RELATIVISTIC EFFECTS OF SLOW PRIMARY ISOTROPIC MASS LOSS ON PLANETARY ORBITS

by

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ABSTRACT

The theory of adiabatic invariants and perturbation techniques are applied to the Schwarzschild exterior solution to determine the effect of primary isotropic mass loss on planetary orbits. As would be expected, the relativistic corrections found are a small fraction of the corresponding Newtonian results.

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INTRODUCTION

Since Einstein published the general theory of relativity, the only application of the theory in planetary motion has been in the advance of perihelia of the planets. This lack of application is due to the fact that Newtonian gravitational theory gives accuracy consistent with almost all observations.

The evolution of planetary orbits as the primary body, e.g. the sun, slowly loses mass will be considered in this thesis. Since we have no observations to compare our results with, we will limit our discussion to the assumptions and techniques used in applying general relativity to planetary motion.

The specific technique used is to approximate relativistic planetary motion as motion in a perturbed Newtonian gravitational field. The theory of adiabatic invariants is applied to motion in this field to obtain relativistic results for the evolution of planetary orbits.

GENERAL RELATIVITY

Einstein's general theory of relativity may be viewed as a postulate and a recipe. The postulate is that all bodies move on geodesics in four dimensional space-time. The recipe tells how to determine the metric (and hence, the geodesics) of space-time, given the distribution of mass and energy. The metric (a tensor quantity) describes the shape of space-time.

For a discussion of the procedure below the reader is referred to texts on general relativity (e.g. Adler, 1965, DeWitt, 1964). Before considering the Einstein field equation which links the metric and the mass energy distribution, some definitions are necessary. The covariant and contravariant representations of the metric tensor are denoted by g_{ij} and g^{ij}, respectively; the Riemann-Christoffel tensor $R^{\ell}_{\cdot jnp}$ is given by

$$R^{\ell}_{\cdot jnp} \triangleq \frac{g^{\ell r}}{2} \left\{ \frac{\partial^2 g_{rp}}{\partial x^j \partial x^n} + \frac{\partial^2 g_{jn}}{\partial x^r \partial x^p} - \frac{\partial^2 g_{rn}}{\partial x^j \partial x^p} - \frac{\partial^2 g_{jp}}{\partial x^r \partial x^n} \right\},$$

where $\triangleq$ means "by definition", and repeated indices indicate the Einstein summation convention (e.g. $\sum_{i=1}^{4} g_{ij} g^{ik} \triangleq g_{ij} g^{ik}$). If the Riemann-Christoffel tensor is contracted (summed) on its first and last indices, the Ricci tensor R_{ij} is obtained:

$$R_{ij} \triangleq R^{\ell}_{\cdot ij\ell}.$$

The double contraction of the product of the metric and Ricci tensor gives the curvature invariant (R).

$$R \triangleq g^{ij} R_{ij}$$

With this groundwork we may define the Einstein tensor ($G^{a}_{\cdot b}$) where δ^{a}_{b} is the Kronecker delta ($\delta^{a}_{b} = 0$ for $a \neq b$; $\delta^{a}_{b} = 1$ for $a = b$).

$$G^{a}_{\cdot b} = g^{ai} R_{bi} - 1/2\ R\ \delta^{a}_{b}$$

Einstein's field equation links the geometrical Einstein tensor $G^{a}_{\cdot b}$ and the physical energy-momentum tensor $T^{a}_{\cdot b}$ as follows:

$$(8\pi\gamma/c^4)\ T^{a}_{\cdot b} + G^{a}_{\cdot b} = 0$$

where γ is the gravitational constant and c is the speed of light in a vacuum.

The Einstein field equation is very profound since it says that the shape of space-time is tied to the distribution of energy and momentum.

For a body moving freely with velocity $|v| << c$ the energy-momentum tensor is

$$T^{\mu\nu} = \rho \begin{bmatrix} 1 & v_x/c & v_y/c & v_z/c \\ v_x/c & v_x^2/c^2 & v_y v_x/c^2 & v_x v_z/c^2 \\ v_y/c & v_y v_x/c^2 & v_y^2/c^2 & v_y v_z/c^2 \\ v_z/c & v_z v_x/c^2 & v_y v_z/c^2 & v_z^2/c^2 \end{bmatrix}$$

where ρ is the density of the body.

