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DYNAMICALLY OPTIMAL MINING STRATEGIES WITH ECOSYSTEM CONSTRAINTS: A CASE STUDY IN CLEARWATER NATIONAL FOREST

by

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ABSTRACT

A microeconomic model is developed describing a mine as a two-product firm which must reclaim waste as well as extract ore. The mine is subject to an ecosystem constraint that imposes an upper limit on the amount of unreclaimed waste. Since waste is generated as a by-product of extraction, the mine must schedule extraction and reclamation in a way that maximizes profits. This model introduces an additional dynamic component, the stock of waste, into the traditional analysis of a mine. Optimal control theory is used to determine the profit-maximizing of levels extraction and reclamation as functions of time. New results from this model include the possibility of production tilting even with a zero discount rate and an optimal level of terminal extraction higher than in models that represent the mine as a single product firm.

The model is used to compare various policy options: taxing production, subsidizing reclamation, taxing the size of the waste pile, subsidizing the amount by which the size of the waste pile is below the maximum size, and command and control standards. Taxing the size of the waste pile is found to work best. All other policies, except command

and control, are able to guarantee only that one of the two functions, extraction or reclamation, would be socially optimal. Command and control policies could guarantee neither socially optimal extraction or reclamation, but could protect against threshold ecosystem effects, which none of the other policies could.

The model is applied to the prefeasibility analysis of a small open-pit gold mine in the Lolo Creek area of Clearwater National Forest in northern Idaho. The Lolo Creek area is the focus of an interdepartmental case study in ecosystem management that originated at the U.S. Bureau of Mines in 1993. Model results are used to evaluate policy impacts on potential mines and existing mines in the Lolo Creek area.

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Chapter 1

INTRODUCTION

This study determines the optimal combination of extraction and reclamation a mine should undertake to comply with an ecosystem constraint. The constraint is the limited amount of damage the mine is allowed to inflict on the ecosystem of a nearby stream. The optimization is dynamic and the result of the analysis is an optimal schedule of extraction and reclamation the mine should comply with to maximize its discounted stream of profits.

The theory is developed to correspond to ore extraction from a disseminated deposit, although with minor changes, it could be applied to mining ore in veins as well. The economic model is applied to the prefeasibility analysis of a small open-pit gold mine in Clearwater National Forest. Prefeasibility analysis is a technique used by the U.S. Bureau of Mines to assess potential mineral supply. This model extends the traditional prefeasibility analysis to include an ecosystem constraint. The model is used to analyze the impact of changing the level of the ecosystem constraint on the profitability of existing and future mines in Clearwater National Forest.

Ecosystem Concepts and Environmental Policy

The idea of ecosystems is ancient. Many Native American beliefs include the idea of the interconnectedness of natural phenomena. The use of ecosystems as a policy concept is one of the newest ideas in the relatively new field of environmental policy. To illustrate where ecosystem constraints fit into environmental policy it will be helpful to briefly summarize major developments in U.S. environmental policy as they relate to mining.

Until 1969, there was no domestic environmental policy that applied to mining. Miners extracted ore without worrying about the environmental consequences of their actions. Mine sites generally were quite distant from population centers, and the science of public health wasn't sufficiently developed to recognize the hazards that mining presented. With no environmental constraints imposed, mines were essentially single-product firms whose operators were concerned only with optimally managing the mine's reserves of ore. As public health science developed and population centers became established around active or inactive mines, public concern developed about the impact that mines had on the environment.

The National Environmental Policy Act (NEPA) was passed in 1969. Although environmental laws had been passed as early as the Refuse Act of 1899, the earlier laws were mainly to protect navigable waters. NEPA was the first in a series of laws passed in response to public concern about the effect of industrial pollution on human-health. Health problems were initially assumed to be linked to visible pollutants. As environmental awareness became more sophisticated, the focus of environmental legislation shifted to toxins. Policy emphasis, however, stayed squarely on human health concerns.

Ecosystem health became an issue when the question of sustainable development was considered and signaled a return to the idea that natural phenomena are highly interconnected: if any single component of the environment is adversely affected, irreversible changes could occur across the range of components. The Science Advisory Committee of the Environmental Protection Agency (EPA) issued a statement in 1990 urging the EPA to consider ecosystem health on an equal basis with human health effects.

Ecosystem Constraints for Mining

Concepts related to ecosystems are still in their infancy. Much current research simply attempts to define ecosystem boundaries. The boundaries appear to vary depending on the scale of the disturbances under consideration. It seems clear however, that the ecosystem concept requires consideration of all the species and their interactions. This is in sharp contrast to the established procedure of considering environmental impacts on a single species. Current environmental management is focused on protecting only those species listed as threatened or endangered. Such policies are reactive since no action is taken until a species becomes endangered. Often, fierce political opposition arises when people are informed that they could no longer use the environment as they had become accustomed.[1] A goal of ecosystem-based policies is to allow more proactive environmental management by formulating constraints that protect species before they become threatened or endangered. Formulating constraints that meet this goal is a difficult task.

Human activity disturbs the ecosystem. To what extent is a subject of current research. Most ecosystem impacts are difficult to characterize since the boundaries of the ecosystem often cannot be rigorously defined, and human activity occurs across a wide spectrum of environmental scales. Most land animals react to encroachment by humans

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[1]For example, the heated 1992-93 debate over protecting the Northern Spotted owl in forests previously used by logging firms in the Pacific Northwest.

simply by moving out of places they had previously used as habitat and into new ones. The difficulty in assessing ecosystem affects comes from comparing interspecies utility changes. If land animals moved to a place that provided a better habitat, they become better off. If a plant species loses an important vehicle for transporting pollen and seeds, it becomes worse off. It is often difficult to say to what extent human encroachment improves or damages the ecosystem.

One interaction between mine and ecosystem is rigorously bounded at all but the largest scales--the ecosystem of streams at or near the mine. Rain falling on piles of mineralized waste material carries positively charged metallic ions (known as metallic cations) which can poison reaches of the stream. Cations first destroy macroinvertebrates, a low member of the food chain, and once these organisms disappear from a reach of a stream, that reach can no longer support higher aquatic life.

The marginal damage to ecosystems from the introduction of metallic cations is unknown. The EPA has set standards for maximum allowable concentrations of common metals and these are used as constraints for the problem in this study. Using a threshold constraint instead of a marginal damage constraint will generally lead to sub-optimal environmental policies from the perspective of maximizing social benefits. The method developed here can be used with either type of constraint to give the miner an optimal program to follow to maximize profits.

Another ecosystem threat, and often a more serious one is produced by water pooling in abandoned mine works, becoming highly mineralized, then leaking into surface and groundwater flows at high pressure. The abandoned mine question has been fairly well explored in mine design literature and is mainly related to proper closure procedures. The procedures may not address all the water pollution problems from abandoned mines but most mines since 1970 are designed to be closed so as to minimize postclosure water contamination. The concept of optimal economic management of mine waste is somewhat newer and may have some surprises for the miner.

Mine design can solve some of the problems associated with mine waste but not all of them unless all mineralized waste is entombed so that it never interacts with the surface waters of the ecosystem. Such entombment, however, would be economically inefficient in all but the worst cases. A better economic solution would be to reclaim just enough of the waste pile to avoid violating federal metal concentration standards and to time the reclamation throughout the mine life to maximize the discounted total profits. This solution would be optimal in the sense of allocative efficiency: spending only what is necessary to for compliance. This solution will be derived theoretically in chapter 3 and applied in chapter 5.

Chapter 2

SURVEY OF RELEVANT LITERATURE

Theoretical literature applicable to this study is from: nonrenewable resources, capital theory, and environmental modeling. Other readings in cost estimation and mineral deposit modeling are briefly touched on as they relate to the prefeasibility analysis in chapter 4. The cost estimation literature is relevant to the application of the model but not to the theoretical model itself. Mineral deposit modeling is used to show how policy conclusions can be drawn from the applied model.

The nonrenewable resources literature contains several principles borrowed from capital theory because ore is traditionally managed as a capital stock. In this study, an additional capital stock must be considered and the optimal management of these two stocks gives a new twist to the traditional capital theoretic results of optimal mining practices. Capital theory literature also contributes in another sense. A traditional problem in capital theory is finding the optimal time path of capital accumulation when the process of accumulating capital causes social disutility. This literature provides a guideline for the modeling in chapter 3 and the policy discussion in chapter 4.

The environmental modeling literature is rich in complex examples of models developed for specific sites. Because the focus of this model is on economic rather than chemical aspects, the environmental modeling of the flow of mineralized rainwater into nearby streams will be greatly simplified. Mineral deposit modeling attempts to associate grade and tonnage distributions with deposit types. The best place to start the literature discussion is with the nonrenewable resources literature.

Nonrenewable Resources

Nonrenewable resources pose unique economic problems because they include (by definition) at least one limiting input. This complication means that the traditional economic analysis of the firm which assumed inputs would always be available is inappropriate. While the micro-economic analysis of a firm with unlimited inputs may be quite complex, the problem is fundamentally static. For nonrenewable resources, the economic problem is fundamentally a dynamic problem focused on how to best manage a finite stock of a nonrenewable resource--in the case of mining, the ore. Mining lends itself to a capital theoretic treatment since the problem of optimal depletion of ore is closely related to the problem of optimal accumulation of capital. Three articles provide the necessary theory.

Harold Hotelling's 1931 article, "The Economics of Exhaustible Resources," is an excellent starting point. Hotelling preformed the earliest rigorous mathematical treatment of the problem of optimal management of a stock of ore, although the unique features of the problem had been considered as early as 1914 by L.C. Gray in "Rent Under the Assumption of Exhaustibility." Hotelling's approach was close to what would now be called a capital theoretic approach. He used the calculus of variations to consider the dynamics of the mining problem. He considered three cases, the mine as a price taker, as a monopolist, and as a social optimizer. Hotelling was concerned more with the mining industry as a whole and the aggregate supply of mined products than with the microeconomics of a single mine, but his framework provided the mathematical basis for future analysis.

Hotelling used a pure depletion model. He assumed that price would always exceed the marginal cost of extraction, resulting in the firm depleting all the stock of ore. He did not consider the possibility of discoveries of new ore bodies or changes in the existing grade of ore. Hotelling's ideas were extended by several others, notably Carlisle (1954). Hotelling's framework is suitable for the problem in this study, a small open-pit gold mine extracting ore from a disseminated deposit such as those commonly found in the western United States. This type of mine is adequately described by Hotelling's mathematical framework with only minor modifications.

One of the major conclusions in Hotelling's paper was the idea of a "user cost" incurred by extracting a unit of resource today and not having it available in the future. Hotelling used the concept to describe price movements in mineral commodities. Later empirical work (Barnett and Morse, 1958) showed that Hotelling's prediction of rising prices had not come true owing to technological changes Hotelling had not included in his model. Despite this, the concept of user cost is still important in dynamic analysis and will be applied in this study to both the stock of ore and the stock of waste that generates social disutility.

Anthony Scott's (1967), "The Theory of the Mine Under Conditions of Certainty," analyzed the microeconomics of a single mine. Scott's development was graphical and intuitive rather than mathematical as in Hotelling, but his conclusions were sound. He introduced the idea of production tilting toward the earlier years of the life of the mine. He demonstrated that although such tilting was often blamed on decreasing ore grades or worn-out equipment, it was in fact optimal for mine managers to tilt production toward the present if they had a rate of time preference (a discount rate). He also considered the effect of discounting on the life of the mine. He demonstrates that without discounting, the mine manager will maximize profits by operating at the minimum cost point, while discounting will encourage him to operate closer to the point of maximum profits. He shows that these two operating points, minimum cost and maximum profit, are distinct and that cost is minimized at a lower rate of extraction than that which maximizes profit. As a consequence, the life of the mine will be affected by the discount rate selected by the mine manager. Scott considered maximizing profits at each point in the life of the mine as well as maximizing profits over the entire life of the mine. The analysis here will concentrate on maximizing profits over the entire life of the mine.

Scott's article completes the major theoretical points from the nonrenewable resources literature. His focus on the economics of a single mine rather than Hotelling's focus on the entire mining industry will be adopted here.

Scott's discussion is applicable to mining in the period before environmental legislation. Scott describes the correct management practices for a mine as a one-product firm. This study will attempt to extend Scott's management practices to cover a two-product mining firm--one that must reclaim as well as extract.

Hotelling and Scott's work are neatly extended and cast in a modern mathematical framework by H. Stuart Burness (1976) in "On the Taxation of Nonreplenishable Natural Resources". Burness provides the analytic framework for the discussion in chapter 4 of policy implications of the model. Burness considers the mine as a single-product firm, as Hotelling and Scott did, but takes full advantage of the mathematical elegance afforded by optimal control theory to analyze the effects of taxes on the mine. Chapter 4 extends Burness' analysis to cover the mine as a two-product firm.

This study will show that once the mine is analyzed as a two-product firm, tilting can occur even at zero discount rates, contrary to Scott's conclusions. Also Hotelling and Scott's optimal rates of extraction in the final period are too low if the mine is considered as a two product firm and the environmental constraint is binding in the final period.

The mining literature extends the work of Hotelling and Scott in directions beyond those considered by Burness. Schulze (1974) reworks Scott's analysis in the case of an industry with free entry and exit and finds that Scott's tilting effect is accomplished by firms leaving the industry as time progresses. Other complications including varying ore quality and discoveries of new deposits have been considered by other authors (Bohi and Toman 1984). In addition to optimal management of a mine, the mining literature also considers questions such as optimal investment in capital (Smith 1968) and optimal exploration (Pindyck 1978). Though interesting, the main theoretic results for the work here are found in Hotelling, Scott, and Burness.

Capital Theory

Capital theoretic literature on optimal growth that includes a stock of pollution that best applies to this problem is by Bruce A. Forster. His 1980 article, "Optimal Energy Use in A Polluted Environment," sets the discussion in the context of a social planner trying to determine optimal energy use given a fixed stock of fuel and the fact the energy use contributes to a stock of pollution as well as increasing utility. He finds it optimal to reduce fuel use in the early years--counter to the production tilting ideas of Scott's mining problem--and then increase fuel consumption in the future. Forster's work is interesting and directly applicable to the work here. His decision to work at the level of greatest generality by not specifying functional forms makes it hard for him to obtain any really striking conclusions. He did obtain a negative shadow price for the stock of pollution which also occurs in this study. This negative shadow price will be re-interpreted as a positive user cost in chapter 3.

Forster's work is anticipated by M.L. Cropper (1976) in "Regulating Activities With Catastrophic Environmental Effects." Cropper uses a stochastic optimal control framework (Forster's was deterministic) to explore the consequences of consuming nuclear energy and stockpiling nuclear waste given a small probability of a catastrophic nuclear accident. Cropper finds that if the size of the stock of nuclear waste can be included in the social utility function, there is an equilibrium level of consumption and an equilibrium level of the nuclear waste stockpile. As Cropper notes, the equilibrium may be non-unique, that is, there may be more than one pair of consumption level and stockpile sizes that maximize social utility.

The major difficulty with Cropper is quantifying the impact on utility of a change in the probability of a nuclear catastrophe. On an environmental level, Cropper's work is relevant to the case where certain ecosystem management practices might result in irreversible effects. Although Cropper's work isn't applied directly here, it points the way for extending the model by making the ecosystem constraint stochastic instead of deterministic. Such an extension is considered in chapter 6.

Environmental Modeling

The relevant environmental modeling literature is mainly concerned with the chemical processes by which metal is absorbed from mineralized waste rock and various mechanism for its transport into the surface and groundwater systems. For this study, the chemical issues will be largely ignored and empirical data from the U.S. Bureau of mines (White and Jeffers, 1992) on artificial weathering of waste rock will be used for the application in chapter 4. Those interested in a preliminary description of the chemical issues on a non-site-specific basis are referred to M. Sengupta's 1993 Environmental Impacts of Mining: Monitoring, Restoration, and Control, especially chapter 5, "Acid Rock Drainage and Metal

Migration" and the references cited there. Some site-specific literature that provides a good description of the chemical processes involved includes Schuiling (1992), Boyles and others (1974), Steele and others (1991), Nimick and Moore (1991), and Sanden (1991).

For this study, a simple theoretical model is used that assumes cation release is proportional to the mass of the waste pile. This model is supported by White and Jeffers (1992) who present data on the artificial weathering of waste rock from iron mines in Minnesota focussing on sulfate release. They use controlled weathering to relate the mass of rock to the amount of sulfates released. These data will be used for generic cation release from disseminated gold bearing rock in northern Idaho--a gross simplification, but acceptable at the prefeasibility level.

Cost Estimation and Mineral Deposit Modeling

The literature from cost estimation and mineral deposit modeling is important to the application of the model but not to its theoretical development. Cost estimation provides the form and parameters for the extraction and reclamation cost functions in the application. Both cases use models from the U.S. Bureau of

Mines. Extraction costs are estimated from Camm's (1992) model of mine cost functions and reclamation costs are estimated from Denton's (1992) HEAPREC model for estimating the cost of reclaiming cyanide leach heaps. Camm's models are currently used in the potential mineral supply models at the Bureau of Mines and this study could be seen as an extension of that model incorporating a stream ecosystem constraint.

