T-2766

A WEIGHTED-CENTROID APPROACH TO MULTIPLE-CRITERIA DECISION ANALYSIS

by

Judith Ann Barlow

A thesis submitted to the Faculty and Board of Trustees of the Colorado School of Mines in partial fulfillment of the requirements for the degree of Master of Science (Mineral Economics).

Golden, Colorado

Date: 13 August 1984

Judith Ann Barlow

Judith Ann Barlow

Approved: Ruth A. Maurer

Dr. Ruth A. Maurer
Thesis Advisor

Golden, Colorado

Date: August 17, 1984

J. A. Cordes

Dr. John A. Cordes
Associate Professor and
Acting Head,
Mineral Economics Department

ABSTRACT

The weighted-centroid compromise programming algorithm is a new technique for the solution of multiple-criteria decision making (MCDM) problems. The method attempts to more closely simulate the human decision making process by incorporating numerous incommensurable evaluation criteria. It combines modelling and computational aspects of linear programming, goal programming and multiobjective linear programming. The algorithm evolved from an attempt to solve problems encountered in applying Milan Zeleny's displaced-ideal compromise method. The major theoretical difference between the two compromise programming algorithms lies in the definition of an equitable compromise. While Zeleny's method defines a compromise solution with respect to each objective's best case, the method presented here defines a compromise solution with respect to each objective's best and worst cases.

This thesis is the first attempt to apply the weighted-centroid compromise programming algorithm to small, but representative problems. In every case, the algorithm yielded a unique solution which was both efficient and equitable. Much further research, especially testing the method using larger, real-life problems, is needed.

TABLE OF CONTENTS

LIST OF TABLES AND FIGURES

List of Tables

List of Figures

ACKNOWLEDGMENTS

I dedicate this thesis to Professor Milan Zeleny of Columbia University and Dr. Ruth A. Maurer, my thesis advisor, mentor, and friend. Professor Zeleny's pioneering work first sparked my interest in multiple-criteria decision analysis and his assistance, both through his many excellent publications and personal communications, helped to guide my research along productive paths. Dr. Maurer's constant support, encouragement and solid advice saw this project to completion.

I would like to express my sincere appreciation for the advice and assistance given by my committee members Dr. R.E.D. Woolsey and Dr. Karen Wiley. I would also like to thank my collegues and friends at the Colorado School of Mines, many of whom applied their expertise and careful thought to the many seemingly insurmountable problems encountered in the course of this work.

Lastly, and mostly, I thank my son Eli for his constant and unquestioning love and support. His sacrifice was the greater.

Chapter 1

MCDM: A SYNTHESIS OF OLD TOOLS BRINGS A NEW AND BROADER PERSPECTIVE TO QUANTITATIVE DECISION MAKING

Increasing complexity in business decision making and advances in operations research methods and computer technology have all contributed to current interest in the multiple-criteria decision making (MCDM) problem. The purpose of this thesis is to present a new technique for solving MCDM problems, the weighted-centroid compromise programming algorithm. This algorithm was developed as a means of identifying an equitable and efficient compromise solution for the MCDM problem.

Chapter 1 presents the MCDM problem and current approaches to its solution. The focus of Chapter 2 is a literature review showing the historical development of the MCDM problem. Chapter 3 includes a presentation of the weighted centroid compromise programming approach to the multiple objective problem as well as an explanation of the algorithm. Chapter 4 will suggest mineral/energy and other problems suited to evaluation by multiple criteria decision techniques. Current solution techniques will be evaluated and compared with the solution method presented in Chapter 3. Chapter 5 is a summary and conclusions section, with potential applications and interpretations presented.

1.1 Introduction

Multiple-criteria decision making (MCDM) methods may offer the decision maker a new and broader perspective in quantitative decision making. Such methods can free the manager from the restrictions of decision making based on a single criterion and allow incorporation of numerous incommensurable criteria into the decision model. Because MCDM methods can simultaneously incorporate numerous incommensurable objectives and goals as criteria for decision making, they can be used to more closely simulate a complex human decision making process. MCDM methods are especially useful when the decision maker is faced with conflicting objectives and/or goals such as maximizing sales while minimizing the probability of incurring monopoly violations or maximizing both productive input efficiency and employee morale.

The MCDM algorithm presented in this thesis is a weighted-centroid variation of compromise programming. These concepts will be defined and explained in Chapter 3. The algorithm combines aspects of both linear programming and goal programming. When one individual criterion conflicts with another, each reaching an optimal value under mutually exclusive conditions (at different feasible region corner points or along different feasible edges), a compromise solution which is both efficient and equitable is identified. Efficiency is evaluated by both the mathematical criterion of nondominance and the economic criteria of Pareto optimality, efficiency and improvement. These evaluation criteria will be defined and explained in Chapter 2 of this thesis. Equity is evaluated in terms of the relative values assigned to each objective or class of commensurate evaluation criteria. An equitable solution is one in which the resource allocation is such that all objectives approach their individual optimal values at a rate proportionate to their relative, user-assigned weights. The algorithm presented here first identifies an equitable allocation of resources and then seeks to make improvements without violating the efficiency criterion.

1.2 Approaches to Decision Making

When a decision problem is uncomplicated enough for solution by simple deductive reasoning, the decision maker seeks to choose the alternative which most closely conforms to the "ideal" solution. The relative closeness to the ideal of any decision alternative is often not directly measurable in any standard units of measurement; but rather, is definable in numerous incommensurable units of measurement. This is the nature of the multiple-criteria decision problem.

In the course of routine day-to-day decision making, the proficient decision maker evaluates each decision alternative by a number of different criteria; however, many large-scale, high-risk, management science-assisted decision making techniques are evaluated by a single criterion. When the problem requires the simultaneous consideration of multiple-criteria, model builders have usually altered the problem to fit a known solution technique rather than designing a solution technique that fits the problem. While a single criterion optimization technique is often useful for evaluation purposes, it should be recognized that simple optimization is not necessarily sufficient or desirable.

Consider the case of a manager faced with the responsibility of choosing the best applicant for a job opening. The manager's objective is to choose the best applicant for the job; however, the criterion of the "best" applicant for the job implies no specific units of measurement. For evaluation purposes, the manager may formally or informally list the measurable criteria which best reflect the "ideal" employee. These measurable criteria may reflect a number of the manager's objectives such as (1) find an employee with the most appropriate level of knowledge and experience; (2) choose the employee who will best "fit" into the workplace and is most likely to stick with the job; (3) hire the applicant who best fits the company image and has the best previous work and health history, etc. It is clear that limiting the evaluation to only one of the above criteria would likely result in a sub-optimal outcome. Our manager is faced with a multiple-criteria decision problem.

1.2.1 Informal Approaches

There are a number of reasonable approaches to the employment decision problem. In certain uncomplicated circumstances, when the number of suitable applicants is small and the manager sufficiently understands the job requirements and the work environment, the manager may be able to make the "optimum" choice through a process of deductive reasoning. It should be noted that, through a sound deductive reasoning process, the manager considers the relative desirability of each applicant on the basis of numerous criteria. The human thought process can reflect a multiple-criteria decision technique.

1.2.2 Formal Approaches

More often, the employment decision problem requires a more standardized approach. The method should be such that any competent individual using the chosen decision analysis technique would reach the same conclusion as any other. A linear programming approach to this problem would likely include sets of constraints representing lower acceptable boundaries on each of the decision criteria and an objective function whose coefficients reflect the relative importance of each of the evaluation criteria.

An alternative goal programming approach would be to formulate a system of constraints which define the "ideal" employee. The objective function would be to minimize the weighted deviations from the ideal. While either of these approaches often yields satisfactory results, each has the potential of yielding sub-optimal results.

1.2.2.1 <u>Linear Programming</u>. In the linear programming case, maximizing the sum of positive attributes subject to minimum requirements, there are two potential problems. The minimum acceptable levels of measurement criteria such as years of experience are somewhat arbitrarily set at a given level. A prospective candidate who falls only slightly below any one element of the constraint set will immediately be rejected, even if all other measurement criteria indicate that this applicant would be a superior choice.

Another problem with the linear programmming approach is that of assigning the objective function coefficients. How does one add incommensurable units like years of experience and positive personality characteristics? While the linear programming proponent may argue that there are means of compensating for this problem, the astute decision maker should be suspicious of any method which attempts to combine incommensurable units of measurement. This is a case of modifying the problem to fit a known solution technique rather than designing a solution technique which accurately represents the problem.

1.2.2.2 <u>Goal</u> <u>Programming</u>. Goal programming (Zeleny, 1982) compensates for the problems associated with a traditional linear programming approach; however, the goal programming model brings with it a whole new set of problems. Because the different evaluation criteria have various levels of importance in the decision process, weights are assigned to the goals to reflect their relative importance. That is, first the highest priority goal is set as closely as possible to the target level and then each succeeding goal is met according to its relative weight. Although the decision maker can be assured of optimization with respect to the highest priority goal, there is no guarantee that the less important goals will ever be considered. For example, if level of knowledge is the highest priority goal in the employment decision problem, followed by years of experience and personality characteristics, the goal programming model will first limit the feasible solution set to all those applicants who either meet or exceed the "optimum" level of knowledge as defined by the model. It may be that this solution set contains only one of the applicants. Suppose further that this applicant is far below all other targeted goals and that there is at least one other applicant who comes close to meeting the highest priority goal and meets or exceeds the remaining lower priority goals.

The goal programming model will always yield the applicant which meets the highest priority goal. Although the algorithm often goes further than meeting the highest priority goal, the lowest priority goals are seldom considered at all. Most criticisms of goal programming relate to the relative weighting of targeted goals and the problem of satisficing rather than optimizing. Goal programming is often a pseudo optimization technique that is especially sensitive to relative data values and somewhat arbitrarily set target levels (Zeleny, 1982).

1.2.2.3 Multiple-Criteria Decision Making (MCDM) Methods. Multiple-criteria decision analysis offers a sensible solution to the problems associated with both the traditional linear and goal programming approaches as well as providing a means of solving certain problems which are not solvable using either of these solution techniques. The multiple-criteria decision technique presented in this thesis simultaneously seeks to optimize numerous objective functions while compensating for the fact that global optimization may be impossible. When objectives are in conflict with each other, compromises are made which reflect the relative importance of the stated objectives. The multiple-criteria compromise programming method in this thesis searches for the solution which is closest to the ideal while insuring that all evaluation criteria are considered.

The multiple-criteria solution method can be applied to the employment decision problem in such a manner as to simulate the manager's deductive reasoning process presented earlier. A separate objective function is formulated for each class of evaluation criteria so that the MCDM model becomes one of individually optimizing level of knowledge, positive personality characteristics, years of experience and/or any other criteria deemed relevant in the employment decision problem. If the individual evaluation criteria yield different optimal solutions, an equitable and efficient compromise is identified from the feasible solution set. This technique assures that all evaluation criteria are considered in the decision making process.