When the field equation is applied to a spherically symmetric mass distribution with Lorentz (special relativity) boundary conditions at infinity, the exterior solution gives the Schwarzschild metric. The explicit form of the

Schwarzschild metric is

$$g_{ij} = \begin{bmatrix} 1-\frac{2\Gamma}{r} & 0 & 0 & 0 \\ 0 & \frac{-1}{1-2\Gamma/r} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}$$

where $\Gamma = \gamma M/c^2$ and M is the mass of the primary.

Having the metric, the differential equation of the geodesics can be determined from

$$d^2x^\ell/ds^2 + \{i\,{}^\ell\,k\} \frac{dx^i}{ds} \cdot \frac{dx^k}{ds} = 0 \quad (1)$$

where $(ds)^2 = g_{ik}dx^i dx^k$, and the Christoffel symbol of the second kind is given by

$$\{i\,{}^\ell\,j\} = g^{\ell k}/2 \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right).$$

Like the other symbols introduced above, the Christoffel symbol of the second kind may appear to have little intuitive meaning, but it occurs implicitly in Lagrangian dynamics. Consider a particle of unit mass subject to a conservative force $F_i = -\partial V/\partial x^i$. The kinetic energy of the particle (T) is $1/2(ds)^2/(dt)^2$ or $1/2 g_{ik}\dot{x}^i\dot{x}^k$. Where $\dot{x}^k = dx^k/dt$ the Lagrangian (L) is T-V and the Lagrange equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i} \; .$$

In explicit form these become

$$g_{ik}\ddot{x}^k + \frac{\partial g_{ik}}{\partial x^\ell}\dot{x}^\ell\dot{x}^k = \frac{1}{2}\frac{\partial g_{\ell k}}{\partial x^i}\dot{x}^\ell\dot{x}^k + F_i$$

or

$$g_{ik}\ddot{x}^k + \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^\ell} + \frac{\partial g_{i\ell}}{\partial x^k} - \frac{\partial g_{\ell k}}{\partial x^i}\right]\dot{x}_\ell\dot{x}_k = F_i .$$

Multiplying by g^{ij}

$$\ddot{x}^j + \{\ell{}^j k\}\dot{x}^\ell\dot{x}^k = F^j .$$

This equation, except for the F^j term, is the geodesic equation. From equation (1), the differential equation of the geodesic (orbit) for the Schwarzschild metric is found to be

$$d^2u/d\phi^2 + u = \frac{\gamma M}{4H^2} + \frac{3\gamma Mu^2}{c^2} \tag{2}$$

where $u = 1/r$, ϕ is the anomaly, and H ($H = 1/2\ r^2\dot{\phi}$) is the constant of areal velocity.

Once the differential equation of the orbit is obtained, the problem may be considered in an alternate light. Binet's formula of classical mechanics states

$$d^2u/d\phi^2 + u = \frac{1}{4H^2u}\left(\frac{\partial V}{\partial r}\right)$$

where V is a classical central force potential.

Applying Binet's formula to equation (2), V is found to be

$$V = -\gamma M/r(1 + 4H^2/c^2r^2) . \tag{3}$$

Once the potential (3) is obtained, no further mention of general relativity is necessary. The problem may be analyzed by classical mechanics. The procedure of treating relativistic motion as motion in a perturbed classical potential should not make us lose sight of the fact that relativistic mechanics has a different basis than classical mechanics. In classical mechanics, gravitational forces influence the motion of bodies, but in relativistic mechanics the shape of space influences the motion of bodies.

ADIABATIC INVARIANTS FOR CENTRAL FORCE MOTION

An adiabatic invariant is invariant because of the "slowness" of changes in the parameters of the problem. Adiabatic invariants were first used by Einstein and Lorentz to study the motion of a pendulum as the length of its suspending thread was slowly changed (Schmidt, 1966; Goldstein, 1950).

For motion of a particle in a slowly varying central force potential the action is an adiabatic invariant as is shown below.

The equations describing planar central force motion for a particle of mass m are

$$m(\ddot{r} - r\dot{\theta}^2) = F \qquad (4a)$$

$$(d/dt)(mr^2\dot{\theta}) = 0 \qquad (4b)$$

Integrating equation (4b) we obtain:

$$r^2\dot{\theta} = 2H \qquad (5)$$

where H is the constant of areal velocity, introduced in Section 2.