The standard reference on mineral deposit modeling is from the U.S. Geologic Survey (1986). It is used here to draw conclusions about the policy impacts of changes in water quality standards on the potential mineral supply in the application discussed in chapter 5.

Chapter 3

A MODEL OF MINE WATER POLLUTION

Controlling the size of the waste pile in order to comply with heavy metal concentration limits is the focus of the model presented in this chapter. Mine waste and tailings are both referred to in this model as "waste". Technically, waste refers to rock that is removed and not processed; tailings refer to rock that has been processed to extract a mineral commodity. The amount of waste generated is assumed to be proportional to the amount of ore extracted. All run-off from the waste pile is assumed to enter the stream and contribute to the dissolved metal concentration. The effects on groundwater are ignored.[2] Water pollution from submerged mine works, typically associated with abandoned mines, may pose significant environmental hazards (Western Governors Association, 1991) but it is ignored here since the focus of this model is on an active mine.

Precipitation of metals out of stream water is also ignored even though studies (for example, Boyles and others

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[2]Ignoring metal migration into groundwater makes the model more sensitive to surface water issues.

1974) point out that precipitation is significant and highly dependent on pH. The reason for ignoring precipitation is that it offers no real solution to the problem of metal concentrations in the stream ecosystem. The precipitate acts as a non-point source which continues to recharge the stream metal concentration depending on pH fluctuations (Khein, 1993). Therefore, even though precipitation does remove metal from the stream, it does not solve the ecosystem problem considered in this study.

Mathematical Representation

The problem facing the mine manager is how to optimally manage the mine waste pile and the reserves of ore. A diagram of the physical situation is shown in figure 3.1.

The mine extracts a mass of ore, $q(t)$, from a known stock of ore, $X(t)$. The ore is separated at the mill. A constant fraction, b ($0 \le b \le 1$), of the ore is sold as a mineral commodity. The b parameter is upper-bounded by the mass fraction of mineral commodity in the ore. The technology of mining and milling determine the value of b. The assumption that b is constant implies that a constant level of technology is used over the life of the mine. This would correspond to a constant level of installed

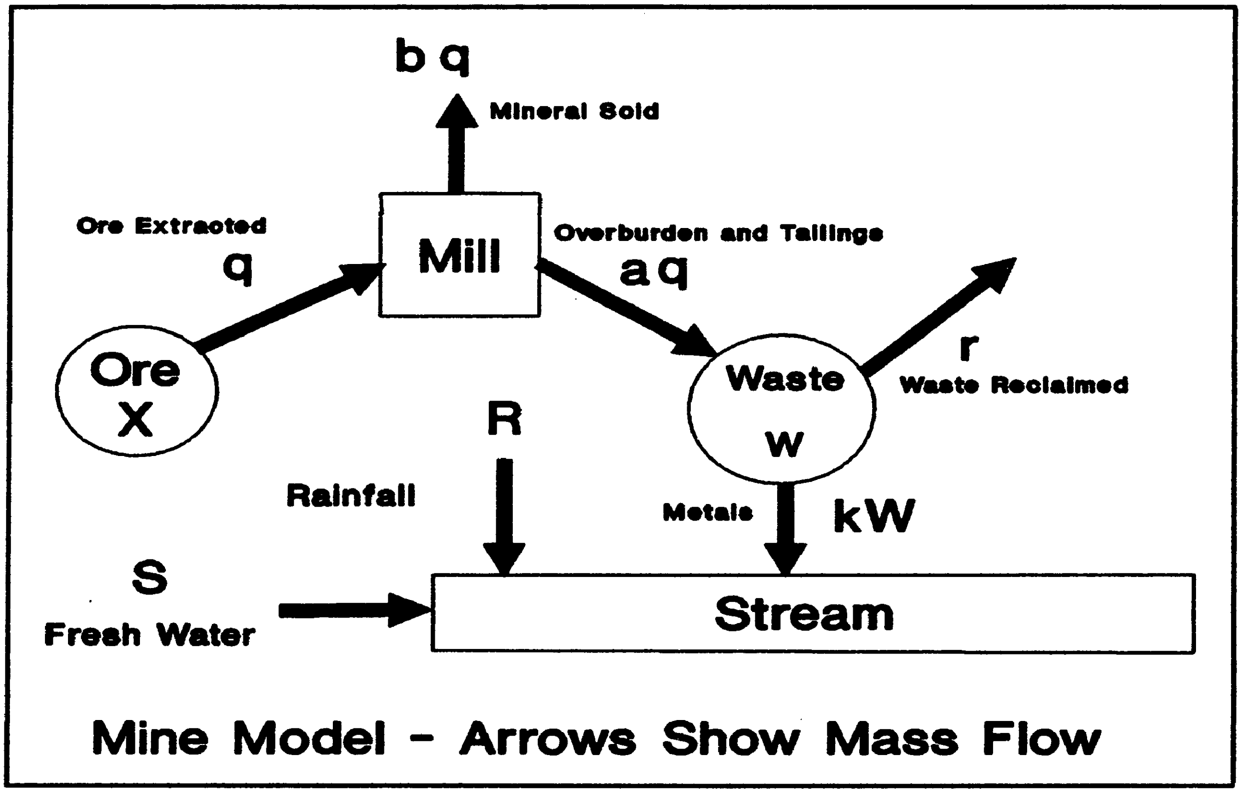

Figure 3.1. The Mine Model

technology. The mill adds material to the waste pile in the amount aq where $a \geq 1-b$. The parameter a accounts for mass gained by the ore in the milling process as well as overburden material exposed when the ore is extracted. If $a = 1-b$ then the milling process would add no mass to the processed ore and no overburden material would be exposed when the ore was extracted. This situation is ideal but not possible with current technology.

The constant, k, relates the size of the waste pile to the amount of metal released into the stream. Since rainfall is the primary mechanism for metal migration, k is a function of the average rainfall for the area in which the mine is located. The parameter k also depends on the characteristics of the rock in the waste pile and the chemical processes used in milling the ore.

Stream flow is a function of time: typically large in the spring when the winter snow accumulation melts and smaller in the summer and late fall. Rainfall is also a function of time.

The mine manager can reclaim a mass r(t) from the waste pile at any time. The reclaimed waste is assumed to be isolated from the drainage into the stream so that it cannot contribute to further stream metal concentration.

The Model

Economic analysis of mining generally requires dynamic methods since, unlike other firms, mines have one limiting input[3] (ore) whose amount is fixed.[4] The classic mining problem involves determining the optimal rate of extraction

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[3]Constant installed technology can also be thought of as a fixed input but does not have the same dynamic implications as the fixed stock of ore.

[4]The diagram in figure 1 shows ore as an output of mining and an input to milling. In most of the mining literature the distinction between mining and milling is ignored and ore is considered to be an input to mining.

of ore. The model considered here has an additional dynamic component, the stock of waste. The most widely used method of dynamic analysis is optimal control theory which was introduced in the 1950s as an improvement on the calculus of variations. Using optimal control theory terminology, the model will have two state variables, the mass of the waste pile and the mass of ore in the ground.[5]

The total metal mass added to the stream is kW (figure 3.1). The volume of fresh water from stream flow (assuming there is no polluter upstream of the mine) added to the stream is S(t). The volume of fresh water from rainfall is R(t). The diluted metal concentration downstream of the run-off point, E(t), is defined as follows:

$$E(t)=\frac{kW(t)}{R(t)+S(t)} \qquad (3.1)$$

As long as $E(t) \leq E_c$, where E_c is the critical concentration of the metal of interest, the environmental constraint is not violated. Critical stream metal concentrations for a variety of metals are published by the EPA. Using equation (3.1), the critical size of the waste pile can be written in terms of the critical stream metal

[5]The concentration of metal in the stream could also be a state variable but in the solution presented here, it is eliminated by relating it to the size of the waste pile.

concentration:

$$W_c(t)=\frac{[R(t)+S(t)]E_c}{k} \tag{3.2}$$

The waste pile grows when ore is extracted and shrinks when waste is reclaimed or when metals migrate from the waste pile into the stream. The change in the mass of the waste pile is thus:

$$\dot{W}=aq(t)-r(t)-kW(t) \tag{3.3}$$

where:

W(t) is the mass of the waste pile (the dot over the W indicates the time derivative of W).

q(t) is mass of ore extracted at any time.

r(t) is the mass of waste reclaimed at any time.

The final term in equation (3.3) represents the mass of waste that leaves the pile as metal and migrates into the stream. The mass flows described in equation (3.3) are shown diagrammatically in figure 3.1.

The second state variable is the stock of ore in the ground which decreases when ore is extracted. This formulation ignores new discoveries of ore and changes in the effective size of the orebody due to changing extraction technology. This is the "pure depletion" model of resource extraction from Hotelling (1931).

Mathematically, the relationship is

$$\dot{X}=-q \tag{3.4}$$

where X(t) is the mass of ore in the ground as a function of time (the dot over the X indicates the time derivative of X).

The control variables are q(t), the amount of ore extracted and r(t), the amount of waste reclaimed. These are the choice variables that the mine manager uses to influence the behavior of the state variables. The time arguments indicate that the mine manager can change the level of either variable at any time. Both control variables are required to be non-negative and limited by initially installed capital which is assumed to be constant.

$$0 \le q \le q_{max} \tag{3.5}$$

$$0 \le r \le r_{max} \tag{3.6}$$

The mine manager is assumed to be a profit maximizer who selects q(t) and r(t) at every time so as to maximize the sum of discounted profits from the mine over a free time horizon using a constant discount rate. Initial and final levels of ore are known, as is the initial level of waste. The final level of waste is free. The problem the mine manager must solve is

$$\max_{\{q,r\}}\int_0^T e^{-\delta t}\{p(t)bq(t)-C[q(t)]-A[r(t)]\}dt \tag{3.7}$$

subject to:

$$\dot{W}=aq-r-kW \tag{3.8}$$

$$\dot{X}=-q \tag{3.9}$$

$$W(t)\le W_c(t) \tag{3.10}$$

$$W(t)\ge 0 \tag{3.11}$$

$$X(t)\ge 0 \tag{3.12}$$

$$0\le r\le r_{max} \tag{3.13}$$

$$0\le q\le q_{max} \tag{3.14}$$

where

T is the operating life of the mine (free).

δ is the discount rate.

p(t) is the price of the mineral resource.

b is the mass fraction of the ore sold as mineral resource ($0\le b\le 1$).

$q(t)$ is the amount of ore extracted at any time.
$C[q(t)]$ is the cost of extracting ore.
$r(t)$ is the amount of waste the mine reclaimed at any time.
$A[r(t)]$ is the abatement cost, the cost of reclaiming waste.
$W(t)$ is the mass of the waste pile. $W(0)$ known, $W(T)$ free.
a is the constant of proportionality between the mass of ore extracted and the mass added to the waste pile ($a \geq 1-b$).
k is the constant of proportionality between the mass of metal released and the mass of the waste pile.
$X(t)$ is the mass of ore in the ground. $X(0)$ and $X(T)$ known.

The extraction and reclamation cost functions are assumed to have neoclassical shapes:

$$C'(q) > 0 \tag{3.15}$$

$$C''(q) \geq 0 \tag{3.16}$$

The extraction cost function depends only on the amount extracted which implies that the orebody is homogenous. Such a cost function is well-suited to mining for disseminated ore such as occurs in many gold deposits in western states, but would be incorrect for mining ore that occurred in veins or that occurred in varying quality as it was extracted. For vein deposits or deposits of varying grade, extraction cost should also be a function of the cumulative amount extracted.

$$A'(r) \geq 0 \tag{3.17}$$

$$A''(r) \geq 0 \tag{3.18}$$

The abatement cost function depends only on the amount of waste reclaimed. This implies that the waste is also homogenous which is consistent with the previous assumption of homogenous ore. The inequality in equation (3.17) implies that the abatement cost function may have flat spots instead of being strictly increasing like the extraction cost function. This is necessary in order to satisfy the transversality conditions below.

The critical size of the waste pile, $W_c(t)$ in equation (3.10), varies over time like an irregular sine wave because of the seasonal variation in stream flow and rainfall. This causes equation (3.10) to be alternately concave and convex which severely complicates the mathematics of the model. To make equation (3.10) more tractable, the critical level of metal in the stream is assumed to be set when stream flow and rainfall are at their historical minimum. From equation (3.2) this will set W_c at the lowest possible level (assuming new historical low stream flow and rainfall are not set). Setting W_c in this manner corresponds to an environmentally cautious policy. It also allows $W_c(t)$ to be replaced by W_c, a constant. This type of constraint is typical of a "command and control" policy (so called because the critical level of metal concentration is set by decree). Such policies are usually inefficient; yet they are quite prevalent because they are easy for the policymaker to implement. The shortcomings will be discussed further in chapter 4.

Economic Interpretation of the Maximum Principle

The system of equations in (3.7)-(3.14) are typically combined into a Hamiltonian. The non-negativity constraints in equations (3.11) and (3.12) can be omitted from the Hamiltonian for different reasons. Equation (3.12) requires the amount of ore in the ground be non-negative. This makes sense for a physical commodity, but including this constraint merely reinforces the idea that once the ore is used up, all activity stops. Equation (3.11) requires that the waste pile be non-negative. It will be shown that the waste pile increases initially and if it subsequently decreases, then equation (3.10) will always become binding before (3.11). The current-valued (as opposed to discounted) Hamiltonian (suppressing time arguments) is

$$\tilde{H}=pbq-C(q)-A(r)-\mu_X q+\mu_W[aq-r-kW]+\lambda(W_c-W) \qquad (3.19)$$

Where:

H is the current-valued stream of profits from the mine.

μ_W is the current-valued shadow price of mine waste.

μ_X is the current-valued shadow price of ore in the ground.

λ is the current-valued shadow price of an additional unit of environmental slack. The multiplier λ is a function of time and required to be non-negative.

The Hamiltonian combines current and future effects of the mine manager's decisions on the current-valued stream of profits generated by the mine. As will be shown below, μ_W is negative; therefore, reclamation has positive impacts on future profits and negative impacts on current profits. Extraction, on the other hand, will have positive and negative impacts on current profits and negative impacts on future profits. The maximum principle yields the following necessary conditions:

$$\frac{\partial\tilde{H}}{\partial q}=pb-C'(q)-\mu_X+a\mu_W=0 \qquad (3.20)$$

$$\frac{\partial \tilde{H}}{\partial r} = -A'(r) - \mu_W = 0 \tag{3.21}$$

$$\frac{\partial \tilde{H}}{\partial \lambda} = W_c - W \geq 0, \ \ \lambda \geq 0, \ \ \lambda (W_c - W) = 0 \tag{3.23}$$

$$\dot{W} = \frac{\partial \tilde{H}}{\partial \mu_W} = aq - r - kW \tag{3.23}$$

$$\dot{\mu}_W = \delta \mu_W - \frac{\partial \tilde{H}}{\partial W} = (\delta + k) \mu_W + \lambda \tag{3.24}$$

$$\dot{X} = \frac{\partial \tilde{H}}{\partial \mu_X} = -q \tag{3.25}$$

$$\dot{\mu}_X = \delta \mu_X - \frac{\partial \tilde{H}}{\partial X} = \delta \mu_X \tag{3.26}$$

Equations (3.20) and (3.21) can be interpreted as marginal rules for extraction and reclamation. Equation (3.20) can be re-written as:

$$pb = C'(q) + \mu_X - a\mu_W \tag{3.27}$$

Equation (3.27) equates marginal revenue from extraction with three other terms. The first two terms on the right hand side of equation (3.27) are familiar from Hotelling. The first is the marginal cost of extraction. The second term is the shadow price of ore in the ground which has its usual interpretation as a user cost, or future wealth forgone by current period extraction. The third term requires a little discussion. Equations (3.17) and (3.21) imply that μ_W is negative. Therefore, the third term in equation (3.27) may be thought of as a marginal cost and interpreted as a user cost. This user cost represents future wealth forgone by using up some of the existing stock of environmental "slack" in the present that will be unavailable in the future. Environmental slack can be thought of as the difference between the current and critical size of the waste pile. Therefore, the marginal rule is to equate the marginal revenue from extraction to the marginal cost of extraction and the sum of the user costs from depleting the stocks of ore and environmental slack.

Similarly, equation (3.21) can be re-written:

$$A'(r) = -\mu_W \tag{3.28}$$

Equation (3.28) sets the marginal cost of reclamation equal to the negative shadow price (the user cost) of consuming environmental slack from existing stock. Thus, at optimality, the marginal cost of reclamation (which can be thought of as the cost of adding a unit to the stock of environmental slack) should be equal to the user cost of consuming environmental slack from stock.

