{bmatrix}1+\omega(X,Y)_2 \ldots \\ +\,\omega(X,Y)_3\end{bmatrix}^{-1}\cdot 2\cdot k2(X,Y)^{-1}\right] \\ \ln\left[\omega(X,Y)_2\cdot\begin{bmatrix}1+\omega(X,Y)_2 \ldots \\ +\,\omega(X,Y)_3\end{bmatrix}^{-1}\cdot 0.02\cdot k2(X,Y)^{-1}\right] \\ \ln\left[\omega(X,Y)_3\cdot\begin{bmatrix}1+\omega(X,Y)_2 \ldots \\ +\,\omega(X,Y)_3\end{bmatrix}^{-1}\cdot\frac{2}{k1(X)\cdot k2(X,Y)}\right]\end{bmatrix}$$

$$A(X,Y) := \overrightarrow{\exp(B(X,Y))}$$

$$N := 149 \qquad i := 0\,..N$$

$$\begin{bmatrix} Z_0 \\ X_0 \\ Y_0 \end{bmatrix} := \begin{bmatrix} 0 \\ 0.0008 \\ 1000 \end{bmatrix} \qquad \begin{bmatrix} Z_{i+1} \\ X_{i+1} \\ Y_{i+1} \end{bmatrix} := A\left[X_i, Y_i\right] \qquad i := 0,5\,..N+1$$

$$f_i := f\left[X_i, Y_i\right]$$

$$F_i := 1 - F\left[X_i, Y_i\right]^{-1}$$

**STEP 6: Iterations.

i	$1 - Z_i$	f_i	F_i
0	1	100000000.8704006	1.002007187981
5	0.611675546275	0.593855815566	0.574193461636
10	0.434507923381	0.419097451997	0.401654251926
15	0.279377953051	0.266182265071	0.251613138192
20	0.155230644415	0.145517340134	0.135241687156
25	0.072541496547	0.066782932341	0.060984592306
30	0.028479339451	0.025775618431	0.023177401198
35	0.009638971531	0.008606953021	0.007651105027
40	0.00292907209	0.002591603864	0.002286745839
45	0.000831020115	0.00073120869	0.00064239001
50	0.000226556368	0.000198725577	0.000174168103
55	0.000060429766	0.000052917826	0.000046319361
60	0.000015932912	0.000013940086	0.000012193742
65	0.00000417562	0.000003651692	0.00000319313
70	0.000001090929	0.000000953825	0.000000833901
75	0.000000284564	0.000000248771	0.000000217474
80	0.000000074167	0.000000064834	0.000000056675
85	0.000000019322	0.00000001689	0.000000014764
90	0.000000005033	0.000000004399	0.000000003846
95	0.000000001311	0.000000001146	0.000000001002
100	0.000000000341	0.000000000298	0.000000000261
105	0.000000000089	0.000000000078	0.000000000068
110	0.000000000023	0.00000000002	0.000000000018
115 120	$5.999201135864 \cdot 10^{-12}$	$5.269503407042 \cdot 10^{-12}$	$4.622968674539 \cdot 10^{-12}$
125 130	$1.515898517823 \cdot 10^{-12}$	$1.372240271075 \cdot 10^{-12}$	$1.189270903978 \cdot 10^{-12}$
135	0	0	0
140	0	0	0
145	0	0	0
150	0	0	0
	0	0	0
	0	0	0

**STEP 7: Finding the optimal solution.

$f_{N+1} = 0$

$X_{N+1} = 0.9999999793$

$Y_{N+1} = 0.9999999586$

Appendix H

FERTILIZER PLANT DESIGN PROBLEM

FERTILIZER PLANT DESIGN PROBLEM

MAXIMIZE:

$$T(m,p) := \left[360 \cdot m^{0.8} - 0.00000627 \cdot m^{0.8} \cdot p^{2.3} - 9920 \cdot m^{0.9} \cdot p^{-0.9} \ldots \right. \\ \left. + (-0.0155 \cdot m \cdot p) - 0.650 \cdot m - 1370000 \right]$$

A 5 - 2 - 1 = 2 dd multi-variable signomial without constraints.

**STEP 1: Preprocessing.

The constant term can be ignored during optimization.

**STEP 2: Transforming into the all constraint form.

The problem becomes:

MINIMIZE: z^{-1} ▫

Subject to:

$$k1 \cdot z \cdot m^{-0.8} + k2 \cdot m^{0.1} \cdot p^{-0.9} + \left[k3 \cdot p^{2.3} + k4 \cdot m^{0.2} \cdot p + k5 \cdot m^{0.2} \right] \le 1 \quad ▫$$

Where

$$k1 := \frac{1}{360} \qquad k2 := \frac{9920}{360} \qquad k3 := \frac{0.00000627}{360}$$

$$k4 := \frac{0.0155}{360} \qquad k5 := \frac{0.650}{360}$$

A 6 - 3 - 1 = 2 dd multi-variable posynomial with a constraint.

RULE 2A:
RULE 2B:

$$\begin{matrix} \\ z: \\ m: \\ p: \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.8 & 0.1 & 0 & 0.2 & 0.2 \\ 0 & 0 & -0.9 & 2.3 & 1 & 0 \end{bmatrix} \cdot \delta := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad ▫$$

**STEP 3: Reducing the d.d.

Condense the terms in brackets above:

$$\delta 1(m,p) := \frac{k3 \cdot p^{2.3}}{k3 \cdot p^{2.3} + k4 \cdot m^{0.2} \cdot p + k5 \cdot m^{0.2}}$$

$$\delta 2(m,p) := \frac{k4 \cdot m^{0.2} \cdot p}{k3 \cdot p^{2.3} + k4 \cdot m^{0.2} \cdot p + k5 \cdot m^{0.2}}$$

$$\delta 3(m,p) := \frac{k5 \cdot m^{0.2}}{k3 \cdot p^{2.3} + k4 \cdot m^{0.2} \cdot p + k5 \cdot m^{0.2}}$$

$$k(m,p) := \left[\frac{k3}{\delta 1(m,p)}\right]^{\delta 1(m,p)} \cdot \left[\frac{k4}{\delta 2(m,p)}\right]^{\delta 2(m,p)} \cdot \left[\frac{k5}{\delta 3(m,p)}\right]^{\delta 3(m,p)}$$

Thus the problem is now:

MINIMIZE: z^{-1}

Subject to:

$$\left[\begin{array}{l} k1 \cdot z \cdot m^{-0.8} + k2 \cdot m^{0.1} \cdot p^{-0.9} \ldots \\ + k(m,p) \cdot p^{2.3 \cdot \delta 1(m,p) + \delta 2(m,p)} \cdot m^{0.2 \cdot \delta 2(m,p) + 0.2 \cdot \delta 3(m,p)} \end{array}\right] \le 1$$

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(m,p) := \begin{bmatrix} 1 & 0 & 0 \\ -0.8 & 0.1 & 0.2 \cdot \delta 2(m,p) + 0.2 \cdot \delta 3(m,p) \\ 0 & -0.9 & 2.3 \cdot \delta 1(m,p) + \delta 2(m,p) \end{bmatrix}^{-1}$$

$$\omega(m,p) := \left[\text{augment}\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, W(m,p)^{T}\right]\right]^{T} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega(1,1) = \begin{bmatrix} 1 \\ 1 \\ 0.1022860499 \\ 3.9488941792 \end{bmatrix}$$

RULE 1:

$$G(m,p) := \left[\frac{k1}{\omega(m,p)_1}\right]^{\omega(m,p)_1} \cdot \left[\frac{k2}{\omega(m,p)_2}\right]^{\omega(m,p)_2} \cdot \left[\frac{k(m,p)}{\omega(m,p)_3}\right]^{\omega(m,p)_3}$$

$$F(m,p) := G(m,p) \cdot \left[\omega(m,p)_1 + \omega(m,p)_2 + \omega(m,p)_3\right]^{\omega(m,p)_1 + \omega(m,p)_2 + \omega(m,p)_3}$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(m,p) := W(m,p)^{T} \cdot \begin{bmatrix} \ln\left[\omega(m,p)_1 \cdot \left[\omega(m,p)_1 + \omega(m,p)_2 + \omega(m,p)_3\right]^{-1} \cdot k1^{-1}\right] \\ \ln\left[\omega(m,p)_2 \cdot \left[\omega(m,p)_1 + \omega(m,p)_2 + \omega(m,p)_3\right]^{-1} \cdot k2^{-1}\right] \\ \ln\left[\omega(m,p)_3 \cdot \left[\omega(m,p)_1 + \omega(m,p)_2 + \omega(m,p)_3\right]^{-1} \cdot k(m,p)^{-1}\right] \end{bmatrix}$$

$$A(m,p) := \overrightarrow{\exp(B(m,p))}$$

SOLUTION:

$$\begin{bmatrix} z_0 \\ m_0 \\ p_0 \end{bmatrix} := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} z_{i+1} \\ m_{i+1} \\ p_{i+1} \end{bmatrix} := A\left[m_i, p_i\right]$$

$N := 15 \quad i := 0 ..N$

$i := 0 ..N + 1$

$T'_i := T\left[m_i, p_i\right]$

$F_i := F\left[m_i, p_i\right]^{-1} - 1370000$

**STEP 6: Iterations.

i	F_i	$z_i - 1370000$	T'_i
		-1369999	-1379560.665506
0	$7.999862 \cdot 10^{11}$		
1	728287992.030757	$7.999862 \cdot 10^{11}$	$-1.5927 \cdot 10^{16}$
2	3463138.866791	728287992.030757	-14098262520.80824
3	3252570.782412	3463138.866791	3245671.013657
4	3252255.376081	3252570.782412	3252219.215499
5	3252251.952596	3252255.376081	3252251.546304
6	3252251.914069	3252251.952596	3252251.909498
7	3252251.913635	3252251.914069	3252251.913584
8	3252251.91363	3252251.913635	3252251.91363
9	3252251.91363	3252251.91363	3252251.91363
10	3252251.91363	3252251.91363	3252251.91363
11	3252251.91363	3252251.91363	3252251.91363
12	3252251.91363	3252251.91363	3252251.91363
13	3252251.91363	3252251.91363	3252251.91363
14	3252251.91363	3252251.91363	3252251.91363
15	3252251.91363	3252251.91363	3252251.91363
16	3252251.91363	3252251.91363	3252251.91363

i	p_i	m_i
0	1	1
1	75481.6653138275	$3.653566 \cdot 10^{12}$
2	706.7214230246	1423087054.540784
3	479.7561867682	1502877.629306
4	470.2011889963	1462038.579777
5	471.0111499317	1459822.711904
6	470.9252540848	1460059.873157
7	470.9343582942	1460034.536377
8	470.9333926368	1460037.222162
9	470.9334950553	1460036.937285
10	470.9334841926	1460036.9675
11	470.9334853448	1460036.964295
12	470.9334852226	1460036.964635
13	470.9334852355	1460036.964599
14	470.9334852341	1460036.964603
15	470.9334852343	1460036.964602
16	470.9334852343	1460036.964602

**STEP 7: Finding the optimal solution.

Wilde's solution:

$m_w := 1476000$

$p_w := 459.654$

$T[m_w, p_w] = 3244646$

Optimal solution:

$m_{N+1} = 1460037$

$p_{N+1} = 470.933$

$T[m_{N+1}, p_{N+1}] = 3252252$

Wilde's solution was obtained by simplifying the problem using engineering considerations, bounding the problem, scaling the variables, condensing some terms, and bounding again. Although the final bounds are within 1% of the optimal solution, the procedure depends on engineering considerations, and cannot be generalized. The Wessels algorithm found the optimal solution to 13 significant figures in 9 iterations.

Appendix I

FLEET DESIGN PROBLEM

FLEET DESIGN PROBLEM

MINIMIZE:

$$C(N,D,V,U) := 200\cdot N\cdot D + 0.5\cdot N\cdot D^{\frac{2}{3}}\cdot V^{3} + 0.8\cdot N\cdot D^{\frac{2}{3}}\cdot V^{3}\cdot U$$

SUBJECT TO:

$$\frac{11}{4}\cdot 10^{6}\cdot N^{-1}\cdot D^{-1}\cdot V^{-1}\cdot U^{-1} + \frac{1}{500}\cdot V^{2}\cdot D^{-\left[\frac{1}{3}\right]} \le 1$$

$$\frac{2}{3}\cdot 10^{-6}\cdot D\cdot V\cdot U + U \le 1$$

**STEP 1: Preprocessing.

Advanced sign table:

$$\begin{matrix} N: \\ D: \\ V: \\ U: \end{matrix} \begin{bmatrix} 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & \frac{2}{3} & \frac{2}{3} & -1 & -\left[\frac{1}{3}\right] & 1 & 0 \\ 0 & 3 & 3 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 1 \end{bmatrix}$$

Density: $\frac{19}{28} = 68\cdot\%$

Do the following substitutions:

$N := X^{-2}\cdot Y^{3}\cdot V^{-6}$ $D := X^{3}\cdot Y^{-3}\cdot V^{6}$

or

$X := N\cdot D$ $Y := N\cdot V^{2}\cdot D^{\frac{2}{3}}$

Note: These substitutions correspond to a series of row operations, the detail of which is not shown here.

The advanced sign table becomes:

$$\begin{matrix} X: \\ Y: \\ V: \\ U: \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 3 & 0 \\ 0 & 1 & 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & 1 & -1 & 0 & 7 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 1 \end{bmatrix}$$

Density: $\frac{16}{28} = 57\cdot\%$

This corresponds to the following problem:

MINIMIZE: $G(X,Y,V,U) := 200\cdot X + 0.5\cdot Y\cdot V + 0.8\cdot Y\cdot V\cdot U$

SUBJECT TO: $\frac{11}{4}\cdot 10^{6}\cdot X^{-1}\cdot V^{-1}\cdot U^{-1} + \frac{1}{500}\cdot X^{-1}\cdot Y \le 1$

AND TO: $\frac{2}{3}\cdot 10^{-6}\cdot X^{3}\cdot Y^{-3}\cdot V^{7}\cdot U + U \le 1$ ▫

A 7 - 4 - 1 = 2 dd multi-variable posynomial with constraints.