Perhaps an even more important feature of the multiple-criteria decision method is its ability to handle problems which cannot be formulated as either strict linear or goal programming models. Consider the following problem from Milan Zeleny's Multiple-Criteria Decision Making (page 214):

Abe Ream is the new manager of a federal funds disbursing system. He has to distribute funds to five political localities. For each federal dollar it receives, locality 1 will add $0.20 as its matching contribution. Localities 2 to 5 will add $0.25, $0.33, $0.40, and $0.50, respectively.

Mr. Ream manages a budget of $10 million. Since he would like to receive at least the same amount for the next year, he feels that he should spend at least 95% of this year's budget, or even exceed it by up to 20% if he can do so without getting into trouble.

Each of the localities faces a separate economic rate of return on funds committed to local projects. These rates are currently estimated to be 0.30, 0.25, 0.22, 0.20, and 0.15 for localities 1 to 5, respectively.

Mr. Ream would like to maximize the total return for every dollar invested. He is also under pressure to obtain the highest rate of return for each federal dollar invested in local projects. Political considerations also require that federal funds be spread more or less evenly among individual localities.

Finally, all funds distributed through this system must be audited. On the basis of historical records, Mr. Ream has estimated the following hours of audit time required for each $1000 invested in localities 1 to 5: 0.02, 0.03, 0.08, 0.10, and 0.12. Ream's audit staff has only 800 hours available per year.

There is one more aspect which Abe Ream cannot neglect: each locality contributes a different amount, and supporting such local activity is of importance. Some bonus for a locality's own contribution is in place, but this policy particularly conflicts with efforts for even distribution. Abe would like to explore this conflict.

Mr. Ream has established two maximization objectives: total dollar return and rate of return on federal dollar invested; five goals: $2 million for each of the five localities; and two minimization objectives: total sum of deviations over the five goals and/or the maximum of all deviations. There are also some constraints: four local support bonus constraints, two budget constraints, and an audit constraint. All functions are linear.

There are several different multiple-criteria optimization techniques that can be applied to this problem such as those suggested by Zionts and Wallenius (1983); Zeleny (1974 & 1982), and Evans and Steuer (1973). Although the problem formulation for each of these methods is similar, the algorithms for the determination of the optimum solution or optimum solution sets are quite different. The problem formulation for Mr. Ream's cash disbursement problem and the optimal solution using the weighted centroid compromise programming algorithm, to be developed below, is included in Chapter 4 of this thesis.

1.3 An MCDM Solution Method

The multiple-criteria solution method presented in this thesis first identifies the corner points which represent optimum levels for each of the objective functions in the problem. The solution method then finds the "centroid" defined by these optimum corner points. The "weighted-centroid" is the point which minimizes the weighted distance from each individual objective's optimum point and represents an equitable allocation of resources. The manager or investigator may assign different weights to the various objective functions, and the effect of these different weights is to pull the centroid point proportionately in the direction representing the relative values of the stated objectives.

The weighted-centroid is then tested for efficiency. Efficiency is evaluated according to both the mathematical criterion of nondominance and the economic criteria of Pareto optimality, efficiency and improvement.

When the weighted-centroid represents an inefficient allocation of resources, an efficient solution which maintains the equity conditions of the weighted-centroid is identified. The criteria for optimality for the MCDM weighted centroid compromise programming algorithm are efficiency with respect to the constraint set and equity with respect to the objective function set.

1.4 The MCDM Compromise Model

The weighted-centroid approach to compromise programming uses a multiple objective/multiple goal problem formulation:

$$\text{OPTIMIZE } Z_h = \sum_{j=1}^{n} (C_{hj} X_j + C_{hk} S_j) \text{ ,for } h = 1,2,\ldots,p$$

such that:

$$\sum_{i=1}^{j} (a_{ij}X_j + a_{ik}S_j) = b_i, \text{ for } i=1,2,\ldots,m$$

and $X_j \geq 0$ for all j;

where p is the number of objective functions, m is the number of constraints, n is the number of decision variables and k = j + n.

C_{hj} cost coefficient on the j-th decision variable in the h-th objective function

X_j the j-th decision variable

C_{hk} cost coefficient on the (k-n)th slack/surplus variable in the h-th objective function

S_j the slack or surplus variable associated with the j-th constraint

a_{ij} the coefficient on the j-th decision variable in the i-th constraint

a_{ik} the coefficient on the (k-n)th slack/surplus variable in the i-th constraint

b_i the Right-Hand-Side value on the i-th constraint

The model differs from the classical linear programming and goal programming models in that it allows for multiple objective functions which may represent either cost/profit functions or functions representing deviations from goals or some combination of the two. Each objective function, however, faces the same set of constraining equations. The constraint set defines the universe of possibilities or feasible region for the problem. Linear programming involves the optimization of a cost/profit vector (c-vector) with respect to a constraint set (b-vector and a-matrix). The MCDM presented here involves "optimization" of a c-matrix composed of multiple cost/profit vectors with respect to the linear programming constraint set.

The slack and surplus variables in the multiple objective problem formulation can be interpreted in several ways. They can represent slack or surplus variables as in a traditional linear programming formulation, or they may represent deviations from targeted goals for those elements in the b-vector which represent goals. A goal programming objective function would have a non-zero cost coefficient on all slack and surplus variables which represent deviations from corresponding b-vector goals; however, the cost coefficients on the decision variables are zero. The linear programming objectives contain non-zero cost coefficients on decision variables and all cost coefficients on slack and surplus variables are zero.

Chapter 2

HISTORICAL DEVELOPMENT OF THE MCDM PROBLEM

The weighted-centroid compromise programming algorithm, presented in detail in chapter 3 of this thesis, has historical and theoretical foundations in mathematics, economics and operations research. This chapter includes a presentation of the historical development of the multiple-criteria decision problem and approaches to its solution.

The concepts of Pareto optimality and mathematical nondominance presented in this chapter provide the theoretical foundation for the algorithm developed in the next chapter of this thesis. Also included in this chapter is a brief review of the development of operations research techniques of goal programming, linear multiobjective programming and compromise programming. This chapter concludes with Milan Zeleny's "displaced-ideal" approach to compromise programming. The similarities and differences between the weighted-centroid and displaced-ideal approaches to compromise programming will be evaluated in chapter 5 of this thesis.

2.1 MCDM Models in Economics

The earliest reference to difficulties encountered in solving the multiple objective problem appeared in 1896 (Pareto, 1906). Pareto recognized the problem of formulating a single optimality criterion from a set of incommensurable criteria. Acknowledging the incommensurable units in different individuals' utility functions, Pareto defined a means of identifying when equilibrium is reached among a group of consumers. Given a finite amount of desired resources and any arbitrary allocation of those resources among the consumers, the allocation can be tested for economic efficiency. Further, if the current allocation is inefficient, a set of Pareto improvements can be identified with respect to the current resource mix. The Pareto optimal solution set contains an infinite number of nondominated solutions. A solution is nondominated if no one consumer's utility function value can be increased without causing a decrease in the value of at least one other individual's utility function.

The Pareto optimality principle is one of the cornerstones of traditional economic theory. The Pareto Principle defines a state of nature A which is preferred to state of nature B if at least one individual is better off in A and there are no losers in a move from state of nature B to A. This concept of Pareto optimality may be used to demonstrate the theoretical workings of the free enterprise system and is further used to explain the rudiments of consumer behavior in market exchange.

The Pareto principle is often demonstrated using a simple two-person, two-good world. Each individual's objective is to maximize utility subject to constraining conditions. The constraining conditions are the availability of each of the two goods.

Each of the two individuals is assigned an objective function which represents the utility derived from consuming each of the two available goods. The constraint set includes two constraining equations: one representing the total availability of each of the two goods. Economic efficiency will be realized when the available resources are fully utilized.

Economists are quick to note that the Pareto optimality principle has nothing to contribute to the equity question. This is because positive economics does not allow for judgments regarding fairness in the allocation of scarce resources. Using the concept of Pareto optimality, an infinite set of efficient or nondominated solutions is defined along the contract curve. Although we cannot choose one alternative on the contract curve above another, if we are given the initial or current allocation of resources, we can test this mix for economic efficiency. If this point does not lie on the contract curve, we can be assured that a more efficient allocation of resources is possible. The economist would identify the point on the contract curve which is nearest the current allocation as an efficient Pareto improvement.

Pareto's efficiency criteria were extended to production by the 1975 Nobel prize winner in economics, T. C. Koopmans (1951). Productive efficiency is achieved along the production-possibilities frontier. If each production commodity is assigned an objective function reflecting its relative value, the efficient points corresponding to the production possibility frontier lie in the region whose corner points are the collection of each objective function's optimal corner point subject to a feasible region bounded by resource constraints.

Both Pareto and Koopmans presented their efficiency criteria as macroeconomic cases. Because positive economics does not allow for moral judgments, the optimum solution was represented by an infinitely large solution set. In the microeconomic case of the individual firm, however, relative valuation of the firm's goals and objectives may be intuitively apparent or empirically available. Relative weights may be assigned to each of the firm's objectives or evaluation criteria. Through the valuation process, the efficiency criteria of Pareto and Koopmans can be used to narrow the feasible solution set to a single optimal solution. This process will be demonstrated in Chapter 3 of this thesis.

2.2 MCDM in Operations Research

The multiple objective problem was a popular topic of theoretical debate in operations research literature throughout the 1950's (Zeleny, 1982). In 1961, Charnes and Cooper presented the "spiral method", a direct extension of Koopmans' ideas, in Volume I of their two volume set, Management Models and Industrial Applications of Linear Programming (1961). The spiral method offered a means of identifying all "efficient" solutions to a linear programming problem. It was treated neither prominently nor extensively by the authors.

Neither Charnes nor Cooper explored the multiple criteria implications of the spiral method. Their goal programming model, however, served as an early prototype for MCDM in that it was the first documented operations research method which allowed for multiple decision criteria (Charnes and Cooper, 1961).

2.2.1 Goal Programming

Goal programming models are formulated much as single objective linear programming problems. Both models' feasible regions are defined by a set of constraining equations which define the universe of possibilities for the problem. While linear programming algorithms search for an optimal value for a cost or profit function, goal programming models' objective function is to optimize the deviation from targeted goals. While the slack and surplus variables in a linear programming model have zero cost/profit coefficients in the objective function, the slack and surplus variables in the strict goal programming model are the only variables with nonzero cost/profit coefficients. The objective of goal programming is to minimize and/or maximize user-specified slack and/or surplus variables' optimum values.

Numerous variations in the approaches to and applications of goal programming have appeared in operations research literature since the technique was first described by Charnes and Cooper (1961). Although goal programming was originally based on Archimedian weights representing relative differential contributions of each class of goals, the use of pre-emptive weights was explored during the 1960's as a means of compensating for incommensurable units of measurement (Ijiri, 1965; Charnes, 1967). Pre-emptive weights are priority weights which determine the hierarchy of the goals. Goals of higher priority levels are satisfied first and only then may the lower priority goals be considered. Lower priority goals cannot alter the goal attainment of higher priority levels.