The shadow prices have results similar to those

Hotelling found in his model. Equation (3.26) implies that the current-valued shadow price of ore in the ground increases at the discount rate; that is, the discounted shadow price is constant. This is a standard result for pure depletion models. Equation (3.24) says that when λ = 0 the current-valued shadow price of waste rises at the discount rate plus the metal release rate from the waste pile. The metal release rate acts as an additional discount rate that inflates the current-valued shadow price of waste. This is similar to Hotelling's result on user cost of ore.

Solving the Model With General Cost Functions

Following Burness (1976), general comments about the solution of the model can be made without specifying the forms of the cost functions (this analysis will be performed in greater depth in the policy analysis of chapter 4). These results can be compared with traditional analysis to determine where the models differ.

Solving equation (3.20) for μ_X and taking the derivative with respect to time:

$$\dot{\mu}_X = -C''(q) + a\dot{\mu}_W \qquad (3.29)$$

Equating (3.29) and (3.26) and using (3.24) gives an expression for the slope of the optimal path of extraction through time:

$$\dot{q}=\frac{-\delta\,[pb-C'(q)]-akA'(r)+a\lambda}{C''(q)} \tag{3.30}$$

Burness arrives at the following formula for the slope of the optimal extraction path:

$$\dot{q}=\frac{-\delta\,[pb-C'(q)]}{C''(q)} \tag{3.31}$$

Invoking the usual assumption that marginal revenue is always greater than or equal to marginal cost, equation (3.31) implies that the slope of extraction through time is negative as long as the discount rate is positive. This implies that extraction is tilted toward the present. Tilting is a common result in mining problems and is usually attributed to a non-zero discount rate (Hotelling, 1931; Scott, 1968). As equation (3.31) shows, a zero discount rate in the traditional mine analysis would result in a level slope of extraction through time (extracting the same amount in each period). However, equation (3.30) shows that in the model developed here, a discount rate of zero still produces a negative slope for the time path of extraction when $\lambda = 0$:

$$\dot{q} = \frac{-akA'(r)}{C''(q)} \tag{3.32}$$

Therefore, a non-zero discount rate contributes to the negativity of the slope of extraction but does not cause it. When $\lambda > 0$, the slope of the optimal extraction path through time is less negative. Thus the constraint may cause the rate of extraction to decrease less rapidly and therefore shorten the life of the mine. Since the control variables are only required to be piecewise continuous, q may have a sharp corner or a jump discontinuity when the constraint becomes binding. Figure 3.2 shows some possible shapes for the extraction function, denoting the time the constraint becomes binding at t_c.

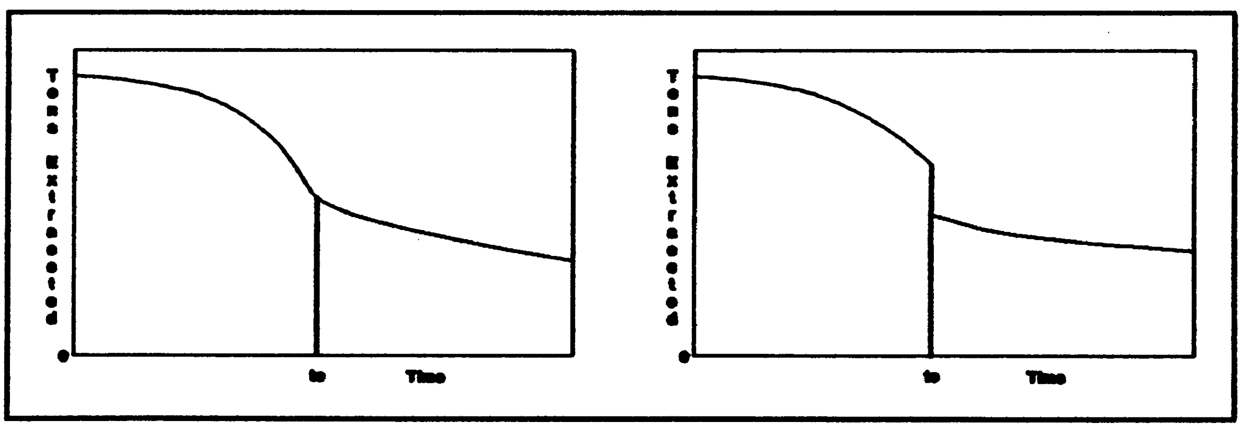

Figure 3.2. Possible Shapes for q(t)

Taking the time derivative of equation (3.21) and equating it to (3.24) yields the following expression for the slope of reclamation through time:

$$\dot{r}=\frac{(\delta+k)A'(r)-\lambda}{A''(r)} \tag{3.33}$$

The slope of reclamation is unambiguously positive when λ = 0, implying that the optimal level of reclamation increases throughout the life of the mine at least until the waste pile hits the critical size. Once again, the discount rate contributes to the positive slope but does not cause it. When λ > 0, the slope of reclamation becomes less positive and may even become negative. Figure 3.3 shows some possible shapes for r(t). Like q(t), r(t) is only required to be piecewise continuous and may have a discontinuity(either a sharp corner or a jump) at t_c.

The condition that W(T) is free implies that the current-valued shadow price of waste at the terminal time will be zero:

$$\mu_W(T)=0 \tag{3.34}$$

The condition that the terminal time is free implies that the current-valued Hamiltonian will be zero at the terminal time:

$$\bar{H}(T)=0 \tag{3.35}$$

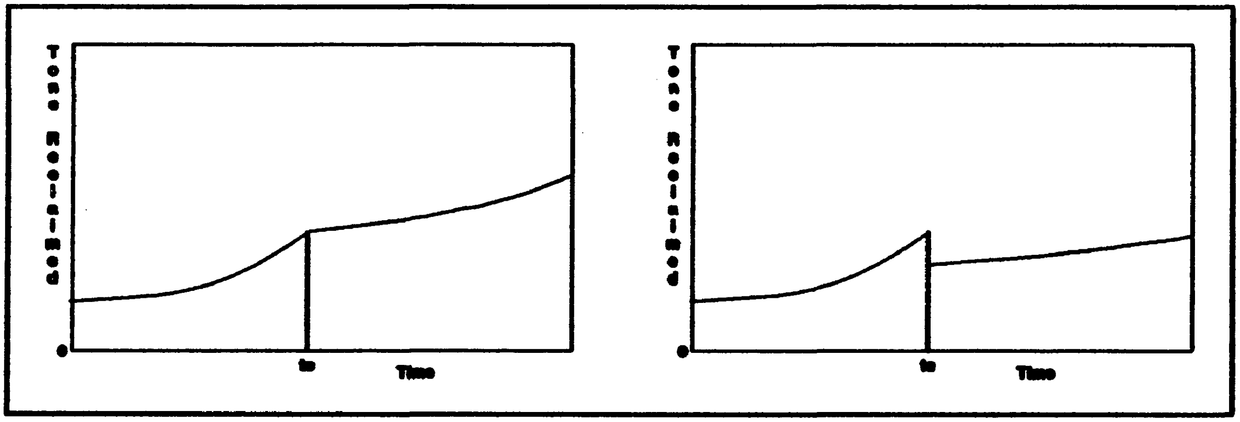

Figure 3.3. Possible Shapes for r(t)

Combining equations (3.34) and (3.35) gives a relationship for the level of extraction at the terminal time:

$$C'[q(T)]=\frac{C[q(T)]}{q(T)}+\frac{A[r(T)]}{q(T)} \tag{3.36}$$

Equation (3.36) sets the marginal cost of extraction at the terminal time equal to the average cost of extraction plus the cost of reclamation divided by the amount extracted at the terminal time. This is similar to the usual result in the mining literature that the marginal cost of extraction equals the average cost at the terminal time. Once again, Burness gives the level of extraction at the terminal time in traditional analysis:

$$C'[q(T)] = \frac{C[q(T)]}{q(T)} \qquad (3.37)$$

The difference in this model is the additional term on the right hand side of equation (3.36). If the cost of reclamation is non-zero at the terminal time, the terminal level of extraction will be greater than in the traditional model. Once the constraint becomes binding, equation (3.36) may not be valid. However, the optimal terminal level of extraction may still be above the level indicated in equation (3.37).

Assuming that the level of extraction remains above the level of reclamation until the constraint becomes binding, the waste pile grows until it reaches W_c. At this time, denoted by t_c, the constraint in equation (3.22) becomes active. When the constraint is active, it becomes much more difficult to characterize the solution without specifying the forms of the cost functions. A few general conclusions may still be drawn if two features of the model are considered. First, the mine receives no payment for any extra environmental slack it has in stock at the terminal time, and the mine incurs additional costs if it creates additional environmental slack once the initial stock is used up. Second, by assumption, the mine will never exceed the critical size of the waste pile; that is, the stock of environmental slack will never be negative. These points strongly suggest that the size of the waste pile will remain at W_c from $t = t_c$ to $t = T$. Figure 3.4 shows a possible shape for W(t).

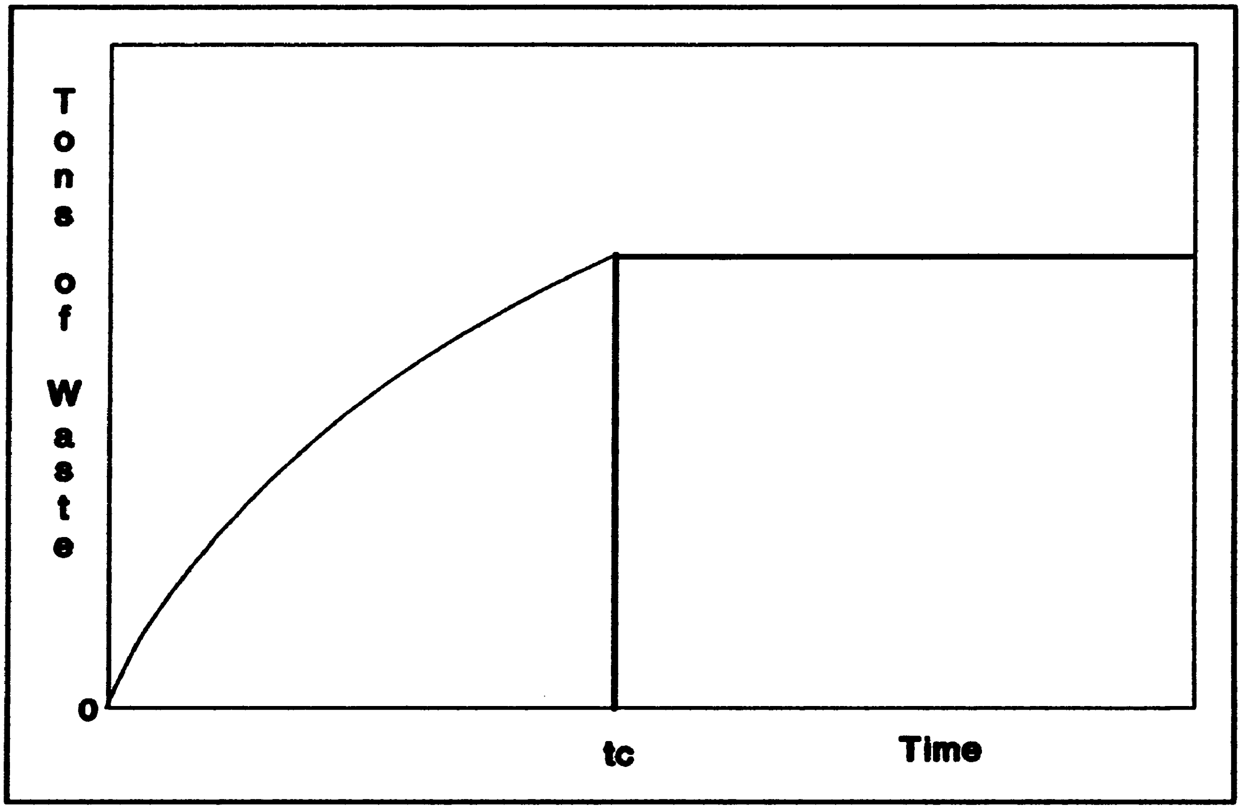

Figure 3.4. Possible Shape for W(t)

The waste pile may not rise to the critical level before the ore is completely extracted. This would correspond to a robust ecosystem where $\lambda = 0$ for the entire time period. If $\lambda = 0$ for the entire period, then by equation (3.34) $\mu_W(0) = 0$, which implies that the current-valued shadow price of waste is zero for the entire period. This simply corresponds to the one-product mine analyzed in the traditional literature.

For less robust ecosystems, λ eventually becomes positive and the constraint in equation (3.22) becomes active. By complementary slackness, the following relationship holds between extraction and reclamation once the critical waste pile size is reached:

$$aq=r+kW_c \tag{3.38}$$

If neither control variable is at its maximum value or zero, an additional equation is needed to determine the optimal levels of each variable. Solving equations (3.20) and (3.21) for μ_W and equating them yields the required relationship:

$$\frac{pb-C'(q)-\mu_X(0)\,e^{\delta t}}{a}=A'(r) \tag{3.39}$$

Solving equations (3.38) and (3.39) simultaneously gives the optimal levels of extraction and reclamation in the period when $\lambda \geq 0$, provided neither r nor q is at its maximum level or zero. If either r or q is known to be at the limits of its range, equation (3.38) gives the level of the other control variable, subject to installed capital constraints. For example, if $r = r_{max}$, then equation (3.38) says

$$q=\frac{r_{max}+kW_c}{a} \qquad (3.40)$$

but if the level of q from equation (3.40) is greater than q_{max}, then q is set to q_{max} and r is determined from equation (3.38):

$$r=aq_{max}+kW_c \qquad (3.41)$$

It turns out that μ_X and μ_W will not have discontinuities (Seierstad and Sydsaeter, 1987). λ will be may be positive or zero until t = T at which point the transversality condition in equation (3.34) implies that it will equal 0 - $\mu_W(T)$ (a positive number, since $\mu_W(T) < 0$). Example time paths for X, μ_X, and μ_W are shown in figures 3.5-3.7.

The policy implications of this model are explored in detail using general cost functions in chapter 4. The model is applied to the prefeasibility analysis of a small open-pit gold mine in chapter 5. Possible extensions of the model are discussed in chapter 6.

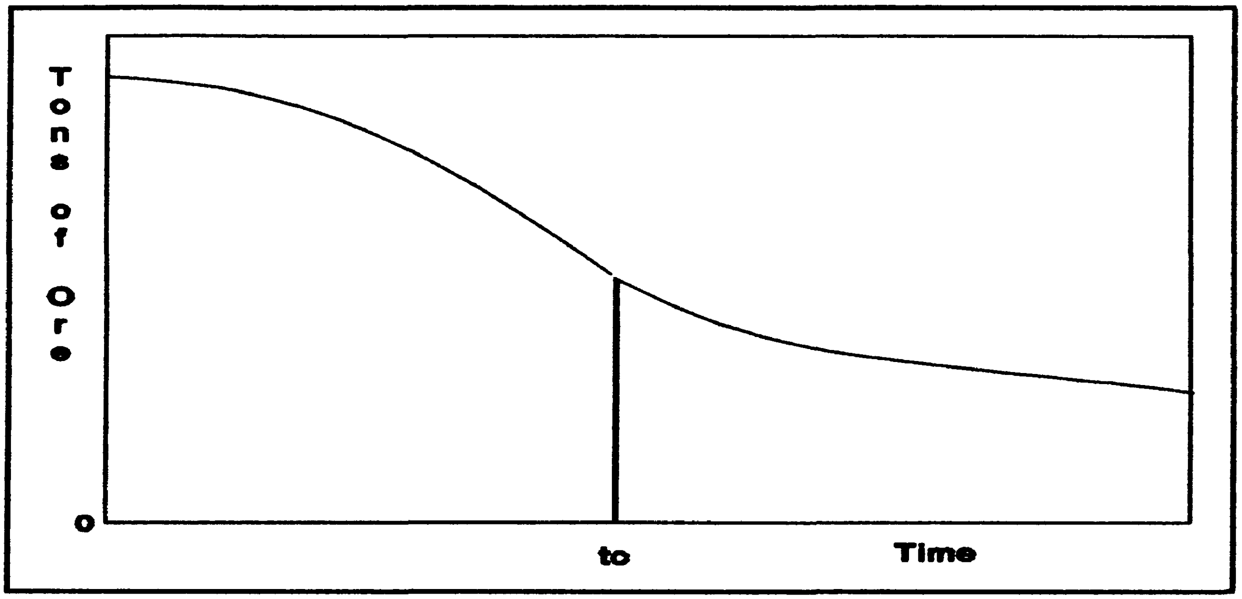

Figure 3.5. Possible Shape for X(t)

