LAPLACE'S EQUATION ON PERTURBED DOMAINS

by

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ABSTRACT

Laplace's equation is a prototypical elliptic PDE that appears in many electromagnetic and fluid dynamics problems. We develop two methods for solving Laplace's equation on domains that are perturbations of a circle. These methods are derived from governing equations and applied to several test cases. Both Dirichlet and Neumann boundary conditions are considered. We verify our methods by constructing exact solutions for the perturbed geometries.

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CHAPTER 1
INTRODUCTION

Laplace's equation appears in many fields of applied mathematics, and its solutions, called potentials, are important in many electromagnetic and fluid dynamics problems. We are able to express solutions of Laplace's equation in terms of elementary and special functions only in very specific cases, and often times numerical calculations are employed. One might be interested in how the solution changes as we vary the solution point, and this behavior is not always discernible from looking at solutions computed numerically. We will develop and demonstrate two methods for calculating the potential for a domain that has been perturbed from a circle.

Solving Laplace's equation on a perturbed domain is not a new idea. Indeed, several approaches for solving it have been developed in the past. Hadamard's variational formula is an integral equation that will compute the Green's function for a perturbed domain; however, it is generally quite difficult to evaluate the necessary integrals. Even once we have the exact Green's function, we still must use it in another integral over our domain.

An approach used by Joseph[4] to solve the inviscid, irrotational Euler equation in two dimensions expands the velocity potential, free surface boundary condition, and the wave speed in terms of a small perturbation parameter. Then, by using these expansions in the governing equations, he develops a hierarchy of problems that relate the solution on the "wavy" domain to solutions on the unperturbed, flat domain. In this process, one still must solve a problem involving wave propagation, but by expanding in a small parameter, solutions become easier to obtain.

A similar process is used by Bruno and Reitich[1][2] to solve the Helmholtz equation. In [1] they investigate the three-dimensional scattering problem for a very specific set of incident waves for a very specific boundary condition. They assume the scatterer is a perturbation of a sphere and obtain numerical results that compare favorably with known results.

Several years later, Bruno and Reitich consider the interior Helmholtz equation on a unit disc[2]. In this situation, they are solving the eigenvalue problem for a two-dimensional membrane. As a result, in addition to expanding the potential in a power series, they also allow for perturbations of the eigenvalue as well.

Both Bruno and Reitich as well as Joseph are very careful to show that the expansion method yields well-posed problems, and that solutions exist. Both authors use proofs based on conformal mapping that utilize analytic continuation. Bruno and Reitich[2] have a particularly rigorous proof for their consideration of the Helmholtz equation. We will not be proving such properties about our problem and its solutions; however, these authors' methods are similar to the ones we develop and small adjustments to the proofs they present would likely satisfy the reader's curiosity.

In this thesis, we develop and apply a method that solves a hierarchy of problems on the unperturbed domain. This method is similar to those employed by Bruno and Reitich, and Joseph. We develop this method from governing equations in detail in chapter 2. We then apply this method to several test cases and construct exact solutions for comparison in chapters 3 and 4. In chapter 5 we develop an alternative solution method using an integral representation for the potential. An example of this method is then shown in chapter 6 and compared with constructed exact solutions.

CHAPTER 2
MATHEMATICAL FORMULATION

We will first consider Laplace's equation with either a Dirichlet or Neumann boundary condition.

$$\nabla^2 u = 0 \quad \text{outside } \Omega$$
$$u = F \qquad \text{on } \partial\Omega$$

or

$$\nabla^2 u = 0 \quad \text{outside } \Omega$$
$$\frac{\partial u}{\partial n} = F \qquad \text{on } \partial\Omega$$

where F is a known function and Ω is our perturbed domain. For the Neumann problem, we consider an outward-facing normal vector. The domain Ω is given by

$$\Omega = \{(r, \theta) : r \leq a(1 + \varepsilon g(\theta; \varepsilon))\}$$

for a positive constant a and some known 2π periodic function $g(\theta; \varepsilon)$. ε is a small positive constant that governs the magnitude of the perturbation. Additionally, we seek solutions that are bounded at infinity.

As Ω is simply a perturbation of a circle of radius a, it is natural to approach this problem using polar coordinates. With this convention, on $\partial\Omega$, we have

$$u(r, \theta)|_{r=a+\varepsilon a g(\theta;\varepsilon)} = F(r, \theta)|_{r=a+\varepsilon a g(\theta;\varepsilon)}.$$

If we expand this boundary condition about $\varepsilon = 0$, we find

$$u(r, \theta)|_{r=a+\varepsilon a g(\theta;\varepsilon)} = u(a, \theta) + \varepsilon a g(\theta; 0)\frac{\partial u}{\partial r}(a, \theta) + \varepsilon^2 \left(\frac{1}{2}a^2 g(\theta; 0)^2 \frac{\partial^2 u}{\partial r^2} + a\frac{\partial g}{\partial \varepsilon}(\theta; 0)\frac{\partial u}{\partial r}(a, \theta)\right) + \cdots \tag{2.1}$$

A similar expression is found when we expand the right hand side.

Now we assume that we can express $u(r, \theta)$ as a power series in terms of ε.

$$u(r, \theta) = \sum_{n=0}^{\infty} \varepsilon^n u_n(r, \theta).$$

Mathematically, we are asserting that the solution to the perturbed problem is simply a perturbation of the solution to the unperturbed problem. If we substitute this series into Laplace's equation, it is easily seen that each $u_n(r,\theta)$ must satisfy $\nabla^2 u_n(r,\theta) = 0$. Now we substitute this power series into our expanded boundary condition (2.1), and match powers of epsilon, we obtain the following series of boundary conditions.

$$\begin{aligned}
&\varepsilon^0 : u_0(a,\theta) = F(a,\theta) \\
&\varepsilon^1 : u_1(a,\theta) = ag(\theta;0)\left(\frac{\partial F}{\partial r}(a,\theta) - \frac{\partial u_0}{\partial r}(a,\theta)\right) \\
&\varepsilon^2 : u_2(a,\theta) = \frac{a^2 g(\theta;0)^2}{2}\left(\frac{\partial^2 F}{\partial r^2}(a,\theta) - \frac{\partial^2 u_0}{\partial r^2}(a,\theta)\right) + a\frac{\partial g}{\partial \varepsilon}(\theta;0)\left(\frac{\partial F}{\partial r}(a,\theta) - \frac{\partial u_0}{\partial r}(a,\theta)\right) \\
&\qquad\qquad - ag(\theta;0)\frac{\partial u_1}{\partial r}(a,\theta) \\
&\varepsilon^3 : \cdots
\end{aligned} \tag{2.2}$$

The Neumann problem on $\partial\Omega$ is given by

$$\frac{\partial u}{\partial n}(r,\theta)|_{r=a+\varepsilon ag(\theta)} = F(r,\theta)|_{r=a+\varepsilon ag(\theta)}.$$

To solve this problem, we adopt a similar process; however, we must make some restrictions on our boundary function $F(r,\theta)$ so that our problem is well-posed.

The boundary of Ω is given by $r - \varepsilon ag(\theta;\varepsilon) - a = 0$. If we call the left hand side of this equation $H(r,\theta;\varepsilon)$, then the boundary is simply $H(r,\theta;\varepsilon) = 0$. Thus, the normal vector to this boundary is given by

$$\begin{aligned}
\vec{n} &= \frac{\nabla H(r,\theta;\varepsilon)}{|\nabla H(r,\theta;\varepsilon)|} \\
&= \frac{1}{|\left\langle \frac{\partial H}{\partial r}, \frac{1}{r}\frac{\partial H}{\partial \theta}\right\rangle|}\left\langle \frac{\partial H}{\partial r}, \frac{1}{r}\frac{\partial H}{\partial \theta}\right\rangle \\
&= \frac{1}{\sqrt{1 + \left(\frac{\varepsilon ag'(\theta;\varepsilon)}{a+\varepsilon ag(\theta;\varepsilon)}\right)^2}}\left\langle 1, -\frac{\varepsilon a\frac{\partial g}{\partial \theta}(\theta;\varepsilon)}{a+\varepsilon ag(\theta;\varepsilon)}\right\rangle.
\end{aligned} \tag{2.3}$$

If we expand each term of the vector in a series about $\varepsilon = 0$, we get a formula for the vector normal to $\partial\Omega$.

$$\vec{n} = \left\langle 1 - \frac{\varepsilon^2}{2}\frac{\partial g}{\partial \theta}(\theta;0) + \cdots, -\varepsilon\frac{\partial g}{\partial \theta}(\theta;0) + \varepsilon^2\left(g(\theta;0)\frac{\partial g}{\partial \theta}(\theta;0) - \frac{\partial^2 g}{\partial\theta\partial\varepsilon}(\theta;0)\right) + \cdots\right\rangle.$$

Using this as our normal vector, we can calculate $\frac{\partial u}{\partial n}(r,\theta) = \nabla u(r,\theta)\cdot \vec{n}$.

$$\begin{aligned}\nabla u(r,\theta)\cdot\vec{n} &= \left\langle \frac{\partial u}{\partial r}(r,\theta), \frac{1}{r}\frac{\partial u}{\partial \theta}(r,\theta)\right\rangle \cdot \vec{n} \\ &= \frac{\partial u}{\partial r}(r,\theta) - \varepsilon\left(\frac{1}{r}\frac{\partial u}{\partial \theta}(r,\theta)\frac{\partial g}{\partial \theta}(\theta;0)\right) + \varepsilon^2\left(-\frac{1}{2}\frac{\partial u}{\partial r}(r,\theta)\frac{\partial g}{\partial \theta}(\theta;0)\right. \\ &\quad \left. + \frac{g(\theta;0)}{r}\frac{\partial g}{\partial \theta}(\theta;0)\frac{\partial u}{\partial \theta}(r,\theta) + \frac{1}{r}\frac{\partial^2 g}{\partial\theta\partial\varepsilon}(\theta;0)\frac{\partial u}{\partial \theta}(r,\theta)\right) + \cdots\end{aligned}$$

In order to obtain the desired series of boundary conditions, we again expand around $\varepsilon = 0$, substitute our power series expansion for $u(r,\theta)$ and evaluate on $r = a$. We then gather terms by powers of ε and set them equal to the expansion of $F(a(1+\varepsilon g(\theta;\varepsilon)),\theta)$ about $\varepsilon = 0$. Doing this gives us a series of Neumann boundary conditions,

$$\begin{aligned}&\varepsilon^0 : \frac{\partial u_0}{\partial r}(a,\theta) = F(a,\theta) \\ &\varepsilon^1 : \frac{\partial u_1}{\partial r}(a,\theta) = ag(\theta;0)\left(\frac{\partial F}{\partial r}(a,\theta) - \frac{\partial^2 u_0}{\partial r^2}(a,\theta)\right) + \frac{1}{a}\frac{\partial g}{\partial \theta}(\theta;0)\frac{\partial u_0}{\partial \theta}(a,\theta) \\ &\varepsilon^2 : \frac{\partial u_2}{\partial r}(a,\theta) = \frac{a^2 g(\theta;0)^2}{2}\left(\frac{\partial^2 F}{\partial r^2}(a,\theta) - \frac{\partial^3 u_0}{\partial r^3}(a,\theta)\right) + a\frac{\partial g}{\partial \varepsilon}(\theta;0)\left(\frac{\partial F}{\partial r}(a,\theta) - \frac{\partial^2 u_0}{\partial r^2}(a,\theta)\right) \\ &\qquad + \frac{1}{a}\left(\frac{\partial u_1}{\partial \theta}(a,\theta)\frac{\partial g}{\partial \theta}(\theta;0) - g(\theta;0)\frac{\partial u_0}{\partial \theta}(a,\theta)\frac{\partial g}{\partial \theta}(\theta;0) - \frac{\partial u_0}{\partial \theta}(a,\theta)\frac{\partial^2 g}{\partial\theta\partial\varepsilon}(\theta;0)\right) \\ &\qquad + g(\theta;0)\left(\frac{\partial g}{\partial \theta}(\theta;0)\frac{\partial^2 u_0}{\partial r\partial\theta}(a,\theta) - a\frac{\partial^2 u_1}{\partial r^2}(a,\theta)\right) + \frac{\partial u_0}{\partial r}(a,\theta)\left(\frac{1}{2}\frac{\partial g}{\partial \theta}(\theta;0)\right. \\ &\qquad \left. + \left(\frac{\partial g}{\partial \theta}(\theta;0)\right)^2\right) \\ &\varepsilon^3 : \cdots\end{aligned} \tag{2.4}$$

Consider Green's first identity (a consequence of the divergence theorem),

$$\int_{\mathbb{R}^2\setminus\Omega} w\nabla^2 v + \nabla w\cdot\nabla v dS = \oint_{\partial\Omega} w\frac{\partial v}{\partial n}ds.$$

where v is a twice continuously differentiable function and w is a once continuously differentiable function. Both w and v are defined on the region of integration $\mathbb{R}^2\setminus\Omega$. Then, if we choose $w = 1$ and $v = u$, we have

$$\int_{\partial\Omega}\frac{\partial u}{\partial n}ds = 0.$$

We now must calculate what our differential length ds is. By noting that in polar coordinates, arc length is given by $\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$, we have

$$\begin{aligned} ds &= \sqrt{a^2\left(1+\varepsilon g(\theta;\varepsilon)\right)^2 + a^2\varepsilon^2\left(\frac{\partial g}{\partial \theta}(\theta;\varepsilon)\right)^2}\,d\theta \\ &= a\sqrt{1+2\varepsilon g(\theta;\varepsilon)+\varepsilon^2\left[g(\theta;\varepsilon)^2+\left(\frac{\partial g}{\partial\theta}(\theta;\varepsilon)\right)^2\right]}\,d\theta. \end{aligned} \tag{2.5}$$

Thus, for the Neumann problem to be solvable, we must require

$$\int_{-\pi}^{\pi} F(a+\varepsilon a g(\theta;\varepsilon),\theta)\sqrt{1+2\varepsilon g(\theta;\varepsilon)+\varepsilon^2\left[g(\theta;\varepsilon)^2+\left(\frac{\partial g}{\partial\theta}(\theta;\varepsilon)\right)^2\right]}\,d\theta = 0.$$

In general, it will be very difficult for this boundary condition to be met for all orders of ε. However, we can develop simpler conditions that we can verify for each order of ε.

Since each of the problems for $u_i(r,\theta)$ is a Neumann problem, we must check that they are well posed. In order for the u_0 problem to be solvable, it is necessary that the integral of its normal derivative over the unperturbed circle be 0.