Pre-emptive goal programming frees the decision maker from the difficulties encountered in assigning meaningful Archimedian weights by requiring only that the decision maker rank the various decision criteria in order of importance. The pre-emptive version of goal programming dominated the 1970's (Lee, 1972). Today practically all references to goal programming refer to the pre-emptive weighting method.

As goal programming models became more complex, the limitations of somewhat subjectively assigned cost/profit coefficients or pre-emptive weights as the sole criterion for decision analysis were seriously questioned (Zeleny, 1982). The majority of the criticism directed at goal programming is related to the technique's tendency to satisfice rather than optimize.

More recently, goal programming ideas have been incorporated as a subset of multiple-criteria decision analysis. Goal attainment can be part of the objective function set in linear multiobjective programming.

2.2.2 Linear Multiobjective Programming

Linear multiobjective programming evolved from early linear programming work by Dantzig. Although there are several different algorithms and some accompanying computer codes for the solution of multiobjective problems (Evans and Steuer, 1973; Zeleny, 1975), the algorithms all yield an infinitely large nondominated solution set. Significant variations in current literature pertain primarily to how the algorithms deal with incommensurable units of measurement and/or if they search the nondominated feasible set for a "best" solution alternative.

Zeleny presents a multiobjective programming algorithm which identifies representative members of the nondominated solution set with respect to multiple objectives and constraints (1974). In his 1982 book, Zeleny acknowledges the problem of presenting the decision maker with a fair representation of an infinitely large nondominated solution set.

2.2.3 Multiparametric Decomposition

Multiparametric decomposition provides a means of narrowing the solution set of the multiobjective programming model. A utility function representing a composite of the multiple objectives and goals is derived, usually interactively with the decision maker.

Multiparametric decomposition techniques seek an *a priori* relative valuation of the different units of measurement. The multiple linear objectives are combined into one representative non-linear objective function. Zionts and Wallenius (1983) provide an interactive algorithm which searches the solution set for optimality after deriving a convex, non-linear utility function.

2.2.4 Compromise Programming

Milan Zeleny introduced the concept of compromise programming in 1973. Compromise programming, according to Zeleny, is based on the idea that the firm strives to achieve a currently unattainable goal. This long range goal, or "displaced ideal", is defined as the intersection of the objective functions at their individual optimum feasible region corner points. It is assumed that management, acknowledging its inability to achieve the ideal in the short run, will instead attempt to operate as close to the ideal as is feasible given its constraining resources. The optimum solution is found at the point on the edge of the feasible region which is nearest to the ideal. An alternative, but not necessarily equivalent approach to compromise programming identifies an "anti-ideal" and searches for the feasible region edge point which is farthest from the anti-ideal (Zeleny, 1982).

Zeleny's displaced ideal approach to compromise programming involves the identification of a nondominated solution or set of nondominated solutions. A solution is nondominated if there exists no other solution which yields a better value for at least one objective function without adversely affecting any of the remaining objectives' values. Zeleny defines nondominance (1982, page 69):

It is useful to express nondominance in terms of a simple vector comparison. Let x and y be two vectors of n components, $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ respectively.

Thus,

$$x = (x_1,\ldots,x_n) \text{ and } y = (y_1,\ldots,y_n).$$

We say that x dominates y if

$$x_i \geq y_i, \quad i = 1,\ldots,n$$

and $x_i > y_i$ for at least one i.

We may compare x and y directly and say that x dominates y if $x \geq y$ and $x \neq y$ (Zeleny, 1982).

Both Zeleny's and the compromise programming method presented here often yield similar results and both rely on the concept of nondominance; however, the intuitive appeal and methodology of each is quite different. While Zeleny's method moves from the super-optimal, but infeasible displaced ideal, the method presented here moves from the feasible, but usually sub-optimal weighted-centroid point.

Zeleny stresses the intuitive appeal of the displaced ideal. A super-optimal yet currently unattainable point such as the displaced ideal is consistent with progress-oriented management decision making. The identification of this point in terms of the decision variables' values and the constraints which prohibit its attainment may provide valuable information for far-sighted managers and aid in long-range planning.

The method presented here has an important economic appeal. The similarities between Zeleny's definition of nondominance and Pareto's optimality criteria are incorporated in the evaluation.

Because compromise programming is a fairly recently proposed decision analysis method, little has been written on the subject. Its application poses some interesting questions. What happens when there are conflicting parallel objectives or if objectives intersect where one or more decision variables take on an unrealistic negative value? Can the decision maker justify an ideal which lies somewhere within the interior of the feasible region where none of the constraints are binding? The implications of these and other aspects of a weighted-centroid extension of compromise programming will be explored in the next chapter of this thesis.

Chapter 3

A WEIGHTED-CENTROID APPROACH TO COMPROMISE PROGRAMMING

The weighted-centroid compromise programming method is presented in this chapter from three different perspectives. First the theoretical justification for the algorithm is presented. This is followed by the mechanics of the five-step algorithm. This chapter concludes with representitive two-dimensional cases of the weighted-centroid compromise programming technique, presented both graphically and numerically.

3.1 The Compromise Programming Method

The compromise programming algorithm presented here was developed to handle problems where evaluation criteria are expressed in incommensurable units of measurement such as nearness to targeted goals for goal programming objective functions and profit and/or utility measures for linear programming objective functions. Rather than seeking an *a priori* relative valuation of the incommensurable evaluation criteria such as dollars/util and utils/goal category, this algorithm first identifies the optimal allocation of resources for each of the individual objective functions.

Often different units of measurement are "incommensurable" in the linear programming case simply because of a non-linear relationship between the different evaluation criteria as is the case in the measurement of utility. Further, the relative valuation is not a constant one, and varies not only from one application to another, but also among different individuals' perceptions of the same application. By examining each linear objective function individually and identifying the set of optimal conditions associated with each of the problem objective functions, the algorithm presented here avoids combining the different units of measurement into a single, non-linear objective function as required for multiparametric decomposition techniques.

If all problem objectives achieve optimality under the same conditions, no compromise is necessary. A global optimal solution and the corresponding decision variable values along with standard linear programming information is available to the user.

When two or more of the problem objective functions reach optimality at different corner points or along different feasible region edges, an equitable compromise is identified. An equitable compromise is one in which each of the problem objectives approach optimality at a rate proportionate to its user-assigned weights.

Although this algorithm does not seek an a priori means of combining the incommensurable units of measurement, weights reflecting each objective's relative importance to the user may be assigned to each of the individual objective functions. Unlike the weights in multiple goal programming which dictate only the order in which the problem objectives are satisfied, the weighting method here is used to determine an equitable compromise only after each objective function has been satisfied in an optimal, but perhaps jointly infeasible manner. The weighted-centroid is a feasible solution which represents an equitable trade-off in the level of optimality of each objective function with respect to both the user-supplied weights and each objective's best and worst cases.

The weighted-centroid is defined as the point in n-space, where n is the number of decision variables, at which equilibrium is reached when the specified weights on each of the objectives is exerted as a force pulling the centroid point in the direction of its optimum point or plane. Because the coordinate system in the linear programming model represents the values of the decision variables, the weighted average of these decision variables with respect to the objectives' optimum points defines the weighted-centroid point. It is from this weighted-centroid point that the search for improvement begins.

The weighted-centroid point represents an equitable allocation of resources in that objectives approach optimality at a rate proportionate to their relative user-assigned weights. The user-assigned weight is assumed to be equivalent to the relative importance of each objective function in the original problem. The search for a better allocation, while maintaining the equity conditions of the weighted-centroid point, begins by testing for nondominance or Pareto optimality. If we can identify a feasible region point at which no objective function would decrease in optimality and at least one objective function would become more nearly optimal (relative to the weighted-centroid point), then the weighted-centroid point is dominated or Pareto inefficient.

The weighted-centroid point fails the economic efficiency test when it lies within the interior of the feasible region. Only when the weighted-centroid lies on a feasible region edge will it represent an optimal allocation of resources with respect to the multiple objectives and/or goals. This is because any point within the feasible region and not on the feasible region edge results in the under-utilization of all resources in the problem. In single objective linear programming and multiple criteria compromise programming, as well as in the economic allocation of scarce resources, a necessary condition for optimality and efficiency is that at least one constraint be binding. At points within the interior of the feasible region, there are no real constraints in that all of the constraining equations are non-binding.

Since at least one constraint must be binding for the necessary condition of nondominance/Pareto optimality to be met, the search for an equitable improvement begins by narrowing the original problem's feasible region to those points which represent Pareto improvement with respect to the weighted-centroid. A Pareto improvement, with respect to the weighted-centroid, would represent an improvement in the value of at least one of the problem objective functions without adversely affecting any of the other objective values. Possible candidates for Pareto improvement are defined by the following criterion: Each candidate must result in the improvement of at least one problem objective funtion (with respect to the weighted-centroid) without adversely affecting the values of any of the other problem objectives. This criterion can be assured by adding additional constraints to the original problem constraint set. For each problem objective, a constraint is defined which requires that the objective function achieve at least the same level of optimality as it would at the weighted-centroid. This assures that all points evaluated meet at least the minimal requirements for Pareto improvement.

Using the original problem's constraint set and the new constraints requiring Pareto improvement, a new problem is formulated using a goal programming model. The goals in the model are to achieve each of the original individual objective's optimum values. The goal objective function is one of minimizing the sum of the deviations from the optimal, but jointly infeasible goals. Because of the nature of the incommensurable units of measurement in the different original objective functions of the MCDM problem, the compromise solution must reflect the fact that there can be no compensation among the criteria. That is, a low value in one objective cannot be offset by a higher value in another objective. This can be accomplished by minimizing the maximum of the deviations from the goals rather than minimizing the sum of their deviations.

It can be shown that the maximum deviation reaches its minimum at the point where the deviations are equal to one another (Zeleny, 1982). Goal deviations are set at a minimum by adding (h-1) constraints, where h is the number of original problem objective functions, which assure that each objective's improvement from the weighted centroid to the compromise solution is at the same rate as the objective's sacrifice in going from its individual optimal value to the weighted centroid.

3.2 The Weighted Centroid Compromise Programming Algorithm

The steps in the weighted-centroid compromise programming algorithm are as follows:

0- Define a multiple objective/multiple goal linear programming model.

The feasible region applies to all objective functions and represents the realm of possibilities. The objectives are the competing and/or complementary evaluation criteria by which the alternatives are evaluated for comparison.

1- Identify the initial individual optima: the feasible region corner points and/or edges at which each of the objectives reaches its optimum value.

These values are found by successive calls to a set of linear programming subroutines; one call for each objective function.

2- If all objective functions are optimal at the same feasible region corner point or edge, a global optimum has been identified and the algorithm stops.