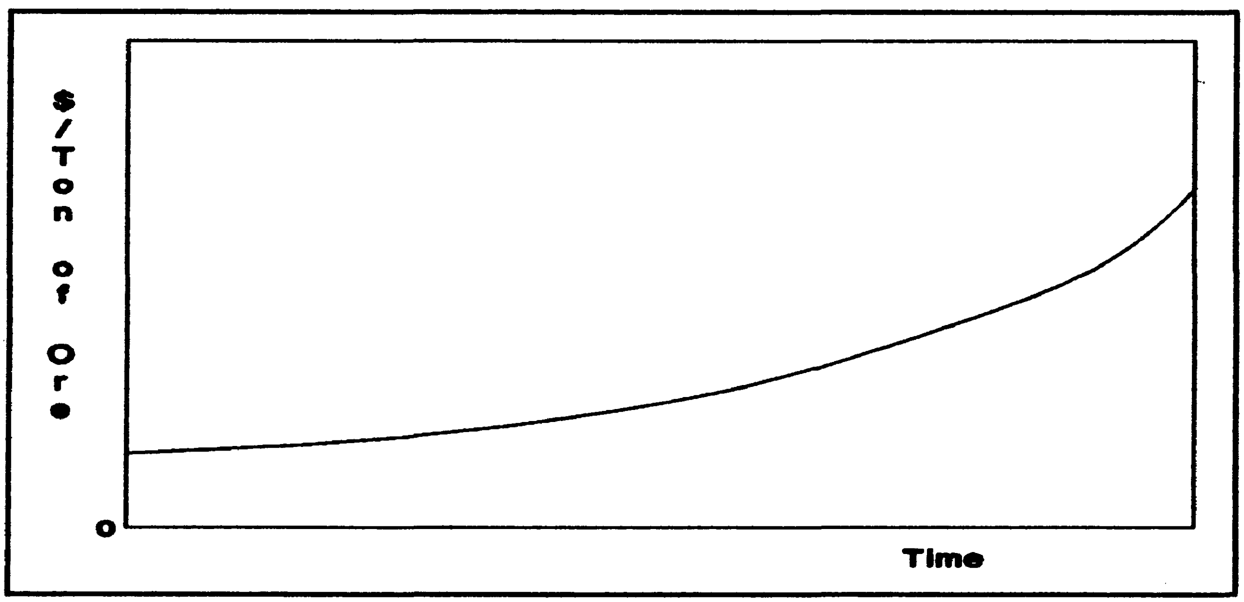

Figure 3.6. Current Valued Shadow Price of Ore

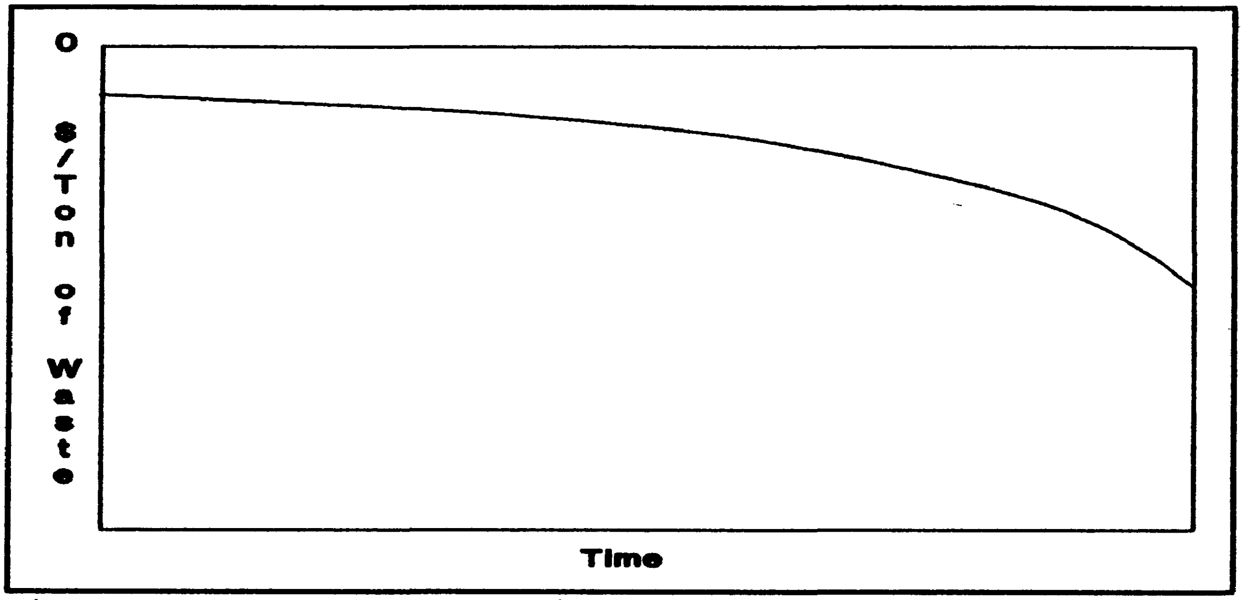

Figure 3.7. Current-Valued Shadow Price of Waste

Chapter 4

POLICY IMPLICATIONS OF THE MODEL

This chapter explores the policy implications of the model developed in chapter 3 without specifying forms for the cost functions. A fictional social optimizer is postulated who can determine the marginal damage caused by increasing stream metal concentrations. The operation of the mine under various regulatory regimes is compared to operation by a social optimizer in order to determine the efficiency of each type of regulation: taxing production, subsidizing reclamation, taxing the size of the waste pile, subsidizing the stock of environmental slack, and command and control standards. The mathematical techniques used here rely heavily on Burness (1976).

The Social Optimizer

A social optimizer is postulated who can measure the total and marginal social damage done by increasing stream metal concentration, and includes this social damage term in his objective function. Social damage, D(W), is assumed to be a function only of the mass of the waste pile and to have a strictly positive first derivative and a non-negative second derivative. Thus social damage can be thought of as a neoclassical cost function with properties analogous to the extraction cost function discussed in chapter 3. The objective function for the social optimizer is

$$\max\int_0^T e^{-\delta t}[pbq-C(q)-A(r)-D(W)]\,dt \tag{4.1}$$

subject to

$$\dot{X}=-q \tag{4.2}$$

$$\dot{W}=aq-r-kW \tag{4.3}$$

The current-valued Hamiltonian is

$$\tilde{H}=pbq-C(q)-A(r)-D(W)-\mu_X q+\mu_W(aq-r-kW) \tag{4.4}$$

The maximum principle gives the following equations:

$$\frac{\partial\tilde{H}}{\partial q}=pb-C'(q)-\mu_X-a\mu_W=0 \tag{4.5}$$

$$\frac{\partial\tilde{H}}{\partial r}=-A'(r)-\mu_W=0 \tag{4.6}$$

$$\dot{\mu}_X=\delta\mu_X-\frac{\partial\tilde{H}}{\partial X}=\delta\mu_X \tag{4.7}$$

The transversality condition for free terminal time is

$$\dot{\mu}_W = \delta\mu_W - \frac{\partial\tilde{H}}{\partial W} = (\delta + k)\,\mu_W + D'(W) \tag{4.8}$$

$$\tilde{H}(T) = 0 \tag{4.9}$$

The transversality condition for free size of the waste pile at the terminal time is

$$\mu_W(T) = 0 \tag{4.10}$$

Substituting equation (4.4) into equation (4.9) and solving for the current-valued shadow price of ore in the ground:

$$\mu_X(T) = p - \frac{C[T]}{q(T)} - \frac{A[r(T)]}{q(T)} - \frac{D[W(T)]}{q(T)} \tag{4.11}$$

Solving equation (4.5) for $\mu_X(T)$ and equating the result with (4.11) gives an equation that can be used to determine the level of extraction at the terminal time. This equation is critical to the policy analysis since we infer the optimal terminal level of reclamation from it as well. Subscripts will be used to differentiate the cases:

$$C'[q_{so}(T)] = \frac{C[q_{so}(T)]}{q_{so}(T)} + \frac{A[r_{so}(T)]}{q_{so}(T)} + \frac{D[W(T)]}{q_{so}(T)} \tag{4.12}$$

Burness's analysis for the level of extraction at the terminal time for an unregulated mine comes up with an equation that only has the first two terms in equation (4.12). In other words, terminal extraction is set where the marginal cost is equal to the average cost. This agrees with traditional analysis of the mine as a one-product firm.

Equation (4.12) can be seen as an extension of the "marginal cost equals average cost" rule with two additional average cost components: the terminal cost of reclamation and the terminal level of social damage, each averaged over the terminal level of extraction. The implications of this rule for the social optimizer is that if the two additional components are non-zero, which appears likely since either terminal social damage and/or terminal reclamation will almost certainly be non-zero, the socially optimal terminal level of reclamation will be greater than the profit-maximizing terminal level for an unregulated mine (figure 4.1). To determine the consequences of this result on the life of the mine, we have to examine the slope of the socially optimal extraction function.

Solving equation (4.5) for μ_X and taking the time derivative:

$$\dot{\mu}_X = -C''(q) + a\dot{\mu}_W \tag{4.13}$$

Substituting equation (4.8) into equation (4.13), and equation (4.5) into equation (4.7) and eliminating μ_X from the two resulting equations gives the slope of the socially optimal extraction path through time:

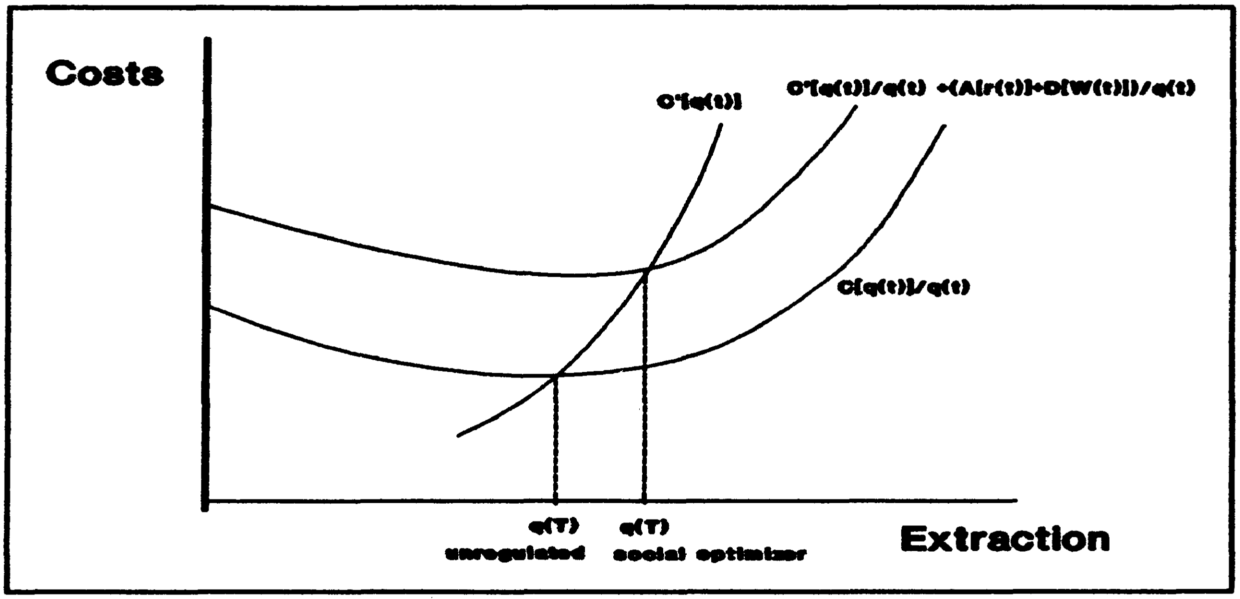

Figure 4.1. Terminal Extraction Levels Compared

$$\dot{q} = \frac{-\delta\,[p - C'(q)] - akA'(r) + aD'(W)}{C''(q)} \tag{4.14}$$

The slope of the extraction path and a single point on it (the terminal value determined above) give the entire path of extraction. These two pieces of information will be at the heart of the analysis in this section. If two policies yield the same time slope of extraction and share a common point, they have the same extraction path. If they share a common point with different slopes, they have intersecting extraction paths. If they have the same slope of the extraction path, but one point is different, then the extraction paths are parallel. Comparisons of reclamation will be carried out similarly. The final useful fact in comparing mine lives for various policies is that in every case, the same amount of ore is extracted; therefore, by equation (4.2), the area under the extraction curves must be equal for each policy. With these facts we can compare the socially optimal mine life and the unregulated mine life.

Whether equation (4.14) gives a slope that is more or less negative, or the same as the Burness analysis of the unregulated mine depends on whether the sum of the last two terms in the numerator is negative, positive, or zero. If the slopes are the same, the fact that the terminal level of extraction is lower for the unregulated mine implies that the unregulated mine will extend the life of the mine beyond the socially optimal life (figure 4.2). If the socially optimal slope is less negative (less steep) then the socially optimal mine life increases. Similarly, if the socially optimal slope is steeper than the unregulated mine life, the socially optimal mine life decreases. The same technique can be used to analyze the socially optimal path of reclamation.

Taking the time derivative of equation (4.6) and eliminating μ_W between the resulting equation and equation (4.8) yields an equation for the slope of the socially optimal time path of reclamation:

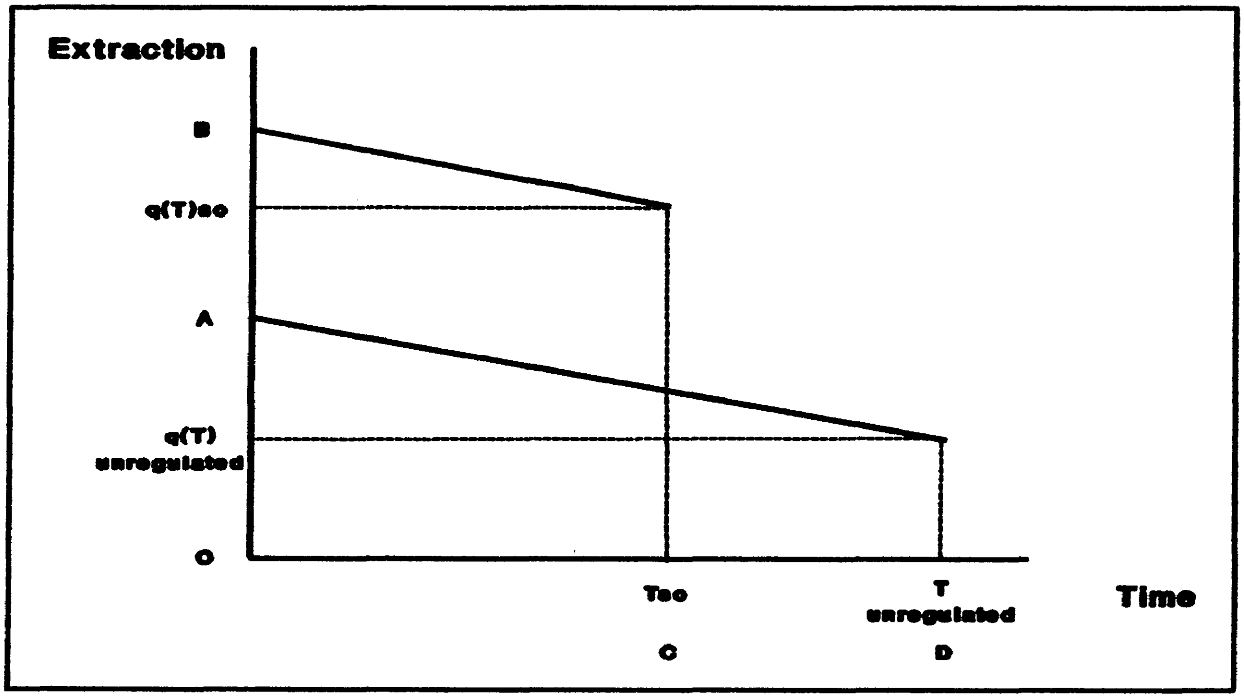

Figure 4.2. Extraction Functions Compared

$$\dot{r} = \frac{(\delta+b)A'(r) - D'(W)}{A''(r)} \tag{4.15}$$

The unregulated mine's reclamation level is always zero so its derivative is always zero. Therefore, if the right-hand side of equation (4.15) is non-zero, the socially optimal slope of reclamation will be different from that for the unregulated mine. Of course, if the socially optimal level of reclamation is ever non-zero it will also be different than the unregulated case.

The preceding discussion illustrates the method that will be used to evaluate various policies in this chapter. Equations (4.12), (4.14), and (4.15) characterize the solution from the perspective of the social optimizer. This is the way the mine would be run to maximize net social benefit. For policy analysis, these three equations will be the standard by which solutions generated by other regulatory regimes will be judged. Often policies will have the potential to generate partial solutions that is, the extraction path could be made socially optimal but the reclamation path cannot or vice-versa. The tools in this chapter can only point out the results of these options and identify the trade-offs the policy maker must consider in choosing between them.