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{\partial u_0}{\partial n}d\theta &= \int_{-\pi}^{\pi}\frac{\partial u_0}{\partial r}(a,\theta)d\theta \\ &= \int_{-\pi}^{\pi} F(a,\theta)d\theta. \end{aligned} \tag{2.6}$$

For the u_1 problem, the integral of the right hand side of (2.4) over $[-\pi,\pi)$ must also be 0. So we must satisfy

$$\int_{-\pi}^{\pi} a g(\theta;0)\frac{\partial F}{\partial r}(a,\theta) + \frac{1}{a}\frac{\partial g}{\partial\theta}(\theta;0)\frac{\partial u_0}{\partial\theta}(a,\theta) - a g(\theta;0)\frac{\partial^2 u_0}{\partial r^2}(a,\theta)d\theta = 0.$$

Since $\nabla^2 u_0 = 0$, we have that $\frac{\partial^2 u_0}{\partial r^2}(a,\theta) = -\frac{1}{a}\frac{\partial u_0}{\partial r}(a,\theta) - \frac{1}{a^2}\frac{\partial^2 u_0}{\partial\theta^2}(a,\theta)$. Substituting this into the previous integral yields

$$\begin{aligned} \int_{-\pi}^{\pi} a g(\theta;0)\frac{\partial F}{\partial r}(a,\theta) + \frac{1}{a}\frac{\partial g}{\partial\theta}(\theta;0)\frac{\partial u_0}{\partial\theta}(a,\theta) + g(\theta;0)\frac{\partial u_0}{\partial r}(a,\theta) + \frac{g(\theta;0)}{a}\frac{\partial^2 u_0}{\partial\theta^2}(a,\theta)d\theta = 0 \\ \frac{1}{a}\int_{-\pi}^{\pi}\frac{\partial}{\partial\theta}\left(g(\theta;0)\frac{\partial u_0}{\partial\theta}\right)d\theta + a\int_{-\pi}^{\pi} g(\theta;0)\left(\frac{\partial F}{\partial r}(a,\theta) + \frac{1}{a}\frac{\partial u_0}{\partial r}(a,\theta)\right)d\theta = 0. \end{aligned}$$

The first integral is simply $\left[g(\theta;0)\frac{\partial u_0}{\partial\theta}\right]|_{-\pi}^{\pi}$ which is 0 since $g(\theta;0)$ is 2π periodic. Now, noting that

$\frac{\partial u_0}{\partial r}(a,\theta) = F(a,\theta)$, we get our requirement for the u_1 problem,

$$\int_{-\pi}^{\pi} g(\theta;0)\left(\frac{\partial F}{\partial r}(a,\theta) + \frac{1}{a}F(a,\theta)\right) d\theta = 0. \tag{2.7}$$

Therefore, the Neumann problem for $u_1(r,\theta)$ is well posed if both (2.6) and (2.7) are satisfied. Higher order Neumann problems, i.e the problems for u_n with $n \geq 2$, will impose more and harsher restrictions on what we consider to be a suitable boundary condition.

CHAPTER 3
THE TRANSLATED CIRCLE

We will now solve Laplace's equation outside of a circle that has been slightly shifted. We begin by employing the methods described in chapter 2 to obtain an approximate solution. We will then derive the exact answer to the problem using coordinate transformations and compare to the previously computed answer. Both Dirichlet and Neumann boundary conditions are considered.

3.1 Finding $g(\theta; \varepsilon)$

Consider a circle that has been perturbed a small distance δ along the positive x-axis away from the origin. This geometry is shown in Figure 1.

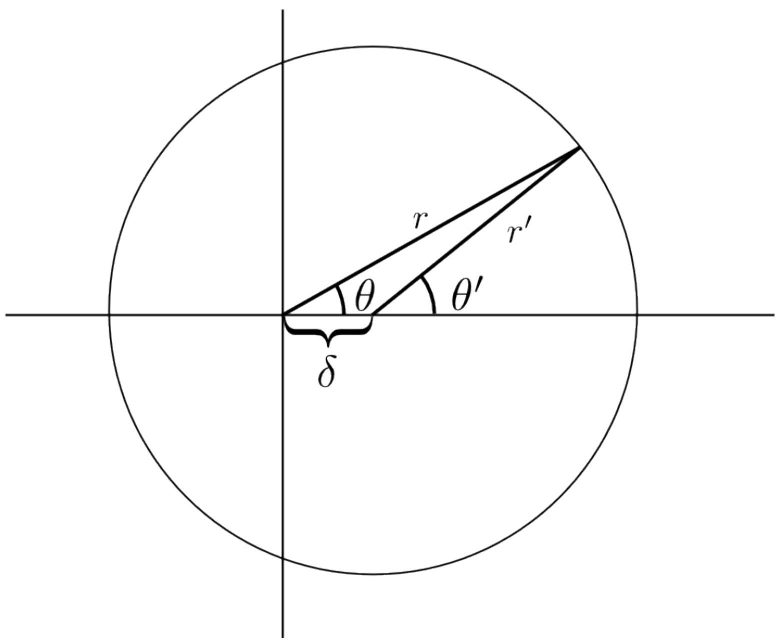

Figure 1: Translated Circle

The equation for such a circle is given by

$$(x - \delta)^2 + y^2 = a^2.$$

Converting to standard polar coordinates we get,

$$\begin{aligned} a^2 &= (r\cos\theta - \delta)^2 + (r\sin\theta)^2 \\ &= r^2 - 2r\delta\cos\theta + \delta^2. \end{aligned}$$

Solving this equation for r gives us

$$\begin{aligned} r &= \frac{1}{2}\left(2\delta\cos\theta \pm \sqrt{4\delta^2\cos^2\theta - 4\left(\delta^2 - a^2\right)}\right) \\ r &= \delta\cos\theta \pm \sqrt{a^2 - \delta^2\sin^2\theta}. \end{aligned}$$

If we take $\delta = 0$, we would have a circle of radius a. Thus, we disregard the negative square root. Since δ is small, $\frac{\delta}{a} \ll 1$. If we manipulate our expression for r slightly, we can get it in the form

$$r = a\left(\frac{\delta}{a}\cos\theta + \sqrt{1 - \left(\frac{\delta}{a}\right)^2\sin^2\theta}\right).$$

Now we expand the square root about $\frac{\delta}{a} = 0$ to get an expression for r in terms of $\frac{\delta}{a}$.

$$r = a\left(1 + \frac{\delta}{a}\cos\theta - \frac{1}{2}\left(\frac{\delta}{a}\right)^2\sin^2\theta + \cdots\right).$$

Thus, if we denote $\frac{\delta}{a}$ by ε, then we can set $g(\theta;\varepsilon) = \cos\theta - \frac{1}{2}\varepsilon\sin^2\theta + \cdots$. Then the domain given by $r \leq a(1 + \varepsilon g(\theta;\varepsilon))$ is the circle of radius a shifted to the right by a small amount δ, where $\delta = \varepsilon a$.

3.2 The Dirichlet Problem

We will begin by solving the Dirichlet problem with the given boundary function $F(r,\theta) = r\cos\theta$. Then, the u_0 problem becomes.

$$\begin{aligned} \nabla^2 u_0(r,\theta) &= 0 \\ u_0(a,\theta) &= F(a,\theta). \end{aligned}$$

Here, our boundary condition is simply $F(a,\theta) = a\cos\theta$. Since $u_0(r,\theta)$ satisfies Laplace's equation on the unperturbed, circular domain, we can immediately write down solutions of the form

$$u_0(r,\theta) = B_{0,0} + \sum_{n=1}^{\infty} \{A_{0,n}\sin(n\theta) + B_{0,n}\cos(n\theta)\}\, r^{-n}.$$

Positive powers of r and the logarithmic solution are omitted as we seek solutions that are finite at infinity. Using simple orthogonality relationships, we can solve for the coefficients.

$$\begin{aligned} B_{0,0} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_0(a,\theta)d\theta \\ A_{0,n} &= \frac{a^n}{\pi}\int_{-\pi}^{\pi} F_0(a,\theta)\sin(n\theta)d\theta \\ B_{0,n} &= \frac{a^n}{\pi}\int_{-\pi}^{\pi} F_0(a,\theta)\cos(n\theta)d\theta \end{aligned}$$

Using the known form of $F(r,\theta)$, we find that all coefficients are 0 except for $B_1 = a^2$. Thus, $u_0(r,\theta) = \frac{a^2}{r}\cos\theta$.

Now that we know $u_0(r,\theta)$, we are able to solve the u_1 problem.

$$\begin{aligned} \nabla^2 u_1(r,\theta) &= 0 \\ u_1(a,\theta) &= F_1(a,\theta). \end{aligned}$$

Here we have, from (2.2), that

$$\begin{aligned} F_1(a,\theta) &= F(a,\theta)\frac{\partial}{\partial r}\left(F(r,\theta) - u_0(r,\theta)\right)|_{r=a} \\ &= a\cos\theta\left(1 + \frac{a^2}{r^2}\cos\theta\right)|_{r=a} \\ &= 2a\cos^2\theta. \end{aligned}$$

Since $u_1(r,\theta)$ also satisfies Laplace's equation on the unbounded domain, we can again immediately write down infinite series solutions.

$$u_1(r,\theta) = B_{1,0} + \sum_{n=1}^{\infty} \{A_{1,n}\sin(n\theta) + B_{1,n}\cos(n\theta)\}\, r^{-n}.$$

We can again use simple orthogonality properties to determine values for all coefficients.

$$
\begin{aligned}
B_{1,0} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_1(a,\theta)d\theta \\
A_{1,n} &= \frac{a^n}{\pi}\int_{-\pi}^{\pi} F_1(a,\theta)\sin(n\theta)d\theta \\
B_{1,n} &= \frac{a^n}{\pi}\int_{-\pi}^{\pi} F_1(a,\theta)\cos(n\theta)d\theta
\end{aligned}
$$

For our known $F_1(a,\theta) = 2a\cos^2\theta$, we find that $B_{1,0} = a$ and $B_{1,2} = a^3$. So our solution is

$$u_1(r,\theta) = a + \frac{a^3}{r^2}\cos(2\theta).$$

Now we turn our attention to the u_2 problem,

$$
\begin{aligned}
\nabla^2 u_2(r,\theta) &= 0 \\
u_2(a,\theta) &= F_2(a,\theta).
\end{aligned}
$$

Where, from (2.2),

$$
\begin{aligned}
F_2(a,\theta) = {} & \frac{a^2 g(\theta;0)^2}{2}\left(\frac{\partial^2 F}{\partial r^2}(a,\theta) - \frac{\partial^2 u_0}{\partial r^2}(a,\theta)\right) + a\frac{\partial g}{\partial \varepsilon}(\theta;0)\left(\frac{\partial F}{\partial r}(a,\theta) - \frac{\partial u_0}{\partial r}(a,\theta)\right) \\
& - a g(\theta;0)\frac{\partial u_1}{\partial r}(a,\theta) \\
= {} & -a\cos^3\theta - a\sin^2\theta\cos\theta + 2a\cos\theta\cos(2\theta) \\
= {} & 4a\cos^3\theta - 3a\cos\theta \\
= {} & a\cos(3\theta).
\end{aligned}
$$

We once again write down an infinite series solution,

$$u_2(r,\theta) = B_{2,0} + \sum_{n=1}^{\infty}\left\{A_{2,n}\sin(n\theta) + B_{2,n}\cos(n\theta)\right\} r^{-n}$$

where the values of the coefficients in the series are obtained by exploiting orthogonality. Specifically,

$$B_{2,0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_2(a,\theta)d\theta$$
$$A_{2,n} = \frac{a^n}{\pi}\int_{-\pi}^{\pi} F_2(a,\theta)\sin(n\theta)d\theta$$
$$B_{2,n} = \frac{a^n}{\pi}\int_{-\pi}^{\pi} F_2(a,\theta)\cos(n\theta)d\theta$$

We find that the only non-zero coefficient is $B_{2,3} = a^4$. So the solution for $u_2(r,\theta)$ is simply

$$u_2(r,\theta) = \frac{a^4}{r^3}\cos(3\theta).$$

Returning to our expansion for $u(r,\theta)$, we now have that

$$\begin{aligned} u(r,\theta) &= u_0(r,\theta) + \varepsilon u_1(r,\theta) + \varepsilon^2 u_2(r,\theta) + \mathcal{O}\left(\varepsilon^3\right) \\ &= \frac{a^2}{r}\cos\theta + \varepsilon a + \frac{\varepsilon a^3}{r^2}\cos(2\theta) + \frac{\varepsilon^2 a^4}{r^3}\cos(3\theta) + \mathcal{O}\left(\varepsilon^3\right). \end{aligned}$$

Using $\varepsilon = \frac{\delta}{a}$, we get

$$u(r,\theta) = \frac{a^2}{r}\cos\theta + \delta\left(1 + \frac{a^2}{r^2}\cos(2\theta)\right) + \delta^2\frac{a^2}{r^3}\cos(3\theta) + \mathcal{O}\left(\delta^3\right).$$

We would like to see how our approximate answer compares to the exact solution for this geometry. Consider the same problem; however, we will employ a shifted coordinate system (x', y') so that our perturbed circle is centered at the origin in this new coordinate system. That is,

$$x = x' + \delta$$
$$y = y'$$

Using this coordinate system, our boundary condition becomes

$$F(r,\theta) = r\cos\theta = x = x' + \delta = r'\cos\theta' + \delta. \tag{3.1}$$

So, in this coordinate system, we have the exact problem

$$\nabla^2 u(r',\theta') = 0$$
$$u(a,\theta') = \delta + a\cos\theta'$$

where $u(r', \theta')$ is bounded at infinity. Using a separation of variables approach, it is easily shown that this problem has solution

$$u(r', \theta') = \delta + \frac{a^2}{r'} \cos\theta'.$$

From (3.1), we have $r'\cos\theta' = r\cos\theta - \delta$. Additionally, using the law of cosines, we have that

$$(r')^2 = \delta^2 + r^2 - 2\delta r\cos\theta$$
$$(r')^2 = r^2\left(1 - 2\frac{\delta}{r}\cos\theta + \frac{\delta^2}{r^2}\right).$$

Examining our exact solution, we have

$$u(r', \theta') = \delta + \frac{a^2}{r'}\cos\theta = \delta + \frac{a^2 r'}{r'^2}\cos\theta.$$

Now we substitute for $r'\cos\theta'$ and r'^2. This gives

$$\delta + \frac{a^2(r\cos\theta - \delta)}{r^2\left(1 - 2\frac{\delta}{r}\cos\theta + \frac{\delta^2}{r^2}\right)}.$$

We now expand the denominator in a series about $\frac{\delta}{r} = 0$. Then the exact solution becomes

$$\begin{aligned} u(r,\theta) &= \delta + \frac{a^2}{r^2}\left[(r\cos\theta - \delta)\left(1 + 2\frac{\delta}{r}\cos\theta + \left(\frac{\delta}{r}\right)^2\left(4\cos^2\theta - 1\right) + \cdots\right)\right] \\ &= \delta + \frac{a^2}{r^2}\left[r\cos\theta - \delta + 2\delta\cos^2\theta + \frac{\delta^2}{r}\left(4\cos^3\theta - 3\cos\theta\right) + \mathcal{O}\left(\delta^3\right)\right] \\ &= \delta + \frac{a^2}{r^2}\left[r\cos\theta - \delta + 2\delta\left(\frac{1+\cos(2\theta)}{2}\right) + \frac{\delta^2}{r}\cos(3\theta) + \mathcal{O}\left(\delta^3\right)\right]. \end{aligned}$$

Thus the solution to the exact problem is given by

$$u(r,\theta) = \frac{a^2}{r}\cos\theta + \delta\left(1 + \frac{a^2}{r^2}\cos(2\theta)\right) + \delta^2\frac{a^2}{r^3}\cos(3\theta) + \mathcal{O}\left(\delta^3\right).$$

which is exactly what we had for the approximate solution.