If all objective functions are optimal at the same point or along the same plane (or hyperplane), then there is no reason for compromise. The optimal, Pareto-efficient, nondominated solution or solution set has been identified.

3- If the individual objective functions are optimal at two or more different points, the weighted-centroid is identified.

The weighted-centroid is calculated by multiplying the normalized weight for each objective function by each of the corresponding optimal decision variable values for that objective.

4- Test the weighted-centroid values for necessary condition for Pareto optimality/nondominance by identifying all binding constraints, if any exist.

If an evaluation of the weighted-centroid point shows that a constraint is binding, then the initial optimal objective coordinates were at adjacent feasible region corner points. If the decision maker were indifferent as to the relative values or weights on the objectives, the entire feasible region edge extending between the corner points would be nondominated.

5- If no constraints are binding with reference to the weighted-centroid, reformulate the problem as a goal programming model.

If there are no binding constraints at the weighted centroid point, the economic efficiency criterion has not been satisfied. The problem is reformulated using the weighted-centroid as a reference point. The reformulation is a goal programming problem where the goals are the attainment of each of the individual, jointly-infeasible optimal values for each of the original problem objective functions.

When weights are assigned to the original problem objectives, these weights are normalized and are used to assure the each objective's compromise value reflects its relative importance. Constraints are formulated which assure that each objective is met according to the weighted value of the attainable level of its objective function.

The goal reformulation is as follows:

$$\text{MINIMIZE} \sum_{h=1}^{p} d_h$$

Such that:

$$\sum_{j=1}^{n} C_{hj}X_j + d_h = Z_h^*, \text{ for h = all maximization objectives,}$$

$$\sum_{j=1}^{n} C_{hj}X_j - d_h = Z_h^*, \text{ for h = all minimization objectives,}$$

$$\sum_{j=1}^{n} C_{hj}X_j \geq Z_h^{wc}, \text{ for } h = \text{all maximization objectives,}$$

$$\sum_{j=1}^{n} C_{hj}X_j \leq Z_h^{wc}, \text{ for } h = \text{all minimization objectives,}$$

$$R_h d_1 - R_1 d_h = 0, \text{ for } h = 2 \text{ to } p;$$

where p is the number of original problem objective functions, n is the number of original problem decision variables and $R_h = |Z_h^* - Z_h^{wc}|$.

The d's to be minimized in the objective function are the amount of deviation from the optimal levels for each of the h original problem objective functions. The first constraint set contains the goal constraints which define the target level for the problem goals:

$$\sum_{j=1}^{n} C_{hj}X_j \pm d_h = Z_h^*$$

The deviations have a +1 coefficient for all maximization objectives and a -1 coefficient on all minimization objectives. The target level for each objective is its individual optimal value.

The second constraint set assures that the solution will result in an economically efficient improvement with respect to the weighted-centroid by requiring that each of the original problem objectives meets the minimal requirement for Pareto improvement:

$$\sum_{j=1}^{n} C_{hj}X_j \geq Z_h^{wc}, \text{ for } h = \text{all maximization objectives and}$$

$$\sum_{j=1}^{n} C_{hj}X_j \leq Z_h^{wc}, \text{ for } h = \text{all minimization objectives.}$$

The third constraint set assures that the incommensurable units of measurement in each of the problem objectives are not directly combined:

$$R_h d_1 - R_1 d_h = 0, \text{ for } h = 2 \text{ to } p.$$

The R-coefficient on the deviations is used to reflect the potential range of variation for the h-th objective function. Each objective function's R value is the absolute value of the difference between its value at optimality and its value at the weighted centroid. The R-coefficient on the deviations assures that each objective improves at a rate which reflects the relative value of its individual optimal solution. The third constraint set's effect is to assure that each objective approaches optimality at a rate proportionate to its weighted potential:

$$\frac{Z_1^* - Z_1^c}{R_1} = \frac{Z_h^* - Z_h^c}{R_h}, \text{ for all h.}$$

The ratios of actual variation in an objective's value to its potential variation are equal for all problem objectives.

The solution of the goal programming reformulation will yield an equitable and efficient compromise for the original multiple objective/multiple goal problem. The solution is efficient because it lies on a feasible region edge where at least one constraint prevents further optimization. The solution is equitable because only candidates for Pareto improvement with respect to the equitable-by-definition weighted-centroid point are feasible in the goal reformulation.

3.3 A Two-Dimensional Graphical Example

The following two-dimensional cases were developed to demonstrate the weighted-centroid compromise programmming algorithm under various conditions. A set of seven constraining equations which defines a seven-sided feasible region in two dimensions is listed in the large problem below. The entire set of constraints defines the feasible universe for a set of multiple objective sample problems. The sample problems include representive combinations of the eight objective functions listed in the large problem below.

The large problem formulation is

$$\text{MAX } Z_1 = 2X_1 + X_2$$

$$\text{MAX } Z_{1A} = X_1$$

$$\text{MAX } Z_2 = X_1 + X_2$$

$$\text{MAX } Z_3 = X_1 + 10X_2$$

$$\text{MAX } Z_4 = -1X_1 + X_2$$

$$\text{MIN } Z_5 = 1X_1$$

$$\text{MIN } Z_6 = 1X_1 + 1X_2$$

$$\text{MAX } Z_7 = 1X_1 - 1X_2$$

Such that:

$$4X_1 + 3X_2 \leq 132$$

$$1X_1 + 7X_2 \leq 158$$

$$-3X_1 + 7X_2 \leq 114$$

$$-3X_1 + 1X_2 \leq 6$$

$$24X_1 + 4X_2 \geq 96$$

$$X_1 + 11X_2 \geq 38$$

$$5X_1 - 4X_2 \leq 72$$

Each of the eight objective functions is optimal at a different feasible region corner point, with the exception of the first two objectives, Z_1 and Z_{1A}, which are both optimal at the same corner point. The first objective function above (Z_1) will be evaluated jointly with the second through sixth objectives using the weighted-centroid compromise programming algorithm.

3.3.1 Case 1: A Global Optimal Solution

Consider the joint optimization of objectives 1 and 1A, the first two objectives in the large problem above. See Figure 1.

$$\text{MAX } Z_1 = 2X_1 + X_2 \text{ and}$$

$$\text{MAX } Z_{1A} = X_1$$

The joint optimization is subject to the entire constraint set:

$$4X_1 + 3X_2 \leq 132$$

$$1X_1 + 7X_2 \leq 158$$

$$-3X_1 + 7X_2 \leq 114$$

$$-3X_1 + 1X_2 \leq 6$$

$$24X_1 + 4X_2 \geq 96$$

$$X_1 + 11X_2 \geq 38$$

$$5X_1 - 4X_2 \leq 72$$

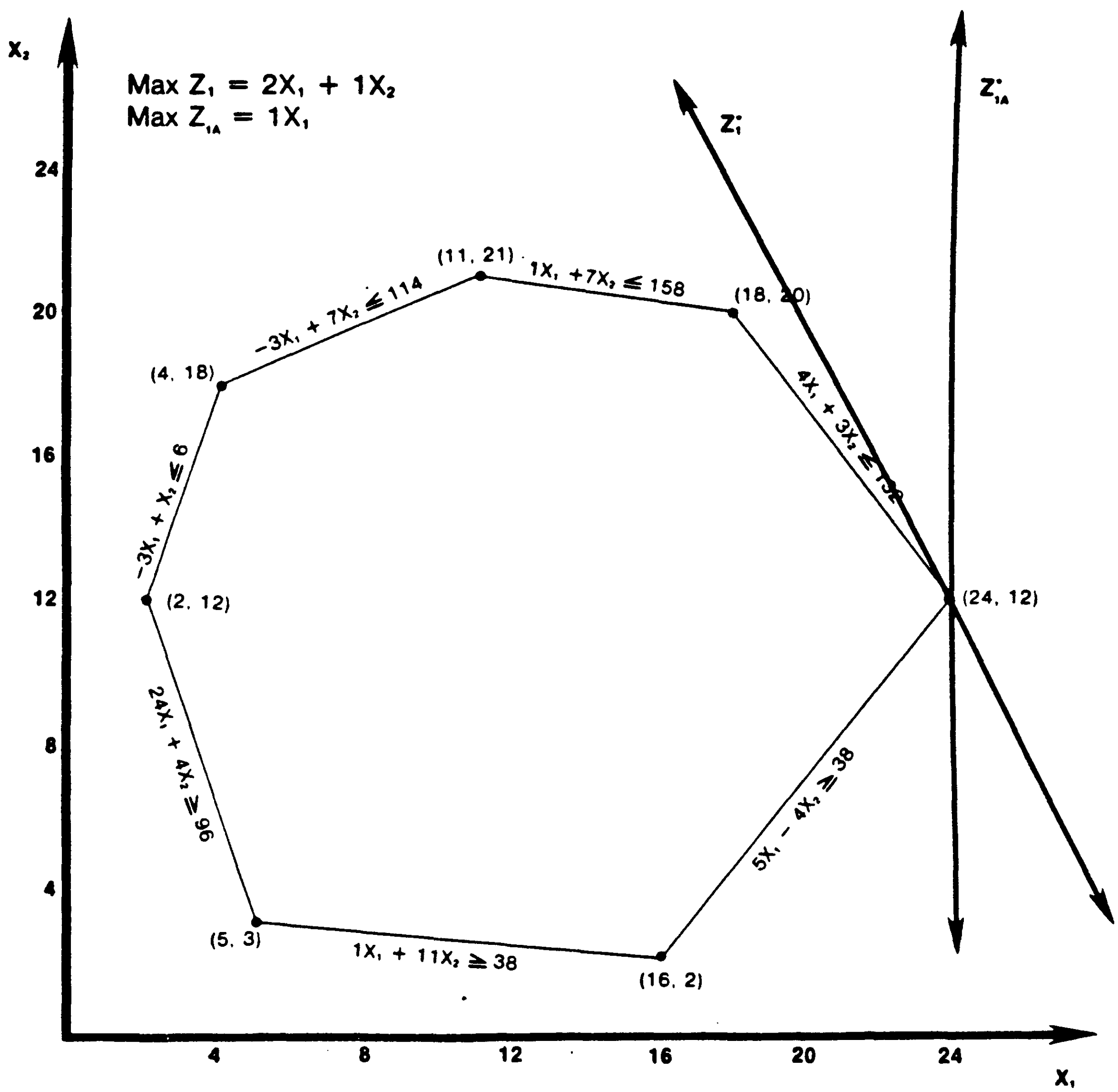

Figure 1

A Global Optimal Solution

Objective 1 achieves optimality at feasible region corner point (24,12) where its value is 60. Objective 1A also reaches optimality at the corner point (24,12) where its value is 24. See Figure 1.