Taxing Production

The objective function with a tax on the value of production is

$$\max \int_0^T e^{-\delta t}[(1-\tau)pbq - C(q) - A(r)]\,dt \qquad (4.16)$$

where τ is the tax rate. The profit-maximizing solution is characterized by three equations analogous to equations (4.12), (4.14) and (4.15) which were obtained in the same manner. The profit-maximizing terminal level of extraction is determined by

Comparing equations (4.17) and (4.12) we see that if the profit-maximizing terminal level of extraction is to be

$$C'[q_{tp}(T)] = \frac{C[q_{tp}(T)]}{q_{tp}(T)} + \frac{A[r_{tp}(T)]}{q_{tp}(T)} \qquad (4.17)$$

socially optimal, the following equation must hold:

$$\frac{A[r_{tp}(T)]}{q_{tp}(T)} = \frac{A[r_{so}(T)]}{q_{so}(T)} + \frac{D[W(T)]}{q_{so}(T)} \qquad (4.18)$$

Since we are assuming $q_{tp}(T) = q_{so}(T)$, equation (4.18) reduces to

$$A[r_{tp}(T)] = A[r_{so}(T)] + D[W(T)] \qquad (4.19)$$

The terminal level of reclamation must be greater than the socially optimal level if terminal social damage is greater than zero. Therefore, this policy will only give an optimal extraction function at the expense of an optimal terminal level of reclamation. To compare the slope of the profit-maximizing extraction function to equation (4.14) we solve for it

$$\dot{q} = \frac{-\delta[p - C'(q)] - akA'(r) - (\dot{t}p + \tau\delta p)}{C''(q)} \qquad (4.20)$$

Comparing equations (4.20) and (4.14) we see that this solution can make the profit-maximizing and socially optimal slopes identical if the tax rate satisfies the following differential equation:

$$\dot{\tau}+\delta\tau=-\frac{aD'(W)}{p} \tag{4.21}$$

If the right-hand side of (4.21) is zero; that is, social damage is constant for all levels of waste, the solution to equation (4.21) is the same as its homogenous solution (sometimes called a long-term or steady-state solution). This would be a tax rate that decreases exponentially at the rate of interest, eventually approaching zero. If the right-hand side of equation (4.21) is non-zero the solution has another part called a transient that may or may not "die out" during the life of the mine. Such a situation would arise if social damage is a linear function of the size of the waste pile, then by equation (4.21) the optimal tax rate would decrease exponentially at the rate of interest down to a constant level.

Assuming the tax rate is set so that equation (4.21) holds, the slopes will be equal. If equation (4.20) holds as well, the socially optimal extraction path will be identical to the profit-maximizing extraction path. We have already seen that the price of making the profit-maximizing terminal level of extraction socially optimal is to make the profit-maximizing level of reclamation different from the social optimum. The equation for the slope of the reclamation function shows that the profit-maximizing slope will always be too steep (too positive), and our policy instrument has no effect on it.

$$\dot{r}=\frac{(\delta+k)A'(r)}{A''(r)} \tag{4.22}$$

Therefore, the best use of this policy would be to use equations (4.20) and (4.21) as guidelines to achieve socially optimal extraction and abandon the goal of optimal reclamation. This assumes the social optimizer has some method of forcing the miner to conform to the socially optimal terminal level of reclamation.

Subsidizing Reclamation

Subsidizing reclamation changes the objective function:

$$\max\int_0^T e^{-\delta t}[pbq-C(q)-(1-\sigma)A(r)]\,dt \tag{4.23}$$

where σ is the subsidy rate ($0 \le \sigma \le 1$). Note that if σ = 1 (reclamation is completely subsidized in all time periods), the problem is the same as the unregulated case and the mine will behave as a single-product firm. The profit-maximizing terminal level of extraction is determined by the following equation:

$$C'[q_{sr}(T)] = \frac{C[q_{sr}(T)]}{q_{sr}(T)} + \frac{(1-\sigma)A[r_{sr}(T)]}{q_{sr}(T)} \tag{4.24}$$

In this case we have a trade-off between socially optimal extraction and reclamation. Assuming the policymaker has some way to mandate the terminal level of extraction and reclamation, he/she must choose between the socially optimal extraction and socially optimal reclamation. Using this policy instrument, equation (4.14) shows that the two goals are mutually exclusive. Later it will be shown that this policy instrument cannot produce the socially optimal reclamation function. Therefore, the policymaker should mandate the following terminal level of reclamation to ensure socially optimal extraction:

$$A[r_{sr}(T)] = \frac{A[r_{so}(T)] + D[W(T)]}{1-\sigma} \tag{4.25}$$

The slope of the profit maximizing extraction function is given by

$$\dot{q} = \frac{-\delta[pb-C'(q)] - akA'(r) + ak\sigma A'(r)}{C''(q)} \tag{4.26}$$

Comparing equations (4.26) and (4.14) we see that this solution will give the correct slope of optimal rate of extraction if the subsidy satisfies the following equation:

$$\sigma=\frac{D'(W)}{kA'(r)} \tag{4.27}$$

The slope of the profit-maximizing reclamation function is:

$$\dot{r}=\frac{(\delta+\sigma)A'(r)}{A''(r)}+\frac{\dot{\sigma}A'(r)}{(1-\sigma)A''(r)} \tag{4.27}$$

Comparing equations (4.28) and (4.15) we see the slope of the extraction function can be socially optimal if the subsidy rate satisfies the following differential equation:

$$A'(r)\dot{\sigma}-D'(W)\sigma=-D'(W) \tag{4.29}$$

If equations (4.27) and (4.29) can be satisfied simultaneously and the terminal level of reclamation is set at the socially optimal terminal level, then the profit-maximizing reclamation function will be the same as the socially optimal reclamation function. Unfortunately, substituting equation (4.27) into equation (4.29) shows that these two requirements are inconsistent with the parameters of our model. After substitution, equation (4.29) can only be satisfied if k = -1 which would require the waste pile to absorb metal from the stream. Thus, a policy of subsidizing reclamation cannot produce the socially optimal solution and in fact, can only do as well as taxing the value of production since it can only ensure that the profit-maximizing extraction function will be socially optimal.

<u>Taxing the Mass of the Waste Pile</u>

Taxing the mass of the waste pile is the clear winner but, as the analysis shows, its success is due in part to the problem formulation. Taxing the waste pile gives the following objective function:

$$\max\int_0^T e^{-\delta t}\left[pbq - C(q) - A(x) - \tau_W W\right] dt \qquad (4.30)$$

where τ_W is the tax rate. The profit-maximizing terminal level of extraction is determined from

$$C'[q_{tw}(T)] = \frac{C[q_{tw}(T)]}{q_{tw}(T)} + \frac{A[x_{tw}(T)]}{q_{tw}(T)} + \frac{\tau_W W(T)}{q_{tw}(T)} \qquad (4.31)$$

Comparing equations (4.31) and (4.12) we see that the profit-maximizing terminal levels of extraction can equal the socially optimal levels if the following equation holds.

$$\tau_W(T) = \frac{D[W(T)]}{W(T)} \tag{4.32}$$

Equation (4.32) says the terminal level of extraction will be socially optimal if the terminal tax rate is equal to the average social damage at the terminal time. The slope of the optimal time path of extraction in this case is:

$$\dot{q} = \frac{-\delta\,[pb - C'(q)] - akA'(x) + a\tau_W}{C''(q)} \tag{4.33}$$

Comparing equations (4.33) and (4.14) we see that this solution will give the correct slope of the optimal rate of extraction if the tax rate satisfies the following equation:

$$\tau_W = D'(W) \tag{4.34}$$

In other words, if the tax rate is equal to the marginal social damage, the slope of the profit-maximizing rate of extraction will be socially optimal. This is a common economic result known as a Pigouvean tax. Such a tax causes a profit-maximizing firm to internalize the externality it creates. If equations (4.31) and (4.29) hold simultaneously, the profit-maximizing rate of extraction will be identical to the socially optimal rate of extraction.

The slope of the profit-maximizing reclamation function is:

$$\dot{r}=\frac{(\delta+k)A'(r)-\tau_W}{A''(r)} \tag{4.35}$$

Comparing equations (4.35) and (4.15) we see the slope of the profit-maximizing extraction function will be socially optimal if the tax rate is equal to the marginal damage function.

These positive results should be taken with a grain of salt. The fact that taxing the mass of the waste pile works so well is a direct result of the assumption that social damage is a function of the mass of the waste pile and the environmental assumption in chapter 3 that the ecosystem interaction is a function of the mass of the waste pile. The agreement among the functional forms of the social damage function, the tax function and the ecosystem interaction term make this policy work so well. If a different form had been specified for any one of the three functions (for example, if metal migration were actually proportional to the exposed surface area of the waste pile instead of to the mass), the results from this policy would not have been so neat.

Subsidizing the Stock of Environmental Slack

We can modify the previous policy slightly by considering a subsidy awarded on the amount of environmental slack the mine has in stock at any time. Note that this is not merely the previous case with a negative tax. The idea of environmental slack implies the existence of a threshold that the policy maker is trying to impose on the miner. Slack is the amount by which the current operating point is below the threshold. The idea of keeping the miner below the threshold by paying him a subsidy has some attractive mathematical implications since it holds out the hope that the state space constraint may be abandoned. Sadly, we'll soon see that this hope is ill-founded.

The objective function in this case is

$$\max\int_0^T e^{-\delta T}[pbq-C(q)-A(r)+h(W_c-W)]\,dt \qquad (4.36)$$

where h is the subsidy and W_c - W is the amount of environmental slack at time t.

The profit-maximizing terminal level of extraction is determined from:

$$C'[q_{se}(T)] = \frac{C[q_{se}(T)]}{q_{se}(T)} + \frac{A[r_{se}(T)]}{q_{se}(T)} - \frac{h[W_c - W(T)]}{q_{se}(T)} \qquad (4.37)$$

From equation (4.37), the terminal values of extraction and reclamation can be at their socially optimal levels as long as the following equation holds:

$$h = -\frac{D[W(T)]}{W_c - W(T)} \qquad (4.38)$$

If h is positive, equation (4.38) implies that the terminal level of waste is greater than the threshold level. This reflects the fact that society receives no value from keeping environmental slack in stock. Continuing the analysis, the following equation gives the slope of the profit-maximizing extraction function:

$$\dot{q} = \frac{-\delta[pb - C'(q)] - akA'(r) + ah}{C''(q)} \qquad (4.39)$$

The profit-maximizing and socially optimal slopes are equal if the following equation holds:

$$h = D'(W) \qquad (4.40)$$

This is a familiar result from the previous policy. The optimal subsidy should be equal to the marginal social damage. The following equation for the slope of the profit-maximizing reclamation function is also identical to the socially optimal slope if equation (4.40) holds:

$$\dot{r}=\frac{(\delta+k)A'(r)-h}{A''(r)} \tag{4.41}$$

Subsidizing environmental slack can produce socially optimal results, but there is nothing to prevent environmental slack from becoming negative; in fact, equation (4.38) implies that the stock of environmental slack will be negative at the terminal time. All the policy tools considered so far have had this drawback. This is just the first one to state it explicitly. Now we turn to one model that will never have this problem, the model developed in chapter 3.

Command and Control Standards

The case of command and control is slightly different from the policy cases discussed above because it incorporates a constraint directly. If the case is not binding at the terminal time, then the command and control case behaves exactly like the unregulated mine analyzed by Burness. If the constraint is binding, then no conclusions can be reached about the terminal level of extraction without specifying the forms of the cost functions (see chapter 5 for analysis of this model with known cost functions). The slopes of the profit maximizing cost functions were calculated in chapter 3 in terms of a multiplier, λ, that was greater than or equal to zero only when the constraint was binding. These equations, (3.30) and (3.33), are reproduced below:

$$\dot{q}=\frac{-\delta[pb-C'(q)]-akA'(r)+a\lambda}{C''(q)}$$

$$\dot{r}=\frac{(\delta+k)A'(r)-\lambda}{A''(r)}$$

The profit-maximizing slope will be equal to the socially optimal slope in each equation if the multiplier is equal to the marginal social damage:

$$\lambda=D'(W) \tag{4.42}$$

If the terminal levels of extraction and reclamation are equal to the socially optimal terminal levels, the command and control standard could produce the socially optimal results. This happy coincidence could occur, but it's much more likely that the socially optimal terminal levels of extraction and or reclamation will be different. In fact, there's no guarantee that λ will equal the marginal social damage. The only thing a command and control standard guarantees is that the threshold level of waste will not be exceeded. Compared to the social optimizer, command and control is the worst of all the policies considered in this chapter since it can't guarantee even one optimal function. On the other hand, if threshold changes in the ecosystem are the policymaker's primary concern, command and control may be the best policy.

Policy Comparison

The best of the unconstrained policies is taxing the mass of the waste pile. This is due to the agreement of the form of the social disutility and the item being taxed. If such an agreement can be identified, the result is a perfect Pigouvean tax. All other unconstrained policies: taxing output, subsidizing reclamation, and subsidizing the stock of environmental slack could only guarantee that one of the two functions, extraction or reclamation, would be optimal. The final policy, command and control, was the worst from the social optimality standpoint since it couldn't guarantee either function would be optimal but it could guarantee the stock of environmental slack would never be negative, something the unconstrained policies could not do. Therefore, command and control policies would be best if the policymaker was primarily concerned with avoiding irreversible changes in the ecosystem. Chapter 5 will apply this model to a case study with known cost functions and further analyze policy implications.

Chapter 5

APPLICATION OF THE MODEL

The microeconomic model developed in chapter 3 will be applied to the pre-feasibility analysis of a small open-pit gold mine in northern Idaho. The site is part of a larger ecosystem project the U.S. Bureau of Mines began evaluating in 1993. The mine is hypothetical but the site data and cost information are as accurate as possible. The purpose of the analysis is to determine the impact various levels of water quality standards will have on mine profitability. This information can be related to statistical hypothesis about the occurrence of ore grades of the required richness (Cox and Singer, 1986) to determine the amount of mineral resources that are no longer economically feasible for any given level of environmental constraint.

Methodology

The mine is assumed to be a small open-pit gold mine using heap leaching to recover gold from ore. The mine is assumed to be located on the western edge of the Clearwater National Forest near U.S. highway 12. Metals from the waste pile are assumed to drain into Lolo Creek. Data on stream flow and rainfall are from the U.S. Forest Service. The constant of proportionality for metal release from waste rock is from work done at the U.S. Bureau of Mines Salt Lake City research center (White and Jeffers, 1992). Cost functions for extraction are taken from those used in the U.S. Bureau of Mines prefeasibility analysis (Camm, 1991). Reclamation cost functions are taken from the HEAPREC model (Denton, 1992). Both extraction and reclamation cost functions are linear.

A question that occurs to people familiar with mining operations and the theory of optimal depletion of exhaustible resources is why mine managers never seem to tilt production toward the earlier portion of the life of the mine as theory says they should. Instead, mine managers generally try to run their operations at as close to maximum capacity as possible. Two explanations for this behavior are consistent with economic theory. First, mine managers may have zero discount rates which would be consistent with uniform extraction rates over the life of the mine. Second, mine managers may act as if they had linear cost functions that would correspond to a "bang-bang" solution in optimal control theory. Mine managers may believe their cost functions are nonlinear, but the uncertainty in measuring the costs in a complex system like a mine may encourage mine managers to adopt the simplifying assumption of linear costs at least over the portion of the operating range the mine manager is concerned with.

The U.S. Bureau of Mines has constructed linear cost models for nine different mining methods and six milling methods. These models are used in the prefeasibility analysis of various mines. The small open-pit mine model and the heap leach mill model will be used for the application in this chapter.

Linear Cost Functions

Consider the following forms for C(q) and A(r):

$$C(q)=C_0+C_1q \tag{5.1}$$

$$A(r)=A_0+A_1r \tag{5.2}$$

Using these cost functions, the Hamiltonian will be linear in the choice variables, and the marginal rules described in equations (3.20) and (3.21) will become switching equations:

$$q=\begin{Bmatrix} q_{max} \\ q* \\ 0 \end{Bmatrix} \; if \; \mu_w(t) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \frac{C_1-p\beta+\mu_x(t)}{a} \tag{5.3}$$

The asterisks indicate a particular feature of optimal control theory known as singular paths. If either of

$$r=\begin{Bmatrix} r_{max} \\ r* \\ 0 \end{Bmatrix} \; if \; -\mu_w(t) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} A_1 \qquad (5.4)$$

equations (5.3) or (5.4) hold with equality for more than an instant of time, a singular path for q or r exists. From equations (5.3) and (5.4), we can easily show that the assumption that a singular path exists for either variable leads to the contradiction that a constant is equal to an exponential for more than one unique instant. Therefore singular paths cannot exist and the behavior of q and r is undefined at the instant when equations (5.3) and (5.4) hold with equality but is defined everywhere else.