3.3 The Neumann Problem

We will now solve a Neumann problem on the shifted circle. In order to have solvable u_0 and u_1 problems, we must have a $F(r, \theta)$ that satisfies both

$$\int_{-\pi}^{\pi} F(a, \theta) d\theta = 0$$

and

$$\int_{-\pi}^{\pi} g(\theta; 0) \left(\frac{\partial F}{\partial r}(a, \theta) + \frac{1}{a} F(a, \theta) \right) d\theta = 0.$$

The boundary condition, $F(r, \theta) = r \sin \theta$ will satisfy both of these conditions. In detail,

$$\begin{aligned} u_0 : &\quad \int_{-\pi}^{\pi} a \sin \theta d\theta \\ u_1 : &\quad \int_{-\pi}^{\pi} \cos \theta \left(\sin \theta + \sin \theta \right) d\theta \end{aligned}$$

These integrals are both clearly 0. Using this boundary condition, our u_0 problem becomes

$$\begin{aligned} \nabla^2 u_0(r, \theta) &= 0 \\ \frac{\partial u_0}{\partial n}(a, \theta) &= a \sin \theta \end{aligned}$$

Since our domain is now the unperturbed circle, the normal derivative is equivalent to a derivative with respect to r.

Using a standard separation of variables approach, we find that the solution to this problem is $u_0(r, \theta) = -\frac{a^3}{r} \sin \theta$. Using this, we can solve the u_1 problem.

$$\begin{aligned} \nabla^2 u_1(r, \theta) &= 0 \\ \frac{\partial u_1}{\partial n}(a, \theta) &= a \cos \theta \left(\frac{\partial F}{\partial r}(a, \theta) - \frac{\partial^2 u_0}{\partial r^2}(a, \theta) \right) - \frac{\sin \theta}{a} \frac{\partial u_0}{\partial \theta}(a, \theta) \end{aligned}$$

Inserting our solution for $u_0(r, \theta)$, the Neumann boundary condition becomes $\frac{\partial u_1}{\partial r}(a, \theta) = 2a \sin(2\theta)$. Again, using a separation of variables approach, we determine that the solution is $u_1(r, \theta) =$

$-\frac{a^4}{r^2}\sin(2\theta)$. We now have

$$\begin{aligned} u(r,\theta) &= u_0(r,\theta) + \varepsilon u_1(r,\theta) + \mathcal{O}\left(\varepsilon^2\right) \\ &= -\frac{a^3}{r}\sin\theta - \varepsilon\frac{a^4}{r^2}\sin(2\theta) + \mathcal{O}\left(\varepsilon^2\right). \end{aligned}$$

Noting that $\varepsilon = \frac{\delta}{a}$ we get

$$u(r,\theta) = -\frac{a^3}{r}\sin\theta - \delta\frac{a^3}{r^2}\sin(2\theta) + \mathcal{O}\left(\delta^2\right).$$

As before, we will verify this solution by considering the exact problem. Based on the perturbation of our circle, we can create a new set of coordinates, defined by

$$\begin{aligned} x &= x' + \delta \\ y &= y \end{aligned}$$

so that in this coordinate system, our problem becomes

$$\begin{aligned} \nabla^2 u(r',\theta') &= 0 \\ \frac{\partial u}{\partial n}(a,\theta') &= r'\sin\theta' \end{aligned}$$

The boundary condition here is due to the fact that we only perturb in the x-direction. Specifically, $F(r,\theta) = r\sin\theta = y = y' = r'\sin\theta'$. This problem is essentially the u_0 problem we solved previously, and so we can readily write down its answer: $u(r',\theta') = -\frac{a^3}{r'}\sin\theta'$.

Now we recall from earlier that $(r')^2 = r^2(1 - 2\frac{\delta}{r}\cos\theta - \frac{\delta^2}{r^2})$ and so, we can transform our solution.

$$\begin{aligned} u(r',\theta') &= -\frac{a^3}{r'}\sin\theta' \\ &= -\frac{a^3 r'}{r'^2}\sin\theta' \\ &= -\frac{a^3 r}{r^2\left(1 - 2\frac{\delta}{r}\cos\theta - \frac{\delta^2}{r^2}\right)}\sin\theta. \end{aligned}$$

We now expand the denominator in a power series about $\frac{\delta}{r} = 0$, to get

$$
\begin{aligned}
u(r,\theta) &\approx -\frac{a^3}{r}\sin\theta\left(1 + 2\frac{\delta}{r}\cos\theta + \mathcal{O}\left(\delta^2\right)\right) \\
&= -\frac{a^3}{r}\sin\theta - 2\delta\frac{a^3}{r^2}\sin\theta\cos\theta + \mathcal{O}(\delta^2) \\
&= -\frac{a^3}{r}\sin\theta - \delta\frac{a^3}{r^2}\sin(2\theta) + \mathcal{O}(\delta^2).
\end{aligned}
$$

This is exactly what we calculated previously (up to order δ^2).

CHAPTER 4
AN ELLIPSE

We will now apply our method to solve Laplace's equation exterior to an elliptical domain. We will proceed as in Chapter 3 where we first found an approximate solution using our method. We will then compare it to an exact solution found by using coordinate transformations. We consider only the Dirichlet problem. Solving the Neumann problem is very similar, and its behavior is not interesting enough to warrant additional investigation.

4.1 Finding $g(\theta; \varepsilon)$

Let us define a coordinate transformation

$$x = f_0 \cosh \mu \cos \nu$$
$$y = f_0 \sinh \mu \sin \nu$$

where μ is a positive real number, and $\nu \in [-\pi, \pi)$. This set of coordinate transforms describes standard elliptic coordinates. Figure 2 depicts this geometry. Note that the figure is not to scale and is exaggerated to highlight the relevant geometry. In this coordinate system, curves with constant

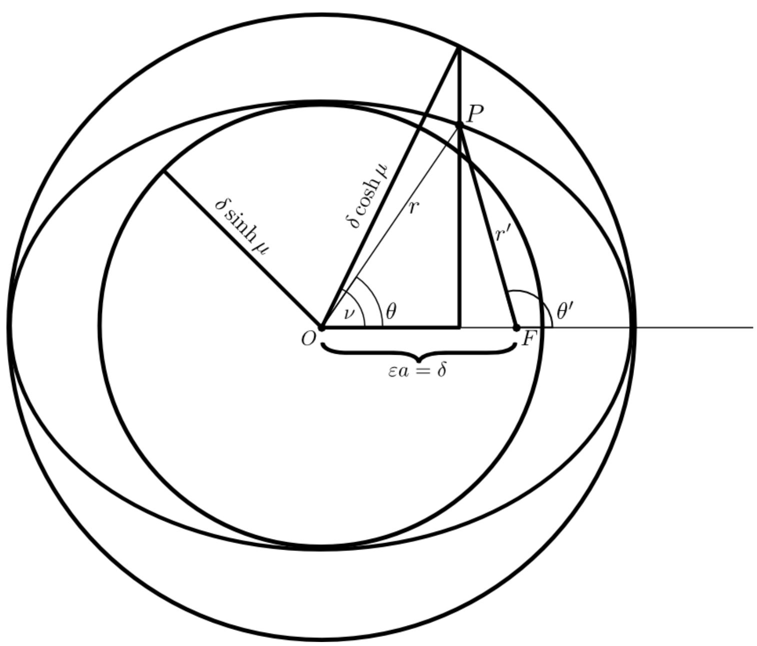

Figure 2: An Ellipse

μ give ellipses centered at the origin with foci at $\pm f_0$, and curves of constant ν give hyperbolae. In this coordinate system, the Laplacian operator is defined to be

$$\nabla^2 u = \frac{1}{f_0^2\left(\sinh^2\mu + \sin^2\nu\right)}\left(\frac{\partial^2 u}{\partial \mu^2} + \frac{\partial^2 u}{\partial \nu^2}\right).$$

Then Laplace's equation simplifies to

$$\frac{\partial^2 u}{\partial \mu^2} + \frac{\partial^2 u}{\partial \nu^2} = 0. \tag{4.1}$$

When we perturb our circle of radius a, into an ellipse, we do so in a way such that the foci of the ellipse are at $\pm\delta$ where $\delta = \varepsilon a$ Then our coordinate transform becomes

$$x = \delta \cosh\mu \cos\nu$$
$$y = \delta \sinh\mu \sin\nu$$

In polar coordinates, if we measure θ from the origin, then the equation for an ellipse is given by

$$r(\theta) = \frac{bc}{\sqrt{(c\cos\theta)^2 + (b\sin\theta)^2}}$$

where b and c are such that in Cartesian coordinates, $\left(\frac{x}{b}\right)^2 + \left(\frac{y}{c}\right)^2 = 1$. Then, since our ellipse is centered at the origin and has major axis along the x-axis, the x-coordinates of the foci of the ellipse are given by

$$f = \pm\sqrt{b^2 - c^2}.$$

We want the foci of our ellipse to be at $\pm\delta$, where as before, $\delta = \varepsilon a$. For ease, we take $c = a$ and $b = \sqrt{a^2+\delta^2}$. Then $f = \pm\sqrt{a^2+\delta^2-a^2} = \pm\delta$, and our equation for r becomes

$$\begin{aligned} r(\theta) &= \frac{a\sqrt{a^2+\delta^2}}{\sqrt{a^2\cos^2\theta + (a^2+\delta^2)\sin^2\theta}} \\ &= a\sqrt{\frac{1+\varepsilon^2}{1+\varepsilon^2\sin^2\theta}}. \end{aligned}$$

If we expand the square root term about $\varepsilon = 0$, we get

$$r(\theta) = a\left(1 + \frac{1}{2}\varepsilon^2\cos^2\theta + \frac{1}{8}\varepsilon^4\left(3\sin^4\theta - 2\sin^2\theta - 1\right) + \mathcal{O}\left(\varepsilon^6\right)\right).$$

We now denote ε^2 as η and proceed as before, keeping in mind that now $\delta = a\sqrt{\eta}$. Using this notation we can describe the domain enclosed by the ellipse as

$$r \leq a(1 + \eta g(\theta; \eta)) \tag{4.2}$$

where

$$g(\theta; \eta) = \frac{1}{2}\cos^2\theta + \frac{1}{8}\eta\left(3\sin^4\theta - 2\sin^2\theta - 1\right) + \mathcal{O}\left(\eta^2\right)$$

.

4.2 The Dirichlet Problem

We again begin by solving the Dirichlet problem with the given boundary function $F(r, \theta) = r\cos\theta$. Since this is the same boundary condition that we had in Chapter 3, the $u_0(r, \theta)$ problem is unchanged and so,

$$u_0(r, \theta) = \frac{a^2}{r}\cos\theta.$$

Since the perturbation function $g(\theta; \varepsilon)$ for an ellipse is different than the perturbation function we encountered for the translated circle, the $u_1(r, \theta)$ problem is slightly different. We have,

$$\begin{aligned} \nabla^2 u_1(r, \theta) &= 0 \\ u_1(a, \theta) &= a g(\theta) \frac{\partial}{\partial r}\left(F - u_0\right)|_{r=a} \end{aligned}$$

The boundary condition thus becomes

$$\begin{aligned} u_1(a, \theta) &= a\frac{\cos^2\theta}{2}\frac{\partial}{\partial r}\left(r\cos\theta - \frac{a^2}{r}\cos\theta\right)|_{r=a} \\ &= a\cos^3\theta \end{aligned}$$

and so we must solve

$$\begin{aligned} \nabla^2 u_1(r, \theta) &= 0 \\ u_1(a, \theta) &= a\cos^3\theta \end{aligned}$$

This boundary condition yields two non-zero Fourier coefficients, and so we are left with

$$u_1(r, \theta) = \frac{3a^2}{4r}\cos\theta + \frac{a^4}{4r^3}\cos(3\theta).$$

Now we can use the solutions for $u_0(r,\theta)$ and $u_1(r,\theta)$ to construct the problem for $u_2(r,\theta)$.

$$\nabla^2 u_2(r,\theta) = 0$$
$$u_2(a,\theta) = F_2(a,\theta)$$

Again, we use (2.2) to construct $F_2(a,\theta)$. In detail,

$$\begin{aligned}
F_2(a,\theta) &= \frac{a^2 g(\theta;0)^2}{2}\left(\frac{\partial^2 F}{\partial r^2}(a,\theta) - \frac{\partial^2 u_0}{\partial r^2}(a,\theta)\right) + a\frac{\partial g}{\partial \varepsilon}(\theta;0)\left(\frac{\partial F}{\partial r}(a,\theta) - \frac{\partial u_0}{\partial r}(a,\theta)\right) \\
&\quad - ag(\theta;0)\frac{\partial u_1}{\partial r}(a,\theta) \\
&= -\frac{a\cos^5\theta}{4} + \frac{a\cos\theta}{4}\left(3\sin^4\theta - 2\sin^3\theta - 1\right) + \frac{3a\cos^2\theta}{8}\left(\cos\theta + \cos(3\theta)\right) \\
&= -\frac{a\cos^5\theta}{4} + \frac{a\cos\theta}{4}\left(3\cos^4\theta - 4\cos^2\theta\right) + \frac{3a\cos^2\theta}{8}\left(2\cos^2\theta - 1\right) \\
&= \frac{a\cos^3\theta}{4}\left(4\cos(2\theta) - 3\right).
\end{aligned}$$

Using this boundary condition, we find three non-zero Fourier coefficients. The solution to the u_2 problem is

$$u_2(r,\theta) = -\frac{a^2}{16r}\cos\theta + \frac{3a^4}{16r^3}\cos(3\theta) + \frac{a^6}{8r^5}\cos(5\theta).$$

Combining the solutions for $u_0(r,\theta)$, $u_1(r,\theta)$, and $u_2(r,\theta)$ we get

$$\begin{aligned}
u(r,\theta) &= u_0(r,\theta) + \eta u_1(r,\theta) + \eta^2 u_2(r,\theta) + \mathcal{O}\left(\eta^3\right) \\
&= \frac{a^2}{r}\cos\theta + \left(\frac{\delta}{a}\right)^2\left(\frac{3a^2}{4r}\cos\theta + \frac{a^4}{4r^3}\cos(3\theta)\right) + \left(\frac{\delta}{a}\right)^4\left(-\frac{a^2}{16r}\cos\theta\right. \\
&\quad \left. + \frac{3a^4}{16r^3}\cos(3\theta) + \frac{a^6}{8r^5}\cos(5\theta)\right) + \mathcal{O}\left(\delta^6\right) \\
&= \frac{a^2}{r}\cos\theta + \delta^2\left(\frac{3}{4r}\cos\theta + \frac{a^2}{4r^3}\left(4\cos^3\theta - 3\cos\theta\right)\right) + \delta^4\left(-\frac{1}{16a^2 r}\cos\theta\right. \\
&\quad \left. + \frac{3}{16r^3}\left(4\cos^3\theta - 3\cos\theta\right) + \frac{a^2}{8r^5}\left(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\right)\right) + \mathcal{O}\left(\delta^6\right) \\
&= \frac{a^2}{r}\cos\theta + \delta^2\left(\frac{3(r^2-a^2)}{4r^3}\cos\theta + \frac{a^2}{r^3}\cos^3\theta\right) + \delta^4\left(\frac{\left(10a^2+r^2\right)\left(a^2-r^2\right)}{16a^2r^5}\cos\theta\right. \\
&\quad \left. + \frac{3r^2 - 10a^2}{4r^5}\cos^3\theta + \frac{2a^2}{r^5}\cos^5\theta\right) + \mathcal{O}\left(\delta^6\right).
\end{aligned} \tag{4.3}$$

As before, we now seek an exact solution with which to compare our approximate solution. In order to obtain an exact solution for this geometry, we first note that (4.1) shows that in elliptic coordinates, Laplace's equation is separable. Thus by imposing our condition at infinity, as well as 2π periodicity, we can use standard separation of variable techniques to write down solutions of the form

$$u(\mu,\nu) = A_0 + \sum_{n=1}^{\infty} e^{-n\mu}\left(A_n\cos(n\nu) + B_n\sin(n\nu)\right). \tag{4.4}$$

When we are dealing with Dirichlet boundary conditions, we will be evaluating on an ellipse, so μ will be constant, μ_0 say. Thus, our boundary condition becomes $u(\mu_0,\nu) = F(\mu_0,\nu)$ for some known function $F(\mu,\nu)$. From this we can derive equations to determine the coefficients present in (4.4).