Since both objectives are optimal under the same resource allocation, no compromise is necessary and a global optimal solution can be reported:

$$X_1^*=24,\ X_2^*=12,\ Z_1^c = Z_1^*,\ Z_2^c= Z_2^*.$$

3.3.2 Case 2: Adjacent Optimal Solutions

Consider the case of attempting to optimize both objectives 1 and 2 above. See Figure 2.

$$\text{MAX } Z_1 = 2X_1 + X_2 \text{ and}$$

$$\text{MAX } Z_2 = X_1 + X_2$$

The two objectives are jointly evaluated using the entire constraint set presented earlier. While objective 1 reaches optimality at corner point (24,12), objective 2 reaches optimality at corner point (18,20). See Figure 2. Objective one's optimal value is 60 and objective two's optimal value is 38.

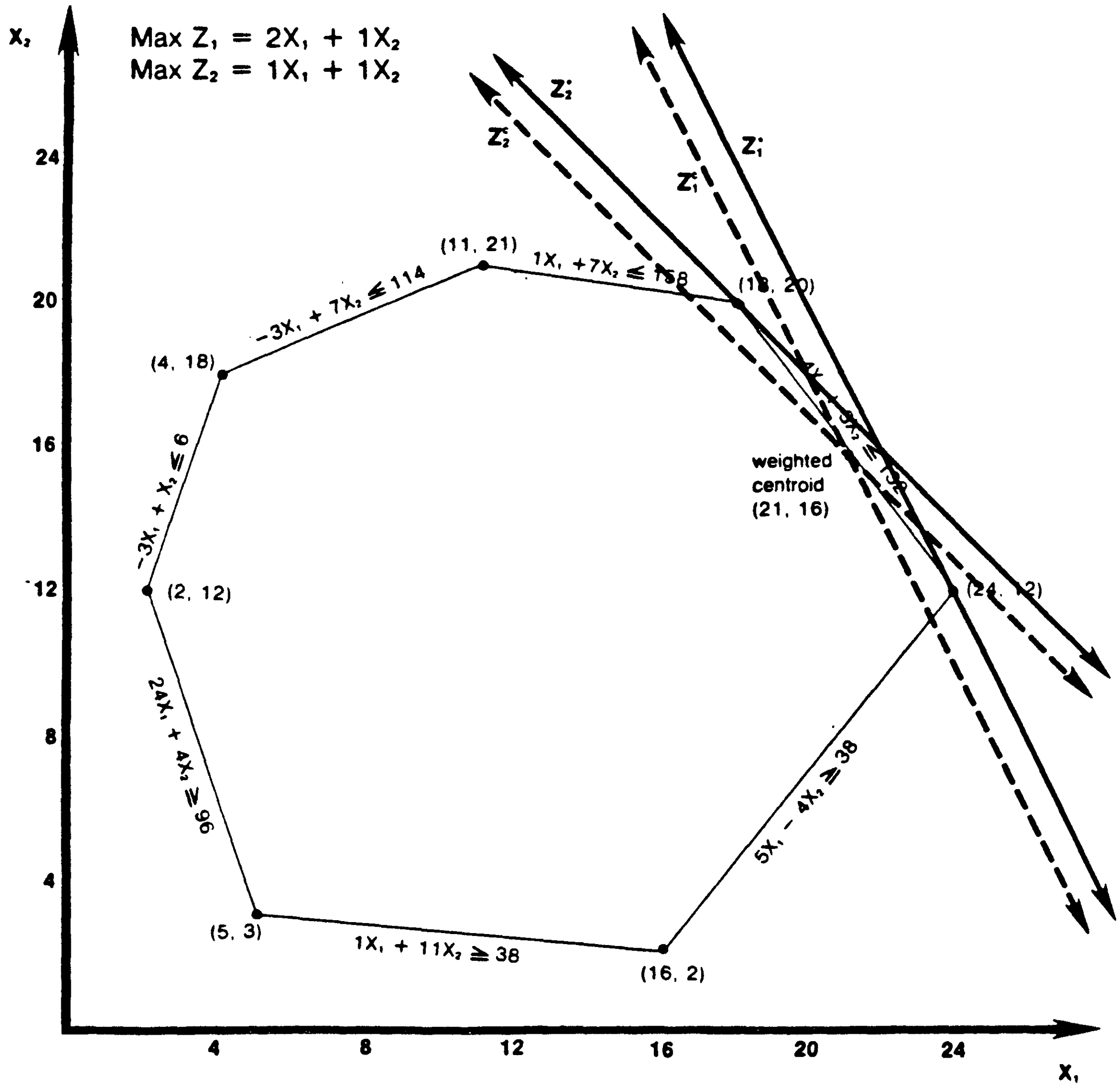

Figure 2

Adjacent Optimal Solutions

Since the two objectives reach optimality at different corner points, the weighted-centroid is calculated as directed in step three of the algorithm. If each of the two objectives is assigned equal weight, then the weighted-centroid is represented by the coordinate pair (21,16). The weighted-centroid value for the first decision variable is calculated (.5)(24) + (.5)(18) = 21. The equal weights here are normalized as directed in in algorithm. The value of the second decision variable at the weighted-centroid is calculated using the same weighting relationship: (.5)(12) + (.5)(20) = 16. Here the value of objective 1 is 58 or 96.7% of its optimal level and the value of objective 2 is 37 or 97.4% or its optimal level.

The weighted-centroid is then evaluated to see if it meets the criterion for Pareto optimality/mathematical nondominance by identifying all binding constraints if any exist. In this case, constraint number 1

$$4X_1 + 3X_2 \leq 132$$

is binding at the weighted centroid.

The algorithm stops. An efficient and equitable compromise solution has been identified where

$X_1^c=21$, $X_2^c=16$, $Z_1^c=58$, and $Z_2^c=37$.

3.3.3 Case 3: Non-Adjacent Optimal Solutions I

Consider the case of attempting to optimize both objectives 1 and 3. See Figure 3.

$\text{MAX } Z_1 = 2X_1 + X_2$ and

$\text{MAX } Z_3 = X_1 + 10X_2$

The joint optimization of these two objective functions is subject to the entire set of seven constraints from the large problem presented earlier.

While objective 1 reaches optimality at corner point (24,12), objective 3 reaches optimality at corner point (11,21). See Figure 3. Objective one's optimal value is 60 and objective three's optimal value is 221.

Since the two objectives reach optimality at different corner points, the weighted-centroid is calculated as follows:

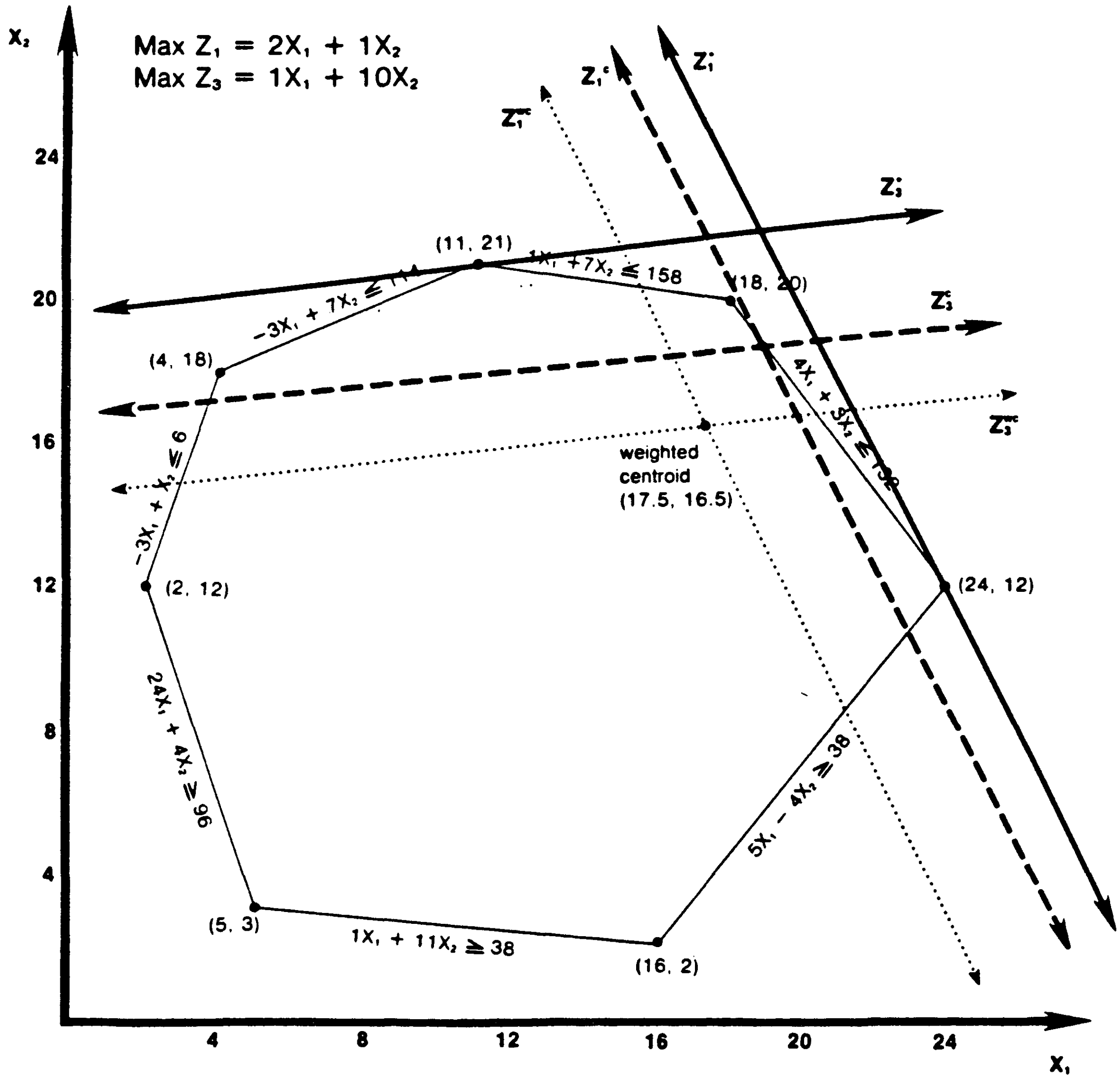

Figure 3

Non-Adjacent Optimal Solutions I

$X_1 = (.5)(24) + (.5)(11)$

$= 17.5$ and

$X_2 = (.5)(12) + (.5)(21)$

$= 16.5.$

The weighted-centroid is represented by the coordinate pair (17.5,16.5) where the value of objective 1 is 51.5 and the value of objective 3 is 182.5.

The weighted-centroid is then evaluated to see if it meets the criterion for Pareto optimality/mathematical nondominance by identifying all binding constraints if any exist. In this case, no constraints are binding at the weighted centroid.