The interpretation of equation (5.3) is similar to the interpretation of its counterpart in chapter 3, equation (3.27). Rearranging equation (5.3) to isolate marginal revenue on the left-hand side:

$$q=\begin{Bmatrix} q_{max} \\ 0 \end{Bmatrix} \; if \; pb \begin{Bmatrix} > \\ < \end{Bmatrix} C_1+\mu_x(t)-a\mu_w(t) \qquad (5.5)$$

The first term on the right-hand side of equation (5.5) is the marginal cost of extraction. The second term is the familiar user cost from Hotelling models: the profits forgone in the future by extracting a unit of resource today. The third term has the same interpretation it had in chapter 3; the user cost associated with consuming the stock of "environmental slack." The constant, a, determines the units of material added to the waste pile for each unit of ore extracted. The minus sign in front of the negative shadow price of waste multiplied by the constant, a, can be thought of as adding a user cost for consuming environmental slack below the level at which the environmental constraint becomes binding. Therefore the marginal rule in equation (5.5) is to extract at the maximum rate as long as the marginal revenue from extraction exceeds the marginal cost of extraction plus the user cost of extraction and the user cost of reducing the stock of environmental slack by adding to the waste pile. If the marginal revenue is insufficient to cover the marginal cost and user costs, then the optimal level of extraction is zero. This is the same as the interpretation of equation (3.27) except that the control variable switches abruptly between its maximum and minimum values.

The interpretation of equation (5.4) is to reclaim at the maximum rate whenever the marginal cost of reclamation is less than the user cost of consuming environmental slack (again, interpreting the negative of a negative shadow price as a positive user cost). In other words, if it is cheaper to make environmental slack than to consume from stock, reclaim at the maximum rate. If the marginal cost of reclamation exceeds the user cost of consuming environmental slack, the optimal level of reclamation is zero.

Assuming the initial value of the current-valued shadow price of mine waste is negative and greater than the switching value for each control variable, we see from equations (5.3) and (5.4) that q will start at q_{max} and later switch to zero; r will start at zero and later switch to r_{max}. Designating the time when r switches from zero to r_{max} as t_r and the time when q switches from q_{max} to zero as t_q, we can identify three distinct cases depending on whether t_r <, =, or > t_q. It turns out that once the constraint is considered, only two unique cases remain.

Case one, $t_r < t_q$, (figure 5.1) can be characterized by the behavior of the control variables in each of its three subperiods. The first subperiod, $0 \le t \le t_r$, is characterized by the following values of the control variables: r = 0 and q = q_{max}. During the first subperiod, the amount of ore in the ground decreases with cumulative extraction

$$X(t) = X_0 - q_{max} t \tag{5.6}$$

The waste pile behaves according to
To determine whether the waste pile grows or shrinks, we take the time derivative of equation (5.7)

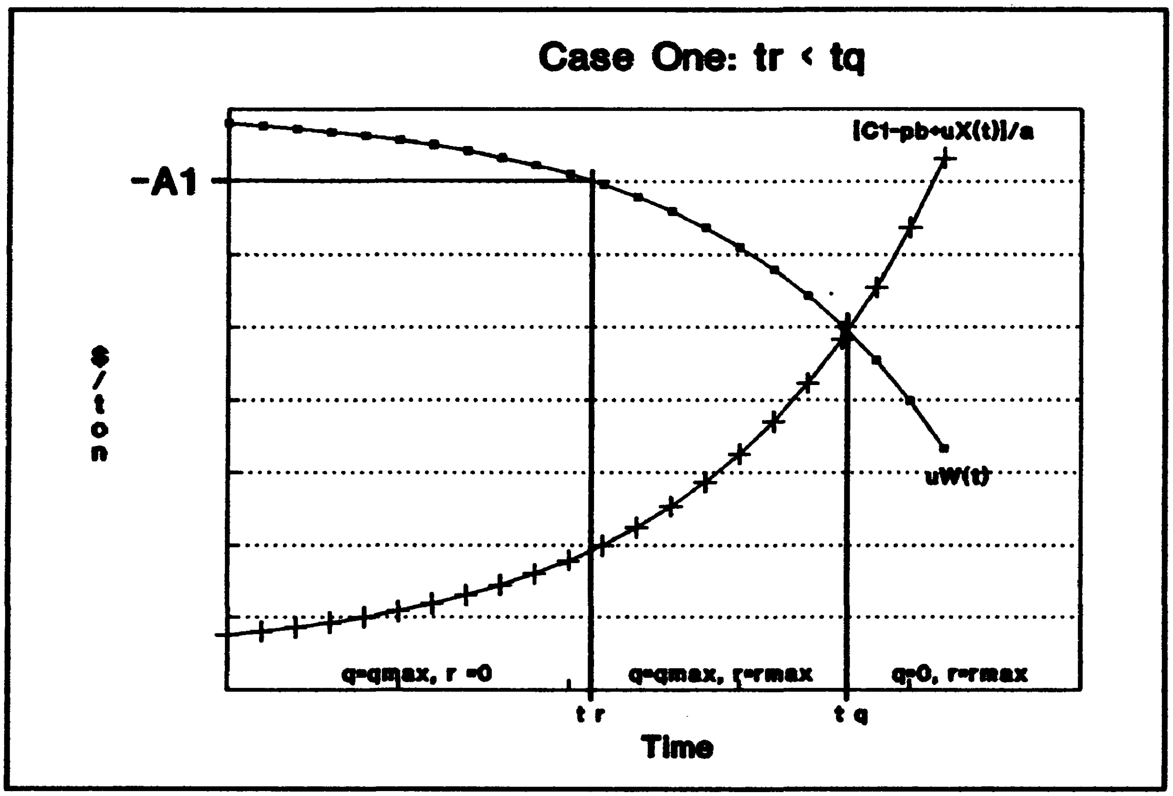

Figure 5.1. Case One tr < tq

$$W(t) = \frac{aq_{max}}{k} + \left(W_0 - \frac{aq_{max}}{k}\right) e^{-kt} \tag{5.7}$$

$$\dot{W} = (aq_{max} - kW_0)\, e^{-kt} \tag{5.8}$$

Equation (5.8) shows that the waste pile will increase, decrease, or remain level in the initial period depending on the initial size of the waste pile. If the initial size of the waste pile is big enough so that the mass lost to metal migration from rainfall is greater than the mass added by extraction at the maximum rate, the waste pile

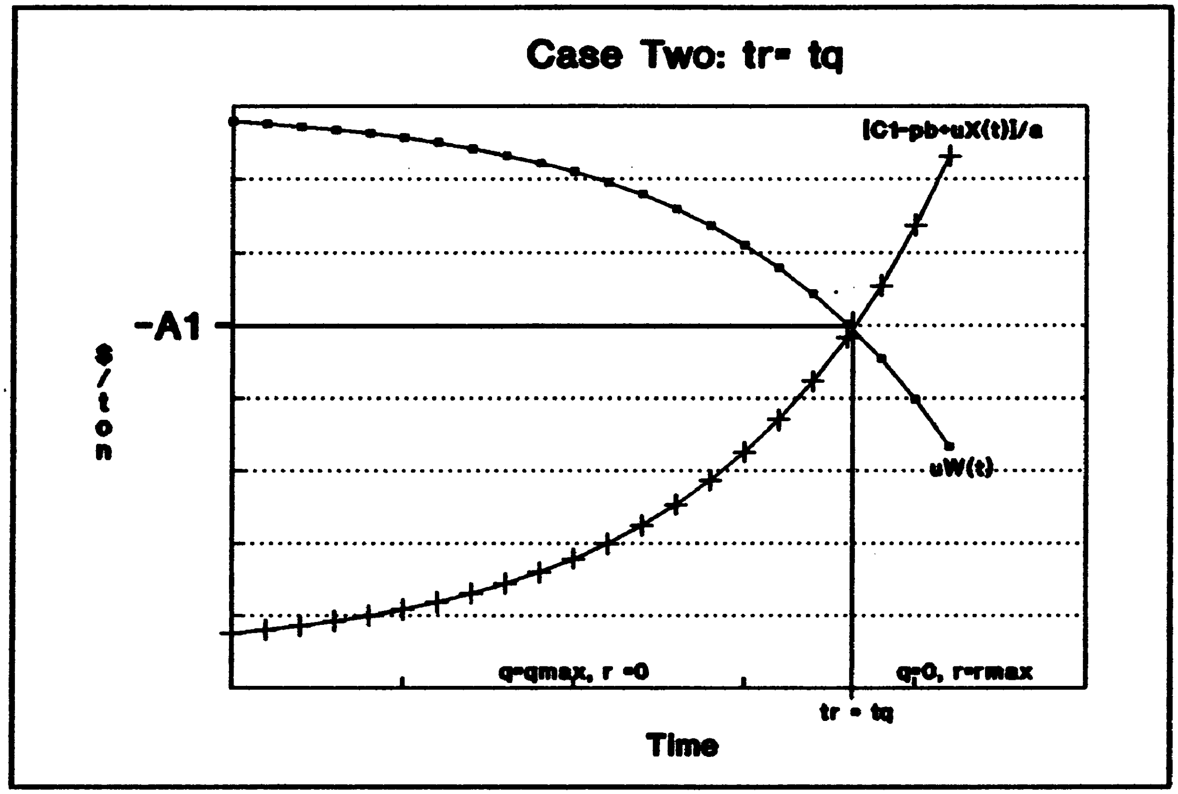

Figure 5.2. Case Two tr = tq

will shrink in the initial period. This seems highly unlikely, and the assumption that the waste pile will grow in the initial period seems more realistic.

Suppose the waste pile reaches the critical size, W_c, in this subperiod at time t_c. Then the constraint becomes active and the solution changes from a bang-bang solution to a constrained solution. When the constraint is active, the control variables can assume values other than the extremes of their control ranges. The question then becomes what values of q and r are optimal during the

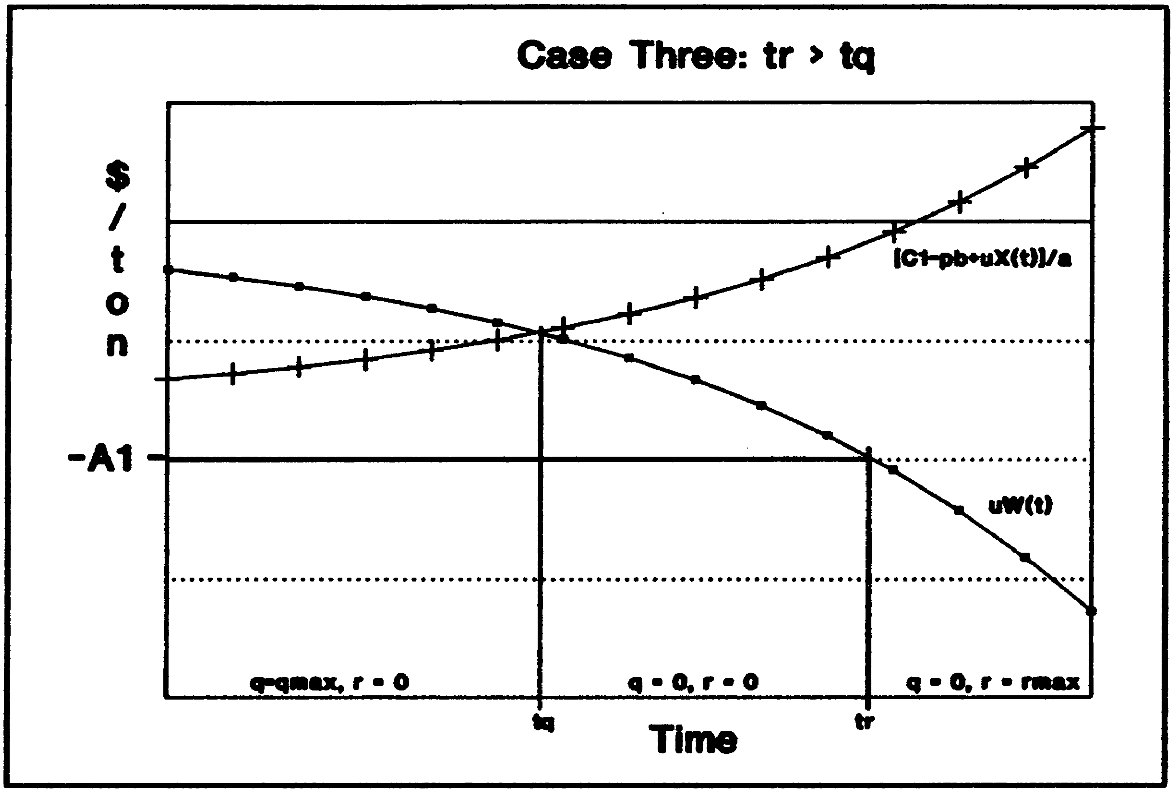

Figure 5.3. Case Three tr > tq

constrained solution. If it is unprofitable to be on the state-space boundary at q_{max}, then, in this model, it will be unprofitable to be on the boundary at any level of extraction, since cost and revenue functions are linear. If it is profitable to stay on the boundary, then the following argument demonstrates why extraction should remain at q_{max}, for the constrained solution and reclamation should be adjusted to maintain the waste pile at its critical size from t_c until the terminal time, T.

The argument for remaining on the state-space boundary is similar to the argument presented in chapter 3 except that it can be presented more sharply when the forms of the cost function are known. Once again, the first point to note is that the mine gets no reward for any environmental slack left in stock at the terminal time. The second point comes from considering the following example.

Suppose q_{max} units of ore remain at t_c. Would it be more profitable to extract it all in the next period and adjust reclamation to ensure the waste pile stayed at the critical level or to let the waste pile fall below the critical level by deferring the extraction of some of the remaining ore? If it turns out to be more profitable to defer extraction, what is the optimal amount to defer?

First, calculate the profits the mine earns from staying on the state-space boundary (i.e., maintaining W at W_c) by extracting q_{max} and setting r as determined from equation (3.38):

$$r_c = aq_{max} - kW_c \tag{5.9}$$

With these choices, the following period would be the terminal period, T, and the mine would realize the following profits while the constraint is active:

$$\pi_1 = (pb - C_1)\, q_{max} - A_1 (aq_{max} - kW_c) \tag{5.10}$$

Next, determine the profits if the mine retreats from the state space boundary by n units. Assuming the mine still sets reclamation as in equation (5.9) but defers production by n units for one period, then profits are

$$\pi_2 = (pb-C_1)\,(q_{max}-n) - A_1\,(aq_{max}-kW_c) + \left(\frac{n\,(pb-C_1)}{(1+\delta)}\right) \quad (5.11)$$

No reclamation is necessary in the final period since the slack created by retreating from the state-space boundary in the previous period, T-1, is exactly sufficient to cover the slack consumed by extraction in T. The difference between the profits realized by staying on the boundary and retreating are

$$\pi_1 - \pi_2 = \frac{\delta}{1+\delta}\,(pb-C_1)\,n \quad (5.12)$$

The difference in profits is positive for all positive values of n. Therefore, it is never more profitable to retreat from the state-space boundary. Furthermore, profits are maximized if we always extract the maximum amount (i.e., set q as large as possible and adjust r to make (3.38) zero when $W = W_c$).

The argument for setting $q = q_{max}$ and $r = r_c$ as defined in equation (5.8) implicitly assumed $r_c < r_{max}$. If $r_c > r_{max}$, then r should be set at r_{max} and q should be set at q_c as defined below:

$$q_c=\frac{r_{max}+kW_c}{a} \tag{5.13}$$

Continuing with the analysis of case one, if the waste pile does not hit the critical level in the initial subperiod, a second subperiod occurs. The second subperiod, $t_r \le t \le t_q$, has the following values for the control variables: $r = r_{max}$ and $q = q_{max}$ (figure 5.1). The amount of ore in the ground continues to decrease:

$$X(t)=X_0-q_{max}t \tag{5.14}$$

The waste pile behaves according to:

$$W(t)=\left(W_0+\frac{r_{max}e^{kt_r}-aq_{max}}{k}\right)e^{-kt}+\frac{aq_{max}-r_{max}}{k} \tag{5.15}$$

Taking the time derivative of equation (5.15) to determine whether the pile grows or shrinks:

$$\dot{W}=(aq_{max}-kW_0)\,e^{-kt}-r_{max}e^{kt_r-t} \tag{102}$$

The waste pile will increase, decrease, or remain constant depending on whether the right-hand side of equation (5.16) is positive, negative, or zero. In the first subperiod we argued that the quantity in parenthesis in equation (5.16) is positive so the question comes down to the magnitudes of the two terms in equation (5.16). Later, it will be shown that if the waste pile decreases in this period, the problem is unconstrained. Assuming the waste pile continues to increase, consider the possibility that it hits the critical level during this subperiod. Since the waste pile is increasing even with r at r_{max}, q will have to be reduced to q_c specified in equation (5.13) in order for the waste pile to stay on the state-space boundary. If the waste pile does not hit the critical level in the second subperiod, then the third and final subperiod in case one occurs.

The third subperiod, $t_q \le t \le T$, has the control variables set at q=0, and $r=r_{max}$ (figure 5.1). The waste pile shrinks in this subperiod since its equation of motion is now:

$$\dot{W}=-r_{max}-kW \tag{5.17}$$

Therefore, if the waste pile doesn't hit it's critical level in the first two subperiods, it never will. If the waste pile never hits the critical level then the solution corresponds to the unconstrained solution which has been thoroughly analyzed in the traditional mining literature (for example, Burness 1976). From equation (3.24) and the transversality condition for free terminal size of the waste pile, it can be seen that in the unconstrained solution $\mu_W(0) = 0$, or the shadow price of mine waste is always zero, thus no switching occurs; $q = q_{max}$ and $r = 0$

for the entire period of case one.