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\mu_0,\nu)d\nu \\ A_n &= \frac{e^{n\mu_0}}{\pi}\int_{-\pi}^{\pi} F(\mu_0,\nu)\cos(n\nu)d\nu \\ B_n &= \frac{e^{n\mu_0}}{\pi}\int_{-\pi}^{\pi} F(\mu_0,\nu)\sin(n\nu)d\nu \end{aligned}$$

For this example, we chose $F(r,\theta) = r\cos\theta$. Making the substitutions required to transform to elliptic coordinates gives us $F(\mu,\nu) = \delta\cosh\mu\cos\nu$. Thus on the boundary of the ellipse defined by constant $\mu = \mu_0$, we have $F(\mu_0,\nu) = \delta\cosh\mu_0\cos\nu$. Thus, our problem for the exact case $u(\mu,\nu)$ is given by

$$\begin{aligned} \nabla^2 u(\mu,\nu) &= 0 \\ u(\mu_0,\nu) &= \delta\cosh\mu_0\cos\nu \end{aligned}$$

We use the formula discussed above to obtain coefficients for the series solution and we find simply that for the exact case,

$$u(\mu,\nu) = \delta e^{\mu_0-\mu}\cosh\mu_0\cos\nu.$$

We now want to find equations that relate (r,θ) and (μ,ν) in order to verify our series solution. To that end, we will examine eccentric anomaly. The eccentric anomaly of an ellipse is given by

$$r' = \delta\cosh\mu(1-\epsilon\cos\nu)$$

where ϵ is the eccentricity of the ellipse. Figure 3 depicts the relevant geometry.

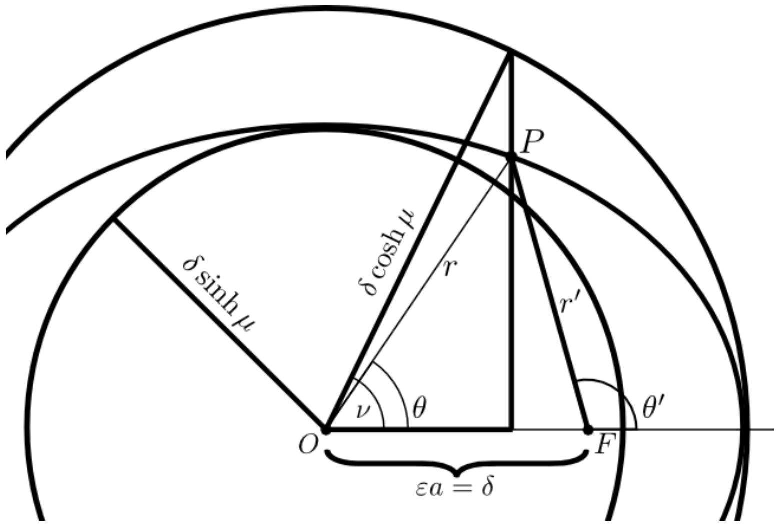

Figure 3: Eccentric Anomaly

The eccentricity of our ellipse is given by

$$\begin{aligned}\epsilon &= \sqrt{1 - \left(\frac{\varepsilon a \sinh \mu}{\varepsilon a \cosh \mu}\right)^2} \\ &= \sqrt{1 - \tanh^2 \mu} \\ &= \operatorname{sech} \mu.\end{aligned}$$

Using this, we get that

$$r' = \delta \cosh \mu (1 - \operatorname{sech} \mu \cos \nu) = \delta(\cosh \mu - \cos \nu). \tag{4.5}$$

Now, from the law of cosines, we have

$$\begin{aligned}(r')^2 &= r^2 + \delta^2 - 2r\delta \cos \theta \\ &= r^2 + \delta^2 - 2\delta^2 \cosh \mu \cos \nu.\end{aligned} \tag{4.6}$$

Now we use (4.5) in (4.6) to get

$$\delta^2 \cos^2 \nu - 2\delta^2 \cos \nu \cosh \mu + \delta^2 \cosh^2 \mu = r^2 + \delta^2 - 2\delta^2 \cosh \mu \cos \nu.$$

Solving for r^2, gives

$$r^2 = \delta^2(\cosh^2 \mu + \cos^2 \nu - 1).$$

We now substitute this into the coordinate transformation for x to obtain

$$\begin{aligned} r^2\cos^2\theta &= \delta^2\cosh^2\mu\cos^2\nu \\ \delta^2\cosh^2\mu\cos^2\theta + \delta^2\cos^2\nu\cos^2\theta - \delta^2\cos^2\theta &= \delta^2\cosh^2\mu\cos^2\nu \\ \cosh^2\mu &= \frac{\cos^2\theta(1-\cos^2\nu)}{\cos^2\theta - \cos^2\nu} \end{aligned}$$

We notice that this equation is singular if $\theta = \nu$. Due to our coordinate transform, we know that $\theta \approx \nu$ so they will never be far apart. If $\theta = \nu = \frac{\pi}{2}$ or if $\theta = \nu = \frac{3\pi}{2}$ then the coordinate transform gives trivially $0 = 0$ and we must use the coordinate transform for y instead. Using this we get $r = \delta\sinh\mu$ as we expect. Similarly, for $\theta = \nu = 0$ and $\theta = \nu = \pi$, the coordinate transform for x gives us that $r = \delta\cosh\mu$. These four values are the only times that θ and ν are equal.

Now if we again consider the coordinate transformation for x and solve for $\cosh^2\mu$, we have

$$\begin{aligned} \cosh^2\mu &= \frac{r^2\cos^2\theta}{\delta^2\cos^2\nu} \\ \frac{\cos^2\theta(1-\cos^2\nu)}{\cos^2\theta-\cos^2\nu} &= \frac{r^2\cos^2\theta}{\delta^2\cos^2\nu} \\ r^2\cos^2\theta - r^2\cos^2\nu &= \delta^2\cos^2\nu - \delta^2\cos^4\nu \end{aligned}$$

This gives us a quadratic equation to solve for $\cos^2\nu$. Applying the quadratic formula gives us

$$\begin{aligned} \cos^2\nu &= \frac{\delta^2 + r^2 \pm \sqrt{\delta^4 + 2\delta^2 r^2 + r^4 - 4\delta^2 r^2\cos^2\theta}}{2\delta^2} \\ &= \frac{1}{2}\left(1 + \left(\frac{r}{\delta}\right)^2 \pm \sqrt{1 + \left(\frac{r}{\delta}\right)^4 + 2\left(\frac{r}{\delta}\right)^2 - 4\left(\frac{r}{\delta}\right)^2\cos^2\theta}\right) \\ &= \frac{1}{2}\left(1 + \left(\frac{r}{\delta}\right)^2\left(1 \pm \sqrt{1 + \left(\frac{\delta}{r}\right)^4 + 2\left(\frac{\delta}{r}\right)^2 - 4\left(\frac{\delta}{r}\right)^2\cos^2\theta}\right)\right) \\ &= \frac{1}{2}\left(\frac{r}{\delta}\right)^2\left(\left(\frac{\delta}{r}\right)^2 + 1 \pm \sqrt{1 + \left(\frac{\delta}{r}\right)^4 + 2\left(\frac{\delta}{r}\right)^2 - 4\left(\frac{\delta}{r}\right)^2\cos^2\theta}\right). \end{aligned}$$

Note that for small δ, the quantity we are multiplying by a very large quantity. $\cos^2\nu \leq 1$ for all values of ν, thus, we expect all terms in $\left(\frac{r}{\delta}\right)^2$ to disappear. To enforce this condition, we only consider the negative square root solution for $\cos^2\nu$.

We now expand the square root about $\frac{\delta}{r} = 0$. When we do this expansion, we must consider terms up to order δ^6 as we are multiplying the series by a factor of $\left(\frac{\delta}{r}\right)^2$ and wish to preserve terms of order δ^4. Generally we consider only terms of order δ^2, but since our power series substitution for $u(r,\theta)$ was in terms of $\eta = \varepsilon^2$, we must consider all terms of up to fourth order. Performing this expansion, we get

$$\begin{aligned}
\cos^2\nu &= \frac{1}{2}\left(\frac{r}{\delta}\right)^2\left(\left(\frac{\delta}{r}\right)^2 + 1 - 1 + \left(\frac{\delta}{r}\right)^2\left(2\cos^2\theta - 1\right) + \left(\frac{\delta}{r}\right)^4\left(2\cos^4\theta - 2\cos^2\theta\right)\right.\\
&\qquad\left. + \left(\frac{\delta}{r}\right)^6\left(4\cos^6\theta - 6\cos^4\theta + 2\cos^2\theta\right) + \mathcal{O}\left(\delta^8\right)\right)\\
&= \cos^2\theta\left(1 + \left(\frac{\delta}{r}\right)^2\left(\cos^2\theta - 1\right) + \left(\frac{\delta}{r}\right)^4\left(2\cos^4\theta - 3\cos^2\theta + 1\right) + \mathcal{O}\left(\delta^6\right)\right).
\end{aligned}$$

Note that at $\theta = 0$ and $\theta = \pi$, we have $\nu = \theta$. At the values $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ we have $\nu = \theta + \mathcal{O}\left(\delta^4\right)$. This agrees with our earlier statement that $\theta = \nu$ at only these four values.

Now that we have an expression for $\cos^2\nu$, we want to obtain an expression for $\cosh^2\mu$. This is accomplished by again using the coordinate transformation for elliptic coordinates,

$$\begin{aligned}
\cosh^2\mu &= \left(\frac{r}{\delta}\right)^2\frac{\cos^2\theta}{\cos^2\nu}\\
&= \left(\frac{r}{\delta}\right)^2\frac{\cos^2\theta}{\cos^2\theta\left(1 + \left(\frac{\delta}{r}\right)^2\left(\cos^2\theta - 1\right) + \left(\frac{\delta}{r}\right)^4\left(2\cos^4\theta - 3\cos^2\theta + 1\right) + \mathcal{O}\left(\delta^6\right)\right)}\\
&= \left(\frac{r}{\delta}\right)^2\frac{1}{1 + \left(\frac{\delta}{r}\right)^2\left(\cos^2\theta - 1\right) + \left(\frac{\delta}{r}\right)^4\left(2\cos^4\theta - 3\cos^2\theta + 1\right) + \mathcal{O}\left(\delta^6\right))}.
\end{aligned}$$

Now that we have expressions for $\cos^2\nu$ and $\cosh^2\mu$, we can substitute them into the exact solution in order to recover the solution on the perturbed boundary. We begin by obtaining an expansion for $\delta\cosh\mu$.

$$\begin{aligned}
\cosh^2\mu &= \left(\frac{r}{\delta}\right)^2\left(1 + \left(\frac{\delta}{r}\right)^2\left(\cos^2\theta - 1\right) + \left(\frac{\delta}{r}\right)^4\left(2\cos^4\theta - 3\cos^2\theta + 1\right) + \mathcal{O}\left(\delta^6\right))\right)^{-1}\\
\cosh\mu &= \pm\frac{r}{\delta}\left(1 + \left(\frac{\delta}{r}\right)^2\left(\cos^2\theta - 1\right) + \left(\frac{\delta}{r}\right)^4\left(2\cos^4\theta - 3\cos^2\theta + 1\right) + \mathcal{O}\left(\delta^6\right))\right)^{-\frac{1}{2}}\\
\delta\cosh\mu &= r\left(1 - \frac{1}{2}\left(\frac{\delta}{r}\right)^2\left(\cos^2\theta - 1\right) - \frac{1}{8}\left(\frac{\delta}{r}\right)^4\left(5\cos^4\theta - 6\cos^2\theta + 1\right) + \mathcal{O}\left(\delta^6\right)\right).
\end{aligned}$$

Here, we are able to ignore the negative square root since $\cosh\mu$ is always positive.