According to the sign table both constraints must be tight.

**STEP 2: Transforming into the all constraint form.

MINIMIZE: E ▫

SUBJECT TO: $$200\cdot E^{-1}\cdot X + \left[0.5\cdot E^{-1}\cdot Y\cdot V + 0.8\cdot E^{-1}\cdot Y\cdot V\cdot U\right] \le 1$$ ▫

$$\left[\frac{11}{4}\cdot 10^{6}\cdot X^{-1}\cdot V^{-1}\cdot U^{-1} + \frac{1}{500}\cdot X^{-1}\cdot Y\right] \le 1$$ ▫

$$\frac{2}{3}\cdot 10^{-6}\cdot X^{3}\cdot Y^{-3}\cdot V^{7}\cdot U + U \le 1$$ ▫

**STEP 3: Reducing the d.d.

Condense the terms in brackets:

$$\delta 1(U) := \frac{0.5}{0.5 + 0.8\cdot U} \qquad \delta 2(U) := \frac{0.8\cdot U}{0.5 + 0.8\cdot U}$$

$$k1(U) := \left[\frac{0.5}{\delta 1(U)}\right]^{\delta 1(U)} \cdot \left[\frac{0.8}{\delta 2(U)}\right]^{\delta 2(U)}$$

$$\delta 3(Y,V,U) := \frac{2.75\cdot 10^{6}\cdot V^{-1}\cdot U^{-1}}{2.75\cdot 10^{6}\cdot V^{-1}\cdot U^{-1} + 0.002\cdot Y}$$

$$\delta 4(Y,V,U) := \frac{0.002\cdot Y}{2.75\cdot 10^{6}\cdot V^{-1}\cdot U^{-1} + 0.002\cdot Y}$$

$$k2(Y,V,U) := \left[\frac{2.75\cdot 10^{6}}{\delta 3(Y,V,U)}\right]^{\delta 3(Y,V,U)} \cdot \left[\frac{0.002}{\delta 4(Y,V,U)}\right]^{\delta 4(Y,V,U)}$$

Thus the problem is now:

MINIMIZE: $F(E) := E$ □

SUBJECT TO:

$$200 \cdot E^{-1} \cdot X + k1(U) \cdot E^{-1} \cdot Y \cdot V \cdot U^{\S 2(U)} \le 1 \ \square$$

$$k2(Y,V,U) \cdot X^{-1} \cdot V^{-\S 3(Y,V,U)} \cdot U^{-\S 3(Y,V,U)} \cdot Y^{\S 4(Y,V,U)} \le 1 \ \square$$

$$\frac{2}{3} \cdot 10^{-6} \cdot X^{3} \cdot Y^{-3} \cdot V^{7} \cdot U + U \le 1 \ \square$$

A 6 - 5 - 1 = 0 dd multi-variable posynomial with constraints.

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(Y,V,U) := \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 3 & 0 \\ 0 & 1 & \S 4(Y,U,V) & -3 & 0 \\ 0 & 1 & -\S 3(Y,V,U) & 7 & 0 \\ 0 & \S 2(U) & -\S 3(Y,V,U) & 1 & 1 \end{bmatrix}^{-1} \quad \begin{bmatrix} E \\ X \\ Y \\ V \\ U \end{bmatrix} \ \square$$

$$\omega(Y,V,U) := \left[\text{augment}\left[\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, W(Y,V,U)^{T} \right] \right]^{T} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \omega(1,1,1) = \begin{bmatrix} 1 \\ 0.7 \\ 0.3 \\ 1 \\ 0.1 \\ 0.71538 \end{bmatrix}$$

RULE 1:

$$P(Y,V,U) := \left[\frac{200}{\omega(Y,V,U)_1} \right]^{\omega(Y,V,U)_1} \cdot \left[\frac{k1(U)}{\omega(Y,V,U)_2} \right]^{\omega(Y,V,U)_2}$$

$$Q(Y,V,U) := \left[\omega(Y,V,U)_1 + \omega(Y,V,U)_2 \right]^{\omega(Y,V,U)_1 + \omega(Y,V,U)_2}$$

$$R(Y,V,U) := k2(Y,V,U)^{\omega(Y,V,U)_3} \cdot \left[\frac{\frac{2}{3} \cdot 10^{-6}}{\omega(Y,V,U)_4} \right]^{\omega(Y,V,U)_4} \cdot \left[\frac{1}{\omega(Y,V,U)_5} \right]^{\omega(Y,V,U)_5}$$

$$S(Y,V,U) := \left[\omega(Y,V,U)_4 + \omega(Y,V,U)_5\right]^{\omega(Y,V,U)_4 + \omega(Y,V,U)_5}$$

$$F'(Y,V,U) := P(Y,V,U)\cdot Q(Y,V,U)\cdot R(Y,V,U)\cdot S(Y,V,U)$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(Y,V,U) := W(Y,V,U)^{T}\cdot\begin{bmatrix} \ln\left[\omega(Y,V,U)_1\cdot\left[\omega(Y,V,U)_1 + \omega(Y,V,U)_2\right]^{-1}\cdot 0.005\right] \\ \ln\left[\omega(Y,V,U)_2\cdot\left[\omega(Y,V,U)_1 + \omega(Y,V,U)_2\right]^{-1}\cdot k1(U)^{-1}\right] \\ \ln\left[k2(Y,V,U)^{-1}\right] \\ \ln\left[\omega(Y,V,U)_4\cdot\left[\omega(Y,V,U)_4 + \omega(Y,V,U)_5\right]^{-1}\cdot 1.5\cdot 10^{6}\right] \\ \ln\left[\omega(Y,V,U)_5\cdot\left[\omega(Y,V,U)_4 + \omega(Y,V,U)_5\right]^{-1}\right] \end{bmatrix}$$

$$A(Y,V,U) := \overrightarrow{\exp(B(Y,V,U))}$$

SOLUTION:

$$M := 14 \qquad i := 0..M$$

$$i := 0..M+1$$

$$\begin{bmatrix} E_0 \\ X_0 \\ Y_0 \\ V_0 \\ U_0 \end{bmatrix} := \begin{bmatrix} 1 \\ 100 \\ 10000 \\ 10 \\ 10 \end{bmatrix} \qquad \begin{bmatrix} E_{i+1} \\ X_{i+1} \\ Y_{i+1} \\ V_{i+1} \\ U_{i+1} \end{bmatrix} := A\left[Y_i, V_i, U_i\right]$$

$$N := \overrightarrow{\left[X^{-2}\cdot Y^{3}\cdot V^{-6}\right]}$$

$$D := \overrightarrow{\left[X^{3}\cdot Y^{-3}\cdot V^{6}\right]}$$

$$F'_i := F'\left[Y_i, V_i, U_i\right] \qquad C_i := C\left[N_i, D_i, V_i, U_i\right]$$

**STEP 6: Iterations.

i	E_i	F'_i	C_i
0	1	66270559.41646004	870000
1	66270559.41646011	73926780.25441992	74336255.84152198
2	73926780.2544201	73959232.0043242	73927826.87396912
3	73959232.00432442	73959260.54865411	73959232.00447325
4	73959260.54865398	73959260.57071066	73959260.54876512
5	73959260.57071026	73959260.57072781	73959260.5707107
6	73959260.57072785	73959260.57072777	73959260.57072796
7	73959260.57072785	73959260.57072757	73959260.57072784
8	73959260.57072759	73959260.57072824	73959260.57072759
9	73959260.57072838	73959260.57072806	73959260.57072817
10	73959260.57072812	73959260.57072784	73959260.5707279
11	73959260.57072812	73959260.57072781	73959260.57072768
12	73959260.57072812	73959260.57072781	73959260.57072768
13	73959260.57072812	73959260.57072781	73959260.57072768
14	73959260.57072812	73959260.57072781	73959260.57072768
15	73959260.57072812	73959260.57072781	73959260.57072768

i	N_i	D_i	V_i	U_i
0	100	1	10	10
1	13.1944431434	17591.9244285157	13.8059911556	0.8606474366
2	15.1998472306	17273.0245041111	12.0593048386	0.8780655701
3	15.2070377355	17215.8285180986	12.100116707	0.8780589569
4	15.2062911203	17218.3118382529	12.0990172251	0.8780532428
5	15.2063207652	17218.2329689734	12.0990468004	0.8780534715
6	15.2063198342	17218.2352900286	12.0990459846	0.8780534643
7	15.2063198615	17218.2352236392	12.0990460073	0.8780534645
8	15.2063198607	17218.235225517	12.0990460067	0.8780534645
9	15.2063198607	17218.235225464	12.0990460067	0.8780534645
10	15.2063198607	17218.2352254658	12.0990460067	0.8780534645
11	15.2063198607	17218.2352254655	12.0990460067	0.8780534645
12	15.2063198607	17218.2352254655	12.0990460067	0.8780534645
13	15.2063198607	17218.2352254655	12.0990460067	0.8780534645
14	15.2063198607	17218.2352254655	12.0990460067	0.8780534645
15	15.2063198607	17218.2352254655	12.0990460067	0.8780534645

**STEP 7: Finding the optimal solution.

C_{M+1} = 73959260.57072768

N_{M+1} = 15.2063198607

D_{M+1} = 17218.2352254655

V_{M+1} = 12.0990460067

U_{M+1} = 0.8780534645

Check the two original constraints:

$$\frac{11}{4}\cdot 10^{6}\cdot N_{M+1}^{-1}\cdot D_{M+1}^{-1}\cdot V_{M+1}^{-1}\cdot U_{M+1}^{-1} + \frac{1}{500}\cdot V_{M+1}^{2}\cdot D_{M+1}^{-\left[\frac{1}{3}\right]} = 1$$

$$\frac{2}{3}\cdot 10^{-6}\cdot D_{M+1}\cdot V_{M+1}\cdot U_{M+1} + U_{M+1} = 1$$

Appendix J

WALSH'S PROBLEM (NO. 2)

WALSH'S PROBLEM (NO. 2)

MINIMIZE: $f(x,y) := x^2 + 3\cdot x\cdot y + 5\cdot y^2 - 5\cdot x$

SUBJECT TO: $x^2 - 2\cdot y \le 0$ ▫ or to: $0.5\cdot x^2\cdot y^{-1} \le 1$ ▫

$y^2 - 2\cdot x \le 0$ ▫ $0.5\cdot x^{-1}\cdot y^2 \le 1$ ▫

A 6 - 2 - 1 = 3 d.d. multivariable signomial with constraints.

**STEP 1: Preprocessing.

Advanced sign table:

$$\begin{matrix} x: \\ y: \end{matrix} \begin{bmatrix} 2 & 1 & 0 & -(1) & 2 & -1 \\ 0 & 1 & 2 & 0 & -1 & 2 \end{bmatrix}$$ ▫

From the sign table it is clear that the first constraint must be active. If both active, x = y = 2, f = 26, or x = y = 0, f = 0. Thus it is also clear that the objective function value must be negative at optimality if it is possible to do better than x = y = 0, f = 0.

To follow the procedure suggested in section 2.4. of the dissertation (to find the Kuhn-Tucker points), the following possibilities must be considered:

1. Objective function only - unbalanced.
2. Objective function and the first constraint.
3. Objective function and the second constraint - unbalanced.
4. Both constraints binding - two simultaneous equations.

Consider possibility no. 2.

**STEP 2: Transforming into the all constraint form.

MINIMIZE: f'^{-1} ▫

SUBJECT TO: $0.2\cdot f'\cdot x^{-1} + \left[0.2\cdot x + 0.6\cdot y + x^{-1}\cdot y^2\right] \le 1$ ▫

$0.5\cdot x^2\cdot y^{-1} \le 1$ ▫

A 6 - 3 - 1 = 2 d.d. multivariable posynomial with constraints.