Cases two and three are similar to case one. Case two, $t_r = t_q$, (figure 5.2) has only two subperiods. During the first subperiod the state variables behave just as they do in the first subperiod of case one. During the second subperiod, they behave just as in the final subperiod of case one. Therefore, in case two, if the constraint does not become binding in the first subperiod, the problem is unconstrained.

Case three, $t_q < t_r$, (figure 5.3) has three subperiods as does case one. The first and third subperiods have the same control and state variable behavior as in case one, but in case three there is a second subperiod between them where r = 0 and q = 0. During this second subperiod the waste pile decreases because its equation of motion becomes

$$\dot{W}=-kW \tag{5.18}$$

Therefore, if the waste pile remains below the critical level in the first subperiod, the constraint will never be active. Thus we have only two constrained solutions to consider, depending on whether the constraint becomes binding in the first or second subperiod. These solutions are considered separately below.

Solution A

Solution A occurs when the constraint becomes binding in the first subperiod. It can arise from either case one, two, or three. It is the only constrained solution possible in cases two and three. An additional constrained solution is possible in case one that will be considered separately as solution B. The behavior of state and control variables in solution A can be summarized as follows:

$$q = q_{max} \qquad 0 \le t \le T \tag{5.19}$$

$$r = \begin{cases} 0 & 0 \le t \le t_c \\ aq_{max} - kW_c & t_c \le t \le T \end{cases} \tag{5.20}$$

$$X(t) = X_0 - q_{max} t \qquad 0 \le t \le T \tag{5.21}$$

$$W(t) = \begin{cases} \dfrac{aq_{max}}{k} + \left(W_0 - \dfrac{aq_{max}}{k} \right) e^{-kt} & 0 \le t \le t_c \\ W_c & t_c \le t \le T \end{cases} \tag{5.22}$$

The instant when the constraint becomes binding, t_c, can be determined from equation (5.22):

$$t_c = \frac{1}{k} \ln \left(\frac{kW_0 - aq_{max}}{kW_c - aq_{max}} \right) \tag{5.23}$$

Because extraction remains at its maximum level in this solution, the terminal time is the same as in the unconstrained case:

$$T=\frac{X_0-X_T}{q_{max}} \tag{5.24}$$

The initial shadow price for ore is determined from the transversality conditions associated with free terminal time and free terminal size of the waste pile:

$$\mu_X(0)=[pbq_{max}-C_0-C_1q_{max}-A_0-A_1(aq_{max}-kW_c)]\,e^{-\delta T} \tag{5.27}$$

The initial shadow price of waste is determined from the fact that, in this case, the shadow price curves will intersect at T:

$$\mu_W(0)=\left(\frac{C_1-pb-\mu_X(0)\,e^{\delta T}}{a}\right)e^{-(\delta+k)T} \tag{5.26}$$

As Seierstad and Sysdsaeter (1987, Ch5, theorem 1 and note 1) point out, in both solution A and solution B, the current-valued shadow price of waste, μ_W, will be continuous until the terminal time, T. At T, μ_W will jump to zero and the value of $\lambda(T)$ will be $0 - \mu_W(T)$.

Solution B

As noted above, solution B applies only to the second subperiod of case one. The behavior of the control and state variables in solution B is summarized below:

$$X(t)=\begin{cases} X_0-q_{\max}t & 0\le t\le t_c \\ X_0-q_{\max}t_c-q_c(t-t_c) & t_c\le t\le T \end{cases} \tag{5.27}$$

$$W(t)=\begin{cases} \left(W_0-\dfrac{aq_{\max}}{k}\right)e^{-kt}+\dfrac{aq_{\max}}{k} & 0\le t\le t_r \\ W_0e^{-kt}+\dfrac{aq_{\max}}{k}(1-e^{-kt})-\dfrac{r_{\max}}{k}(1-e^{k(t_r-t)} & t_r\le t\le t_c \\ W_c & t_c\le t\le T \end{cases} \tag{5.30}$$

$$q=\begin{cases} q_{\max} & 0\le t\le t_c \\ q_c & t_c\le t\le T \end{cases} \tag{5.29}$$

$$r=\begin{cases} 0 & 0\le t\le t_r \\ r_{\max} & t_r\le t\le T \end{cases} \tag{5.28}$$

Recovering the initial shadow prices and determining the switching times involves solving the following simultaneous system of five equations (three of which are non-linear) for the five variables: t_r, t_c, T, $\mu_X(0)$, and $\mu_W(0)$.

$$t_r=\left(\frac{1}{\delta+k}\right)\ln\left(\frac{-A_1}{\mu_W(0)}\right) \tag{5.31}$$

$$t_c=\frac{1}{k}\ln\left(\frac{kW_0+aq_{max}+r_{max}e^{kt_r}}{kW_c+r_{max}-aq_{max}}\right) \tag{5.32}$$

$$T=\frac{X_0-X(t_c)}{q_{max}}+\frac{X(t_c)-X_T}{q_c} \tag{5.33}$$

$$\mu_X(0)=\left(pb-\frac{C_0}{q_c}-C_1-\frac{A_0-A_1r_{max}}{q_c}\right)e^{kT} \tag{5.34}$$

$$\mu_W(0)=\left(\frac{C_1-pb-\mu_x(0)\,e^{kt_q}}{a}\right)e^{-(\delta+k)\,t_q} \tag{5.35}$$

where t_q is defined as

$$t_q=\frac{X_0-X_T}{q_{max}} \tag{5.36}$$

A solution strategy for this system is given in the appendix to this chapter. This system cannot be solved analytically due to the non-linearities in equations (5.31)-(5.35). Therefore, numerical methods must be used to determine its solution.

Potential Supply Implications of the Model

Solution A is the relevant solution for the application in this chapter and it will dominate the policy discussion. The profits generated by the mine, in solution A, are given by the following equation:

$$\begin{aligned}\pi_A = &\int_0^{t_c} e^{-\delta t}(pbq_{max} - C_1 q_{max})\,dt \\ &+ \int_{t_c}^{T} e^{-\delta t}(pbq_c - C_1 q_c - A_1 r_c)\,dt - C_0\end{aligned} \qquad (5.37)$$

where q_c and r_c refer to the optimal choices of q and r when the constraint is binding as given by equations (5.9) or (5.13) and C_0 is the sum of the fixed costs of reclamation and extraction.

Integrating equation (5.37) gives the following equation for profits:

$$\begin{aligned}\pi_A = &\frac{(pb - C_1)\,q_{max}(1 - e^{-\delta t_c})}{\delta} \\ &- \frac{[(pb - C_1)\,q_{max} - A_1 r_c]\,(e^{-\delta t_c} - e^{-\delta T}) - \delta C_0}{\delta}\end{aligned} \qquad (5.38)$$

The lowest acceptable ore grade; that is, the grade of ore that just sets the sum of discounted profits to zero, is of particular relevance in potential supply analysis since it determines the lowest grade that will be developed under existing policies. If the existing policies change, the change in minimum ore grades can be related by mineral deposit models to the proportion of national mineral wealth that is no longer economically feasible. In solution A, T and t_c are not functions of b; therefore equation (5.38) can be easily solved for the value of b that sets $\pi_A = 0$:

$$b_0 = \frac{\delta C_0 + C_1 q_{max}(1-e^{-\delta t_c}) + (C_1 q_c + A_1 r_c)(e^{-\delta t_c} - e^{-\delta T})}{p[q_{max}(1-e^{-\delta t_c}) + q_c(e^{-\delta t_c} - e^{-\delta T})]} \qquad (5.39)$$

Equation (5.39) can be graphed for the application in this chapter to demonstrate the required value of ore that must be discovered for each level of environmental constraint at any given discount rate (figure 5.4).

Figure 5.4 shows the impact of various policies on the initial grade of ore the mine must find to be profitable. The impact is a function of the expected commodity price (assumed constant) and the discount rate used by the mine manager. For example, with a limiting stream metal concentration of 0.05 mg/L, at \$300/oz and 15%, the mine manager would be interested in grades of 0.089 oz/ton and better. With a higher commodity price, \$350/oz, the mine manager would be interested in grades as low as 0.076 oz/ton. At \$300/oz and a higher discount rate, 20%, the mine manger is only interested in grades of 0.103 oz/ton or

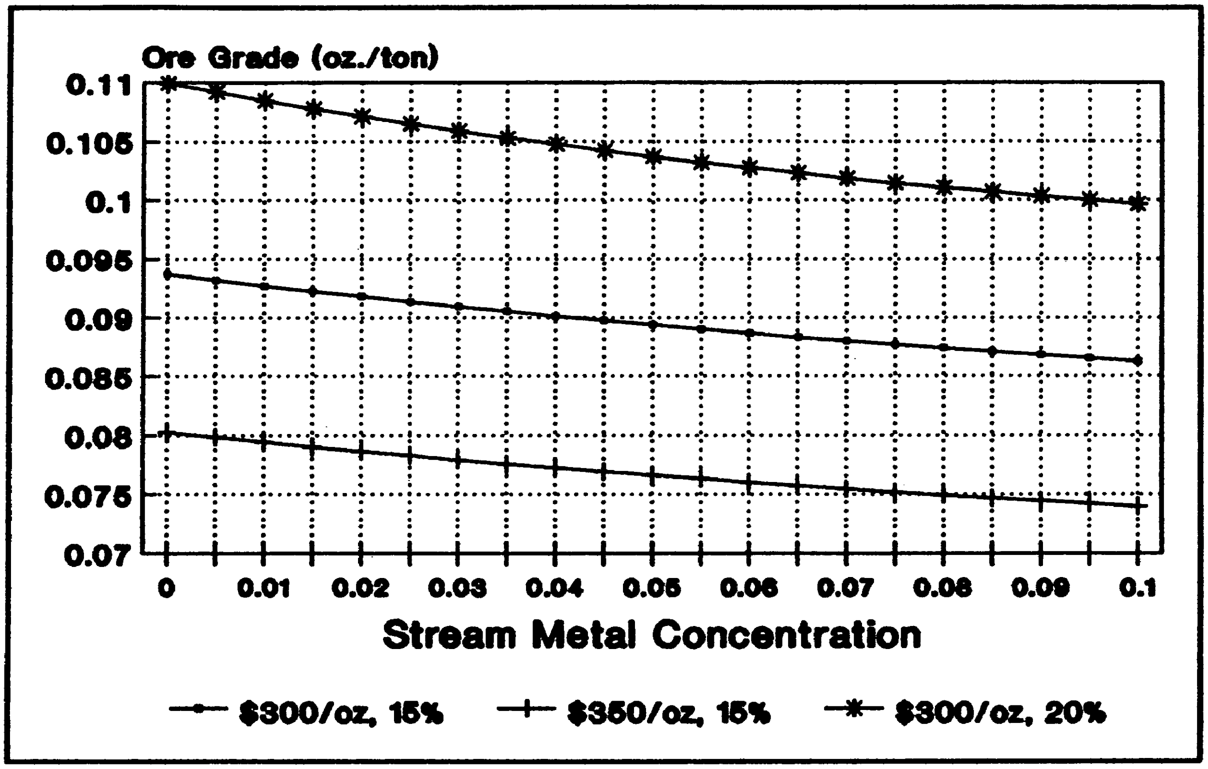

Figure 5.4. Minimum Economical Ore Grades

better.

Figure 5.4 also shows the impact of policy changes on proposed mines. For example, at $300/oz and a discount rate of 20%, changing from 0.095 mg/L to 0.04 mg/L would require an increase in grade from 0.1 oz/ton to 0.105 oz/ton. Thus the exclusion of mineral deposits by policy changes can be quantified by the use of geostatistical inference about grade and tonnage relationships.

Using mineral deposit models (Cox and Singer, 1986), changes in ore grades can be related to the tonnage excluded by such policies. For example if the deposit in this model is of the type that Cox and Singer identify as carbonate hosted gold-silver (model 26a, represented by the disseminated deposits in Carlin, Nevada) then the shift from 0.100 troy ounces per ton to 0.105 troy ounces per ton in the preceding paragraph will eliminate about 3 percent of the occurrences of this type of deposit (Cox and Singer, figure 133).

Implications on Profitability of Existing Mines

Substituting equations (3.2), (5.1), and (5.3) into (5.38) gives profits as a function of the limiting amount of metal in the stream (figure 14).

Figure 14 shows profits increasing at a slightly decreasing rate as the allowable level of stream metal concentration is increased from 0 to 0.1 mg/L. Even at a tolerance of 0 mg/L, the mine is profitable--it earns discounted profits of slightly over 2.5 million dollars.

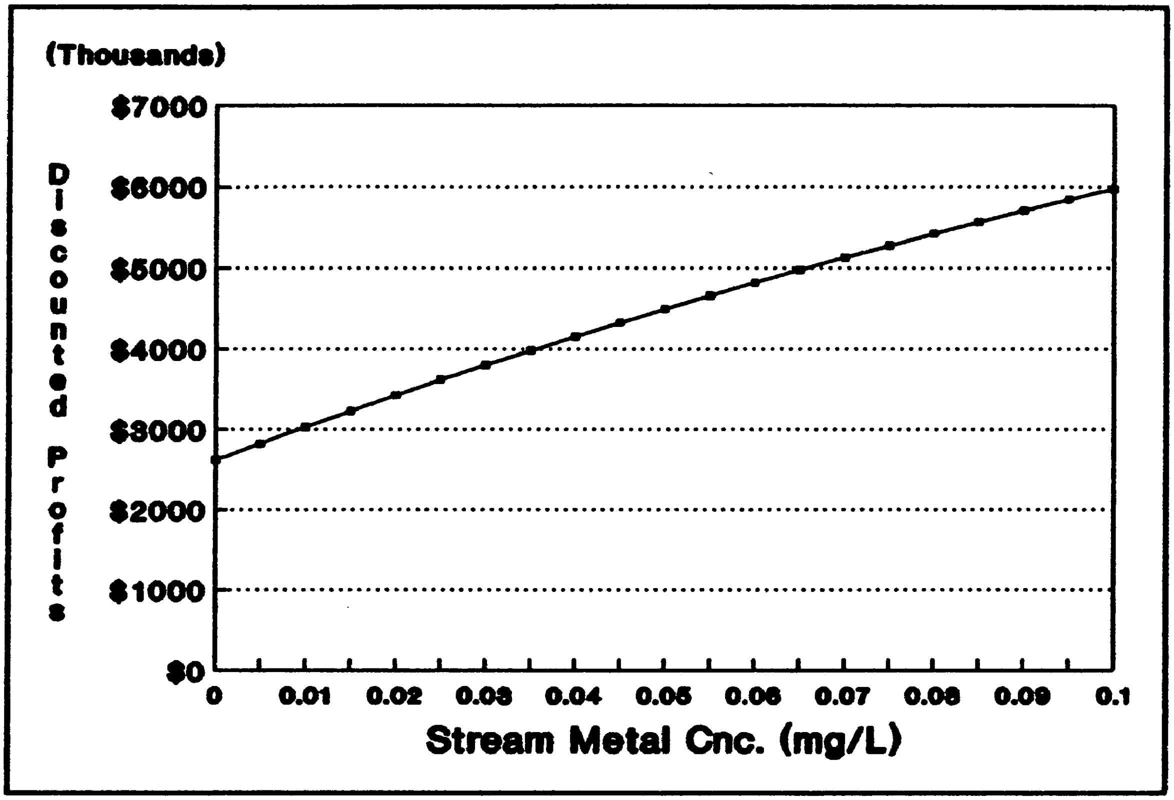

Figure 5.5. Profits at Various Constraint Levels

The equation for profits in solution B has an additional term:

$$\pi_b=\int_0^{t_r} e^{-\delta t}(pbq_{max}-C_1q_{max})\,dt + \int_{t_r}^{t_c} e^{-\delta t}(pbq_{max}-C_1q_{max}-A_1r_{max})\,dt + \int_{t_c}^{T} e^{-\delta t}(pbq_c-C_1q_c-A_1r_c)\,dt-C_0 \tag{5.40}$$

Here, T, t_r, and t_c are complex functions of b and thus equation (5.40) cannot be easily solved for the minimum required ore grade. Integrating equation (5.40) and determining the dependence of profits on the limiting stream metal concentration is also complicated. Since solution B is not relevant to the application in this chapter, its analysis is left as a question for further research.

Chapter 6

CONCLUSIONS, MODEL EXTENSIONS, AND PROBLEMS FOR FURTHER STUDY

The major points in this study are developed through the consideration of a microeconomic model of a mine as a two product firm. Optimal control theory is used to solve the model with an ecosystem constraint to determine the profit-maximizing program of extraction and reclamation for the mine. The solution could be used to extend prefeasibility analysis of mineral deposits currently carried out by the U.S. Bureau of Mines to include an ecosystem constraint.