The problem is reformulated as a goal programming model. The goals in the formulation are the simultaneous attainment of the mutually exclusive optimal levels for objectives 1 and 3 as directed in step 5 of the algorithm:

MINIMIZE $d_1 + d_3$

Such that:

$$2X_1 + X_2 + d_1 = 60$$

$$X_1+10X_2 \qquad + d_3=221$$

$$2X_1 + X_2 \qquad \geq 51.5$$

$$X_1 \quad 10X_2 \qquad \geq 182.5$$

$$38.5d_1 - 8.5d_3 = 0$$

$$4X_1 + 3X_2 \qquad \leq 132$$

$$1X_1 + 7X_2 \qquad \leq 158$$

$$-3X_1 + 7X_2 \qquad \leq 114$$

$$-3X_1 + 1X_2 \qquad \leq 6$$

$$24X_1 + 4X_2 \qquad \geq 96$$

$$X_1 + 11X_2 \qquad \geq 38$$

$$5X_1 - 4X_2 \qquad \leq 72$$

The solution of the goal programming reformulation yields a compromise solution where:

$X_1^c=18.985$, $X_2^c=18.687$, $Z_1^c=56.667$, and $Z_3^c=205.855$.

The binding constraint here is

$4X_1 + 3X_2 \leq 132$.

3.3.4 Case 4: Non-Adjacent Optimal Solutions II

Consider the case of attempting to optimize both objectives 1 and 4 above. See Figure 4.

$$MAX\ Z_1 = 2X_1 + X_2 \text{ and}$$

$$MAX\ Z_4 = -1X_1 + X_2$$

While objective 1 reaches optimality at corner point (24,12), objective 4 reaches optimality at corner point (4,18). See Figure 4. Objective one's optimal value is 60 and objective four's optimal value is 14.

Since the two objectives reach optimality at different corner points, the weighted-centroid is calculated as directed in step three of the algorithm. If each of the two objectives is assigned equal weight, then the weighted-centroid is represented by the coordinate pair (14,15) where the value of objective 1 is 43 and the value of objective 4 is 1.

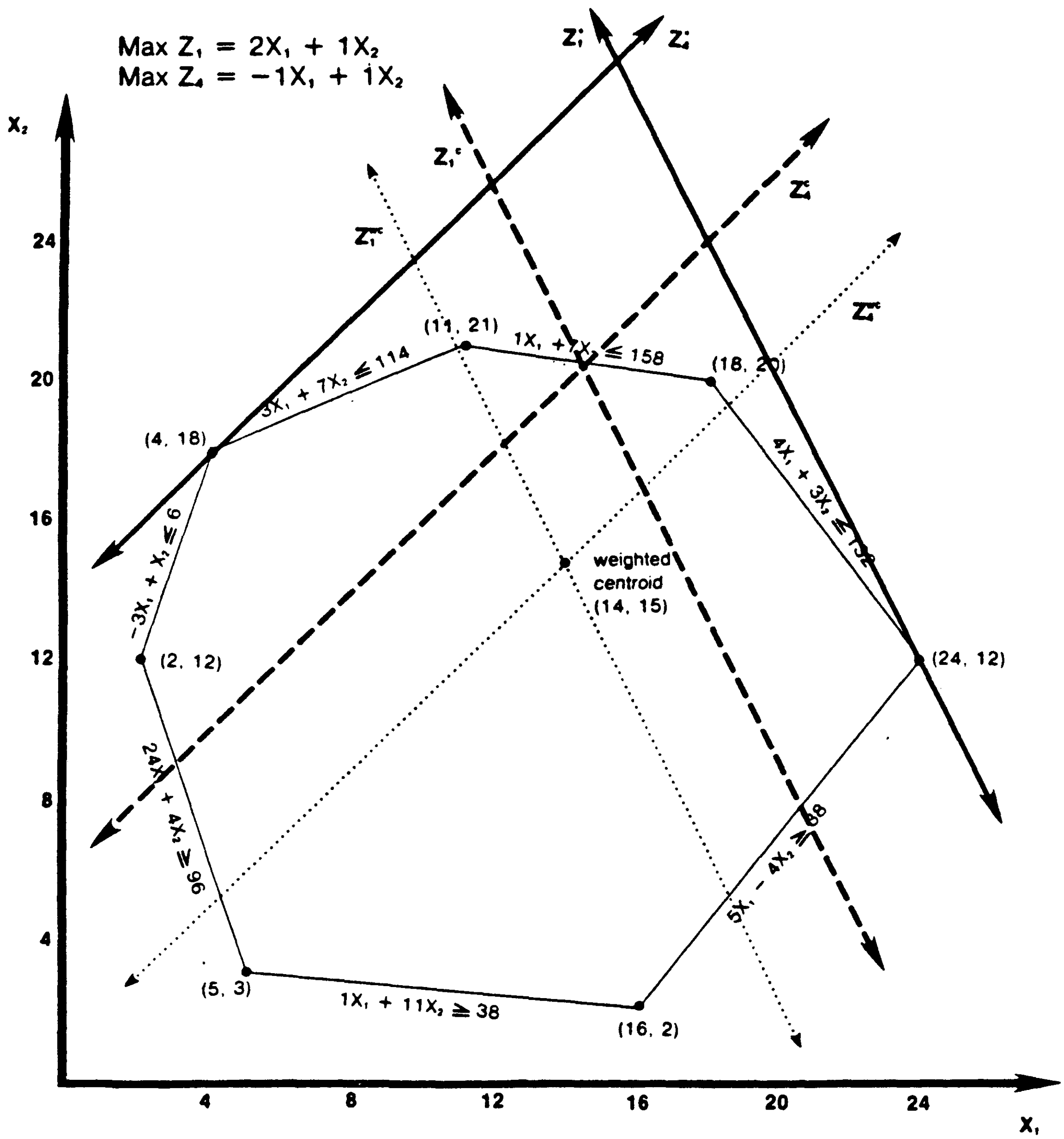

Figure 4

Non-Adjacent Optimal Solutions II

The weighted-centroid is then evaluated to see if it meets the criterion for Pareto optimality/mathematical nondominance by identifying all binding constraints if any exist. In this case, no constraints are binding at the weighted centroid.

The problem is reformulated as a goal programming model. The goals in the formulation are the simulataneous attainment of the mutually exclusive optimal levels for objectives 1 and 4 as directed in step 5 of the algorithm.

MINIMIZE $d_1 + d_4$

Such that:

$$
\begin{aligned}
2X_1 + X_2 + d_1 &= 60 \\
-1X_1 + X_2 + d_4 &= 14 \\
2X_1 + X_2 &\geq 43 \\
-1X_1 + X_2 &\geq 1 \\
13d_1 - 17d_4 &= 0 \\
4X_1 + 3X_2 &\leq 132 \\
1X_1 + 7X_2 &\leq 158 \\
-3X_1 + 7X_2 &\leq 114
\end{aligned}
$$

$$-3X_1 + 1X_2 \leq 6$$

$$24X_1 + 4X_2 \geq 96$$

$$X_1 + 11X_2 \geq 38$$

$$5X_1 - 4X_2 \leq 72$$

The solution of the goal reformulation yields a compromise solution where:

X_1^c=14.511, X_2^c=20.498, Z_1^c=49.521, and Z_4^c=5.987.

The binding constraint here is

$$1X_1 + 7X_2 \leq 158.$$

3.3.5 Case 5: Non-Adjacent Optimal Solutions III

Consider the case of attempting to optimize both objectives 1 and 5 above. See Figure 5.

$$\text{MAX } Z_1 = 2X_1 + X_2$$

$$\text{MIN } Z_5 = 1X_1$$

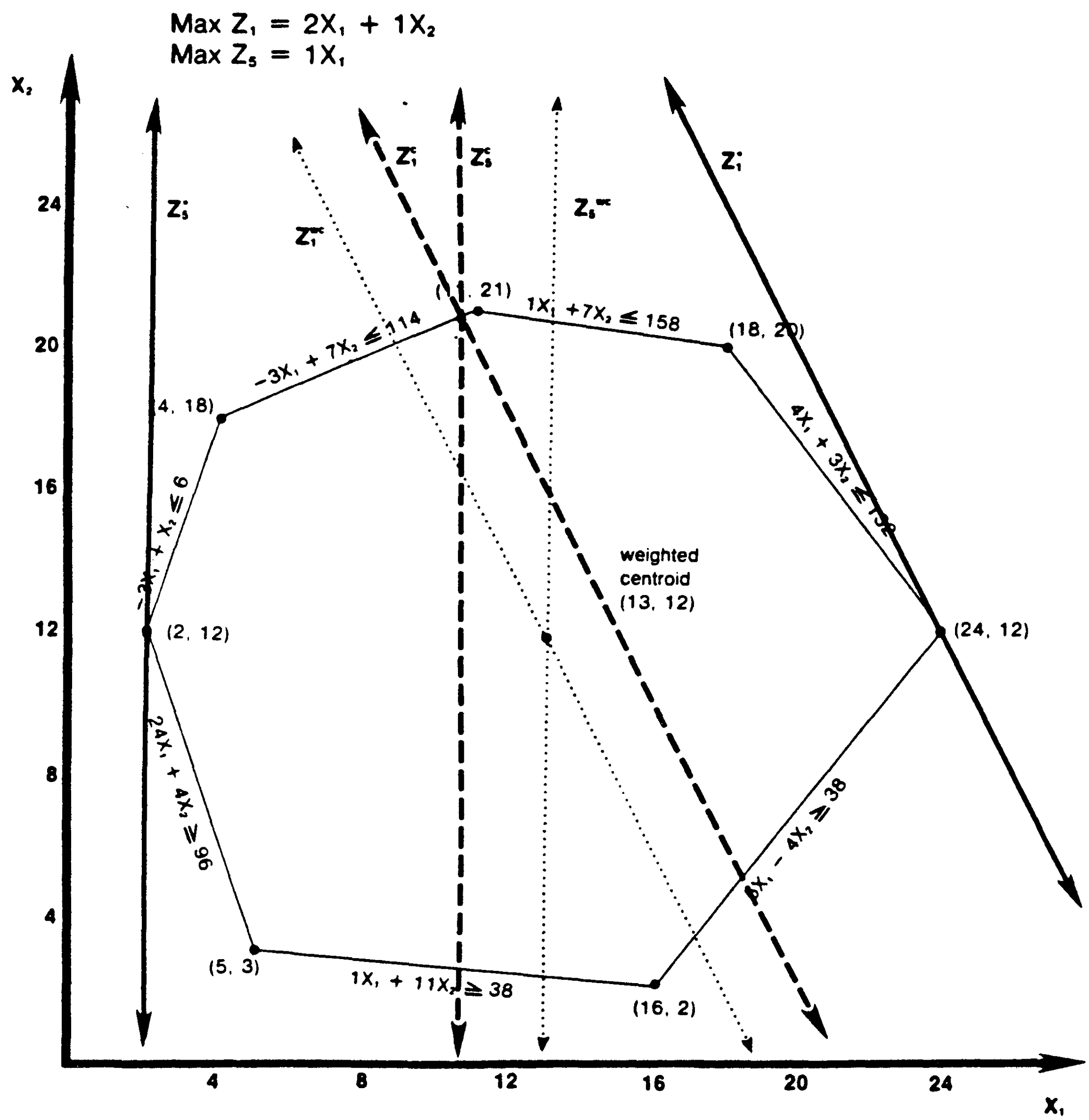

Figure 5

Non-Adjacent Optimal Solutions III

While objective 1 reaches optimality at corner point (24,12), objective 5 reaches optimality at corner point (2,12). See Figure 5. Objective one's optimal value is 60 and objective five's optimal value is 2.