In our exact solution, $\cosh\mu_0$ is simply a number, thus we expect that the terms involving θ

and r in this expansion will also be simply numbers after proper substitution. Noting that when $\theta = 0$ and $r = \sqrt{a^2 + \delta^2}$, $\mu = \mu_0$. This gives us an expression for $\delta \cosh \mu_0$,

$$\begin{aligned}\delta \cosh \mu_0 &\approx \sqrt{a^2 + \delta^2} \\ &= a \left(1 + \frac{1}{2} \left(\frac{\delta}{a} \right)^2 - \frac{1}{8} \left(\frac{\delta}{a} \right)^4 + \mathcal{O}\left(\delta^6\right) \right).\end{aligned}$$

We obtain an expression for $\cos \nu$ in similar fashion

$$\begin{aligned}\cos^2 \nu &= \cos^2 \theta \left(1 + \left(\frac{\delta}{r} \right)^2 \left(\cos^2 \theta - 1\right) + \left(\frac{\delta}{r} \right)^4 \left(2 \cos^4 \theta - 3 \cos^2 \theta + 1\right) + \mathcal{O}\left(\delta^6\right) \right) \\ \cos \nu &= \pm \cos \theta \sqrt{1 + \left(\frac{\delta}{r} \right)^2 (\cos^2 \theta - 1) + \left(\frac{\delta}{r} \right)^4 (2 \cos^4 \theta - 3 \cos^2 \theta + 1) + \mathcal{O}\left(\delta^6\right)} \\ \cos \nu &= \cos \theta \left(1 + \frac{1}{2} \left(\frac{\delta}{r} \right)^2 \left(\cos^2 \theta - 1\right) + \frac{1}{8} \left(\frac{\delta}{r} \right)^4 \left(7 \cos^4 \theta - 10 \cos^2 \theta + 3\right) + \mathcal{O}\left(\delta^6\right) \right).\end{aligned}$$

Note that since $\cos \nu \approx \cos \theta$, we are able to ignore the negative root and consider only the positive solution. Now we turn our attention to the $e^{-\mu}$ term. We have an expression for $\cosh \mu$ so we must now invert it. A useful expression for $\cosh^{-1} x$ is (for $x \geq 1$)

$$\cosh^{-1} x = \ln \left(x + \sqrt{x^2 - 1} \right).$$

Thus, we can express e^{μ} simply as $x + \sqrt{x^2 - 1}$ for $x = \cosh \mu$. Thus, we have

$$x = \frac{r}{\delta} \left(1 + \left(\frac{\delta}{r} \right)^2 \left(\cos^2 \theta - 1\right) + \left(\frac{\delta}{r} \right)^4 \left(2 \cos^4 \theta - 3 \cos^2 \theta + 1\right) + \mathcal{O}\left(\delta^6\right) \right)^{-\frac{1}{2}}.$$

Using this and expanding the resulting expression for e^{μ} about $\delta = 0$ gives

$$\begin{aligned}e^{\mu} = &\frac{r}{\delta} \left(1 + \left(\frac{\delta}{r} \right)^2 \left(\cos^2 \theta - 1\right) + \left(\frac{\delta}{r} \right)^4 \left(2 \cos^4 \theta - 3 \cos^2 \theta + 1\right) + \mathcal{O}\left(\delta^6\right) \right)^{-\frac{1}{2}} \\ &+ \sqrt{\left(\frac{r}{\delta} \right)^2 \left(1 + \left(\frac{\delta}{r} \right)^2 (\cos^2 \theta - 1) + \left(\frac{\delta}{r} \right)^4 (2 \cos^4 \theta - 3 \cos^2 \theta + 1) + \mathcal{O}\left(\delta^6\right) \right)^{-1} - 1}\end{aligned}$$

$$\begin{aligned}
&= \frac{r}{\delta}\left(\left(1+\frac{1}{2}\left(\frac{\delta}{r}\right)^2\left(1-\cos^2\theta\right)-\frac{1}{8}\left(\frac{\delta}{r}\right)^4\left(5\cos^4\theta-6\cos^2\theta+1\right)+\mathcal{O}\left(\delta^6\right)\right)\right. \\
&\quad \left.+\left(1-\frac{1}{2}\left(\frac{\delta}{r}\right)^2\cos^2\theta-\frac{1}{8}\left(\frac{\delta}{r}\right)^4\left(5\cos^4\theta-4\cos^2\theta\right)+\mathcal{O}\left(\delta^6\right)\right)\right) \qquad (4.7) \\
&= 2\frac{r}{\delta}+\frac{1}{2}\frac{\delta}{r}\left(1-2\cos^2\theta\right)-\frac{1}{8}\left(\frac{\delta}{r}\right)^3\left(10\cos^4\theta-10\cos^2\theta+1\right)+\mathcal{O}\left(\delta^5\right).
\end{aligned}$$

From here, we can simply get an expression for $e^{-\mu}$.

$$\begin{aligned}
e^{-\mu} &= \left(2\frac{r}{\delta}+\frac{1}{2}\frac{\delta}{r}\left(1-2\cos^2\theta\right)-\frac{1}{8}\left(\frac{\delta}{r}\right)^3\left(10\cos^4\theta-10\cos^2\theta+1\right)+\mathcal{O}\left(\delta^5\right)\right)^{-1} \\
&= \frac{r}{\delta}\left(2+\frac{1}{2}\left(\frac{\delta}{r}\right)^2\left(1-2\cos^2\theta\right)-\frac{1}{8}\left(\frac{\delta}{r}\right)^4\left(10\cos^4\theta-10\cos^2\theta+1\right)+\mathcal{O}\left(\delta^6\right)\right)^{-1} \\
&= \frac{\delta}{2r}-\frac{1}{8}\left(\frac{\delta}{r}\right)^3\left(1-2\cos^2\theta\right)+\frac{1}{16}\left(\frac{\delta}{r}\right)^5\left(7\cos^4\theta-7\cos^2\theta+1\right)+\mathcal{O}\left(\delta^7\right).
\end{aligned}$$

If we refer back to (4.7), we can again set $\theta = 0$ and $r = \sqrt{a^2+\delta^2}$ to obtain an expression for e^{μ_0}.

$$\begin{aligned}
e^{\mu_0} &= 2\frac{\sqrt{a^2+\delta^2}}{\delta}-\frac{1}{2}\frac{\delta}{\sqrt{a^2+\delta^2}}-\frac{1}{8}\left(\frac{\delta}{\sqrt{a^2+\delta^2}}\right)^3+\mathcal{O}\left(\delta^6\right) \\
&= 2\frac{\sqrt{a^2+\delta^2}}{\delta}\left(1-\frac{1}{4}\frac{\delta^2}{a^2+\delta^2}-\frac{1}{16}\frac{\delta^4}{(a^2+\delta^2)^2}+\mathcal{O}\left(\delta^6\right)\right) \\
&= \frac{2a}{\delta}+\frac{\delta}{2a}-\frac{\delta^3}{8a^3}+\mathcal{O}\left(\delta^5\right).
\end{aligned}$$

Now, putting together the terms involving μ_0, we get

$$\begin{aligned}
\delta\cosh\mu_0 e^{\mu_0} &= a\left(1+\frac{1}{2}\left(\frac{\delta}{a}\right)^2-\frac{1}{8}\left(\frac{\delta}{a}\right)^4+\mathcal{O}\left(\delta^6\right)\right)\left(\frac{2a}{\delta}+\frac{\delta}{2a}-\frac{\delta^3}{8a^3}+\mathcal{O}\left(\delta^5\right)\right) \\
&= \frac{2a^2}{\delta}+\frac{3\delta}{2}-\frac{\delta^3}{8a^2}+\mathcal{O}\left(\delta^5\right).
\end{aligned} \qquad (4.8)$$

Now we expand the harmonic part of the exact solution

$$\begin{aligned}
e^{-\mu}\cos\nu &= \left(\frac{\delta}{2r}-\frac{1}{8}\left(\frac{\delta}{r}\right)^3\left(1-2\cos^2\theta\right)+\frac{1}{16}\left(\frac{\delta}{r}\right)^5\left(7\cos^4\theta-7\cos^2\theta+1\right)+\mathcal{O}\left(\delta^7\right)\right) \\
&\quad\times\left(\cos\theta\left(1+\frac{1}{2}\left(\frac{\delta}{r}\right)^2\left(\cos^2\theta-1\right)+\frac{1}{8}\left(\frac{\delta}{r}\right)^4\left(7\cos^4\theta-10\cos^2\theta+3\right)+\mathcal{O}\left(\delta^6\right)\right)\right) \\
&\approx \frac{\delta}{2r}\cos\theta+\delta^3\left(\frac{1}{2r^3}\cos^3\theta-\frac{3}{8r^3}\cos\theta\right)+\delta^5\left(\frac{1}{r^5}\cos^5\theta-\frac{5}{4r^5}\cos^3\theta+\frac{5}{16r^5}\cos\theta\right).
\end{aligned} \qquad (4.9)$$

Since the expansion of constant terms contains a term in δ^{-1}, we must consider terms of order δ^5 in the expansion of the harmonic terms. Finally, we combine equations (4.8) and (4.9) to arrive at

$$\begin{aligned}
\delta \cosh \mu_0 e^{\mu_0} e^{\mu} \cos \nu = & \left(\frac{2a^2}{\delta} + \frac{3\delta}{2} - \frac{\delta^3}{8a^2} + \mathcal{O}\left(\delta^5\right) \right) \left(\frac{\delta}{2r} \cos\theta + \delta^3 \left(\frac{1}{2r^3} \cos^3\theta - \frac{3}{8r^3} \cos\theta \right) \right. \\
& \left. + \delta^5 \left(\frac{1}{r^5} \cos^5\theta - \frac{5}{4r^5} \cos^3\theta + \frac{5}{16r^5} \cos\theta \right) + \mathcal{O}\left(\delta^7\right) \right) \\
= & \frac{a^2}{r} \cos\theta + \delta^2 \left(\frac{a^2}{r^3} \cos^3\theta - \frac{3a^2}{4r^3} \cos\theta + \frac{3}{4r} \cos\theta \right) + \delta^4 \left(\frac{5a^2}{8r^5} \cos\theta \right. \\
& \left. - \frac{5a^2}{2r^5} \cos^3\theta + \frac{2a^2}{r^5} \cos^5\theta - \frac{9}{16r^3} \cos\theta + \frac{3}{4r^3} \cos^3\theta - \frac{1}{16a^2 r} \cos\theta \right) + \mathcal{O}\left(\delta^6\right) \\
= & \frac{a^2}{r} \cos\theta + \delta^2 \left(\frac{3(r^2 - a^2)}{4r^3} \cos\theta + \frac{a^2}{r^3} \cos^3\theta \right) \\
& + \delta^4 \left(\frac{(10a^2 + r^2)(a^2 - r^2)}{16a^2 r^5} \cos\theta + \frac{3r^2 - 10a^2}{4r^5} \cos^3\theta + \frac{2a^2}{r^5} \cos^5\theta \right) + \mathcal{O}\left(\delta^6\right).
\end{aligned}$$

This is exactly what we obtained from our solution to the Dirichlet problem on the perturbed boundary in (4.3).

CHAPTER 5

AN INTEGRAL REPRESENTATION

We now present an alternative approach to solve $\nabla^2 u = 0$ on the exterior of a closed curve using an integral method as opposed to the separation of variables approach we used earlier. Denote Ω_C as the region exterior to a given closed domain Ω. Let $G(P,Q)$ be the free space Green's function, $-\frac{1}{2\pi}\ln(|P-Q|)$. Then, for $P \neq Q$, $G(P,Q)\nabla^2 u(Q) = 0$ in Ω_C. Then,

$$\begin{aligned}\int_{\Omega_C \setminus P=Q} G(P,Q)\nabla^2 u(Q) dS_Q &= \int_{\Omega_C \setminus P=Q} \nabla \cdot (G(P,Q)\nabla u(Q)) - \nabla_Q G(P,Q) \cdot \nabla u(Q) dS_Q \\ &= \int_{\Omega_C \setminus P=Q} \nabla \cdot (G(P,Q)\nabla u(Q) - u(Q)\nabla_Q G(P,Q)) + \\ &\qquad u(Q)\nabla_Q^2 G(P,Q) dS_Q.\end{aligned} \tag{5.1}$$

Now, noting that $\nabla_Q^2 G(P,Q) = 0$, and that the original integral is identically 0, we apply the divergence theorem to the remaining integral. This gives us three separate contour integrals, the first is over the boundary of Ω, denoted $\partial\Omega$, the second is around a circle of radius ε centered at the point P, denoted $\mathcal{S}_\varepsilon$ and the third is a circle of radius R, denoted $\mathcal{S}_R$, that encloses both Ω and the point P. In order to evaluate over all of the original integral's domain, we take limits $\varepsilon \to 0$, and $R \to \infty$. To properly apply the divergence theorem, the normal vector for the first two domains points outward and that the normal vector for the circle of radius R points inward.

$$\oint_{\partial\Omega} u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} ds + \oint_{\mathcal{S}_\varepsilon} u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} ds + \oint_{\mathcal{S}_R} u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} ds = 0. \tag{5.2}$$

The integral around $\mathcal{S}_\varepsilon$ is easily integrable for the limit $\varepsilon \to 0$.

$$\begin{gathered}\lim_{\varepsilon\to 0} -\frac{1}{2\pi}\oint_{S_\varepsilon} u(P)\frac{1}{\varepsilon} - \ln(\varepsilon)\frac{\partial u}{\partial n}(P) ds = \\ \lim_{\varepsilon\to 0} -\frac{1}{2\pi}\left[\int_{-\pi}^{\pi} u(P)\frac{1}{\varepsilon} - \ln(\varepsilon)\frac{\partial u}{\partial n}(P)\right]\varepsilon d\theta = \\ -u(P) + \frac{\partial u}{\partial n}(P)\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi} \ln(\varepsilon)\varepsilon d\theta = -u(P)\end{gathered}$$

In order to evaluate the integral over $\mathcal{S}_R$ as $R \to \infty$, we must make an assumption about the behavior of u at infinity. As long as $u(P)$ is $O(1)$ as $P \to \infty$, then, $\frac{\partial u}{\partial n}(P) \to 0$ as $P \to \infty$. Then, the integral around $\mathcal{S}_R$ becomes

$$\lim_{R\to\infty} -\frac{1}{2\pi}\oint_{\mathcal{S}_R} u(R)\frac{1}{R-P} - \ln(R-P)\frac{\partial u}{\partial n}(R)ds =$$
$$\lim_{R\to\infty} -\frac{1}{2\pi}\int_0^{2\pi} u(R)\frac{1}{R-P}(-R)d\theta =$$
$$\lim_{R\to\infty} u(R) = u_\infty.$$

where u_∞ is the constant value that $u(P)$ takes at infinity. Thus, we have that

$$\oint_{\partial\Omega} u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}ds = u(P) - u_\infty. \tag{5.3}$$

As with the Neumann problem for separation of variables, we need to evaluate normal derivatives of both $u(Q)$ and $G(P,Q)$ on the boundary. Recalling, (2.3), we have that the unit normal vector for a given boundary $r = a(1+\varepsilon g(\theta;\varepsilon))$ is

$$\begin{aligned}\vec{n} &= \frac{1}{\sqrt{1+\left(\frac{\varepsilon a g'(\theta;\varepsilon)}{a+\varepsilon a g(\theta;\varepsilon)}\right)^2}}\left\langle 1, -\frac{\varepsilon a\frac{\partial g}{\partial\theta}(\theta;\varepsilon)}{a+\varepsilon a g(\theta;\varepsilon)}\right\rangle \\ &= \sqrt{\frac{(1+\varepsilon g(\theta;\varepsilon))^2}{1+2\varepsilon g(\theta;\varepsilon)+\varepsilon^2\left[g(\theta;\varepsilon)^2+\left(\frac{\partial g}{\partial\theta}(\theta;\varepsilon)\right)^2\right]}}\left\langle 1, -\frac{\varepsilon a\frac{\partial g}{\partial\theta}(\theta;\varepsilon)}{a+\varepsilon a g(\theta;\varepsilon)}\right\rangle \\ &= \sqrt{\frac{1}{1+2\varepsilon g(\theta;\varepsilon)+\varepsilon^2\left[g(\theta;\varepsilon)^2+\left(\frac{\partial g}{\partial\theta}(\theta;\varepsilon)\right)^2\right]}}\left\langle 1+\varepsilon g(\theta;\varepsilon), -\varepsilon\frac{\partial g}{\partial\theta}(\theta;\varepsilon)\right\rangle.\end{aligned}$$

Now, we can write (5.3), with point P exterior to the circle, as

$$\begin{aligned}u(r,\theta) - u_\infty &= \oint u(r',\theta')\frac{\partial G}{\partial n_Q}(r,\theta,r',\theta') - G(r,\theta,r',\theta')\frac{\partial u}{\partial n_Q}(r',\theta')ds' \\ &= \oint u(r',\theta')\left(\hat{n}\cdot\nabla_Q G(r,\theta,r',\theta')\right) - G(r,\theta,r',\theta')\left(\hat{n}\cdot\nabla_Q u(r',\theta')\right)ds'.\end{aligned}$$

where ds' is the arc length that we calculated earlier in (2.5).