The model was also used to extend the policy analysis of Burness (1976). Four market-based alternative policy options were compared with the current system of command and control policy for the mine's interaction with the ecosystem of a stream. All of the market-based policies were found to be more efficient than command and control policies.

In chapter 5, the model was applied to the prefeasibility analysis of a small open-pit gold mine in the Lolo Creek area of Clearwater National Forest. A function was developed that related the minimum economic ore grade a new mine would require to each level of the ecosystem constraint. For existing mines, a function was developed relating the change in profitability to each level of the ecosystem constraint.

Extending the Model

The model could be extended in two principle directions: the ecosystem constraints and the mine model. The ecosystem constraint set could be richer both in terms of the number of constraints and the realism of the constraints; that is, including stochastic variables to more realistically model complex ecosystem interactions. The mine model could be improved by including features from any of a number of the sophisticated improvements made since Hotelling's basic model appeared 60 years ago. An important feature to note is that increasing the complexity of the model also increases the difficulty of solving it. Optimal control theory problems usually become impossible to solve analytically with more than three state variables and this limit may be lowered considerably if some of the variables are stochastic. Numerical solutions have the disadvantage of often not being globally optimal and the obvious interplay of the variables in an analytic solution is lost in a numeric solution. For these reasons, the desired level of realism in model extensions should be weighed carefully against the computational requirements of their solution.

Improving the Ecosystem Constraints

As indicated in chapter 2, M.L. Cropper's article provides a guideline for stochastic constraints to replace command and control constraints in the model developed in chapter 3. This might be an extremely realistic extension to the existing model; instead of the unyielding threshold in chapter 3, the constraint would not incorporate the probability of irreversibly changing the ecosystem if the threshold level of waste were exceeded.

A further stochastic extension to the model would be to include a stochastic variable to account for the probabilistic success of entombing the reclaimed waste. Although the model in chapter 3 conveniently assumed the waste could be entombed and removed from the drainage, experience with entombment suggests its success is stochastic (Khein, 1993).

As the science of ecosystems progresses, other ecosystem interactions will be easier to identify and develop constraints for. A possible direction for further ecosystem constraints would be a system of constraints that equate the marginal level of ecosystem damage across ecosystems in a general equilibrium fashion.

Improving the Mine Model

The mine model could be extended to include non-disseminated deposits, the possibilities of new discoveries, the stochastic elements of geologic uncertainty, and stochastic cost functions to account for the complexity of mining operations. Most of these extensions have been explored in other models but not in an ecosystem context.

The economics of the mining model could be improved by incorporating state and federal taxes to put the analysis on a more realistic after-tax basis. A stochastic disturbance could be included in the discount rate to account for uncertainty in the mine manger's time preference.

Problems for Further Study

The main areas for further study in this model have to do with extending the analysis in chapter 5 to encompass solution B. This appears to be a non-trivial mathematical exercise but is necessary if the model is to be extended to all types of deposits.

All the extensions discussed above increase the reality at the expense of computational tractability and/or certainty. Including stochastic variables may accurately reflect reality but it inevitably limits the precision of the conclusions that can be drawn from the model. It may be preferable to use deterministic models as a reliable first approximation than to use more realistic stochastic models that offer no firm conclusions.

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APPENDIX: Solving the Solution B Equations

Solution B yields a system of five tightly coupled equations, three of which are non-linear. Solving such a system is generally quite difficult and, unlike linear systems, no hard and fast rules can be applied. Instead, each system is analyzed independently. Solving the system usually requires finding clever variable substitutions that exploit the structure of the equations and their inter-relations. Often, even with a series of apparently clever substitutions, the system cannot be solved analytically.

Substitutions are used to reduce the solution B system to two non-linear equations in two variables. The existence of a solution to this system can be verified graphically but graphical solutions often cannot provide the required precision. Therefore, numerical methods are used to solve the reduced system to the required level of precision.

The numeric method used is a technique known as geometric programming which has been shown (Burns and Locascio, 1991) to offer significantly greater performance in solving systems of non-linear equations than algorithms based on the more-commonly used Newton-Rapheson method.

An iterative method is outlined for implementing a geometric programming solution. Since solution B does not apply to the application investigated in this chapter, the implementation of the geometric programming solution is left as a problem for further study.

Reducing to Two Equations in Two Variables

The original system of equations is reproduced and renumbered below:

$$t_x=\frac{1}{(\delta+k)}\ln\left(\frac{-A_1}{\mu_W(0)}\right) \tag{A.1}$$

$$t_c=\frac{1}{k}\ln\left(\frac{kW_0+aq_{max}+xe^{kt_x}}{kW_c+x_{max}-aqmax}\right) \tag{A.2}$$

$$T=\frac{X_0-(X_0-q_{max}t_c)}{q_{max}}+\frac{X_0-q_{max}t_c-X_T}{q_c} \tag{A.3}$$

$$\mu_X(0)=\left(pb-\frac{C_0}{q_c}-C_1-A_0-\frac{A_1x_{max}}{q_c}\right)e^{kT} \tag{A.4}$$

$$\mu_W(0)=\left(\frac{C_1-pb-\mu_X(0)\,e^{kt_q}}{a}\right)e^{-(\delta+k)\,t_q} \tag{A.5}$$

The first step in the substitution scheme is to simplify the notation by collecting constants and giving them shorter names. Throughout the substitution scheme subscripted lowercase letter d's will be used to identify constants. An initial substitution of a positive variable for the negative of the current-valued shadow price of waste will be made. This has the double benefit of removing the negative sign in the argument of the natural logarithm function and making the variables conceptually clearer. The initial substitution and new constants are:

$$u=-\mu_W(0) \tag{A.6}$$

$$d_1=\frac{1}{\delta+k} \tag{A.7}$$

$$d_2=\ln(A_1) \tag{A.8}$$

$$d_3=\frac{1}{k} \tag{A.9}$$

$$d_4=kW_0+aq_{max} \tag{A.10}$$

$$d_5=\ln(kW_c+r_{max}-aq_{max}) \tag{A.11}$$

$$d_6=\frac{q_{max}}{q_c} \tag{A.12}$$

$$d_7=\frac{X_0-X_T}{q_c} \tag{A.13}$$

$$d_8=\left(pb-\frac{C_0}{q_c}-C_1-A_0-\frac{A_1 r_{max}}{q_c}\right) \tag{A.14}$$

$$d_9=e^{-(\delta+k)t_q}\left(\frac{C_1-pb}{a}\right) \tag{A.15}$$

$$d_{10}=e^{-\delta t_q} \tag{A.16}$$

The original system now becomes:

$$t_r=d_1d_2-d_1\ln u \tag{A.17}$$

$$t_c=d_3\ln(d_4+re^{kt_r})-d_3d_5 \tag{A.18}$$

$$T=(1-d_6)t_c+d_7 \tag{A.19}$$

$$\mu_x(0)=d_8e^{kT} \tag{A.20}$$

$$u=d_{10}\mu_x(0)-d_9 \tag{A.21}$$

A second substitution defines a single variable, t_2, for the argument of the logarithm function in equation (A.18) above.

$$t_2=d_4+re^{kt_r} \tag{A.22}$$

The structure of the problem can be exploited by making substitutions that involve taking exponentials of natural logarithm terms in order to reduce the system to a series of polynomial terms without either exponentials or natural logarithms.

Using the equation for t_r:

$$t_2=d_4+re^k e^{(d_1 d_2-d_1 \ln u)}=d_4+\frac{re^k e^{d_1 d_2}}{e^{d_1 \ln u}}=d_4+d_{11}u^{-d_1} \qquad (A.23)$$

$$d_{11}=re^k e^{d_1 d_2} \qquad (A.24)$$

The substituted system is now:

$$t_2=d_4+d_{11}u^{-d_1} \qquad (A.25)$$

$$t_c=d_3 \ln t_2-d_3 d_5 \qquad (A.26)$$

$$T=(1-d_6)\, t_c+d_7 \qquad (A.27)$$

$$\mu_x(0)=d_8 e^{kT} \qquad (A.28)$$

$$u=d_{10}\mu_x(0)-d_9 \qquad (A.29)$$

Using equation (A.26) to eliminate T in equation (A.28)

$$\begin{aligned}\mu_X(0) &= d_8 e^{k[(1-d_6)t_c+d_7]} \\ &= d_8 e^{kd_7} e^{k(1-d_6)t_c} \\ &= d_{12} e^{d_{13}t_c}\end{aligned} \qquad (A.30)$$

where:

$$d_{12}=d_8 e^{kd_7} \qquad (A.31)$$

$$d_{13}=k(1-d_6) \qquad (A.32)$$

Using (A.30) to eliminate $\mu_X(0)$ in (A.28) reduces the system to three non-linear equations in three variables:

$$t_2=d_4+d_{11}u^{-d_1} \qquad (A.32)$$

$$t_c=d_3\ln t_2-d_3d_5 \qquad (A.34)$$

$$u=d_{10}d_{12}e^{d_{13}t_c}-d_9 \qquad (A.35)$$

One remaining substitution can be made to eliminate the final exponential/logarithmic pair in equations (A.34) and (A.35). Substituting equation (A.34) into equation (A.35):

After the final substitution a system of two equations

$$\begin{aligned} u &= d_{10}d_{12}e^{d_{13}}e^{(d_3 \ln t_2 - d_3 d_5)} - d_9 \\ &= \frac{d_{10}d_{12}e^{d_{13}}t_2^{d_3}}{e^{d_3 d_5}} - d_9 \\ &= d_{15}t_2^{d_3} - d_9 \end{aligned} \tag{A.36}$$

with polynomial terms (no exponentials or logarithms) remains:

$$t_2 = d_4 + d_{11}u^{-d_1} \tag{A.37}$$

$$u = d_{15}t_2^{d_3} - d_9 \tag{A.38}$$

Up to this point, the solution has been analytical. Substitutions were made to eliminate variables without significantly the resulting equation system. Now we must abandon the analytical approach. Equations (A.37) and (A.38) cannot be solved by substitution. The reason substitution fails is that it will require raising a sum or difference to a power and no analytic formula exists that allows us to resolve the resulting expression (except in the unlikely event of an integer power - in which case the binomial formula comes to the rescue). Solving this system requires numerical methods of non-linear programming.

Prior to using numerical techniques, graphical methods should be used to verify the existence of a solution. If (A.37) were solved for u, both equations could be put on

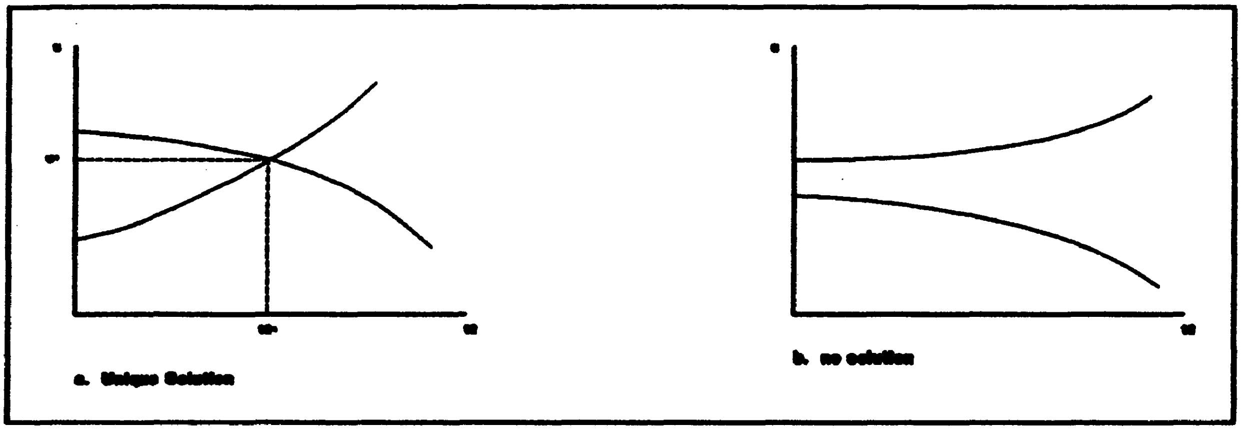

Figure A.1. Uniqueness of the Optimal Solution

the same graph. Equation (A.37) is monotonically decreasing for increasing values of t_2 and (A.38) is monotonically increasing for increasing values of t_2. Figure 15 shows the two possible cases when these functions are superimposed, either a unique solution exists, or no solution exists.

Geometric Programming Solution

Assuming the existence of a solution has been verified graphically, the next step is to determine the values of the variables at the solution. Graphical methods seldom give these values with the necessary precision. Therefore, numeric methods must be used. The method chosen here is geometric programming. Due to its relative unfamiliarity, the method will be described in sufficient detail to allow a reader to create a computer code for its implementation, if he or she is so inclined. More detail on this method can be found in Burns and Locascio (1991).

The geometric programming solution collects the positive and negative terms in each equation, then uses the geometric-arithmetic inequality to substitute a monomial term for sums of polynomial terms. Each equation then consists of a monomial representing the positive terms minus a monomial representing the negative terms equalling zero. Adding the negative monomial to both sides of each equation equates the two monomials. Dividing by one of the two monomials leaves each equation in the form of a monomial equal to one. Taking the natural logarithms of each equation results in a linear system of equations in the logs of the original variables. Solving the linear system gives the next guess for the variables.

To solve equations (A.37) and (A.38), first separate positive and negative terms:

$$t_2=d_{11}u^{-d_1}+d_4 \tag{A.39}$$

$$u+d_9=d_{15}t_2^{d_3} \tag{A.40}$$

In equation (A.39) the terms on the right hand side will be combined into a monomial, in equation (A.40) the terms on the left hand side will be combined. To combine terms, an initial guess for either u or t_2 must be made (and the other variable solved for using either equation (A.37) or (A.38). This is also what would be called, in computer algorithm terms, the start of the iteration process. For the benefit of the code-writing reader algorithm steps will be given in all capital letters throughout the discussion.

PICK AN INITIAL VALUE OF U AND T2.

START THE ITERATION.

Combining the terms in (A.39) depends on the following application of the arithmetic-geometric mean inequality:

$$d_{11}u^{-d_1}+d_4 \geq \left(\frac{d_{11}u^{-d_1}}{\delta_{11}}\right)^{\delta_{11}}\left(\frac{d_4}{\delta_{12}}\right)^{\delta_{12}} \tag{A.41}$$

The key to geometric programming is that the arithmetic-geometric inequality holds with equality if the deltas are chosen correctly. In this case, the correct choices are:

$$\delta_{11}=\frac{d_{11}u^{-d_1}}{d_{11}u^{-d_1}+d_4} \tag{A.42}$$

$$\delta_{12}=\frac{d_4}{d_{11}u^{-d_1}+d_4} \tag{A.63}$$

When equation (A.41) holds with equality, the right hand side can be substituted for the right-hand side of equation (A.39). Both sides of equation (A.39) can be divided by the new right hand side to yield the following equation:

$$K_1 t_2 u^{-d_1 \delta_{11}} = 1 \tag{A.63}$$

where K_1 is given by

$$K_1 = \frac{1}{\left(\frac{d_{11}}{\delta_{11}}\right)^{\delta_{11}} \left(\frac{d_4}{\delta_{12}}\right)^{\delta_{12}}} \tag{A.45}$$

Similarly, (A.40) can be transformed to:

$$K_2 u^{\delta_{21}} t_2^{-d_3} = 1 \tag{A.46}$$

where the constant term and the deltas are given by:

$$K_2 = \frac{\left(\frac{1}{\delta_{21}}\right)^{\delta_{21}} \left(\frac{d_9}{\delta_{22}}\right)^{\delta_{22}}}{d_{15}} \tag{A.47}$$

$$\delta_{21} = \frac{u}{u+d_9} \tag{A.48}$$

$$\delta_{22} = \frac{d_9}{u+d_9} \tag{A.49}$$

The computer algorithm steps are:

COMPUTE THE DELTAS.

COMPUTE THE CONSTANTS.

Taking natural logarithms of equations (A.44) and (A.46) results in a system of two linear equations which can be easily solved for the logarithms of the variables:

$$\ln u = \frac{d_3 \ln K_1 + \ln K_2}{d_3 d_1 \delta_{11} - \delta_{21}} \tag{A.50}$$

$$\ln t_2 = d_1 \delta_{11} \ln u - \ln K_1 \tag{A.51}$$

Updated values of t_2 and u can be calculated from equations (A.50) and (A.51). At equilibrium, the updated values will be the same as the previous values. Therefore a stopping criterion is to iterate until the updated values are as close as desired to the previous value. In computer terms:

CALCULATE THE UPDATED T2 AND U VALUES.

STOP IF THE NEW VALUES ARE CLOSE ENOUGH TO THE PREVIOUS VALUES, ELSE GO TO THE START OF THE ITERATION STEP.

This completes the description of the methods used to solve the solution B equations.