Since the two objectives reach optimality at different corner points, the weighted-centroid is calculated as directed in step three of the algorithm. If each of the two objectives is assigned equal weight, then the weighted-centroid is represented by the coordinate pair (13,12) where the value of objective 1 is 38 and the value of objective 5 is 13.

The weighted-centroid is then evaluated to see if it meets the criterion for Pareto optimality/mathematical nondominance by identifying all binding constraints if any exist. In this case, no constraints are binding at the weighted centroid.

The problem is reformulated as a goal programming model. The goals in the formulation are the simulataneous attainment of the mutually exclusive optimal levels for objectives 1 and 5 as directed in step 5 of the algorithm.

MINIMIZE $d_1 + d_5$

Such that:

$$
\begin{aligned}
2X_1 + X_2 + d_1 \quad &= 60 \\
X_1 \quad - d_5 &= 2 \\
2X_1 + X_2 \quad &\geq 38 \\
X_1 \quad &\leq 13 \\
11d_1 - 22d_5 &= 0 \\
4X_1 + 3X_2 \quad &\leq 132 \\
1X_1 + 7X_2 \quad &\leq 158 \\
-3X_1 + 7X_2 \quad &\leq 114 \\
-3X_1 + 1X_2 \quad &\leq 6 \\
24X_1 + 4X_2 \quad &\geq 96 \\
X_1 + 11X_2 \quad &\geq 38 \\
5X_1 - 4X_2 \quad &\leq 72
\end{aligned}
$$

The solution of the goal reformulation yields a compromise solution where:

X_1^c=10.774, X_2^c=20.903, Z_1^c=42.452, and Z_5^c=10.774.

The binding constraint here is

$-3X_1 + 7X_2 \leq 114.$

In each of the five two-dimensional cases presented, the weighted-centroid compromise programming algorithm yielded a unique compromise solution which was both efficient and equitable. The algorithm can be used with maximization objective functions, minimization objective functions, or some combination of the two. In all cases, each problem objective function sacrificed 50% of it's potential variation in moving from its optimum corner-point to the equitable weighted-centroid. Each objective function recovered the same proportion of that loss in moving from the weighted-centroid to the efficient compromise solution.

Chapter 4

APPLICATION OF THE WEIGHTED-CENTROID COMPROMISE PROGRAMMING ALGORITHM TO SELECTED MINERAL PROBLEMS

This chapter begins with a demonstration of how the weighted-centroid compromise programming algorithm can be applied to classical linear programming models like the product-mix problem. The problem is first presented and solved in a generic form suited to the minerals industry before extended applications are discussed. Later in the chapter the weighted-centroid compromise programming algorithm is used to solve Mr. Ream's funds disbursement problem presented in Chapter 1 of this thesis. The chapter concludes with a discussion of how the weighted-centroid compromise programming algorithm might be applied to public policy decision making, specifically in the area of natural resource economics.

4.1 A Product-Mix Model With Two Objectives

Before discussing some real-life applications of compromise programming, consider the following simple numerical exercise adapted from Saska (1968) and Zeleny (1982). The original model has been redefined as a mineral processing problem to demonstrate how the weighted-centroid compromise programming algorithm can be applied to decision analysis in the mineral industry.

Four different profit-generating mineral products are processed at three different sites with capacities (in tons of ore input per planning period) of 3780, 4410, and 350 respectively. Final product one can be processed at either site one or site two and requires 45 tons of ore per ton of final product at both processing sites. Final product two can be processed at all three sites; however, each site has its own recovery rate. Processing one ton of final product two requires 20 tons of ore at site one, 25 tons of ore at site two or 5 tons of ore at site three. Final product three can be processed at either site one or site two. One ton of final product three requires 20 tons of ore at site one or 25 tons of ore at site two. Final product four can only be processed at site one where 9 tons of ore yields one ton of final product four. If X_i represents the number of tons of final product i processed during the planning period, the capacity constraints for the three sites are written as follows:

$$45X_1 + 20X_2 + 20X_3 + 9X_4 \leq 3780$$

$$45X_1 + 25X_2 + 25X_3 \leq 4410$$

$$5X_2 \leq 350$$

Due to recent income requirements, upper management is interested in maximizing the revenue generated from a given production. Final product one sells for $1,260 per ton. Final product two sells for $1,960 per ton. Final product three sells for $700 per ton and final product four sells for $1,000 per ton. Subject to the capacity constraints above, the objective function is

$$\text{MAXIMIZE } Z_1 = 1260X_1 + 1960X_2 + 700X_3 + 1000X_4$$

Maximizing objective function number 1 above yields the following result:

$$X_1 = 0,\ X_2 = 0,\ X_3 = 0,\ X_4 = 420.$$

That is only final product four is processed. Moreover, product 4 can be processed only at the first site--none of the remaining sites would participate in such a production mix.

Site managers argue that the profitability on product 4 at \$50 per per ton is not high enough and that overall profitability of the operations would suffer in the long run. The site managers point out that profit on product one is \$120 per ton, while final product two yields a profit of \$180 per ton and final product three yields \$140 per ton. Maximization of profits has been suggested by the site managers as an alternative goal. The objective function becomes

$$\text{MAXIMIZE } Z_2 = 120X_1 + 180X_2 + 140X_3 + 50X_4$$

Maximizing this second objective with respect to the capacity constraints yields the solution

$$X_1 = 0,\ X_2 = 70,\ X_3 = 106.4,\ X_4 = 28.$$

This solution provides a larger variety of production, but still none of mineral product 1 would be processed. Comparing the two situations shows that

(1) Objective 1 reaches optimality at coordinates (0,0,0,420) where its value is 420,000 sales dollars. Here objective two's value is $21,000.

(2) Objective 2 reaches optimality at coordinates (0,70,106.4,28) where its value is $28,896. Here objective one's value is $239,680.

(3) The range of variation of objective function 1 is $180,320 in sales and the range on objective function 2 is $7,896 of profit.

Although both of the problem objective functions are expressed in dollars, the two evaluation criteria may not be combined into a single objective function. This is because the first objective function is evaluated with respect to dollar value of revenue generated from a given production and the second objective uses dollar value of profits generated from a given production. Because the two objective functions are incommensurable and different managers within the firm have interest in the optimization of each of the objectives, compromise programming may be used to resolve the conflict. While upper management is especially concerned with meeting income requirements, site managers are concerned with profitability. The weighted-centroid compromise programming algorithm is applied to the resolution of this problem because both levels of management have knowledge as to the best and worst situations that they may face.

Because these two objective functions are optimal at different feasible region corner-points, the weighted-centoid is calculated as directed in step 3 of the weighted-centroid compromise programming algorithm. If the two objectives are assigned equal weights, the weighted centroid is defined by

$X_1^{wc} = 0$, $X_2^{wc} = 35$, $X_3^{wc} = 53.2$, $X_4^{wc} = 224$,

$Z_1^{wc} = 329840$, and $Z_2^{wc} = 24948$.

Note that at the weighted-centroid, both objectives decreased by 50% of their potential variation. It is from this equitable compromise point that a search for Pareto improvement begins.

The problem is reformulated using a goal programming model:

MINIMIZE $d_1 + d_2$

Such that:

$$1260X_1 + 1960X_2 + 700X_3 + 1000X_4 + d_1 = 420000$$

$$120X_1 + 180X_2 + 140X_3 + 50X_4 + d_2 = 28896$$

$$1260X_1 + 1960X_2 + 700X_3 + 1000X_4 \geq 329840$$

$$120X_1 + 180X_2 + 140X_3 + 50X_4 \geq 24948$$

$$45X_1 + 20X_2 + 20X_3 + 9X_4 \leq 3780$$

$$45X_1 + 25X_2 + 25X_3 \leq 4410$$

$$5X_2 \leq 350$$

$$3948d_1 - 90160d_2 = 0$$

The compromise solution resulting from the above goal formulation yields the following mix:

$X_1^c = 0$, $X_2^c = 70$, $X_3^c = 23.758$, $X_4^c = 211.648$, $Z_1^c = 365478.745$,

and $Z_2^c = 26508.579$.

Here two of the original problem capacity constraints are binding:

$$45X_1 + 20X_2 + 20X_3 + 9X_4 \leq 3780$$

$$5X_2 \leq 350$$

Note that at the weighted-centroid, which was the initial equitable and feasible solution, each objective has sacrificed 50% of the difference between its best and worst cases. See Table 1. The weighted-centroid represents an equitable compromise here because each of the two competing objectives has met the other half way. It was from the weighted-centroid that the search for an efficient improvement began. The most efficient improvement which maintains the equitable relationship defined at the weighted-centroid is the compromise solution.

At the compromise solution, each objective regains 39.53% of the loss it incurred in going from its most optimum point to the weighted-centroid. See Table 1. At the compromise solution, each objective has sacrificed the same proportion of its potential variation with respect to its best and worst cases. At the compromise solution, objective one's value is $54,521.255 less than its best case, a loss of 30.236% of its potential variation of $180,320. At the compromise solution, objective two's value is $2,387.42 less than its best case, a loss of 30.236% of its potential variation of $7,896.

	symbol	Linear Objective Functions MAX Z_h for h=1	MAX Z_h for h=2
Best Case	Z^{*}_{h}	420,000	28,896
Worst Case	$Z^{*}_{\not{h}}$	239,680	21,000
Best − Worst	$R_h = Z^{*}_{h} - Z^{*}_{\not{h}}$	180,320	7,896
Weighted Centroid	Z^{wc}_{h}	329,840	24,948
Best − Weighted Centroid	$R^{wc}_{h} = Z^{*}_{h} - Z^{wc}_{h}$	90,160	3,948
Compromise	Z^{c}_{h}	365,478.745	26,508.579
Best Case − Compromise	$R^{c}_{h} = Z^{*}_{h} - Z^{c}_{h}$	54,521.255	2,387.421
Compromise − Wgtd. Centroid	$Z^{c}_{h} - Z^{wc}_{h}$	35,638.745	1,560.579
Percentage of Range Lost at Wgtd. Centroid	$(Z^{*}_{h} - Z^{wc}_{h})/R_h$	50%	50%
Percentage of Range Lost at Compromise	$(Z^{*}_{h} - Z^{c}_{h})/R_h$	30.236%	30.236%
Percentage of Loss Recovered at Compromise	$(Z^{*}_{h} - Z^{c}_{h})/R^{c}_{h}$	39.53%	39.53%

Table 1

Effect of the Weighted-Centroid Compromise Programming Algorithm on Two Objective Functions

Milan Zeleny uses the same numerical example to demonstrate his displaced-ideal approach to compromise programming (1982, page 333). Zeleny's compromise solution is

$$X_1^c = 0, \; X_2^c = 70, \; X_3^c = 13, \; X_4^c = 234,$$

$$Z_1^c = 380{,}300 \text{ and } Z_2^c = 26{,}120.$$

Zeleny defines his compromise solution where each objective function achieves the same proportion of its optimal level:

$$\frac{Z_{h-1}^c}{Z_{h-1}^*} = \frac{Z_h^c}{Z_h^*} \text{ for } h = 2 \text{ to number of objectives.}$$

At the compromise solution using the displaced-ideal approach to compromise programming, each objective achieves 90.6% of its optimal value.