For ease of notation, we will treat each of the integral terms in (5) separately for the moment. Denote the first term as I_1 and the second I_2. First we examine I_1 and assume as before that we are able to express $u(r,\theta)$ as a power series in ε. Noting that the square root terms from ds' and $\hat{n}$

cancel exactly, we have,

$$\begin{aligned} I_1 &= a\int_{-\pi}^{\pi} u\left(\left\langle 1+\varepsilon g(\theta';\varepsilon), -\varepsilon\frac{\partial g}{\partial\theta'}(\theta';\varepsilon)\right\rangle \cdot \left\langle \frac{\partial G}{\partial r'}, \frac{1}{r'}\frac{\partial G}{\partial\theta'}\right\rangle\right) d\theta' \\ &= a\int_{-\pi}^{\pi} u\left[\frac{\partial G}{\partial r'} + \varepsilon\left(g(\theta';\varepsilon)\frac{\partial G}{\partial r'} - \frac{1}{r'}\frac{\partial g}{\partial\theta'}(\theta';\varepsilon)\frac{\partial G}{\partial\theta'}\right)\right] d\theta'. \end{aligned}$$

This equation is exact, and as such, both G and u depend on ε as $r' = a(1+\varepsilon g(\theta';\varepsilon))$ on the boundary. Expanding $G(r,\theta,r',\theta')$ about $\varepsilon = 0$, gives

$$G(r,\theta,a(1+\varepsilon g(\theta';\varepsilon)),\theta') = G(r,\theta,a,\theta') + \varepsilon a g(\theta';0)\frac{\partial G}{\partial r'}(r,\theta,a,\theta') + \mathcal{O}(\varepsilon^2).$$

Doing the same for $u(r',\theta')$ gives

$$u(a(1+\varepsilon g(\theta';\varepsilon)),\theta') = u(a,\theta') + \varepsilon a g(\theta';0)\frac{\partial u}{\partial r'}(a,\theta') + \mathcal{O}(\varepsilon^2).$$

Now we insert our power series expansion, $u(a,\theta) = \sum_{n=0}^{\infty}\varepsilon^n u_n(a,\theta)$ and obtain

$$u(a(1+\varepsilon g(\theta';\varepsilon)),\theta') = u_0(a,\theta') + \varepsilon\left(a g(\theta;0)\frac{\partial u_0}{\partial r'}(a,\theta') + u_1(a,\theta')\right) + \mathcal{O}\left(\varepsilon^2\right). \tag{5.4}$$

Since $g(\theta';\varepsilon)$ appears in the integral, we must also expand it around $\varepsilon = 0$; however, this expansion is trivial and is given simply by

$$g(\theta';\varepsilon) = g(\theta';0) + \varepsilon\frac{\partial g}{\partial\varepsilon}(\theta';0) + \mathcal{O}\left(\varepsilon^2\right). \tag{5.5}$$

Additionally, the constant u_∞ can depend on ε, so we simply write it as

$$u_\infty = \kappa_0 + \varepsilon\kappa_1 + \mathcal{O}\left(\varepsilon^2\right).$$

Here we consider only first order effects as terms of order ε^2 or higher lead to extremely unwieldy integral equations. Using our expansions, we find for the first orders of ε,

$$
\begin{aligned}
&\varepsilon^0 : a \int_{-\pi}^{\pi} u_0 \frac{\partial G}{\partial r'} d\theta' \\
&\varepsilon^1 : a \int_{-\pi}^{\pi} u_1 \frac{\partial G}{\partial r'} + g(\theta';0) u_0 \frac{\partial G}{\partial r'} + a g(\theta';0) u_0 \frac{\partial^2 G}{\partial r'^2} + \\
&\qquad\qquad a g(\theta';0) \frac{\partial u_0}{\partial r'} \frac{\partial G}{\partial r'} - \frac{1}{a} \frac{\partial g}{\partial \theta'}(\theta';0) u_0 \frac{\partial G}{\partial \theta'} d\theta' \\
&\varepsilon^2 : \cdots
\end{aligned}
\tag{5.6}
$$

where G is a function of r, θ, a, and θ', and both u_0 and u_1 are functions of a and θ'. A similar procedure for I_2 gives the following expressions for each order of ε.

$$
\begin{aligned}
&\varepsilon^0 : -a \int_{-\pi}^{\pi} G \frac{\partial u_0}{\partial r'} d\theta' \\
&\varepsilon^1 : -a \int_{-\pi}^{\pi} G \frac{\partial u_1}{\partial r'} + g(\theta';0) G \frac{\partial u_0}{\partial r'} + a g(\theta';0) G \frac{\partial^2 u_0}{\partial r'^2} + \\
&\qquad\qquad a g(\theta';0) \frac{\partial u_0}{\partial r'} \frac{\partial G}{\partial r'} - \frac{1}{a} \frac{\partial g}{\partial \theta'}(\theta';0) G \frac{\partial u_0}{\partial \theta'} d\theta' \\
&\varepsilon^2 : \cdots
\end{aligned}
\tag{5.7}
$$

Now we combine the expressions for I_1 and I_2 to get a series of integral equations on the unperturbed circle $r' = a$ that we must solve.

$$
\begin{aligned}
&\varepsilon^0 : u_0 - \kappa_0 = a \int_{-\pi}^{\pi} u_0 \frac{\partial G}{\partial r'} - G \frac{\partial u_0}{\partial r'} d\theta' \\
&\varepsilon^1 : u_1 - \kappa_1 = a \int_{-\pi}^{\pi} u_1 \frac{\partial G}{\partial r'} - G \frac{\partial u_1}{\partial r'} + g(\theta';0) \left(u_0 \frac{\partial G}{\partial r'} - G \frac{\partial u_0}{\partial r'} \right) + \\
&\qquad a g(\theta';0) \left(u_0 \frac{\partial^2 G}{\partial r'^2} - G \frac{\partial^2 u_0}{\partial r'^2} \right) - \frac{1}{a} \frac{\partial g}{\partial \theta'}(\theta';0) \left(u_0 \frac{\partial G}{\partial \theta'} - G \frac{\partial u_0}{\partial \theta'} \right) d\theta' \\
&\varepsilon^2 : \cdots
\end{aligned}
$$

CHAPTER 6

TRANSLATED CIRCLE - INTEGRAL METHOD

6.1 Free Space Green's Function

For our integral representation, the two dimensional free-space Green's function is given by

$$G(\vec{u},\vec{v}) = -\frac{1}{2\pi}\ln(|\vec{u}-\vec{v}|).$$

Since our domain is a perturbation of a circle, we would like to express this Green's function in standard polar coordinates. Noting that

$$\begin{aligned}
-\frac{1}{2\pi}\ln\left(|\vec{u}-\vec{v}|\right) &= -\frac{1}{2\pi}\ln\left(\sqrt{|\vec{u}-\vec{v}|^2}\right)\\
&= -\frac{1}{4\pi}\ln\left((\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})\right)\\
&= -\frac{1}{4\pi}\ln\Big(\big\langle r\cos\theta - r'\cos\theta', r\sin\theta - r'\sin\theta'\big\rangle\\
&\qquad\qquad \cdot\big\langle r\cos\theta - r'\cos\theta', r\sin\theta - r'\sin\theta'\big\rangle\Big)\\
&= -\frac{1}{4\pi}\ln\left(r^2 + r'^2 - 2rr'\cos\theta\cos\theta' - 2rr'\sin\theta\sin'\theta'\right)\\
&= -\frac{1}{4\pi}\ln\left(r^2 + r'^2 - 2rr'\cos(\theta-\theta')\right).
\end{aligned}$$

Since every term in the integral we are evaluating contains $G(r,\theta,r',\theta')$ or one of its derivatives, we will pull $-\frac{1}{4\pi}$ out of the integral, and consider other constants as they arise on a term-by-term basis.

6.2 The Dirichlet Problem

Though our solution approach is different, the boundary conditions for our hierarchy of problems remains unchanged from (2.2),

$$\begin{aligned}
u_0(a,\theta) &= F(a,\theta)\\
u_1(a,\theta) + ag(\theta;0)\frac{\partial u_0}{\partial r}(a,\theta) &= ag(\theta;0)\frac{\partial F}{\partial r}(a,\theta)\\
\vdots \qquad &= \qquad \vdots
\end{aligned}$$

From earlier, we have that for this type of perturbation, $g(\theta;\varepsilon) = \cos\theta - \frac{1}{2}\varepsilon\sin^2\theta + \cdots$. We first examine the u_0 problem using $F(r,\theta) = r\cos\theta$.

$$u_0(r,\theta) - \kappa_0 = -\frac{a}{4\pi}\int_{-\pi}^{\pi} a\cos\theta' \frac{2a - 2r\cos(\theta-\theta')}{a^2+r^2-2ar\cos(\theta-\theta')} - \ln\left(a^2+r^2-2ar\cos\left(\theta-\theta'\right)\right)\frac{\partial u_0}{\partial r'}(a,\theta')d\theta'. \tag{6.1}$$

Let us consider each integral separately. Denote

$$I_1 = -\frac{a^2}{2\pi}\int_{-\pi}^{\pi}\frac{a - r\cos(\theta-\theta')}{a^2+r^2-2ar\cos(\theta-\theta')}\cos\theta' d\theta'$$
$$I_2 = \frac{a}{4\pi}\int_{-\pi}^{\pi}\ln\left(a^2+r^2-2ar\cos\left(\theta-\theta'\right)\right)\frac{\partial u_0}{\partial r'}(a,\theta')d\theta'$$

so that $u_0(r,\theta) = I_1 + I_2$. Let us first evaluate I_1. In order to make this integral easier to evaluate, we perform a change of variable. Let $\phi = \theta' - \theta$, then $d\phi = d\theta'$ and we can rewrite our integral as

$$I_1 = -\frac{a^2}{2\pi}\int_{-\pi}^{\pi}\frac{a - r\cos\phi}{a^2+r^2-2ar\cos\phi}\cos(\phi+\theta)d\phi.$$

Note that the integration bounds have not changed since ϕ is simply an offset of θ'; we are still integrating around the entire circle. We now multiply the top and bottom by $\frac{1}{r^2}$ and denote $\beta = \frac{a}{r}$. This gives

$$I_1 = -\frac{a^2}{2\pi r}\int_{-\pi}^{\pi}\frac{\beta - \cos\phi}{1+\beta^2-2\beta\cos\phi}\left(\cos\theta\cos\phi - \sin\theta\sin\phi\right)d\phi.$$

The fraction

$$\frac{\beta - \cos\phi}{1+\beta^2-2\beta\cos\phi}$$

is clearly even, and so when we multiply it by an odd function, such as $\sin\phi$, and integrate over an interval symmetric about 0, we get 0. Thus, we can reduce I_1 to

$$I_1 = -\frac{a^2}{2\pi r}\cos\theta\int_{-\pi}^{\pi}\frac{(\beta - \cos\phi)\cos\phi}{1+\beta^2-2\beta\cos\phi}d\phi. \tag{6.2}$$

Evaluating this integral is not trivial; however, it is similar to a known integral given by [3] for $a^2 < 1$ and $n \geq 0$,

$$\int_0^{\pi}\frac{\cos(nx)}{1-2a\cos x + a^2}dx = \frac{\pi a^n}{1-a^2}.$$

Noting that the integrand is an even function, we can deduce that

$$\int_{-\pi}^{\pi} \frac{\cos(n\phi)}{1+\beta^2-2\beta\cos\phi} d\phi = \frac{2\pi\beta^n}{1-\beta^2}. \tag{6.3}$$

Now, after expanding the numerator in (6.2), we have

$$I_1 = -\frac{a^2}{2\pi r}\cos\theta \int_{-\pi}^{\pi} \frac{\beta\cos\phi - \frac{1}{2} - \frac{1}{2}\cos(2\phi)}{1+\beta^2-2\beta\cos\phi} d\phi.$$

Applying (6.3) three times, we get

$$\begin{aligned} I_1 &= -\frac{a^2}{2\pi r}\frac{2\pi}{1-\beta^2}\cos\theta\left(\beta^2 - \frac{1}{2} - \frac{1}{2}\beta^2\right) \\ &= \frac{a^2}{2r}\cos\theta. \end{aligned}$$

Now we turn our attention to I_2. Since each of the $u_i(r,\theta)$ solve Laplace's equation on the unperturbed circle, we can take a derivative of the series solution

$$u_0(r',\theta') = B_0 + \sum_{n=1}^{\infty}\left\{A_n\sin(n\theta') + B_n\cos(n\theta')\right\} r'^{-n}.$$

Taking this derivative and substituting $r' = a$, gives us

$$\frac{\partial u_0}{\partial r'}(a,\theta') = \sum_{n=1}^{\infty}(-n)\left\{A_n\sin(n\theta') + B_n\cos(n\theta')\right\} a^{-n-1}.$$