The advanced sign table:

$$\begin{matrix} f': \\ x: \\ y: \end{matrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$$ ▫

**STEP 3: Reducing the d.d.

Condense the terms in brackets above:

$$\delta_1(x,y) := \frac{0.2\cdot x}{0.2\cdot x + 0.6\cdot y + x^{-1}\cdot y^{2}}$$

$$\delta_2(x,y) := \frac{0.6\cdot y}{0.2\cdot x + 0.6\cdot y + x^{-1}\cdot y^{2}}$$

$$\delta_3(x,y) := \frac{x^{-1}\cdot y^{2}}{0.2\cdot x + 0.6\cdot y + x^{-1}\cdot y^{2}}$$

$$k(x,y) := \left[\frac{0.2}{\delta_1(x,y)}\right]^{\delta_1(x,y)}\cdot\left[\frac{0.6}{\delta_2(x,y)}\right]^{\delta_2(x,y)}\cdot\left[\frac{1}{\delta_3(x,y)}\right]^{\delta_3(x,y)}$$

The problem becomes:

MINIMIZE: f'^{-1}

SUBJECT TO:

$$0.2\cdot f'\cdot x^{-1} + k(x,y)\cdot x^{\delta_1(x,y)-\delta_3(x,y)}\cdot y^{\delta_2(x,y)+2\cdot\delta_3(x,y)} \le 1$$

$$0.5\cdot x^{2}\cdot y^{-1} \le 1$$

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(x,y) := \begin{bmatrix} 1 & 0 & 0 \\ -1 & \delta_1(x,y) - \delta_3(x,y) & 2 \\ 0 & \delta_2(x,y) + 2\cdot\delta_3(x,y) & -1 \end{bmatrix}^{-1}$$

$$\omega(x,y) := \left[\text{augment}\left[\begin{bmatrix}1\\0\\0\end{bmatrix}, W(x,y)^{T}\right]\right]^{T}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \omega(1,1) = \begin{bmatrix}1\\1\\0.4090909091\\0.5909090909\end{bmatrix}$$

RULE 1:

$$f''(x,y) := 0.2\cdot\left[\frac{k(x,y)}{\omega(x,y)_2}\right]^{\omega(x,y)_2}\cdot\left[1 + \omega(x,y)_2\right]^{1+\omega(x,y)_2}\cdot 0.5^{\omega(x,y)_3}$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(x,y) := W(x,y)^T \cdot \begin{bmatrix} \ln\left[\omega(x,y)_1 \cdot \left[\omega(x,y)_1 + \omega(x,y)_2\right]^{-1} \cdot 5\right] \\ \ln\left[\omega(x,y)_2 \cdot \left[\omega(x,y)_1 + \omega(x,y)_2\right]^{-1} \cdot k(x,y)^{-1}\right] \\ \ln(2) \end{bmatrix}$$

$$A(x,y) := \overrightarrow{\exp(B(x,y))}$$

SOLUTION: $N := 9$ $i := 0 ..N$

$$\begin{bmatrix} f'_0 \\ x_0 \\ y_0 \end{bmatrix} := \begin{bmatrix} 1 \\ 0.1 \\ 100 \end{bmatrix} \qquad \begin{bmatrix} f'_{i+1} \\ x_{i+1} \\ y_{i+1} \end{bmatrix} := A\left[x_i, y_i\right]$$

$i := 0 ..N + 1$

$$f''_i := f''\left[x_i, y_i\right]^{-1}$$

**STEP 6: Iterations.

i	$-f''_i$	$-f'_i$	$f\left[x_i, y_i\right]$
0	-3.7435531956004	-1	50029.51
1	-2.2514388157744	-3.7435531956004	-1.2608137415806
2	-2.1987911662159	-2.2514388157744	-2.1978095349276
3	-2.1986916999733	-2.1987911662159	-2.1986785131427
4	-2.1986903973147	-2.1986916999733	-2.1986902349325
5	-2.1986903812269	-2.1986903973147	-2.1986903792072
6	-2.1986903810269	-2.1986903812269	-2.1986903810018
7	-2.1986903810244	-2.1986903810269	-2.1986903810241
8	-2.1986903810243	-2.1986903810244	-2.1986903810243
9	-2.1986903810243	-2.1986903810243	-2.1986903810243
10	-2.1986903810243	-2.1986903810243	-2.1986903810243

i	x_i	y_i
0	0.1	100
1	0.9983307795	0.4983321727
2	0.6682738905	0.2232949964
3	0.6803946191	0.2314684189
4	0.6789990585	0.2305198607
5	0.6791540388	0.2306251042
6	0.6791367559	0.2306133666
7	0.6791386824	0.2306146749
8	0.6791384676	0.2306145291
9	0.6791384916	0.2306145454
10	0.6791384889	0.2306145435

**STEP 7: Finding the optimal solution.

Check the original constraints:

$x_{N+1}^2 - 2 \cdot y_{N+1} = 0$ As expected

$y_{N+1}^2 - 2 \cdot x_{N+1} = -1.3050939101$ Satisfied

Walsh's solution:

$x' := 0.7590$

$y' := .2880$

$f(x',y') = -2.148$

$x'^2 - 2 \cdot y' = 0.000081$

Violated

$y'^2 - 2 \cdot x' = -1.435056$

Satisfied

This solution to 4 significant figures:

$x_{N+1} = 0.6791$

$y_{N+1} = 0.2306$

$f\left[x_{N+1}, y_{N+1}\right] = -2.199$

$0.6791^2 - 2 \cdot 0.2306 = -0.000023$

Satisfied

$0.2306^2 - 2 \cdot 0.6791 = -1.305024$

Satisfied

Appendix K

UNDERWOOD'S PROBLEM

UNDERWOOD'S PROBLEM

MINIMIZE: $f(x,y) := (x-2)^4 + (x-2\cdot y)^2$

SUBJECT TO: $x^2 - y \le 0$ □ or to: $x^2 \cdot y^{-1} \le 1$ □

**STEP 1: Preprocessing.

Expand the expressions in brackets.

The objective function becomes:

$$x^4 - 8\cdot x^3 + 25\cdot x^2 - 32\cdot x + 4\cdot y^2 - 4\cdot x\cdot y + 16$$ □

Ignore the constant term while minimizing.

A 7 - 2 - 1 = 4 d.d. multivarable signomial with a constraint.

Since f(0,0) - 16 = 0 the objective function of the problem we must minimize must be negative at optimality if it is possible to improve on the trivial solution.

Thus we have to multiply the objective function with -1 and maximize.

The problem becomes:

MAXIMIZE: $y' := 8\cdot x^3 + 32\cdot x + 4\cdot x\cdot y - x^4 - 25\cdot x^2 - 4\cdot y^2$ □

SUBJECT TO: $x^2 \cdot y^{-1} \le 1$ □

**STEP 2: Transforming into the all constraint form.

The problem becomes:

MINIMIZE: z^{-1} □

SUBJECT TO: $z + \left[x^4 + 25\cdot x^2 + 4\cdot y^2\right] \le \left[8\cdot x^3 + 32\cdot x + 4\cdot x\cdot y\right]$ □

$x^2 \cdot y^{-1} \le 1$ □

To simplify the algebraic manipulations, the terms on the right hand side of the constraint will be condensed at the same time as the condensation to reduce the degrees of difficulty.

**STEP 3: Reducing the d.d.

Condense the terms in brackets above.

$$\delta 1(x,y) := \frac{8\cdot x^{3}}{8\cdot x^{3} + 32\cdot x + 4\cdot x\cdot y} \qquad \beta 1(x,y) := \frac{x^{4}}{x^{4} + 25\cdot x^{2} + 4\cdot y^{2}}$$

$$\delta 2(x,y) := \frac{32\cdot x}{8\cdot x^{3} + 32\cdot x + 4\cdot x\cdot y} \qquad \beta 2(x,y) := \frac{25\cdot x^{2}}{x^{4} + 25\cdot x^{2} + 4\cdot y^{2}}$$

$$\delta 3(x,y) := \frac{4\cdot x\cdot y}{8\cdot x^{3} + 32\cdot x + 4\cdot x\cdot y} \qquad \beta 3(x,y) := \frac{4\cdot y^{2}}{x^{4} + 25\cdot x^{2} + 4\cdot y^{2}}$$

$$k1(x,y) := \left[\frac{1}{\beta 1(x,y)}\right]^{\beta 1(x,y)} \cdot \left[\frac{25}{\beta 2(x,y)}\right]^{\beta 2(x,y)} \cdot \left[\frac{4}{\beta 3(x,y)}\right]^{\beta 3(x,y)}$$

$$k2(x,y) := \left[\frac{\delta 1(x,y)}{8}\right]^{\delta 1(x,y)} \cdot \left[\frac{\delta 2(x,y)}{32}\right]^{\delta 2(x,y)} \cdot \left[\frac{\delta 3(x,y)}{4}\right]^{\delta 3(x,y)}$$

Let: $\alpha 1(x,y) := 3\cdot \delta 1(x,y) + \delta 2(x,y) + \delta 3(x,y)$

$\alpha 2(x,y) := 4\cdot \beta 1(x,y) + 2\cdot \beta 2(x,y)$

The problem is now:

MINIMIZE: z^{-1} ▫

SUBJECT TO:

$$\left[\begin{array}{l} k2(x,y)\cdot z\cdot x^{-\alpha 1(x,y)}\cdot y^{-\delta 3(x,y)} \ \ldots \\ + \ k1(x,y)\cdot k2(x,y)\cdot x^{\alpha 2(x,y)-\alpha 1(x,y)}\cdot y^{2\cdot \beta 3(x,y)-\delta 3(x,y)} \end{array}\right] \le 1$$

$$x^{2}\cdot y^{-1} \le 1$$

A 4 - 3 - 1 = 0 d.d. posynomial with two constraints.

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(x,y) := \begin{bmatrix} 1 & 0 & 0 \\ -\alpha 1(x,y) & \alpha 2(x,y) - \alpha 1(x,y) & 2 \\ -\delta 3(x,y) & 2\cdot\beta 3(x,y) - \delta 3(x,y) & -1 \end{bmatrix}^{-1}$$

$$\omega(x,y) := \left[\text{augment}\left[\begin{bmatrix}1\\0\\0\end{bmatrix}, W(x,y)^{T}\right]\right]^{T} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \omega(1,1) = \begin{bmatrix}1\\1\\1.9615384615\\0.2538461538\end{bmatrix}$$

RULE 1:

$$F(x,y) := \left[\frac{1}{k2(x,y)}\right] \cdot \left[\frac{\omega(x,y)_2}{k1(x,y)\cdot k2(x,y)}\right]^{\omega(x,y)_2} \cdot \left[\frac{1}{1+\omega(x,y)_2}\right]^{1+\omega(x,y)_2}$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(x,y) := W(x,y)^{T} \cdot \begin{bmatrix} \ln\left[\omega(x,y)_1 \cdot \left[\omega(x,y)_1 + \omega(x,y)_2\right]^{-1} \cdot k2(x,y)^{-1}\right] \\ \ln\left[\omega(x,y)_2 \cdot \left[\omega(x,y)_1 + \omega(x,y)_2\right]^{-1} \cdot (k1(x,y)\cdot k2(x,y))^{-1}\right] \\ \ln(1) \end{bmatrix}$$

$$A(x,y) := \overrightarrow{\exp(B(x,y))}$$

SOLUTION $\qquad N := 19 \qquad i := 0 \,..N$

$$i := 0 \,..N + 1$$

$$\begin{bmatrix} z_0 \\ x_0 \\ y_0 \end{bmatrix} := \begin{bmatrix} 1000000 \\ 1000000 \\ 1000000 \end{bmatrix} \qquad \begin{bmatrix} z_{i+1} \\ x_{i+1} \\ y_{i+1} \end{bmatrix} := A\left[x_i, y_i\right]$$

**STEP 6: Iterations.

i	$-F[x_i, y_i] + 16$	$-z_i + 16$	$f[x_i, y_i]$
0	-416.012799997729	-999984	$9.99992000025 \cdot 10^{23}$
1	3.755205269222	-416.012799997729	4612.146307752019
2	3.262867574314	3.755205269222	8.706819425474
3	2.048425593214	3.262867574314	2.274639614554
4	1.957137606685	2.048425593214	1.978841196767
5	1.94745021974	1.957137606685	1.949881263816
6	1.946333605844	1.94745021974	1.946618372997
7	1.946201589202	1.946333605844	1.946235436195
8	1.946185848583	1.946201589202	1.946189891522
9	1.946183966379	1.946185848583	1.946184450119
10	1.946183741088	1.946183966379	1.946183799002
11	1.946183714113	1.946183741088	1.946183721047
12	1.946183710882	1.946183714113	1.946183711713
13	1.946183710495	1.946183710882	1.946183710595
14	1.946183710449	1.946183710495	1.946183710461
15	1.946183710444	1.946183710449	1.946183710445
16	1.946183710443	1.946183710444	1.946183710443
17	1.946183710443	1.946183710443	1.946183710443
18	1.946183710443	1.946183710443	1.946183710443
19	1.946183710443	1.946183710443	1.946183710443
20	1.946183710443	1.946183710443	1.946183710443

**STEP 7: Finding the optimal solution.

Optimal solution

$f[x_{N+1}, y_{N+1}] = 1.9461837104$

$x_{N+1} = 0.9455829952$

$y_{N+1} = 0.8941272007$

Starting point

$f[x_0, y_0] = 9.9999200003 \cdot 10^{23}$

$x_0 = 1000000$

$y_0 = 1000000$

Note how far the starting point was chosen from the optimal values, instead of starting at the usual x = y = 1. It is not very interesting to start too close to the answer.

Appendix L

VERMA'S GRINDING PROBLEM

VERMA'S GRINDING PROBLEM

MINIMIZE:

$$T(w,s,a) := \left[0.002 \cdot w^{-1} \cdot a^{-1} + 10130000 \cdot w^{-0.4} \cdot s^{0.4} \cdot a^{-0.4} \ldots + 2030 \cdot w^{-0.176} \cdot s^{0.176} \cdot a^{-0.176} \right]$$

SUBJECT TO:

C1: $70790 \cdot w^{0.68} \cdot s^{-0.68} \cdot a^{0.78} \le 1$ ▫

C2: $63711 \cdot w^{0.68} \cdot s^{-0.68} \cdot a^{0.78} \le 1$ ▫

C3: $0.0046648 \cdot w^{0.21} \cdot s^{0.42} \cdot a^{0.545} \le 1$ ▫

C4: $816 \cdot w^{0.4} \cdot s^{-0.4} \cdot a^{0.4} \le 1$ ▫

C5: $2.65 \cdot w^{-0.42} \cdot s^{0.42} \cdot a^{0.28} \le 1$ ▫

C6: $15182 \cdot w^{0.68} \cdot s^{0.42} \cdot a^{0.78} \le 1$ ▫ C7: $2.22 \cdot w \le 1$ ▫

C8: $0.0306 \cdot s \le 1$ ▫ C9: $25.88 \cdot s^{-1} \le 1$ ▫

A 12 - 3 - 1 = 8 dd multi-variable posynomial with constraints.

Note: This is a modification of the grinding problem supplied by A. P. Verma.

**STEP 1: Preprocessing.

It is clear that constraint 2 dominates constraint 3, therefore constraint 3 can be ignored.