We define the angular action per unit mass j_θ as

$$\begin{aligned} j_\theta &= J_\theta/m = \int r^2\dot{\theta}d\theta = 4\pi H \\ &= \int r^2\dot{\theta}(d\theta/dt)dt = \int (r\dot{\theta})^2 dt \\ &= \langle v_\theta^2 \rangle_t \, P \end{aligned}$$

where P is the period of revolution and $\langle x \rangle_t$ denotes the expected value of x with respect to time.

We now consider the radial equation of motion

$$\ddot{r} - r\dot{\theta}^2 = F/m = F'. \tag{6}$$

Using equation (5) to eliminate the angle, we obtain

$$\ddot{r} - 4H^2/r^3 = F'. \tag{7}$$

We define the "radial" potential energy per unit mass V_r by:

$$-(\partial V_r/\partial r) \triangleq F'' \triangleq F' + 4H^2 r^3. \tag{8}$$

Substituting back into equation (7) we obtain

$$\ddot{r} + (\partial V_r/\partial r) = 0 \;. \tag{9}$$

Multiplying equation (9) by $\dot{r}$ and integrating, we obtain the "radial" energy per unit mass, W, as a constant of integration

$$\dot{r}^2/2 + V_r = W \;. \tag{10}$$

The radial action per unit mass (j_r) is given by

$$j_r \triangleq J_r/m = \int \dot{r} dr = \int [2(W-V)]^{1/2}\, dr \;. \tag{11}$$

Now equation (11) is the expression for the action per unit mass for a non-linear harmonic oscillator. It has been shown in the literature (Lenard, 1959; Gardner, 1959) that for a slowly and continuously varying potential the action for a non-linear harmonic oscillator is constant.

Since j_r is an adiabatic invariant and j_θ is an invariant from the conservation of angular momentum, the action per unit mass j is an adiabatic invariant also, as is shown below.

$$\begin{aligned} j &= \langle v^2 \rangle_t \; P \\ &= \langle \dot{r}^2 + (r\dot{\theta})^2 \rangle_t \; P \\ &= \langle \dot{r}^2 \rangle_t \; P + \langle (r\dot{\theta})^2 \rangle_t \; P \\ &= j_r + j_\theta \, . \end{aligned} \tag{12}$$

RELATIVISTIC EFFECTS OF ISOTROPIC MASS LOSS ON PLANETARY ORBITS

We have found (Equation 3) a classical potential which gives the same orbits as predicted by general relativity and have established that for slowly varying central force potentials, the action is an adiabatic invariant. We now combine these two results to study relativistic effects of primary isotropic mass loss on planetary orbits.

We make two assumptions idealizing the mass loss from the primary: 1) The mass loss is isotropic, and 2) the lost mass does not interact with the planetary system. These two assumptions imply that the angular momentum of the planetary system is conserved.

To define the planetary orbit we take the sideral period, P, the eccentricity, e, and the mass of the primary, M. Since the orientation of the orbit changes due to the relativistic advance of periastron, we do not specify the orientation of the orbit.

We treat relativistic effects as perturbations and calculate all results to first order in ε where

$$\varepsilon = 3(\gamma M/2cH)^2 \ .$$

Having established that H is a constant and j is an adiabatic invariant, we wish to determine these for the Schwarzschild potential.

We first find j, so we apply the virial theorem (Richards, 1959) which states:

$$\langle mv^2\rangle_t = \langle r\cdot\nabla V\rangle_t .$$

For a homogeneous function of degree N

$$r\cdot\nabla V(r) = NV(r) .$$

If V(r) is the Schwarzschild potential, then:

$$\langle v^2\rangle_t = \langle \gamma M/r + 12\ \gamma MH^2/(c^2 r^3)\rangle_t$$

j is therefore given by

$$j = \int(\gamma M/r)dt + 12\int(\gamma MH^2/(c^2 r^3))dt \qquad (13a)$$

$$= \gamma M \int dt/r + (16H^4/\gamma M)\varepsilon \int dt/r^3 . \qquad (13b)$$