Substituting this into I_2 gives us

$$\begin{aligned} I_2 &= \frac{a}{4\pi}\int_{-\pi}^{\pi} \ln\left(a^2+r^2-2ar\cos\left(\theta-\theta'\right)\right)\sum_{n=1}^{\infty}(-n)\left\{A_n\sin(n\theta') + B_n\cos(n\theta')\right\} a^{-n-1} d\theta' \\ &= -\sum_{n=1}^{\infty}\frac{n}{4a^n\pi}\int_{-\pi}^{\pi} \ln\left(a^2+r^2-2ar\cos\left(\theta-\theta'\right)\right)\left\{A_n\sin(n\theta') + B_n\cos(n\theta')\right\} d\theta' \\ &= -\sum_{n=1}^{\infty}\frac{n}{4a^n\pi}\int_{-\pi}^{\pi} \ln\left(\left(a - re^{\imath(\theta-\theta')}\right)\left(a - re^{-\imath(\theta-\theta')}\right)\right)\left\{A_n\sin(n\theta') + B_n\cos(n\theta')\right\} d\theta' \\ &= -\sum_{n=1}^{\infty}\frac{n}{4a^n\pi}\int_{-\pi}^{\pi} \left\{\ln\left(a - re^{\imath(\theta-\theta')}\right) + \ln\left(a - re^{-\imath(\theta-\theta')}\right)\right\}\left\{A_n\sin(n\theta') + B_n\cos(n\theta')\right\} d\theta'. \end{aligned}$$

We now have four relevant integrals we must calculate,

$$
\begin{aligned}
I_{21} &= A_n \int_{-\pi}^{\pi} \ln\left(a - r\mathrm{e}^{\imath(\theta-\theta')}\right)\sin(n\theta')d\theta' \\
I_{22} &= A_n \int_{-\pi}^{\pi} \ln\left(a - r\mathrm{e}^{-\imath(\theta-\theta')}\right)\sin(n\theta')d\theta' \\
I_{23} &= B_n \int_{-\pi}^{\pi} \ln\left(a - r\mathrm{e}^{\imath(\theta-\theta')}\right)\cos(n\theta')d\theta' \\
I_{24} &= B_n \int_{-\pi}^{\pi} \ln\left(a - r\mathrm{e}^{-\imath(\theta-\theta')}\right)\cos(n\theta')d\theta'
\end{aligned}
$$

We employ a similar process to evaluate each of the four integrals, so we will describe it only once then present results for all four integrals.

$$
\begin{aligned}
I_{23} &= B_n \int_{-\pi}^{\pi} \ln\left(r\mathrm{e}^{\imath(\theta-\theta')}\left(\frac{a}{r}\mathrm{e}^{-\imath(\theta-\theta')} - 1\right)\right)\cos(n\theta')d\theta' \\
&= B_n \int_{-\pi}^{\pi} \left\{\ln(r\mathrm{e}^{\imath(\theta-\theta')}) + \ln(-1) + \ln\left(1 - \frac{a}{r}\mathrm{e}^{-\imath(\theta-\theta')}\right)\right\}\cos(n\theta')d\theta'.
\end{aligned}
\tag{6.4}
$$

The middle of these three integrals is clearly zero. The first integral can be reduced simply to

$$
B_n \int_{-\pi}^{\pi} \ln(r)\cos(n\theta') + \imath\theta\cos(n\theta') - \imath\theta'\cos(n\theta')d\theta'.
$$

All three of these integrals are zero when integrated around the entire boundary. So the value of I_{23} depends only on the third term in (6.4). Since we are outside of the circular region, $r > a$, so we are able to write the logarithm as a Taylor series. This turns our integral into

$$
\begin{aligned}
I_{23} &= B_n \int_{-\pi}^{\pi} -\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{a}{r}\right)^m \mathrm{e}^{-\imath m(\theta-\theta')}\cos(n\theta')d\theta' \\
&= -B_n \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{a}{r}\right)^m \mathrm{e}^{-\imath m\theta} \int_{-\pi}^{\pi} \mathrm{e}^{\imath m\theta'}\cos(n\theta')d\theta' \\
&= -B_n \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{a}{r}\right)^m \mathrm{e}^{-\imath m\theta} \int_{-\pi}^{\pi} \left\{\cos(m\theta') + \imath\sin(m\theta')\right\}\cos(n\theta')d\theta'.
\end{aligned}
$$

Over the interval $[-\pi, \pi)$, $\sin(m\theta')$ and $\cos(m\theta')$ are orthogonal to $\cos(n\theta')$, so for $m \neq n$, the integral evaluates to 0. Thus, the value of the integral is simply

$$
I_{23} = -\frac{B_n\pi}{n}\left(\frac{a}{r}\right)^n \mathrm{e}^{-\imath n\theta}.
$$

A similar process is used to evaluate the other relevant integrals.

$$\begin{aligned}
I_{21} &= -\imath\frac{A_n\pi}{n}\left(\frac{a}{r}\right)^n \mathrm{e}^{-\imath n\theta} + \frac{2A_n\pi}{n} \\
I_{22} &= \imath\frac{A_n\pi}{n}\left(\frac{a}{r}\right)^n \mathrm{e}^{\imath n\theta} - \frac{2A_n\pi}{n} \\
I_{24} &= -\frac{B_n\pi}{n}\left(\frac{a}{r}\right)^n \mathrm{e}^{\imath n\theta}
\end{aligned} \tag{6.5}$$

Plugging these values into I_2 gives us

$$\begin{aligned}
I_2 &= -\sum_{n=1}^{\infty}\frac{1}{4a^n}\left(\frac{a}{r}\right)^n\left\{\imath A_n(\mathrm{e}^{\imath n\theta} - \mathrm{e}^{-\imath n\theta}) - B_n(\mathrm{e}^{\imath n\theta} + \mathrm{e}^{-\imath n\theta})\right\} \\
&= -\sum_{n=1}^{\infty}\frac{1}{4r^n}\left\{-2A_n\sin(n\theta) - 2B_n\cos(n\theta)\right\} \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\left\{A_n\sin(n\theta) + B_n\cos(n\theta)\right\} r^{-n} \\
&= \frac{1}{2}u_0(r,\theta).
\end{aligned}$$

Now if we substitute I_1 and I_2 into (6.1), we get

$$\begin{aligned}
u_0(r,\theta) - \kappa_0 &= \frac{a^2}{2r}\cos\theta + \frac{1}{2}u_0(r,\theta) \\
u_0(r,\theta) &= 2\kappa_0 + \frac{a^2}{r}\cos\theta.
\end{aligned}$$

Since κ_0 is a constant, we can use our boundary condition to determine its value. Setting $r = a$ at enforcing $u_0(a,\theta) = F(a,\theta)$ gives us $\kappa_0 = 0$. Thus,

$$u_0(r,\theta) = \frac{a^2}{r}\cos\theta. \tag{6.6}$$

Now that we have a solution for the u_0 problem, we are able to solve the u_1 problem,

$$\begin{aligned}
u_1 - \kappa_1 = -\frac{a}{4\pi}\int_{-\pi}^{\pi} u_1\frac{\partial G}{\partial r'} - G\frac{\partial u_1}{\partial r'} + g(\theta';0)\left(u_0\frac{\partial G}{\partial r'} - G\frac{\partial u_0}{\partial r'}\right) + ag(\theta';0)\left(u_0\frac{\partial^2 G}{\partial r'^2}\right. \\
\left. -G\frac{\partial^2 u_0}{\partial r'^2}\right) - \frac{1}{a}\frac{\partial g}{\partial \theta'}(\theta';0)\left(u_0\frac{\partial G}{\partial \theta'} - G\frac{\partial u_0}{\partial \theta'}\right) d\theta'.
\end{aligned}$$

If we rearrange these terms, we get

$$\begin{aligned} u_1 - \kappa_1 = -\frac{a}{4\pi}\int_{-\pi}^{\pi} \frac{\partial G}{\partial r'}\big(u_1 + g(\theta';0)u_0\big) + ag(\theta';0)u_0\frac{\partial^2 G}{\partial r'^2} - \frac{1}{a}\frac{\partial g}{\partial \theta'}(\theta';0)u_0\frac{\partial G}{\partial \theta'} \\ + G\left(\frac{1}{a}\frac{\partial g}{\partial \theta'}(\theta';0)\frac{\partial u_0}{\partial \theta'} - g(\theta';0)\frac{\partial u_0}{\partial r'} - ag(\theta';0)\frac{\partial^2 u_0}{\partial r'^2} - \frac{\partial u_1}{\partial r'}\right) d\theta'. \end{aligned}$$

Since u_0 satisfies Laplace's equation on our domain, we have

$$\begin{aligned} \frac{\partial^2 u_0}{\partial r^2} + \frac{1}{r}\frac{\partial u_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_0}{\partial \theta^2} = 0 \\ rg(\theta;0)\frac{\partial^2 u_0}{\partial r^2} + g(\theta;0)\frac{\partial u_0}{\partial r} + \frac{g(\theta;0)}{r}\frac{\partial^2 u_0}{\partial \theta^2} = 0. \end{aligned}$$

Using this, we can simplify our integral to

$$\begin{aligned} u_1 - \kappa_1 = -\frac{a}{4\pi}\int_{-\pi}^{\pi} \frac{\partial G}{\partial r'}\big(u_1 + g(\theta';0)u_0\big) + ag(\theta';0)u_0\frac{\partial^2 G}{\partial r'^2} \\ - \frac{1}{a}\frac{\partial g}{\partial \theta'}(\theta';0)u_0\frac{\partial G}{\partial \theta'} + G\left(\frac{1}{a}\frac{\partial}{\partial \theta'}\left(g(\theta';0)\frac{\partial u_0}{\partial \theta'}\right) - \frac{\partial u_1}{\partial r'}\right) d\theta'. \end{aligned}$$

Now we add and subtract

$$\frac{g(\theta';0)}{a}\frac{\partial G}{\partial \theta'}\frac{\partial u_0}{\partial \theta'}$$

which allows us to write

$$\begin{aligned} u_1 - \kappa_1 = -\frac{a}{4\pi}\int_{-\pi}^{\pi} \frac{\partial G}{\partial r'}\big(u_1 + g(\theta';0)u_0\big) + ag(\theta';0)u_0\frac{\partial^2 G}{\partial r'^2} - \frac{1}{a}\frac{\partial g}{\partial \theta'}(\theta';0)u_0\frac{\partial G}{\partial \theta'} \\ - \frac{1}{a}g(\theta';0)\frac{\partial u_0}{\partial \theta'}\frac{\partial G}{\partial \theta'} + \frac{1}{a}\frac{\partial}{\partial \theta'}\left(Gg(\theta';0)\frac{\partial u_0}{\partial \theta'}\right) - G\frac{\partial u_1}{\partial r'}d\theta'. \end{aligned} \tag{6.7}$$

We are able to evaluate the last two integrals immediately since one simply negates the derivative, and the second we recognize as I_2 from the u_0 problem, but with u_1 instead.

$$\begin{aligned} u_1 - \kappa_1 = \big(\ln(a^2 + r^2 - 2ar\cos(\theta-\theta'))\cos\theta'\sin\theta'\big)\,|_{-\pi}^{\pi} + \frac{1}{2}u_1 \\ - \frac{a}{4\pi}\int_{-\pi}^{\pi} \frac{\partial G}{\partial r'}\big(u_1 + g(\theta';0)u_0\big) + ag(\theta';0)u_0\frac{\partial^2 G}{\partial r'^2} - \frac{1}{a}\frac{\partial g}{\partial \theta'}(\theta';0)u_0\frac{\partial G}{\partial \theta'} - \frac{1}{a}g(\theta';0)\frac{\partial u_0}{\partial \theta'}\frac{\partial G}{\partial \theta'}d\theta'. \end{aligned}$$

Then after we substitute the boundary value for $u_1(a,\theta)$,

$$u_1 - 2\kappa_1 = -\frac{a}{2\pi}\int_{-\pi}^{\pi} \frac{\partial G}{\partial r'}\left(ag(\theta';0)\left(\frac{\partial F}{\partial r'} - \frac{\partial u_0}{\partial r'}\right) + g(\theta';0)u_0\right) + ag(\theta';0)u_0\frac{\partial^2 G}{\partial r'^2} \\ - \frac{1}{a}\frac{\partial G}{\partial \theta'}\left(\frac{\partial g}{\partial \theta'}(\theta';0)u_0 + g(\theta';0)\frac{\partial u_0}{\partial \theta'}\right) d\theta'.$$

Substituting known quantities, we get three integrals we must evaluate,

$$u_1 - 2\kappa_1 = -\frac{a}{2\pi}\int_{-\pi}^{\pi} 6a\cos^2\theta' \frac{a - r\cos(\theta-\theta')}{a^2 + r^2 - 2ar\cos(\theta-\theta')} - 4a\cos\theta'\sin\theta' \frac{r\sin(\theta-\theta')}{a^2 + r^2 - 2ar\cos(\theta-\theta')} \\ - 2a^2\cos^2\theta' \frac{a^2 + r^2\cos(2(\theta-\theta')) - 2ar\cos(\theta-\theta')}{(a^2 + r^2 - 2ar\cos(\theta-\theta'))^2} d\theta'.$$

We once again consider each integral individually, denote

$$\begin{aligned} I_3 &= -\frac{3a^2}{2\pi}\int_{-\pi}^{\pi} \frac{a - r\cos(\theta-\theta')}{a^2 + r^2 - 2ar\cos(\theta-\theta')}\left(1 + \cos(2\theta')\right) d\theta' \\ I_4 &= \frac{a^2}{\pi}\int_{-\pi}^{\pi} \frac{r\sin(\theta-\theta')}{a^2 + r^2 - 2ar\cos(\theta-\theta')}\sin(2\theta') d\theta' \\ I_5 &= \frac{a^3}{2\pi}\int_{-\pi}^{\pi} \frac{a^2 + r^2\cos(2(\theta-\theta')) - 2ar\cos(\theta-\theta')}{(a^2 + r^2 - 2ar\cos(\theta-\theta'))^2}\left(1 + \cos(2\theta')\right) d\theta' \end{aligned}$$

We evaluate I_3 in a similar way to I_1. Using the same change of variables and introducing β again, we get

$$I_3 = -\frac{3a^2}{2\pi r}\int_{-\pi}^{\pi} \frac{\beta - \cos\phi}{1 + \beta^2 - 2\beta\cos\phi}\left(1 + \cos(2\phi + 2\theta)\right) d\phi.$$

Expanding $\cos(2\phi + 2\theta)$ and disregarding the odd part, we get

$$I_3 = -\frac{3a^2}{2\pi r}\left(\int_{-\pi}^{\pi} \frac{\beta - \cos\phi}{1 + \beta^2 - 2\beta\cos\phi} d\phi + \cos(2\theta)\int_{-\pi}^{\pi} \frac{\beta\cos(2\phi) - \frac{1}{2}\cos\phi - \frac{1}{2}\cos(3\phi)}{1 + \beta^2 - 2\beta\cos\phi} d\phi\right).$$

We now apply (6.3) to each term to get

$$\begin{aligned} I_3 &= -\frac{3a^2}{2\pi r}\frac{2\pi}{1-\beta^2}\left(\beta - \beta + \cos(2\theta)\left(\beta^3 - \frac{1}{2}\beta - \frac{1}{2}\beta^3\right)\right) \\ &= \frac{3a^3}{2r^2}\cos(2\theta). \end{aligned}$$