Look at the advanced sign table:

$$X := \begin{bmatrix} -1 & -0.4 & -0.176 & 0.68 & 0.21 & 0.4 & -0.42 & 0.68 & 1 & 0 & 0 \\ 0 & 0.4 & 0.176 & -0.68 & 0.42 & -0.4 & 0.42 & 0.42 & 0 & 1 & -1 \\ -1 & -0.4 & -0.176 & 0.78 & 0.545 & 0.4 & 0.28 & 0.78 & 0 & 0 & 0 \end{bmatrix}$$

Let $x := w \cdot s^{-1} \cdot a$ ▫ Density: $\frac{26}{33} = 79 \cdot \%$

$$Y := \text{augment}\left[(X^T)^{<0>}, (X^T)^{<1>} + (X^T)^{<0>} \right]$$

$$Z := \left[\text{augment}\left[Y, (X^T)^{<2>} - (X^T)^{<0>} \right] \right]^T$$

$$Z = \begin{bmatrix} -1 & -0.4 & -0.176 & 0.68 & 0.21 & 0.4 & -0.42 & 0.68 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0.63 & 0 & 0 & 1.1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0.1 & 0.335 & 0 & 0.7 & 0.1 & -1 & 0 & 0 \end{bmatrix}$$

Density: $\frac{20}{33} = 61 \cdot \%$

Thus term 10 is necessary to balance the problem, therefore constraint 7 must be binding.

Thus $w := 2.22^{-1}$

Let $k1 := 0.002 \cdot w^{-1}$ $\quad k2 := 10130000 \cdot w^{-0.4}$ $\quad k3 := 2030 \cdot w^{-0.176}$

$k4 := 70790 \cdot w^{0.68}$ $\quad k6 := 0.0046648 \cdot w^{0.21}$ $\quad k7 := 816 \cdot w^{0.4}$

$k8 := 2.65 \cdot w^{-0.42}$ $\quad k9 := 15182 \cdot w^{0.68}$

The problem becomes:

MINIMIZE: $U(s,a) := k1 \cdot a^{-1} + k2 \cdot s^{0.4} \cdot a^{-0.4} + k3 \cdot s^{0.176} \cdot a^{-0.176}$

SUBJECT TO:

C1: $k4 \cdot s^{-0.68} \cdot a^{0.78} \le 1$ ▫ $\quad$ C3: $k6 \cdot s^{0.42} \cdot a^{0.545} \le 1$ ▫

C4: $k7 \cdot s^{-0.4} \cdot a^{0.4} \le 1$ ▫ $\quad$ C5: $k8 \cdot s^{0.42} \cdot a^{0.28} \le 1$ ▫

C6: $k9 \cdot s^{0.42} \cdot a^{0.78} \le 1$ ▫ $\quad$ C8: $0.0306 \cdot s \le 1$ ▫

C9: $25.88 \cdot s^{-1} \le 1$ ▫

Advanced sign table of the reduced problem:

$$X := \begin{bmatrix} 0 & 0.4 & 0.176 & -0.68 & 0.42 & -0.4 & 0.42 & 0.42 & 1 & -1 \\ -1 & -0.4 & -0.176 & 0.78 & 0.545 & 0.4 & 0.28 & 0.78 & 0 & 0 \end{bmatrix}$$

Let $y := s \cdot a^{-1}$ ▫ $\quad$ or $\quad s := y \cdot a$ ▫

MINIMIZE: $V(y,a) := k1 \cdot a^{-1} + k2 \cdot y^{0.4} + k3 \cdot y^{0.176}$

SUBJECT TO:

C1: $k4 \cdot y^{-0.68} \cdot a^{0.1} \le 1$ ▫ $\quad$ C3: $k6 \cdot y^{0.42} \cdot a^{0.965} \le 1$ ▫

C4: $k7 \cdot y^{-0.4} \le 1$ ▫ $\quad$ C5: $k8 \cdot y^{0.42} \cdot a^{0.7} \le 1$ ▫

C6: $k9 \cdot y^{0.42} \cdot a^{1.2} \le 1$ ▫ $\quad$ C8: $0.0306 \cdot y \cdot a \le 1$ ▫

AND TO: C9: $25.88 \cdot y^{-1} \cdot a^{-1} \le 1$

Sign table:

$$Y := \left[\mathrm{augment}\left[(X^T)^{<0>}, (X^T)^{<1>} + (X^T)^{<0>}\right]\right]^T$$

$$Y = \begin{bmatrix} 0 & 0.4 & 0.176 & -0.68 & 0.42 & -0.4 & 0.42 & 0.42 & 1 & -1 \\ -1 & 0 & 0 & 0.1 & 0.965 & 0 & 0.7 & 1.2 & 1 & -1 \end{bmatrix}$$

To find the Kuhn-Tucker points, we have to consider the following possibilities:
1. Objective function only - unbalanced
2. Objective function and C1
3. Objective function and any other single constraint - unbalanced
4. Combinations of two simultaneous equations.

Consider possibility #2:

**STEP 2: Transforming into the all constraint form.

The problem becomes:

MINIMIZE: t

SUBJECT TO: $k1 \cdot t^{-1} \cdot a^{-1} + \left[k2 \cdot t^{-1} \cdot y^{0.4} + k3 \cdot t^{-1} \cdot y^{0.176}\right] \le 1$

$k4 \cdot y^{-0.68} \cdot a^{0.1} \le 1$

A 5 - 3 - 1 = 1 dd multi-variable posynomial with constraints.

**STEP 3: Reducing the d.d.

RULE 2A:
RULE 2B:

$$\begin{matrix} \\ t: \\ y: \\ a: \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0.4 & 0.176 & -0.68 \\ 0 & -1 & 0 & 0 & 0.1 \end{bmatrix} \cdot \S := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Condense the third and fourth term:

$$\S 1(y) := \frac{k2 \cdot y^{0.4}}{k2 \cdot y^{0.4} + k3 \cdot y^{0.176}} \qquad \S 2(y) := \frac{k3 \cdot y^{0.176}}{k2 \cdot y^{0.4} + k3 \cdot y^{0.176}}$$

$$k(y) := \left[\frac{k2}{\S 1(y)}\right]^{\S 1(y)} \cdot \left[\frac{k3}{\S 2(y)}\right]^{\S 2(y)}$$

Thus the problem is now:

MINIMIZE: t ▫

SUBJECT TO: $k1 \cdot t^{-1} \cdot a^{-1} + k(y) \cdot t^{-1} \cdot y^{0.176+0.224 \cdot \Phi 1(y)} \le 1$ ▫

$k4 \cdot y^{-0.68} \cdot a^{0.1} \le 1$ ▫

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(y) := \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0.176 + 0.224 \cdot \Phi 1(y) & -0.68 \\ -1 & 0 & 0.1 \end{bmatrix}^{-1}$$

$$\omega(y) := \left[\text{augment}\left[\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, W(y)^{T}\right]\right]^{T} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \qquad \omega(1) = \begin{bmatrix} 1 \\ 0.0555506315 \\ 0.9444493685 \\ 0.5555063148 \end{bmatrix}$$

RULE 1:

$$G(y) := \left[\frac{k1}{\omega(y)_1}\right]^{\omega(y)_1} \cdot \left[\frac{k(y)}{\omega(y)_2}\right]^{\omega(y)_2} \cdot \left[\omega(y)_1 + \omega(y)_2\right]^{\omega(y)_1 + \omega(y)_2} \cdot k4^{\omega(y)_3}$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(y) := W(y)^{T} \cdot \begin{bmatrix} \ln\left[\omega(y)_1 \cdot \left[\omega(y)_1 + \omega(y)_2\right]^{-1} \cdot k1^{-1}\right] \\ \ln\left[\omega(y)_2 \cdot \left[\omega(y)_1 + \omega(y)_2\right]^{-1} \cdot k(y)^{-1}\right] \\ \ln\left[k4^{-1}\right] \end{bmatrix}$$

$$A(y) := \overrightarrow{\exp(B(y))}$$

SOLUTION: $N := 3 \qquad i := 0 \,..N$

$$\begin{bmatrix} t_0 \\ y_0 \\ a_0 \end{bmatrix} := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} t_{i+1} \\ y_{i+1} \\ a_{i+1} \end{bmatrix} := A\left[y_i\right]$$

$i := 0 \,..N + 1$

$s_i := y_i \cdot a_i$

$w_i := 2.22^{-1}$

**STEP 6: Iterations.

i	$G[y_i]$	$T[w_i, s_i, a_i]$	t_i
0	1875862060.2038	13938736.9391	1
1	1876390521.2548	1876390528.5837	1875862060.2038
2	1876390521.2548	1876390521.2548	1876390521.2548
3	1876390521.2548	1876390521.2548	1876390521.2548
4	1876390521.2548	1876390521.2548	1876390521.2548

**STEP 7: Finding the optimal solution.

$t := t_{N+1}$ $t = 1.8763905213 \cdot 10^{9}$

$w := w_{N+1}$ $w = 4.504504504505 \cdot 10^{-1}$

$s := s_{N+1}$ $s = 7.76596868611 \cdot 10^{-6}$

$a := a_{N+1}$ $a = 4.259266334389 \cdot 10^{-11}$

Check the original constraints:

C1: $70790 \cdot w^{0.68} \cdot s^{-0.68} \cdot a^{0.78} = 1$ As expected

C2: $63711 \cdot w^{0.68} \cdot s^{-0.68} \cdot a^{0.78} = 0.9$ Satisfied

C3: $0.0046648 \cdot w^{0.21} \cdot s^{0.42} \cdot a^{0.545} = 6.2801285739 \cdot 10^{-11}$ Satisfied

C4: $816 \cdot w^{0.4} \cdot s^{-0.4} \cdot a^{0.4} = 4.6644950227$ VIOLATED

C5: $2.65 \cdot w^{-0.42} \cdot s^{0.42} \cdot a^{0.28} = 0.0000330225$ Satisfied

C6: $15182 \cdot w^{0.68} \cdot s^{0.42} \cdot a^{0.78} = 0.0000005135$ Satisfied

C7: $2.22 \cdot w = 1$ As expected

C8: $0.0306 \cdot s = 0.0000002376$ Satisfied

C9: $25.88 \cdot s^{-1} = 3332488.327732647$ VIOLATED

Consider possibility no. 4.

NOTE: Since this MathCAD file is too full to finish this problem, it will be continued on the next file. In order to have the problem and definitions available in the next file, the initial information will be repeated. The next page will not contain anything new.

VERMA'S GRINDING PROBLEM (CONTINUED)

MINIMIZE:

$$T(w,s,a) := \left[0.002 \cdot w^{-1} \cdot a^{-1} + 10130000 \cdot w^{-0.4} \cdot s^{0.4} \cdot a^{-0.4} \ldots \atop + 2030 \cdot w^{-0.176} \cdot s^{0.176} \cdot a^{-0.176} \right]$$

SUBJECT TO:

C1: $70790 \cdot w^{0.68} \cdot s^{-0.68} \cdot a^{0.78} \le 1$ ▫

C2: $63711 \cdot w^{0.68} \cdot s^{-0.68} \cdot a^{0.78} \le 1$ ▫

C3: $0.0046648 \cdot w^{0.21} \cdot s^{0.42} \cdot a^{0.545} \le 1$ ▫

C4: $816 \cdot w^{0.4} \cdot s^{-0.4} \cdot a^{0.4} \le 1$ ▫

C5: $2.65 \cdot w^{-0.42} \cdot s^{0.42} \cdot a^{0.28} \le 1$ ▫

C6: $15182 \cdot w^{0.68} \cdot s^{0.42} \cdot a^{0.78} \le 1$ ▫ C7: $2.22 \cdot w \le 1$ ▫

C8: $0.0306 \cdot s \le 1$ ▫ C9: $25.88 \cdot s^{-1} \le 1$ ▫

A 12 - 3 - 1 = 8 dd multi-variable posynomial with constraints.

Note: This is a modification of the grinding problem supplied by A. P. Verma.

It is clear that constraint 2 dominates constraint 3, therefore constraint 3 can be ignored.

Term 10 is necessary to balance the problem, therefore constraint 7 must be binding. (See analysis in previous file.)

Thus $w := 2.22^{-1}$

Let $k1 := 0.002 \cdot w^{-1}$ $k2 := 10130000 \cdot w^{-0.4}$ $k3 := 2030 \cdot w^{-0.176}$

$k4 := 70790 \cdot w^{0.68}$ $k6 := 0.0046648 \cdot w^{0.21}$ $k7 := 816 \cdot w^{0.4}$

$k8 := 2.65 \cdot w^{-0.42}$ $k9 := 15182 \cdot w^{0.68}$ $k10 := 0.0306$

The problem becomes:

MINIMIZE: $U(s,a) := k1 \cdot a^{-1} + k2 \cdot s^{0.4} \cdot a^{-0.4} + k3 \cdot s^{0.176} \cdot a^{-0.176}$

SUBJECT TO:

C1: $k4 \cdot s^{-0.68} \cdot a^{0.78} \le 1$ ▫

C3: $k6 \cdot s^{0.42} \cdot a^{0.545} \le 1$ ▫

C4: $k7 \cdot s^{-0.4} \cdot a^{0.4} \le 1$ ▫

C5: $k8 \cdot s^{0.42} \cdot a^{0.28} \le 1$ ▫

C6: $k9 \cdot s^{0.42} \cdot a^{0.78} \le 1$ ▫

C8: $0.0306 \cdot s \le 1$ ▫

C9: $25.88 \cdot s^{-1} \le 1$ ▫

**STEP 1: Preprocessing - Continue the analysis with possibility no. 4.

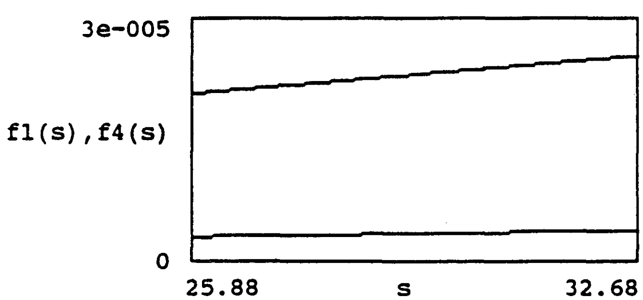
From C8 and C9: $25.88 \le s \le 0.0306^{-1}$ ▫ $0.0306^{-1} = 32.6797385621$

From C1 and C4:

$a \le \left[k4^{-1} \cdot s^{0.68}\right]^{\frac{1}{0.78}}$ ▫ $f1(s) := \left[k4^{-1} \cdot s^{0.68}\right]^{\frac{1}{0.78}}$

$a \le \left[k7^{-1} \cdot s^{0.4}\right]^{\frac{1}{0.4}}$ ▫ $f4(s) := \left[k7^{-1} \cdot s^{0.4}\right]^{\frac{1}{0.4}}$

s := 25.88,25.92 ..32.68



f1(32.68) = 0.0000253618

f1(25.88) = 0.0000206943

f4(32.68) = 0.0000038143

f4(25.88) = 0.0000030206

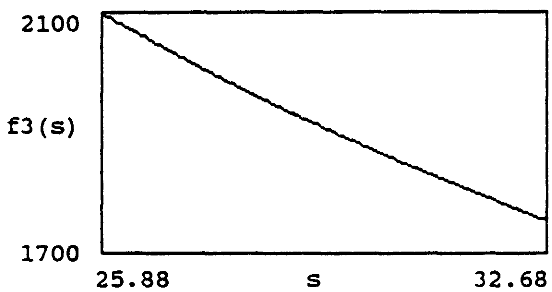
Thus C4 dominates C1 over the possible range of s.