Since we work to first order in ε we can use the Newtonian results to evaluate $\int dt/r^3$. From the well known Newtonian result (Moulton, 1914) we have

$$\mu = 2\pi/P$$

$$r = a(1-e \cos E)$$

$$\mu dt = (1-e \cos E)dE$$

where a is the semi-major axis of the orbit, μ is the mean motion (average angular velocity), and E is the eccentric anomaly. So,

$$\int_o^P (dt/r^3) = \int_o^{2\pi} dE/(\mu a^3(1-e \cos E)^2) .$$

Consulting a table of integrals (e.g. Petit Bois, 1961) we find

$$\int_o^P (dt/r^3) = P/(a^3(1-e^2)^{3/2}) . \qquad (14)$$

We turn now to the evaluation of $\int dt/r$ in (13b)

$$\int_o^P (dt/r) = (1/2H) \int_o^{2\pi} r d\phi . \qquad (15)$$

Hence we consider

$$I_1 = \int_0^{2\pi} r d\phi \; . \tag{16}$$

From the Schwarzschild exterior solution (2) we have that

$$u = 1/r = A + B\cos(\phi - \varepsilon\phi) + \varepsilon(A + [B^2/2A] - B^2\cos 2\phi/6A) \tag{17}$$

where $A = \gamma M/(4H^2)$ and $B = eA$.

Define:

$$C_1 \triangleq A + \varepsilon(A + B^2/(2A)) \tag{18}$$

$$C_2 \triangleq \varepsilon B^2/6A \; . \tag{19}$$

Substituting equation (17) into (16) and using (18) and (19), we have

$$I_1 = \int_0^{2\pi} d\phi/(C_1 + B\cos(\phi - \varepsilon\phi) + C_2\cos 2\phi). \tag{20}$$

Expanding (20) to first order in ε we obtain

$$I_1 = \int_0^{2\pi} d\phi/(C_1 + B\cos z) - \int_0^{2\pi} C_2\cos 2\phi d\phi/(C_1 + B\cos\phi)^2 \tag{21}$$

where $z = (1-\varepsilon)\phi$.

We first consider the first integral on the right hand side of (21), namely

$$I_{11} = \int_0^{2\pi} (d\phi/(C_1 + B\cos z)).$$

Then,

$$I_{11} = 1/(1-\varepsilon) \int_0^{2\pi(1-\varepsilon)} (dz/(C_1 + B\cos z)).$$

This integral may be found from tables.

$$I_{11} = 1/(1-\varepsilon)\{2/(C_1^2-B^2)^{1/2}\tan^{-1}\{(C_1^2-B^2)\tan\pi(1-\varepsilon)/(C_1+B)\}\}.$$

Substituting in values of C_1 and B we obtain

$$I_{11} = 2\pi/(A(1-e^2)^{1/2})\{1+\varepsilon(1-(1+e^2/2)/(1-e^2)-\sqrt{(\tfrac{1-e}{1+e})})\}. \quad (22)$$

Having found I_{11}, we turn to the evaluation of the second integral on the right hand side of equation (21). We write $\cos 2\phi$ as $1-2\sin^2\phi$ and the second integral becomes

$$C_2\{\int(d\phi/(C_1+B\cos\phi)^2) - \int 2\sin^2\phi d\phi/(C_1+B\cos\phi)^2\}. \quad (23)$$

The first of the integrals in equation (23) can be found in tables.

$$\int C_2 d\phi/(C_1+B\cos\phi)^2 = 2\pi\ C_2 B/(C_1^2-B^2)^{3/2}. \quad (24)$$

The evaluation of the second integral in equation (23) is somewhat more involved. We evaluate it by parts:

$$\int \frac{\sin^2\phi d\phi}{(C_1+B\cos\phi)^2} = \frac{\sin\phi}{(B(C_1+B\cos\phi))}\Big]_0^{2\pi} - \int_0^{2\pi} \frac{\cos\phi d\phi}{(B(C_1+B\cos\phi))}\ . \quad (25)$$

The first term on the right hand side drops out, and the second may be found in tables.

We combine equations (16), (21), (22), (24), and (25) to obtain:

$$\int d\phi/u = \{2\pi/A(1-e^2)^{1/2}\}\{1+\varepsilon(2/3 - \frac{3+e^2}{3(1-e)^2} - (\frac{1-e}{1+e})^{1/2}$$
$$+ \frac{(1+e^2)^{1/2}}{3}))\}\ . \quad (26)$$

To find j we must also find H. By definition, H is given by:

$$H \stackrel{\Delta}{=} \int 1/2\ r^2 d\phi/P = \frac{1}{2P}\int \frac{d\phi}{u^2}\ . \quad (27)$$