We employ the same substitution for I_4 and again introduce β. This gives us

$$I_4 = -\frac{a^2}{\pi r}\int_{-\pi}^{\pi} \frac{\sin\phi}{1+\beta^2-2\beta\cos\phi}\sin(2\phi+2\theta)d\phi.$$

The fraction $\frac{\sin\phi}{1+\beta^2-2\beta\cos\phi}$ is an odd function of ϕ, so we expand $\sin(2\phi+2\theta)$ and neglect the even part to get

$$I_4 = -\frac{a^2}{\pi r}\cos(2\theta)\int_{-\pi}^{\pi} \frac{\sin\phi\sin(2\phi)}{1+\beta^2-2\beta\cos\phi}d\phi.$$

We again turn to integration tables to evaluate this integral. [3] gives the following identity for $a^2 < 1$ and $n \geq 0$.

$$\int_{0}^{\pi} \frac{\sin(nx)\sin x}{1-2a\cos x+a^2}dx = \frac{\pi}{2}a^{n-1}.$$

The integrand is even, so we can use the same trick as before and double the result when we change the integration range to $[-\pi,\pi)$. Thus, I_4 becomes

$$\begin{aligned} I_4 &= -\frac{a^2}{\pi r}\cos(2\theta)(\pi\beta) \\ &= -\frac{a^3}{r^2}\cos(2\theta). \end{aligned}$$

Finally, we consider I_5. Proceeding as before, we get

$$I_5 = \frac{a^3}{2\pi r^2}\int_{-\pi}^{\pi} \frac{\beta^2+\cos(2\phi)-2\beta\cos\phi}{(1+\beta^2-2\beta\cos\phi)^2}\left(1+\cos(2\phi+2\theta)\right)d\phi.$$

Once again, the fractional part of the integral is even, so we expand $\cos(2\phi+2\theta)$ and disregard the odd part. This gives us

$$\begin{aligned} I_5 = \frac{a^3}{2\pi r^2}\Bigg(&\int_{-\pi}^{\pi} \frac{\beta^2+\cos(2\phi)-2\beta\cos\phi}{(1+\beta^2-2\beta\cos\phi)^2}d\phi \\ &+\cos(2\theta)\int_{-\pi}^{\pi} \frac{\frac{1}{2}-\beta\cos\phi+\beta^2\cos(2\phi)-\beta\cos(3\phi)+\frac{1}{2}\cos(4\phi)}{(1+\beta^2-2\beta\cos\phi)^2}d\phi\Bigg). \end{aligned}$$

We once again consult [3] to find that for $a^2 < 1$,

$$\int_{-\pi}^{\pi} \frac{\cos(nx)}{(1-2a\cos x+a^2)^m}dx = \frac{2\pi a^{2m+n-2}}{(1-a^2)^{2m-1}}\sum_{k=0}^{m-1}\binom{m+n-1}{k}\binom{2m-k-2}{m-1}\left(\frac{1-a^2}{a^2}\right)^k.$$

We apply this identity to each term in I_5 with $m = 2$, and we find that

$$\begin{aligned}
I_5 &= \frac{a^3}{2\pi r^2}\frac{2\pi\beta^2}{(1-\beta^2)^3}\left(\left(\beta^2+1\right)-4+\left(3-\beta^2\right)+\cos(2\theta)\left(\left(\frac{1}{2}+\frac{1}{2\beta^2}\right)-2+\left(3\beta^2-\beta^4\right)\right.\right.\\
&\quad\left.\left.+\left(2\beta^4-4\beta^2\right)+\left(\frac{5\beta^2}{2}-\frac{3\beta^4}{2}\right)\right)\right)\\
&= \frac{a^3}{2\pi r^2}\frac{2\pi\beta^2}{(1-\beta^2)^3}\cos(2\theta)\frac{\left(1-\beta^2\right)^3}{2\beta^2}\\
&= \frac{a^3}{2r^3}\cos(2\theta).
\end{aligned}$$

Thus when we combine all three of these integrals, we have

$$u_1(r,\theta) = 2\kappa_1 + \frac{a^3}{r^2}\cos(2\theta). \tag{6.8}$$

We again use the boundary condition in order to determine the value of κ_1. Our boundary condition is

$$u_1(a,\theta) = 2a\cos^2\theta = a + a\cos(2\theta).$$

If we set this equal to (6.8) with $r = a$ we find that $\kappa_1 = \frac{a}{2}$. This gives us the solution

$$u_1(r,\theta) = a + \frac{a^3}{r^2}\cos(2\theta). \tag{6.9}$$

Now, we combine our solutions for the u_0 and u_1 problems ((6.6) and (6.9)) to get

$$u(r,\theta) = \frac{a^2}{r}\cos\theta + \varepsilon\left(a + \frac{a^3}{r^2}\cos(2\theta)\right) + \mathcal{O}\left(\varepsilon^2\right).$$

which is exactly what we obtained from the first approach we employed.

6.3 The Neumann Problem

Unlike the first approach we used, we do not require an entirely new formulation in order to solve the Neumann problem. We simply use the Neumann boundary data to evaluate different integrals than we did for the Dirichlet problem. For the Neumann problem, we will use the same boundary condition as we did in our first approach, $F(r,\theta) = r\sin\theta$. This boundary condition satisfies the solvability conditions for the u_0 and u_1 problems. Using (2.4) as our heirarchy of

Neumann boundary values, we begin again with the u_0 problem,

$$\begin{aligned} u_0(r,\theta) - \kappa_0 &= -\frac{a}{4\pi}\int_{-\pi}^{\pi} u_0(a,\theta')\frac{2a - 2r\cos(\theta-\theta')}{a^2+r^2-2ar\cos(\theta-\theta')} \\ &\qquad -a\sin\theta' \ln\left(a^2+r^2-2ar\cos\left(\theta-\theta'\right)\right)d\theta'. \end{aligned}$$

We will again consider each integral separately, let

$$\begin{aligned} N_1 &= -\frac{a}{2\pi}\int_{-\pi}^{\pi} u_0(a,\theta')\frac{a - r\cos(\theta-\theta')}{a^2+r^2-2ar\cos(\theta-\theta')}d\theta' \\ N_2 &= \frac{a^2}{4\pi}\int_{-\pi}^{\pi} \sin\theta' \ln(a^2+r^2-2ar\cos(\theta-\theta'))d\theta' \end{aligned}$$

Since $u_0(r,\theta)$ solves Laplace's equation on a circle, we can use its series expansion to evaluate N_1.

$$N_1 = -\frac{a}{2\pi}\int_{-\pi}^{\pi}\left(B_0 + \sum_{n=1}^{\infty} a^{-n}\left\{A_n\cos(n\theta') + B_n\sin(n\theta')\right\}\right)\frac{a - r\cos(\theta-\theta')}{a^2+r^2-2ar\cos(\theta-\theta')}d\theta'.$$

We use the same change of variables as we did for the Dirichlet problem, and we again introduce β to get

$$N_1 = -\frac{a}{2\pi r}\int_{-\pi}^{\pi}\left(B_0 + \sum_{n=1}^{\infty} a^{-n}\left\{A_n\cos(n\phi+n\theta) + B_n\sin(n\phi+n\theta)\right\}\right)\frac{\beta-\cos\phi}{1+\beta^2-2\beta\cos\phi}d\phi.$$

Examining the term with B_0 and applying (6.3) gives us that

$$-\frac{aB_0}{2\pi r}\int_{-\pi}^{\pi}\frac{\beta-\cos\phi}{1+\beta^2-2\beta\cos\phi}d\phi = 0.$$

We can now interchange the sum and integral and eliminate the odd terms in ϕ. This gives us

$$\begin{aligned} N_1 &= -\sum_{n=1}^{\infty}\frac{a^{1-n}}{2\pi r}\left\{A_n\cos(n\theta) + B_n\sin(n\theta)\right\}\int_{-\pi}^{\pi}\cos(n\phi)\frac{\beta-\cos\phi}{1+\beta^2-2\beta\cos\phi}d\phi \\ &= -\sum_{n=1}^{\infty}\frac{a^{1-n}}{2\pi r}\left\{A_n\cos(n\theta) + B_n\sin(n\theta)\right\}\int_{-\pi}^{\pi}\frac{\beta\cos(n\phi) - \frac{1}{2}\cos((n+1)\phi) - \frac{1}{2}\cos((n-1)\phi)}{1+\beta^2-2\beta\cos\phi}d\phi. \end{aligned}$$

We now apply (6.3) and find

$$\begin{aligned}
N_1 &= -\sum_{n=1}^{\infty} \frac{a^{1-n}}{2\pi r} \{A_n \cos(n\theta) + B_n \sin(n\theta)\} \frac{\pi\beta^{n+1} - \pi\beta^{n-1}}{1-\beta^2} \\
&= \sum_{n=1}^{\infty} \frac{a^{1-n}\beta^{n-1}}{2\pi r} \{A_n \cos(n\theta) + B_n \sin(n\theta)\} \\
&= \frac{1}{2}\sum_{n=1}^{\infty} r^{-n} \{A_n \cos(n\theta) + B_n \sin(n\theta)\} \\
&= \frac{1}{2}u_0(r,\theta) - \frac{B_0}{2}.
\end{aligned}$$

Now we turn our attention towards N_2. We have already evaluated an integral of this form for the Dirichlet problem. If we take $n = 1$ and set $A_1 = 1$ and all other A_n and B_n to be 0 in (6.5), we get

$$\begin{aligned}
N_2 &= \frac{a^2}{4\pi}\left(-\imath\pi\left(\frac{a}{r}\right)e^{-\imath\theta} + \imath\pi\left(\frac{a}{r}\right)e^{\imath\theta}\right) \\
&= -\frac{a^3}{2r}\sin\theta.
\end{aligned}$$

Combining this with the solution for N_1 gives

$$u_0(r,\theta) - \kappa_0 = \frac{1}{2}u_0(r,\theta) - \frac{B_0}{2} - \frac{a^3}{2r}\sin\theta u_0(r,\theta) \quad = \gamma_0 - \frac{a^3}{r}\sin\theta$$

where γ_0 is a constant term we determine by using the boundary condition. Here we want $\frac{\partial u_0}{\partial r}(a,\theta) = F(a,\theta) = a\sin\theta$. Enforcing this give $\gamma_0 = 0$. Thus,

$$u_0(r,\theta) = -\frac{a^3}{r}\sin\theta.$$

For the u_1 problem, we begin by recalling (6.7). We can once again integrate out the second to last term, and since we now have a Neumann problem, the first term is exactly N_1 but with u_1 in place of u_0. Substituting in our boundary terms for $\frac{\partial u_1}{\partial r}(a,\theta)$ and simplifying gives,

$$\begin{aligned}
u_1 - \gamma_1 = -\frac{a}{2\pi}\int_{-\pi}^{\pi} &\frac{\partial G}{\partial r'}g(\theta';0)u_0 + ag(\theta';0)u_0\frac{\partial^2 G}{\partial r'^2} - \frac{1}{a}\frac{\partial G}{\partial \theta'}\left(\frac{\partial g}{\partial \theta'}(\theta';0)u_0 + g(\theta';0)\frac{\partial u_0}{\partial \theta'}\right) \\
&+ G\left(ag(\theta';0)\frac{\partial^2 u_0}{\partial r'^2} - ag(\theta';0)\frac{\partial F}{\partial r'} - \frac{1}{a}\frac{\partial u_0}{\partial \theta'}\frac{\partial g}{\partial \theta'}(\theta';0)\right)d\theta'
\end{aligned}$$

where we have again combined all constant terms into γ_1. When we plug in known quantities, we get four integrals we must evaluate,

$$u_1 - \gamma_1 = -\frac{a}{2\pi}\int_{-\pi}^{\pi} -a^2\sin(2\theta)\frac{a - r\cos(\theta-\theta')}{a^2+r^2-2ar\cos(\theta-\theta')} + 2a^2\cos(2\theta)\frac{r\sin(\theta-\theta')}{a^2+r^2-2ar\cos(\theta-\theta')}$$
$$+ a^3\sin(2\theta)\frac{a^2+r^2\cos(2(\theta-\theta'))-2ar\cos(\theta-\theta')}{(a^2+r^2-2ar\cos(\theta-\theta'))^2} - 2a\sin(2\theta)\ln(a^2+r^2-2ar\cos(\theta-\theta'))d\theta'.$$

These first three integrals are all very similar to ones we evaluated for the Dirichlet problem. We simply make the standard change of variables, introduce β and ignore the odd part of the resulting function. For the fourth integral, we again use (6.5). Once we evaluate all four of these integrals, we get

$$\begin{aligned} u_1(r,\theta) - \gamma_1 &= -\frac{a^4}{2r^2}\sin(2\theta) + \frac{a^4}{r^2}\sin(2\theta) - \frac{a^4}{2r^2}\sin(2\theta) - \frac{a^4}{r^2}\sin(2\theta) \\ &= -\frac{a^4}{r^2}\sin(2\theta). \end{aligned}$$

We again enforce our boundary condition for $u_1(r,\theta)$ and we find that $\gamma_1 = 0$. Combining our $u_0(r,\theta)$ and $u_1(r,\theta)$ solutions, we get

$$u(r,\theta) = -\frac{a^3}{r}\sin\theta - \varepsilon\frac{a^4}{r^2}\sin(2\theta) + \mathcal{O}\left(\varepsilon^2\right).$$

This is exactly what we had for our solution in chapter 3.3.

CHAPTER 7
CONCLUSION

This thesis presents two methods for solving Laplace's equation on a perturbed domain. Instead of solving the exact problem, we express our domain as a perturbation of a circle and then using matched expansions, we develop a series of boundary conditions. Then, assuming a power series form of the potential $u(r, \theta)$, we develop a hierarchy of problems to solve on an unperturbed domain. These problems must be solved in order as the boundary condition of the n^{th} problem depends on the solutions of all previous problems.

In both approaches, we exploit the fact that Laplace's equation is separable on the unperturbed domain. This allows us to write a solution by using the standard series solution for Laplace's equation in polar coordinates. In the first method, coefficients of this series are calculated using simple orthogonality properties. In the second method, we utilize this series in order to evaluate integrals over the boundary of our domain.

After applying this method to several test cases, we then constructed solutions to the exact problem by solving Laplace's equation and then using coordinate transforms. Both approaches yielded solutions that were equivalent to the exact solution (to the order we calculated) for both Dirichlet and Neumann boundary conditions.

This method could easily be adapted to three dimensional problems where we consider a domain as a perturbation of a sphere. The integral method would benefit from this increase in dimension as integrals of the three dimensional free-space Green's function are generally easier to perform than those with the two dimensional function.

The separation of variables method lends itself well for being programmed in a symbolic software package such as Mathematica. Much of the difficulty of this method lies in the sheer number of terms to compute. Automation of this process would lead to being able to calculate terms of almost arbitrary order.

Finally, this approach could be adapted to other elliptic PDEs such as the Helmholtz equation. Expanding about the small perturbation parameter could give rise to a heirarchy of scattering problems involving circle-like or sphere-like objects.

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