From C3, C5, and C6:

$$a \le \left[k6^{-1} \cdot s^{-0.42}\right]^{\frac{1}{0.545}} \qquad f3(s) := \left[k6^{-1} \cdot s^{-0.42}\right]^{\frac{1}{0.545}}$$

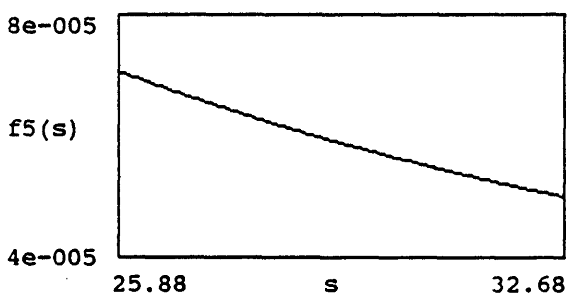
$$a \le \left[k8^{-1} \cdot s^{-0.42}\right]^{\frac{1}{0.28}} \qquad f5(s) := \left[k8^{-1} \cdot s^{-0.42}\right]^{\frac{1}{0.28}}$$

$$a \le \left[k9^{-1} \cdot s^{-0.42}\right]^{\frac{1}{0.78}} \qquad f6(s) := \left[k9^{-1} \cdot s^{-0.42}\right]^{\frac{1}{0.78}}$$



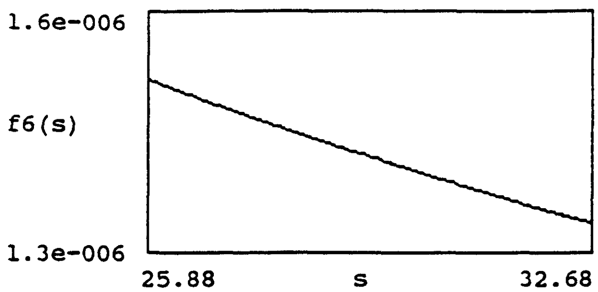
f3(25.88) = 2098.6393020316

f3(32.68) = 1753.3073278795



f5(25.88) = 0.000070702

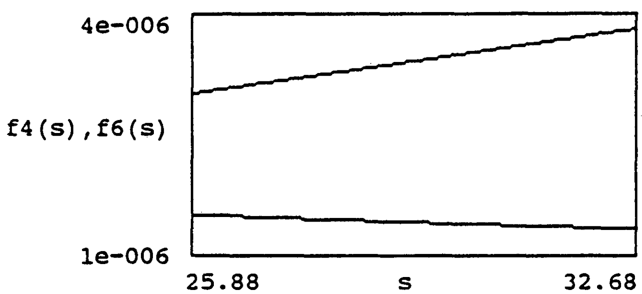
f5(32.68) = 0.0000498259



$f6(25.88) = 0.0000015151$

$f6(32.68) = 0.0000013363$

Thus C6 dominates C5 and C3 over the possible range of s.



Thus C6 dominates C4 and the problem reduces to:

MINIMIZE: $U(s,a) := k1 \cdot a^{-1} + k2 \cdot s^{0.4} \cdot a^{-0.4} + k3 \cdot s^{0.176} \cdot a^{-0.176}$

SUBJECT TO:

C6: $k9 \cdot s^{0.42} \cdot a^{0.78} \le 1$ ▫ C8: $0.0306 \cdot s \le 1$ ▫

C9: $25.88 \cdot s^{-1} \le 1$ ▫

Examine C6 and C8 binding:

$U\left[0.0306^{-1}, f6\left[0.0306^{-1}\right]\right] = 1.2574996564 \cdot 10^{10}$

Examine C6 and C9 binding:

$U(25.88, f6(25.88)) = 1.0893332239 \cdot 10^{10}$

SOLUTION:

$w := 2.22^{-1}$ $w = 0.4504504505$

$s := 25.88$ $s = 25.88$

$a := f6(s)$ $a = 0.000001515108179$

$T(w,s,a) = 10893332239.3814$

Check constraints:

C1:	$70790 \cdot w^{0.68} \cdot s^{-0.68} \cdot a^{0.78} = 0.1301314371$	Satisfied
C2:	$63711 \cdot w^{0.68} \cdot s^{-0.68} \cdot a^{0.78} = 0.1171182934$	Satisfied
C3:	$0.0046648 \cdot w^{0.21} \cdot s^{0.42} \cdot a^{0.545} = 0.0000104204$	Satisfied
C4:	$816 \cdot w^{0.4} \cdot s^{-0.4} \cdot a^{0.4} = 0.7588234042$	Satisfied
C5:	$2.65 \cdot w^{-0.42} \cdot s^{0.42} \cdot a^{0.28} = 0.340944224$	Satisfied
C6:	$15182 \cdot w^{0.68} \cdot s^{0.42} \cdot a^{0.78} = 1$	As expected
C7:	$2.22 \cdot w = 1$	As expected
C8:	$0.0306 \cdot s = 0.791928$	Satisfied
C9:	$25.88 \cdot s^{-1} = 1$	As expected

OBJECTIVE FUNCTION VALUE:

$T(w,s,a) = 1.0893332239 \cdot 10^{10}$

Appendix M

BRIDGE DESIGN PROBLEM

BRIDGE DESIGN PROBLEM

MINIMIZE: $f(x,y,z) := 5992.5 \cdot x + 12063 \cdot y + 50726 \cdot z$

SUBJECT TO:

$$6.62 \cdot x^{2} - 54 \cdot x \cdot z + 26.72 \cdot z + 69.3 \le 0 \quad \square$$

$$6.62 \cdot y^{2} - 11.83 \cdot y - 54 \cdot y \cdot z + 15.6 \cdot z + 71.96 \le 0 \quad \square$$

$$-z + 1.41 \le 0 \quad \square$$

$$-0.22 \cdot y - y \cdot z + 1.99 \le 0 \quad \square$$

**STEP 1: Preprocessing.

Reformulate the constraints in the correct form for geometric programming:

$$\text{C1:} \quad 54^{-1} \cdot \left[6.62 \cdot x \cdot z^{-1} + 26.72 \cdot x^{-1} + 69.3 \cdot x^{-1} \cdot z^{-1}\right] \le 1 \quad \square$$

$$\text{C2:} \quad 54^{-1} \cdot \left[6.62 \cdot y \cdot z^{-1} - 11.83 \cdot z^{-1} + 15.6 \cdot y^{-1} + 71.96 \cdot y^{-1} \cdot z^{-1}\right] \le 1 \quad \square$$

$$\text{C3:} \quad 1.41 \cdot z^{-1} \le 1 \quad \square$$

$$\text{C4:} \quad -0.22 \cdot z^{-1} + 1.99 \cdot y^{-1} \cdot z^{-1} \le 1 \quad \square$$

A $13 - 3 - 1 = 9$ d.d. multivariable signomial with constraints.

Advanced sign table:

$$\begin{matrix} x: \\ y: \\ z: \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & -1 & -1 & -(-1) & 0 & -1 & -1 & -(-1) & -1 \end{bmatrix}$$

Thus C1 must be active, C2 or C4 (or both) must be active, and the following combinations of constraints should be examined:

(a) C1 and C2
(b) C1 and C4
(c) C1, C2, and C3
(d) C1, C2, and C4
(e) C1, C3, and C4.

Consider combination (e). The analysis to know that this is the right combination will not be shown here. Since there are three constraints and three unknowns, the Wessels algorithm will serve as a simultaneous nonlinear equation solver in this case.

**STEP 2: Transforming into the all constraint form.

MINIMIZE: g ▫

SUBJECT TO: $\left[5992.5 \cdot x \cdot g^{-1} + 12063 \cdot y \cdot g^{-1} + 50726 \cdot z \cdot g^{-1}\right] \le 1$ ▫

$$54^{-1} \cdot \left[6.62 \cdot x \cdot z^{-1} + 26.72 \cdot x^{-1} + 69.3 \cdot x^{-1} \cdot z^{-1}\right] \le 1$$ ▫

$$1.41 \cdot z^{-1} \le 1$$ ▫

$$1.99 \cdot y^{-1} \cdot z^{-1} \le \left[1 + 0.22 \cdot z^{-1}\right]$$ ▫

A 10 - 4 - 1 = 5 d.d. multivariable signomial with constraints.

To simplify the algebraic manipulations, the condensation of the terms on the right hand side of the inequality and the condensation to reduce the degree of difficulty will be done simultaneously.

**STEP 3: Reducing the d.d.

Condense the terms in brackets above:

$$\delta 1(x,y,z) := \frac{5992.5 \cdot x}{5992.5 \cdot x + 12063 \cdot y + 50726 \cdot z}$$

$$\delta 2(x,y,z) := \frac{12063 \cdot y}{5992.5 \cdot x + 12063 \cdot y + 50726 \cdot z}$$

$$\delta 3(x,y,z) := \frac{50726 \cdot z}{5992.5 \cdot x + 12063 \cdot y + 50726 \cdot z}$$

$$\delta 4(x,z) := \frac{6.62 \cdot x \cdot z^{-1}}{6.62 \cdot x \cdot z^{-1} + 26.72 \cdot x^{-1} + 69.3 \cdot x^{-1} \cdot z^{-1}}$$

$$\delta 5(x,z) := \frac{26.72 \cdot x^{-1}}{6.62 \cdot x \cdot z^{-1} + 26.72 \cdot x^{-1} + 69.3 \cdot x^{-1} \cdot z^{-1}}$$

$$\delta 6(x,z) := \frac{69.3 \cdot x^{-1} \cdot z^{-1}}{6.62 \cdot x \cdot z^{-1} + 26.72 \cdot x^{-1} + 69.3 \cdot x^{-1} \cdot z^{-1}}$$

$$\delta 7(z) := \frac{1}{1 + 0.22 \cdot z^{-1}}$$

$$\delta 8(z) := \frac{0.22 \cdot z^{-1}}{1 + 0.22 \cdot z^{-1}}$$

$$k1'(x,y,z) := \left[\frac{5992.5}{\delta 1(x,y,z)}\right]^{\delta 1(x,y,z)} \cdot \left[\frac{12063}{\delta 2(x,y,z)}\right]^{\delta 2(x,y,z)}$$

$$k1(x,y,z) := k1'(x,y,z) \cdot \left[\frac{50726}{\delta 3(x,y,z)}\right]^{\delta 3(x,y,z)}$$

$$k2(x,z) := 54^{-1} \cdot \left[\frac{6.62}{\delta 4(x,z)}\right]^{\delta 4(x,z)} \cdot \left[\frac{26.72}{\delta 5(x,z)}\right]^{\delta 5(x,z)} \cdot \left[\frac{69.3}{\delta 6(x,z)}\right]^{\delta 6(x,z)}$$

$$k3(z) := 1.99 \cdot \delta 7(z)^{\delta 7(z)} \cdot \left[\frac{\delta 8(z)}{0.22}\right]^{\delta 8(z)}$$

The problem is now:

MINIMIZE: g

SUBJECT TO:

$$k1(x,y,z) \cdot g^{-1} \cdot x^{\delta 1(x,y,z)} \cdot y^{\delta 2(x,y,z)} \cdot z^{\delta 3(x,y,z)} \le 1$$

$$k2(x,z) \cdot \left[x^{\delta 4(x,z) - \delta 5(x,z) - \delta 6(x,z)} \cdot z^{-\delta 4(x,z) - \delta 6(x,z)}\right] \le 1$$

$$1.41 \cdot z^{-1} \le 1$$

$$k3(z) \cdot y^{-1} \cdot z^{-1 + \delta 8(z)} \le 1$$

A 5 - 4 - 1 = 0 d.d. multivariable posynomial with constraints.