We therefore consider $\int d\phi/u^2 = I_3$

$$I_3 = \int d\phi/(C_1+B\cos + C_2\cos 2\phi)^2$$

$$= \int d\phi/\{(C_1+B\cos z)^2(1+C_2\cos 2\phi/(C_1+B\cos z))^2\}. \quad (28)$$

Expanding equation (28) we obtain:

$$I_3 = \int d\phi/(C_1+B\cos z)^2 - 2C_2\int \cos 2\phi d\phi/(C_1+B\cos\phi)^3. \quad (29)$$

The first integral on the right hand side, I_{31}, may be found in tables, to first order in ε:

$$I_{31} = (2\pi/A^2(1-e^2)^{3/2})\{1+\varepsilon(1+\frac{e^2}{2} - \frac{3(1+e^2/2)}{1-e^2} + $$

$$+\sqrt{\frac{1-e}{1+e}}(1+e)\}. \quad (30)$$

To evaluate the second integral on the right hand side of equation (29), I_{32}, we use the same substitution as in the evaluation of $\int d\phi/r$, namely,

$$\cos 2\phi = 1 - 2\sin^2\phi$$

$$I_{32} = \int \cos 2\phi d\phi/(C_1+B\cos\phi)^3$$

$$= \frac{d\phi}{(C_1+B\cos\phi)^3} - 2\int \frac{\sin^2\phi d\phi}{(C_1+B\cos\phi)^3}. \quad (31)$$

To evaluate the first integral on the right hand side of equation (31), we use tables.

$$\int \frac{d\phi}{(C_1+B\cos\phi)^3} = \pi(2C_1^2 + B^2)/(C_1^2 - B^2)^{5/2}.$$

We integrate the second term on the right hand side of equation (31) by parts.

$$\int \sin^2\phi/(C_1+B\cos\phi)^3 = -\pi/(C_1^2 - B^2)^{3/2}. \tag{32}$$

Combining equations (29), (30), (31), and (32) we obtain to first order in ε

$$\int r^2 d\phi = (2\pi/A^2(1-e^2)^{3/2}.\{1+\varepsilon\, f_H(e)\} \tag{33}$$

where

$$f_H(e) = 2+e^2/2 - 3(1+e^2/2)/(1-e^2) - \left(\frac{1-e}{1+e}\right)^{1/2}(1-e) + \frac{e^2(4-e^2)}{6(1-e^2)} . \tag{34}$$

Having obtained the values for $\int d\phi/u$ and $\int d\phi/u^2$, we return to the physics of the problem. As shown earlier, for slow mass loss H is a constant and j is an adiabatic invariant. To find j we combine equations (13b), (14), (26),

$$j = \{4\pi H/(1-e^2)^{1/2}\}\{1+12\pi H\, f_j(e)/Pc^2(1-e^2)^{3/2}\} \tag{35}$$

where

$$f_j(e) = \left(2/3 - \frac{3+e^2}{3(1-e^2)} - \left(\frac{1-e}{1+e}\right)^{1/2} + \frac{4(1-e^2)^{1/2}}{3}\right) .$$

The equation for H follows directly from equations (33) and (21).

$$H^{-3} = (16\pi/P(\gamma M)^2(1-e^2)^{3/2})(1+\varepsilon f_H(e)) . \tag{36}$$

where $f_H(e)$ is given by equation (34).

Since $f_j(0) = 0$, $e = 0$ initially implies $e = 0$ for all later times. This result is valid for any potential under the assumption that H and j are invariant.

The invariants, H and j, can be determined from equations (35) and (36), respectively, in terms of the initial P, e, and M. The values of H and j thus obtained, when substituted back into equations (35) and (36) allow P and e to be specified as a function of M.

NUMERICAL RESULTS

The Colorado School of Mines PDP-10 was programmed by the algorithm given in the last paragraph of the preceding section for a planet with period 0.10 year and eccentricity 0.10 orbiting a star 10 times as massive as the sun. A 10 percent slow mass loss by the star gave a final period of 0.12345682 years and an eccentricity of 0.10000015. The corresponding Newtonian results are 0.12345679 years and an eccentricity of 0.10000000. For a planet with period 0.2408 years and eccentricity 0.2668 (i.e. Mercury) orbiting a star as massive as the sun, a 10 percent slow mass loss by the star gave a final period of 0.29728395 and an eccentricity of 0.26680000. The corresponding Newtonian results are 0.29728395 years and an eccentricity of 0.26680000.

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