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(x,y,z) := \begin{bmatrix} -1 & 0 & 0 & 0 \\ \delta 1(x,y,z) & \delta 4(x,z) - \delta 5(x,z) - \delta 6(x,z) & 0 & 0 \\ \delta 2(x,y,z) & 0 & 0 & -1 \\ \delta 3(x,y,z) & -\delta 4(x,z) - \delta 6(x,z) & -1 & -1 + \delta 8(z) \end{bmatrix}^{-1}$$

$$\omega(x,y,z) := \left[\text{augment}\left[\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}, W(x,y,z)^T\right]\right]^T \cdot \begin{bmatrix}-1\\0\\0\\0\end{bmatrix}$$

$$\omega(1,1,1) = \begin{bmatrix}1\\1\\0.100026604\\0.5197525817\\0.1753814616\end{bmatrix}$$

RULE 1:

$$F(x,y,z) := k1(x,y,z)\cdot k2(x,z)^{\omega(x,y,z)_2}\cdot 1.41^{\omega(x,y,z)_3}\cdot k3(z)^{\omega(x,y,z)_4}$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(x,y,z) := W(x,y,z)^T\cdot\begin{bmatrix}\ln\left[k1(x,y,z)^{-1}\right]\\ \ln\left[k2(x,z)^{-1}\right]\\ \ln\left[1.41^{-1}\right]\\ \ln\left[k3(z)^{-1}\right]\end{bmatrix}$$

$$A(x,y,z) := \overrightarrow{\exp(B(x,y,z))}$$

SOLUTION: $N := 4$ $i := 0\,..N$

$$\begin{bmatrix}g_0\\x_0\\y_0\\z_0\end{bmatrix} := \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad \begin{bmatrix}g_{i+1}\\x_{i+1}\\y_{i+1}\\z_{i+1}\end{bmatrix} := A\left[x_i, y_i, z_i\right]$$

$i := 0\,..N + 1$

$F_i := F\left[x_i, y_i, z_i\right]$

$f_i := f\left[x_i, y_i, z_i\right]$

**STEP 6: Iterations.

i	F_i	g_i	f_i
0	95542.31204598	1	68781.5
1	96050.959202	95542.31204598	95727.26207325
2	96068.75220687	96050.959202	96061.57203118
3	96068.75636174	96068.75220687	96068.75456647
4	96068.75636174	96068.75636174	96068.75636174
5	96068.75636174	96068.75636174	96068.75636174

i	x_i	y_i	z_i
0	1	1	1
1	1.5613904483	1.2307858585	1.41
2	1.6371616474	1.2208588957	1.41
3	1.6383602349	1.2208588957	1.41
4	1.6383605345	1.2208588957	1.41
5	1.6383605345	1.2208588957	1.41

**STEP 7: Finding the optimal solution.

$$f_{N+1} = 96068.756361737$$

$$x_{N+1} = 1.6383605345$$

$$y_{N+1} = 1.2208588957$$

$$z_{N+1} = 1.41$$

Check the original constraints:

$$6.62 \cdot x_{N+1}^2 - 54 \cdot x_{N+1} \cdot z_{N+1} + 26.72 \cdot z_{N+1} + 69.3 = 0$$ As expected

$$\begin{bmatrix} 6.62 \cdot y_{N+1}^2 - 11.83 \cdot y_{N+1} - 54 \cdot y_{N+1} \cdot z_{N+1} \ldots \\ + 15.6 \cdot z_{N+1} + 71.96 \end{bmatrix} = -3.58$$ Satisfied

$$-z_{N+1} + 1.41 = 0$$ As expected

$$-0.22 \cdot y_{N+1} - y_{N+1} \cdot z_{N+1} + 1.99 = 0$$ As expected

Appendix N

TWO-PHASE SAMPLING PROBLEM

TWO-PHASE SAMPLING PROBLEM

MINIMIZE: $Z(X,Y) := 5\cdot X + 0.5\cdot Y$ or $T'(Q,R) := 5\cdot Q^{-1} + 0.5\cdot R^{-1}$ ▫

SUBJECT TO:

C1: $6\cdot X^{-1} + 100\cdot Y^{-1} \le 0.5$ ▫ or $12\cdot Q + 200\cdot R \le 1$ ▫

C2: $40\cdot X^{-1} + 175\cdot Y^{-1} \le 1$ ▫ or $40\cdot Q + 175\cdot R \le 1$ ▫

C3: $6\cdot X^{-1} + 30\cdot Y^{-1} \le 0.2$ ▫ or $30\cdot Q + 150\cdot R \le 1$ ▫

C4: $2\cdot X^{-1} + 17\cdot Y^{-1} \le 0.1$ ▫ or $20\cdot Q + 170\cdot R \le 1$ ▫

C5: $3\cdot X^{-1} + 21\cdot Y^{-1} \le 0.12$ ▫ or $25\cdot Q + 175\cdot R \le 1$ ▫

C6: $X \le Y$ ▫ or $Q^{-1}\cdot R \le 1$ ▫

where $X := Q^{-1}$ ▫ and $Y := R^{-1}$ ▫

A 13 - 2 - 1 = 10 dd multivariable posynomial with constraints.

**STEP 1: Preprocessing.

It is clear from the constraints that: $Q < \frac{1}{40}$ ▫ and $R < \frac{1}{200}$ ▫

It is also clear that C2 dominates C3, C4, and C5.

The problem becomes:

MINIMIZE: $T'(Q,R) := 5\cdot Q^{-1} + 0.5\cdot R^{-1}$

SUBJECT TO: C1 $12\cdot Q + 200\cdot R \le 1$ ▫

C2 $40\cdot Q + 175\cdot R \le 1$ ▫

C6 $Q^{-1}\cdot R \le 1$ ▫

To find the Kuhn-Tucker points, we have to consider the following possibilities:
1. Objective function only - unbalanced
2. Objective function and C1
3. Objective function and C2
4. Objective function and C6 - unbalanced
5. C1 and C2 (two simultaneous equations)
6. C1 and C6 (two simultaneous equations)
7. C2 and C6 (two simultaneous equations)
8. C1,C2 and C6 - overspecified

Consider possibility no. 2:

**STEP 2: Transforming into the all constraint form.

The problem becomes:

MINIMIZE: T ▫

Subject to: $5 \cdot T^{-1} \cdot Q^{-1} + 0.5 \cdot T^{-1} \cdot R^{-1} \le 1$ ▫

$12 \cdot Q + 200 \cdot R \le 1$ ▫

A 5 - 3 - 1 = 1 dd multi-variable posynomial with constraints.

RULE 2A:
RULE 2B:

$$\begin{matrix} \\ T: \\ Q: \\ R: \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \cdot \delta := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ ▫

**STEP 3: Reducing the d.d.

Condense the fourth and fifth term:

$$\delta 1(Q,R) := \frac{12 \cdot Q}{12 \cdot Q + 200 \cdot R} \qquad \delta 2(Q,R) := \frac{200 \cdot R}{12 \cdot Q + 200 \cdot R}$$

$$k(Q,R) := \left[\frac{12}{\delta 1(Q,R)}\right]^{\delta 1(Q,R)} \cdot \left[\frac{200}{\delta 2(Q,R)}\right]^{\delta 2(Q,R)}$$

Thus the problem is now:

MINIMIZE: T ▫

Subject to: $5 \cdot T^{-1} \cdot Q^{-1} + 0.5 \cdot T^{-1} \cdot R^{-1} \le 1$ ▫

$k(Q,R) \cdot Q^{\delta 1(Q,R)} \cdot R^{\delta 2(Q,R)} \le 1$ ▫

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(Q,R) := \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & \delta 1(Q,R) \\ 0 & -1 & \delta 2(Q,R) \end{bmatrix}^{-1} \qquad \omega(Q,R) := \left[\text{augment}\left[\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, W(Q,R)^T\right]\right]^T \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega(1,1) = \begin{bmatrix} 1 \\ 0.0566037736 \\ 0.9433962264 \\ 1 \end{bmatrix}$$

RULE 1:

$$G(Q,R) := \left[\frac{5}{\omega(Q,R)_1}\right]^{\omega(Q,R)_1} \cdot \left[\frac{0.5}{\omega(Q,R)_2}\right]^{\omega(Q,R)_2} \cdot k(Q,R)^{\omega(Q,R)_3}$$

$$F'(Q,R) := G(Q,R) \cdot \left[\omega(Q,R)_1 + \omega(Q,R)_2\right]^{\omega(Q,R)_1 + \omega(Q,R)_2}$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(Q,R) := W(Q,R)^{T} \cdot \begin{bmatrix} \ln\left[\omega(Q,R)_1 \cdot \left[\omega(Q,R)_1 + \omega(Q,R)_2\right]^{-1} \cdot 0.2\right] \\ \ln\left[\omega(Q,R)_2 \cdot \left[\omega(Q,R)_1 + \omega(Q,R)_2\right]^{-1} \cdot 2\right] \\ \ln\left[k(Q,R)^{-1}\right] \end{bmatrix}$$

$$A(Q,R) := \overrightarrow{\exp(B(Q,R))}$$

SOLUTION: $N := 9 \quad i := 1 \,..N$

$$\begin{bmatrix} T_0 \\ Q_0 \\ R_0 \end{bmatrix} := \begin{bmatrix} 314.9193338483 \\ 0.036374306091976 \\ 0.002817541634481 \end{bmatrix} \quad \begin{bmatrix} T_1 \\ Q_1 \\ R_1 \end{bmatrix} := \begin{bmatrix} 314.9193338483 \\ 0.036374306091976 \\ 0.002817541634481 \end{bmatrix}$$

These starting values were found after many iterations from (1,1,1).

$$\begin{bmatrix} T_{i+1} \\ Q_{i+1} \\ R_{i+1} \end{bmatrix} := 0.5 \cdot \left[A\left[Q_i, R_i\right] + \begin{bmatrix} T_{i-1} \\ Q_{i-1} \\ R_{i-1} \end{bmatrix} \right]$$

Adapted method to avoid cycling.

$i := 0 \,..N + 1$

$$T'_i := T'\left[Q_i, R_i\right]$$

$$F'_i := F'\left[Q_i, R_i\right]$$

**STEP 6: Iterations.

i	F'_i	T'_i	T_i
0	314.9193338483	314.9193338483	314.9193338483
1	314.9193338483	314.9193338483	314.9193338483
2	314.9193338483	314.9193338483	314.9193338483
3	314.9193338483	314.9193338483	314.9193338483
4	314.9193338483	314.9193338483	314.9193338483
5	314.9193338483	314.9193338483	314.9193338483
6	314.9193338483	314.9193338483	314.9193338483
7	314.9193338483	314.9193338483	314.9193338483
8	314.9193338483	314.9193338483	314.9193338483
9	314.9193338483	314.9193338483	314.9193338483
10	314.9193338483	314.9193338483	314.9193338483

i	Q_i	R_i
0	0.036374306092	0.00281754163448
1	0.036374306092	0.00281754163448
2	0.036374306092	0.00281754163448
3	0.036374306092	0.00281754163448
4	0.036374306092	0.00281754163448
5	0.036374306092	0.00281754163448
6	0.036374306092	0.00281754163448
7	0.036374306092	0.00281754163448
8	0.036374306092	0.00281754163448
9	0.036374306092	0.00281754163448
10	0.036374306092	0.00281754163448

**STEP 7: Finding the optimal solution.

Check the constraints:

C1: $12 \cdot Q_{N+1} + 200 \cdot R_{N+1} = 1$ As expected

C2: $40 \cdot Q_{N+1} + 175 \cdot R_{N+1} = 1.9480420297$ VIOLATED

C6: $Q_{N+1}^{-1} \cdot R_{N+1} = 0.0774596669$ Satisfied

Consider possibility no. 3:

MINIMIZE: T □

Subject to: $5 \cdot T^{-1} \cdot Q^{-1} + 0.5 \cdot T^{-1} \cdot R^{-1} \le 1$ □

$40 \cdot Q + 175 \cdot R \le 1$ □

Continued on the next file. The problem and the definitions will be repeated to make them available for the next file.

TWO-PHASE SAMPLING PROBLEM (CONTINUED)

MINIMIZE: $Z(X,Y) := 5 \cdot X + 0.5 \cdot Y$ or $T'(Q,R) := 5 \cdot Q^{-1} + 0.5 \cdot R^{-1}$ ▫

SUBJECT TO:

C1: $6 \cdot X^{-1} + 100 \cdot Y^{-1} \le 0.5$ ▫ or $12 \cdot Q + 200 \cdot R \le 1$ ▫

C2: $40 \cdot X^{-1} + 175 \cdot Y^{-1} \le 1$ ▫ or $40 \cdot Q + 175 \cdot R \le 1$ ▫

C3: $6 \cdot X^{-1} + 30 \cdot Y^{-1} \le 0.2$ ▫ or $30 \cdot Q + 150 \cdot R \le 1$ ▫

C4: $2 \cdot X^{-1} + 17 \cdot Y^{-1} \le 0.1$ ▫ or $20 \cdot Q + 170 \cdot R \le 1$ ▫

C5: $3 \cdot X^{-1} + 21 \cdot Y^{-1} \le 0.12$ ▫ or $25 \cdot Q + 175 \cdot R \le 1$ ▫

C6: $X \le Y$ ▫ or $Q^{-1} \cdot R \le 1$ ▫

where $X := Q^{-1}$ ▫ and $Y := R^{-1}$ ▫

A 13 - 2 - 1 = 10 dd multi-variable posynomial with constraints.

It is clear from the constraints that: $Q < \frac{1}{40}$ ▫ and $R < \frac{1}{200}$ ▫

It is also clear that C2 dominates C3, C4, and C5.

The problem becomes:

MINIMIZE: $T'(Q,R) := 5 \cdot Q^{-1} + 0.5 \cdot R^{-1}$

SUBJECT TO: C1 $12 \cdot Q + 200 \cdot R \le 1$ ▫

C2 $40 \cdot Q + 175 \cdot R \le 1$ ▫

C6 $Q^{-1} \cdot R \le 1$ ▫

To find the Kuhn-Tucker points, we have to consider the following possibilities:
1. Objective function only - unbalanced
2. Objective function and C1
3. Objective function and C2
4. Objective function and C6 - unbalanced
5. C1 and C2 (two simultaneous equations)
6. C1 and C6 (two simultaneous equations)
7. C2 and C6 (two simultaneous equations)
8. C1,C2 and C6 - overspecified

Consider possibility no. 3:

**STEP 2: Transforming into the all constraint form.

MINIMIZE: T ▫

Subject to: $5 \cdot T^{-1} \cdot Q^{-1} + 0.5 \cdot T^{-1} \cdot R^{-1} \le 1$ ▫

$40 \cdot Q + 175 \cdot R \le 1$ ▫

A 5 - 3 - 1 = 1 dd multi-variable posynomial with constraints.

RULE 2A:
RULE 2B:

$$\begin{matrix} \\ T: \\ Q: \\ R: \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \cdot \delta := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ ▫

**STEP 3: Reducing the d.d.

Condense the fourth and fifth term:

$$\delta 1(Q,R) := \frac{40 \cdot Q}{40 \cdot Q + 175 \cdot R} \qquad \delta 2(Q,R) := \frac{175 \cdot R}{40 \cdot Q + 175 \cdot R}$$

$$k(Q,R) := \left[\frac{40}{\delta 1(Q,R)}\right]^{\delta 1(Q,R)} \cdot \left[\frac{175}{\delta 2(Q,R)}\right]^{\delta 2(Q,R)}$$

Thus the problem is now:

MINIMIZE: T ▫

Subject to: $5 \cdot T^{-1} \cdot Q^{-1} + 0.5 \cdot T^{-1} \cdot R^{-1} \le 1$ ▫

$k(Q,R) \cdot Q^{\delta 1(Q,R)} \cdot R^{\delta 2(Q,R)} \le 1$ ▫

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(Q,R) := \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & \delta 1(Q,R) \\ 0 & -1 & \delta 2(Q,R) \end{bmatrix}^{-1} \qquad \omega(Q,R) := \left[\text{augment}\left[\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, W(Q,R)^{T}\right]\right]^{T} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega(1,1) = \begin{bmatrix} 1 \\ 0.1860465116 \\ 0.8139534884 \\ 1 \end{bmatrix}$$

RULE 1:

$$G(Q,R) := \left[\frac{5}{\omega(Q,R)_1}\right]^{\omega(Q,R)_1} \cdot \left[\frac{0.5}{\omega(Q,R)_2}\right]^{\omega(Q,R)_2} \cdot k(Q,R)^{\omega(Q,R)_3}$$

$$F'(Q,R) := G(Q,R) \cdot \left[\omega(Q,R)_1 + \omega(Q,R)_2\right]^{\omega(Q,R)_1 + \omega(Q,R)_2}$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(Q,R) := W(Q,R)^{T} \cdot \begin{bmatrix} \ln\left[\omega(Q,R)_1 \cdot \left[\omega(Q,R)_1 + \omega(Q,R)_2\right]^{-1} \cdot 0.2\right] \\ \ln\left[\omega(Q,R)_2 \cdot \left[\omega(Q,R)_1 + \omega(Q,R)_2\right]^{-1} \cdot 2\right] \\ \ln\left[k(Q,R)^{-1}\right] \end{bmatrix}$$

$$A(Q,R) := \overrightarrow{\exp(B(Q,R))}$$

SOLUTION: $N := 9 \quad i := 1 \,..N$

$$\begin{bmatrix} T_0 \\ Q_0 \\ R_0 \end{bmatrix} := \begin{bmatrix} 552.0751311055 \\ 0.015047207655076 \\ 0.002274923964644 \end{bmatrix} \quad \begin{bmatrix} T_1 \\ Q_1 \\ R_1 \end{bmatrix} := \begin{bmatrix} 552.0751311055 \\ 0.015047207655076 \\ 0.002274923964644 \end{bmatrix}$$

These starting values were found after 100 iterations from (1,1,1).

$$\begin{bmatrix} T_{i+1} \\ Q_{i+1} \\ R_{i+1} \end{bmatrix} := 0.5 \cdot \left[A\left[Q_i, R_i\right] + \begin{bmatrix} T_{i-1} \\ Q_{i-1} \\ R_{i-1} \end{bmatrix} \right]$$

Adapted method to avoid cycling.

$i := 0 \,..N + 1$

$$T'_i := T'\left[Q_i, R_i\right]$$

$$F'_i := F'\left[Q_i, R_i\right]$$

**STEP 6: Iterations.

i	F'_i	T'_i	T_i
0	552.0751311065	552.0751310978	552.0751311055
1	552.0751311065	552.0751310978	552.0751311055
2	552.0751311065	552.0751311021	552.075131106
3	552.0751311065	552.0751311021	552.075131106
4	552.0751311065	552.0751311043	552.0751311062
5	552.0751311065	552.0751311043	552.0751311062
6	552.0751311065	552.0751311054	552.0751311063
7	552.0751311065	552.0751311054	552.0751311063
8	552.0751311065	552.0751311059	552.0751311064
9	552.0751311065	552.0751311059	552.0751311064
10	552.0751311065	552.0751311062	552.0751311064

i	Q_i	R_i
0	0.01504720765508	0.00227492396464
1	0.01504720765508	0.00227492396464
2	0.01504720765496	0.00227492396463
3	0.01504720765496	0.00227492396463
4	0.0150472076549	0.00227492396462
5	0.0150472076549	0.00227492396462
6	0.01504720765487	0.00227492396461
7	0.01504720765487	0.00227492396461
8	0.01504720765485	0.00227492396461
9	0.01504720765485	0.00227492396461
10	0.01504720765484	0.00227492396461

**STEP 7: Finding the optimal solution.

Check constraints:

C1: $12 \cdot Q_{N+1} + 200 \cdot R_{N+1} = 0.6355512848$ Satisfied

C2: $40 \cdot Q_{N+1} + 175 \cdot R_{N+1} = 1$ As expected

C6: $Q_{N+1}^{-1} \cdot R_{N+1} = 0.1511857892$ Satisfied

Conclusion: This is the optimal solution

$X' := Q_{N+1}^{-1}$ $X' = 66.4575131106$

$Y' := R_{N+1}^{-1}$ $Y' = 439.5751311062$

$Z' := Z(X',Y')$ $Z' = 552.0751311062$

Check original constraints:

$6 \cdot X'^{-1} + 100 \cdot Y'^{-1} = 0.3177756424$ (< 0.5)

$40 \cdot X'^{-1} + 175 \cdot Y'^{-1} = 1$ (= 1.0)

$6 \cdot X'^{-1} + 30 \cdot Y'^{-1} = 0.1585309649$ (< 0.2)

$2 \cdot X'^{-1} + 17 \cdot Y'^{-1} = 0.0687681227$ (< 0.1)

$3 \cdot X'^{-1} + 21 \cdot Y'^{-1} = 0.0929150262$ (< 0.12)

and it is clear that $X' < Y'$.

Appendix O

HELICAL SPRING DESIGN PROBLEM

HELICAL SPRING DESIGN PROBLEM

MINIMIZE: $y(f,m,c,d) := f \cdot c^{0.86} \cdot d^{-2.86} - m \cdot c^{0.86} \cdot d^{-2.86}$

SUBJECT TO:

C1: $0.0000605 \cdot f \cdot c^{0.86} \cdot d^{-2.715} \le 1$

C2: $0.0355 \cdot c^{2} \cdot n \cdot d^{-1} \le 1$

C3: $m \cdot f^{-1} + 0.1256 \cdot n \cdot c^{2} \cdot d \le 1$

C4: $0.1 \cdot m^{-1} \le 1$ C5: $0.05 \cdot m \le 1$

C6: $0.05 \cdot c \cdot d^{-1} \le 1$ C7: $4 \cdot d \cdot c^{-1} \le 1$

C8: $\frac{2}{3} \cdot c + \frac{2}{3} \cdot d \le 1$ C9: $d \cdot c^{-1} + 0.75 \cdot c^{-1} \le 1$

C10: $1.12 \cdot n \cdot d + 1.6 \cdot d \le 1$ C11: $3 \cdot n^{-1} \le 1$

A 17 - 5 - 1 = 11 dd multi-variable signomial with constraints.

**STEP 1: Preprocessing.

It is clear from the sign table that constraint 3 and constraint 11 must be binding, thus $n = 3$.

From constraint 3 we have that: $0.1256 \cdot n \cdot c^{2} \cdot d \cdot f := f - m$

Let $f - m := x$ or $f := x + m$ It is clear that $x > 0$.

The problem becomes:

MINIMIZE: $y(x,m,c,d) := x \cdot c^{0.86} \cdot d^{-2.86}$

SUBJECT TO:

C1: $0.0000605 \cdot x \cdot c^{0.86} \cdot d^{-2.715} + 0.0000605 \cdot m \cdot c^{0.86} \cdot d^{-2.715} \le 1$

C2: $0.1065 \cdot c^{2} \cdot d^{-1} \le 1$

C3: $0.3768 \cdot c^{2} \cdot d + 0.3768 \cdot x^{-1} \cdot m \cdot c^{2} \cdot d \le 1$

C4 - C9: Same as above

C10: $4.96 \cdot d \le 1$ C11: $n := 3$

Thus C4 must be binding and C5 not binding, m = 0.1

**STEP 2: Transforming into the all constraint form.

Let $z := x \cdot c^{0.86} \cdot d^{-2.86}$ or $x := z \cdot c^{-0.86} \cdot d^{2.86}$

The problem becomes:

MINIMIZE: z

SUBJECT TO:

C1: $0.0000605 \cdot z \cdot d^{0.145} + 0.00000605 \cdot c^{0.86} \cdot d^{-2.715} \le 1$

C2: $0.1065 \cdot c^{2} \cdot d^{-1} \le 1$

C3: $0.3768 \cdot c^{2} \cdot d + 0.03768 \cdot z^{-1} \cdot c^{2.86} \cdot d^{-1.86} \le 1$

C6: $0.05 \cdot c \cdot d^{-1} \le 1$ C7: $4 \cdot d \cdot c^{-1} \le 1$

C8: $\frac{2}{3} \cdot c + \frac{2}{3} \cdot d \le 1$ C9: $d \cdot c^{-1} + 0.75 \cdot c^{-1} \le 1$

C10: $4.96 \cdot d \le 1$

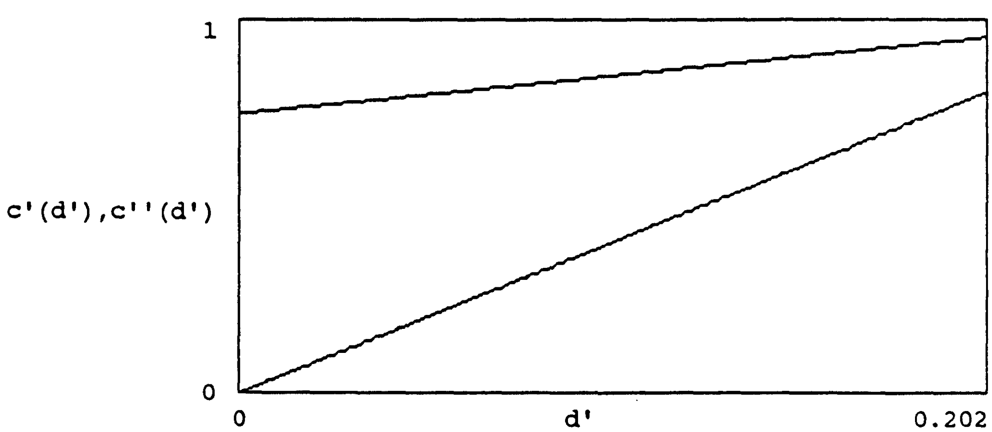
Either C7 or C9 (or both) must be binding.

From C10 we know that: $d \le \frac{1}{4.96}$

Examine C7 and C9 if: $0 \le d \le 0.201613$

From C7: $c'(d) := 4 \cdot d$

From C9: $c''(d) := d + 0.75$ $d' := 0, 0.001 .. 0.202$



Thus C9 dominates C7, C7 can not be binding and C9 must be binding.

To find the Kuhn-Tucker points, we have to consider the following possibilities:

1. Objective function only - unbalanced
2. Objective function and any one constraint - unbalanced
3. Objective function and C3 and C9
4. Objective function and any two constraints (except C3 and C9) - unbalanced
5. Three constraints - three simultaneous equations
6. Four or more constraints - overspecified

Consider possibility no. 3 first.

MINIMIZE: z

SUBJECT TO:

$$0.3768 \cdot c^{2} \cdot d + 0.03768 \cdot z^{-1} \cdot c^{2.86} \cdot d^{-1.86} \le 1$$

$$d \cdot c^{-1} + 0.75 \cdot c^{-1} \le 1$$

A 5 - 3 - 1 = 1 dd multi-variable posynomial with constraints.

**STEP 3: Reducing the d.d.

RULE 2A:
RULE 2B:

$$\begin{matrix} \\ z: \\ c: \\ d: \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & 2.86 & -1 & -1 \\ 0 & 1 & -1.86 & 1 & 0 \end{bmatrix} \cdot \delta := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Condense the fourth and fifth term:

$$\delta 1(d) := \frac{d}{d + 0.75} \qquad \delta 2(d) := \frac{0.75}{d + 0.75}$$

$$k(d) := \left[\frac{1}{\delta 1(d)}\right]^{\delta 1(d)} \cdot \left[\frac{0.75}{\delta 2(d)}\right]^{\delta 2(d)}$$

Thus the problem is now:

MINIMIZE: z

Subject to:

$$0.3768 \cdot c^{2} \cdot d + 0.03768 \cdot z^{-1} \cdot c^{2.86} \cdot d^{-1.86} \le 1$$

$$k(d) \cdot c^{-1} \cdot d^{\delta 1(d)} \le 1$$

**STEP 4: Finding a dual feasible solution.

RULE 2:

$$W(d) := \begin{bmatrix} 0 & -1 & 0 \\ 2 & 2.86 & -1 \\ 1 & -1.86 & \S 1(d) \end{bmatrix}^{-1} \qquad \omega(d) := \left[\mathrm{augment}\left[\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, W(d)^{T}\right]\right]^{T} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega(1) = \begin{bmatrix} 1 \\ 0.1053333333 \\ 1 \\ 3.0706666667 \end{bmatrix}$$

RULE 1:

$$G(d) := \left[\frac{0.3768}{\omega(d)_1}\right]^{\omega(d)_1} \cdot \left[\frac{0.03768}{\omega(d)_2}\right]^{\omega(d)_2} \cdot \left[\omega(d)_1 + \omega(d)_2\right]^{\omega(d)_1 + \omega(d)_2} \cdot k(d)^{\omega(d)_3}$$

**STEP 5: Finding the values of the primal variables.

RULE 4:

$$B(d) := W(d)^{T} \cdot \begin{bmatrix} \ln\left[\omega(d)_1 \cdot \left[\omega(d)_1 + \omega(d)_2\right]^{-1} \cdot 0.3768^{-1}\right] \\ \ln\left[\omega(d)_2 \cdot \left[\omega(d)_1 + \omega(d)_2\right]^{-1} \cdot 0.03768^{-1}\right] \\ \ln\left[k(d)^{-1}\right] \end{bmatrix}$$

$$A(d) := \overrightarrow{\exp(B(d))}$$

SOLUTION:

$N := 23 \quad i := 0 \,..N$

$i := 0 \,..N + 1$

$$\begin{bmatrix} z_0 \\ c_0 \\ d_0 \end{bmatrix} := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} z_{i+1} \\ c_{i+1} \\ d_{i+1} \end{bmatrix} := A\left[d_i\right] \qquad G_i := G\left[d_i\right]$$

$$m_i := 0.1 \quad f_i := x_i + m_i \quad y_i := y\left[f_i, m_i, c_i, d_i\right] \quad x_i := z_i \cdot c_i^{-0.86} \cdot d_i^{2.86}$$

**STEP 6: Iterations.

i	G_i	y_i	z_i
0	0.268399086823	1	1
1	0.32034996254	0.268399086823	0.268399086823
2	0.355403405247	0.32034996254	0.32034996254
3	0.363982923973	0.355403405247	0.355403405247
4	0.366213473505	0.363982923973	0.363982923973
5	0.366721078721	0.366213473505	0.366213473505
6	0.366840537821	0.366721078721	0.366721078721
7	0.366868017112	0.366840537821	0.366840537821
8	0.366874397648	0.366868017112	0.366868017112
9	0.366875871976	0.366874397648	0.366874397648
10	0.366876213413	0.366875871976	0.366875871976
11	0.366876292398	0.366876213413	0.366876213413
12	0.36687631068	0.366876292398	0.366876292398
13	0.36687631491	0.36687631068	0.36687631068
14	0.366876315889	0.36687631491	0.36687631491
15	0.366876316116	0.366876315889	0.366876315889
16	0.366876316168	0.366876316116	0.366876316116
17	0.36687631618	0.366876316168	0.366876316168
18	0.366876316183	0.36687631618	0.36687631618
19	0.366876316184	0.366876316183	0.366876316183
20	0.366876316184	0.366876316184	0.366876316184
21	0.366876316184	0.366876316184	0.366876316184
22	0.366876316184	0.366876316184	0.366876316184
23	0.366876316184	0.366876316184	0.366876316184
24	0.366876316184	0.366876316184	0.366876316184

i	c_i	d_i	f_i	m_i
0	1	1	1.1	0.1
1	0.899927522	0.3122817461	0.1105333333	0.1
2	1.2973541013	0.6164192329	0.1641859863	0.1
3	1.1762186351	0.442158307	0.129953966	0.1
4	1.2664650843	0.5204496839	0.1458874645	0.1
5	1.2303543406	0.4812618562	0.1378372239	0.1
6	1.2495662144	0.4997788356	0.1416512059	0.1
7	1.2407422235	0.4907911909	0.1397993436	0.1
8	1.2450851442	0.4950965051	0.1406866253	0.1
9	1.2430185305	0.4930211565	0.1402589234	0.1
10	1.2440179264	0.4940185344	0.1404644756	0.1
11	1.2435383719	0.4935385126	0.1403655475	0.1
12	1.2437693448	0.4937693773	0.1404131269	0.1
13	1.2436582988	0.4936583063	0.1403902361	0.1
14	1.2437117331	0.4937117349	0.1404012473	0.1
15	1.2436860317	0.4936860321	0.1403959502	0.1
16	1.2436983963	0.4936983964	0.1403984983	0.1
17	1.2436924484	0.4936924485	0.1403972725	0.1
18	1.2436953098	0.4936953098	0.1403978622	0.1
19	1.2436939333	0.4936939333	0.1403975785	0.1
20	1.2436945955	0.4936945955	0.140397715	0.1
21	1.2436942769	0.4936942769	0.1403976493	0.1
22	1.2436944302	0.4936944302	0.1403976809	0.1
23	1.2436943565	0.4936943565	0.1403976657	0.1
24	1.2436943919	0.4936943919	0.140397673	0.1

**STEP 7: Finding the optimal solution.

$n := 3 \quad m := 0.1 \quad c := c_{N+1} \quad d := d_{N+1} \quad f := f_{N+1}$

$c = 1.2436943919$

$d = 0.4936943919$

$f = 0.140397673$

$y_{N+1} = 0.3668763162$

Check constraints:

C1:	$0.0000605 \cdot f \cdot c^{0.86} \cdot d^{-2.715} = 0.0000696354$	Satisfied
C2:	$0.0355 \cdot c^{2} \cdot n \cdot d^{-1} = 0.3336712328$	Satisfied
C3:	$m \cdot f^{-1} + 0.1256 \cdot n \cdot c^{2} \cdot d = 1$	As expected
C4:	$0.1 \cdot m^{-1} = 1$ C5: $0.05 \cdot m = 0.005$	As expected
C6:	$0.05 \cdot c \cdot d^{-1} = 0.1259579218$	Satisfied
C7:	$4 \cdot d \cdot c^{-1} = 1.5878318504$	VIOLATED
C8:	$\frac{2}{3} \cdot c + \frac{2}{3} \cdot d = 1.1582591892$	VIOLATED
C9:	$d \cdot c^{-1} + 0.75 \cdot c^{-1} = 1$	As expected
C10:	$1.12 \cdot n \cdot d + 1.6 \cdot d = 2.4487241839$	VIOLATED
C11:	$3 \cdot n^{-1} = 1$	As expected

Continued on the next file. Note that to provide the next file with the problem and necessary definitions, some material will be repeated.

Possibility no. 5 will be considered.

HELICAL SPRING DESIGN PROBLEM (CONTINUED)

MINIMIZE: $y(f,m,c,d) := f\cdot c^{0.86}\cdot d^{-2.86} - m\cdot c^{0.86}\cdot d^{-2.86}$

SUBJECT TO:

C1: $0.0000605\cdot f\cdot c^{0.86}\cdot d^{-2.715} \le 1$

C2: $0.0355\cdot c^{2}\cdot n\cdot d^{-1} \le 1$

C3: $m\cdot f^{-1} + 0.1256\cdot n\cdot c^{2}\cdot d \le 1$

C4: $0.1\cdot m^{-1} \le 1$ C5: $0.05\cdot m \le 1$

C6: $0.05\cdot c\cdot d^{-1} \le 1$ C7: $4\cdot d\cdot c^{-1} \le 1$

C8: $\frac{2}{3}\cdot c + \frac{2}{3}\cdot d \le 1$ C9: $d\cdot c^{-1} + 0.75\cdot c^{-1} \le 1$

C10: $1.12\cdot n\cdot d + 1.6\cdot d \le 1$ C11: $3\cdot n^{-1} \le 1$

Let $f - m := x$ or $f := x + m$

Let $z := x\cdot c^{0.86}\cdot d^{-2.86}$ or $x := z\cdot c^{-0.86}\cdot d^{2.86}$

The problem becomes:

MINIMIZE: z

SUBJECT TO:

C1: $0.0000605\cdot z\cdot d^{0.145} + 0.00000605\cdot c^{0.86}\cdot d^{-2.715} \le 1$

C2: $0.1065\cdot c^{2}\cdot d^{-1} \le 1$

C3: $0.3768\cdot c^{2}\cdot d + 0.03768\cdot z^{-1}\cdot c^{2.86}\cdot d^{-1.86} \le 1$

C6: $0.05\cdot c\cdot d^{-1} \le 1$ C7: $4\cdot d\cdot c^{-1} \le 1$

C8: $\frac{2}{3}\cdot c + \frac{2}{3}\cdot d \le 1$ C9: $d\cdot c^{-1} + 0.75\cdot c^{-1} \le 1$

C10: $4.96\cdot d \le 1$

To find the Kuhn-Tucker points, we have to consider the following possibilities:
1. Objective function only - unbalanced
2. Objective function and any one constraint - unbalanced
3. Objective function and C3 and C9
4. Objective function and any two constraints (except C3 and C9) - unbalanced
5. Three constraints - three simultaneous equations
6. Four or more constraints - overspecified

Consider possibility no. 5 now.

**STEP 1: Preprocessing.

MINIMIZE: z

SUBJECT TO:

C3: $0.3768 \cdot c^{2} \cdot d + 0.03768 \cdot z^{-1} \cdot c^{2.86} \cdot d^{-1.86} \le 1$

C9: $d \cdot c^{-1} + 0.75 \cdot c^{-1} \le 1$

AND: C1: $0.0000605 \cdot z \cdot d^{0.145} + 0.00000605 \cdot c^{0.86} \cdot d^{-2.715} \le 1$

OR C2: $0.1065 \cdot c^{2} \cdot d^{-1} \le 1$

OR C6: $0.05 \cdot c \cdot d^{-1} \le 1$

OR C8: $\frac{2}{3} \cdot c + \frac{2}{3} \cdot d \le 1$

OR C10: $4.96 \cdot d \le 1$

From C9: $c := d + 0.75$

C2 implies:

$$0.1065 \cdot \left[d^{2} + 1.5 \cdot d + 0.5625\right] \le d$$

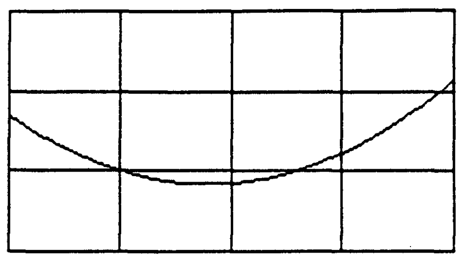
$$0.1065 \cdot d^{2} - 0.84025 \cdot d + 0.05990625 \le 0$$

$$\frac{0.84025 + \sqrt{0.84025^{2} - 4 \cdot 0.1065 \cdot 0.05990625}}{2 \cdot 0.1065} = 7.8177194317$$

$$\frac{0.84025 - \sqrt{0.84025^{2} - 4 \cdot 0.1065 \cdot 0.05990625}}{2 \cdot 0.1065} = 0.0719519298$$

$f(d) := 0.1065 \cdot d^{2} - 0.84025 \cdot d + 0.05990625$

$d := -5, -4.9 .. 15$



Thus C2 implies that: $0.071952 \le d \le 7.8177$ ▫

C6 implies:

$0.05 \cdot (d + 0.75) \cdot d^{-1} \le 1$ ▫

$0.05 \cdot d + 0.0375 \le d$ ▫

$-0.95 \cdot d \le -0.0375$ ▫

$d \ge 0.039474$ ▫

C8 implies:

$\frac{2}{3} \cdot (d + 0.75) + \frac{2}{3} \cdot d \le 1$ ▫

$\frac{4}{3} \cdot d + 0.5 \le 1$ ▫

$d \le 0.375$ ▫

C10 implies:

$4.96 \cdot d \le 1$ ▫

$d \le 4.96^{-1}$ ▫ $4.96^{-1} = 0.2016129032$

C1 implies: (Worst case: z = 0)

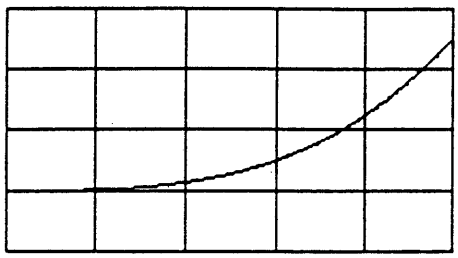
$0.00000605 \cdot (d + 0.75)^{0.86} \cdot d^{-2.715} \le 1$ ▫

$0.00000605 \cdot (d + 0.75)^{0.86} \le d^{2.715}$ ▫

$$d^{\frac{2.715}{0.86}} - 0.00000605 \cdot d - 0.0000045375 \geq 0$$

$$f(d) := d^{\frac{2.715}{0.86}} - 0.00000605 \cdot d - 0.0000045375$$

d := 0,0.001 ..0.25



f(0.020474107942) = 0

Thus $d \geq 0.02047410794$

Thus we know that: $0.071952 \leq d \leq 0.20161$

or either C2 or C10 is a binding constraint.

**STEP 7: Finding the optimal solution.

Solution with C10 a binding constraint:

$d := 4.96^{-1}$ $\qquad d = 0.2016129032$

$c := d + 0.75$ $\qquad c = 0.9516129032$

$z := \left[\dfrac{0.03768}{1 - 0.3768 \cdot c^{2} \cdot d}\right] \cdot c^{2.86} \cdot d^{-1.86}$ $\qquad z = 0.6903312228$

$x := z \cdot c^{-0.86} \cdot d^{2.86}$ $\qquad x = 0.0073876114$

$m := 0.1$ $\qquad m = 0.1$

$f := x + m$ $\qquad f = 0.1073876114$

$n := 3$ $\qquad n = 3$

y(f,m,c,d) = 0.6903312228

Check the original constraints:

C1:	$0.0000605 \cdot f \cdot c^{0.86} \cdot d^{-2.715} = 0.0004813026$	Satisfied
C2:	$0.0355 \cdot c^{2} \cdot n \cdot d^{-1} = 0.4783567742$	Satisfied
C3:	$m \cdot f^{-1} + 0.1256 \cdot n \cdot c^{2} \cdot d = 1$	As expected
C4:	$0.1 \cdot m^{-1} = 1$	As expected
C5:	$0.05 \cdot m = 0.005$	Satisfied
C6:	$0.05 \cdot c \cdot d^{-1} = 0.236$	Satisfied
C7:	$4 \cdot d \cdot c^{-1} = 0.8474576271$	Satisfied
C8:	$\frac{2}{3} \cdot c + \frac{2}{3} \cdot d = 0.7688172043$	Satisfied
C9:	$d \cdot c^{-1} + 0.75 \cdot c^{-1} = 1$	As expected
C10:	$1.12 \cdot n \cdot d + 1.6 \cdot d = 1$	As expected
C11:	$3 \cdot n^{-1} = 1$	As expected

NOTE: The solution with C2 binding is: $y = 2.9275295015$ and is feasible but not optimal.