Zeleny's method defines a compromise solution with respect to the best case for each objective function; however, an objective's worst case is not considered in the evaluation. The weighted-centroid compromise programming algorithm presented in this thesis defines a compromise with respect to both the best and worst cases for each objective function.

4.2 A Cash Disbursement Model With Minimization and Maximization Objectives

The weighted-centroid compromise programming algorithm can be applied to Abe Ream's Federal funds disbursement problem presented in Chapter 1 (page 11) of this thesis. The MCDM formulation of this problem yields two maximization objectives, one minimization objective, and 12 constraining equations.

Let X_i = the number of dollars allocated to area i, for i = 1 to 5. The first objective in the formulation below is to maximize total return on all dollars allocated. The second objective is to maximize federal return on all dollars allocated. The third objective is a goal function which seeks to distribute the available funds equally among the five areas. The d_i^+'s in the goal function are the dollars over the equal allocation goal of \$2,000,000 allocated to area i, for i = 1 to 5. The d_i^-'s in the goal function are the dollars below the equal allocation goal of \$2,000,000 allocated to area i, for i = 1 to 5. The problem formulation is:

$$\text{MAX } Z_1 = .36X_1 + .3125X_2 + .2926X_3 + .28X_4 + .225X_5$$

$$\text{MAX } Z_2 = .30X_1 + .2500X_2 + .2200X_3 + .20X_4 + .150X_5$$

$$\text{MIN } Z_3 = (d_1^+ + d_1^-) + (d_2^+ + d_2^-) + (d_3^+ + d_3^-) + (d_4^+ + d_4^-) + (d_5^+ + d_5^-)$$

$$X_1 - d_1^+ + d_1^- = 2{,}000{,}000$$

$$X_2 - d_2^+ + d_2^- = 2{,}000{,}000$$

$$X_3 - d_3^+ + d_3^- = 2{,}000{,}000$$

$$X_4 - d_4^+ + d_4^- = 2{,}000{,}000$$

$$X_5 - d_5^+ + d_5^- = 2{,}000{,}000$$

$$X_1 - .8X_2 \leq 0$$

$$X_2 - .758X_3 \leq 0$$

$$X_3 - .825X_4 \leq 0$$

$$X_4 - .80X_5 \leq 0$$

$$X_1 + X_2 + X_3 + X_4 + X_5 \geq 9{,}500{,}000$$

$$X_1 + X_2 + X_3 + X_4 + X_5 \leq 12{,}000{,}000$$

$$0.02X_1 + .03X_2 + .08X_3 + .10X_4 + .12X_5 \leq 800{,}000$$

The first step of the weighted-centroid compromise programming algorithm includes successive calls to a set of linear programming subroutines, one call for each of the three objective functions in the funds disbursement problem. The first objective in the problem formulation above, that of maximizing total return on all dollars allocated, reaches optimality as follows:

X_1 = \$1,160,857.0, X_2 = \$1,451,071.0, X_3 = \$1,914,341.0,

X_4 = \$2,320,414.0, X_5 = \$2,900,517.0, $d_1^+ = 0$, $d_1^- = 839{,}140$,

$d_2^+ = 0$, $d_2^- = 548{,}940$, $d_3^+ = 0$, $d_3^- = 85{,}660$, $d_4^+ = 320{,}410$,

$d_4^- = 0$, $d_5^+ = 900{,}520$, $d_5^- = 0$, $Z_1^* = 2{,}733{,}836$,

$Z_2 = 2{,}031{,}340$, and $Z_3 = 2{,}694{,}660$.

The second objective function in the model, that of maximizing federal return on all dollars allocated, reaches optimality under the same conditions as the first objective function. The third objective in the model, that of minimizing the deviations both above and below the equal allocation levels of $2,000,000 per locality, also achieves optimality under the same resource allocation. A global optimal solution has been identified and the algorithm stops.

The fact that the three problem objectives all reach optimality at the same feasible region corner point was not apparent prior to completing step one of the algorithm. If a multiparametric decomposition technique (like those described in Chapter 2 of this thesis) had been applied to this problem, the incommensurable units of measurement in the three problem objectives would have been combined into one representitive, non-linear function before any further evaluation was attempted. Multiparametric decomposition techniques can overcomplicate problems such as Mr. Ream's funds disbursement problem where a global optimal feasible solution exists.

4.3 Potential Applications in Public Policy Decision Making

Public policy issues, such as whether to open a wilderness area to development, are often resolved when an acceptable compromise with respect to all concerned special interest groups is identified. The process of identifying an acceptable compromise begins after each of the concerned parties has expressed a position or objective. After the public officials are made aware of all positions, they seek the course of action which they believe is best. The criterion of "best" here may be in terms of pleasing the greatest number of citizens, appeasing the most vocal private interest groups and/or financial considerations such as maximizing the local tax base or protecting important local industries. This is especially important in mining communities or areas where a large portion of the community earns a living in tourism-related areas.

The weighted-centroid compromise programming method is philosophically consistent with decision making in the public sector where decisions must be made based on numerous incommensurable evaluation criteria. If all concerned individuals want the same outcome and agree on a means for achieving that outcome, the best course of action is obvious. This is the case in compromise programming where a global optimal solution exists. The optimum compromise is the one where none of the interested parties is forced to make a compromise. The usual case, however, is where numerous individuals or groups define individual best cases which are by their nature mutually exclusive. The joint optimization of the objective which achieves optimality under conditions which require that a public area remain undeveloped and the objective which reaches optimality where the same area is developed as a gravel pit are jointly unattainable. This ideal, but infeasible situation is what Milan Zeleny defines as his displaced ideal in his compromise programming method. Zeleny would define a compromise solution where each of the competing individuals or groups attained the same proportion of their optimum value. In this example that may mean to develop a portion of the area and leave the rest in its natural state.

Zeleny's approach to compromise programming may be appropriate for decisions where the individuals with competing objectives are ignorant as to their competitor's best and worst cases. The final evaluation is then how "close" to optimal or what proportion of the pie each party got. When the competing parties are aware of each other's best and worst cases, they have more information with which to bargain. The weighted-centroid compromise programming method takes advantage of this additional information by using a marginal rather than real analysis in defining an equitable compromise solution.

Chapter 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Summary

This thesis began with a discussion of how operations research tools are used to simulate the human decision making process. It was noted that although human decision making is a MCDM process utilizing numerous evaluation criteria, many quantitative tools such as linear programming and goal programming offer optimization with respect to a single criterion. The restriction of single objective optimization techniques as business decision making tools appeared to be unrealistically limiting.

The second chapter includes a presentation of the historical development of the MCDM problem in both economics and operations research. The economic and mathematical efficiency criteria which define a necessary, but not sufficient condition for compromise in the weighted-centroid compromise programming algorithm, are defined and presented. The computational aspects of the thesis algorithm incorporate the use of both linear and goal programming algorithms. These early operations research tools were important in the evolution of the newer MCDM solution methods multiobjective linear programming and compromise programming.

The third chapter of this thesis is a presentation of the weighted-centroid compromise programming algorithm. This chapter includes a discussion of the philosophy behind the algorithm as well as a numerical and graphical demonstration of how the algorithm is used.

The algorithm is applied to the solution of several sample problems in Chapter 4. The compromise solution for each of the sample problems is evaluated and compared with those generated using alternative solution methods.

The weighted-centroid programming algorithm offers a number of advantages in the solution of MCDM problems. The algorithm was specifically designed to solve problems where decision alternatives are best evaluated using numerous incommensurable units of measurement. It frees the decision maker from the tedious task of determining an equitable relative valuation of the incommensurable criteria prior to optimization as is required using multiparametric decomposition methods. Other documented MCDM methods yield a solution set rather than one best solution. Milan Zeleny and other researchers in MCDM have discussed the problem of presenting the decision maker with a useful representation of an infinite solution set. The weighted-centroid compromise programming algorithm yields a single solution. This solution is defined as the best because it is the most efficient solution which is as equitable as the equitable-by-definition weighted-centroid point. This condition is assured by the goal reformulation model which has constraining equations making all inequitable solutions infeasible. Further optimization is with respect to the new feasible region.

The displaced-ideal approach to compromise programming (Zeleny, 1982) overcomes the problems associated with the more popular multiobjective linear programming and multiparametric decomposition methods. Compromise programming evaluates each linear objective function individually using linear or goal programming solution methods. Only when there is a need for compromise, when objectives reach optimality at different feasible region corner points, is any type of compensation attempted between incommensurable objectives. Zeleny moves from the usually infeasible, but super-optimal displaced ideal. The displaced ideal is defined as the point where the problem objectives all intersect with each other while maintaining intersection with each of their individual optimum corner points. The displaced-ideal is feasible only when there is a global optimal solution. When the displaced ideal is infeasible, the feasible edge closest to the ideal is defined as the best compromise. Here each objective attains the same proportion of its best level.

Zeleny's compromise programming method considers optimization with respect to each objectives best case; however, no consideration is given for an objective's worst case. Zeleny's compromise programming method also has problems when there are both maximization and minimization objectives in the model. Constraints assuring that both minimization and maximization objectives achieve the same proportion of optimality are non-linear and are not solveable using linear or goal programming. The application of geometric programming to the solution of this situation was attempted in the course of the research for this thesis. Due to the high degree of difficulty required to solve any of these problems which have many more terms than variables, the use of geometric programming in the reformulation model was abandoned.

The weighted-centroid compromise programming algorithm evolved from an attempt to solve problems encountered in applying the displaced-ideal compromise method. Rather than moving from an infeasible, but super-optimal point toward a compromise as the displaced-ideal methods does, the weighted-centroid compromise programming algorithm defines a feasible, but probably suboptimal starting point from which to move toward optimality. Under the conditions where Zeleny's displaced-ideal is feasible, the weighted-centroid is optimal. The weighted-centroid is also optimal when the individual problem objectives reach optimality at two adjacent feasible region corner points. The weighted-centroid will then lie on the feasible region edge connecting the adjacent optimum corner points.

The primary theoretical difference between the weighted-centroid and displaced-ideal approaches to compromise programming lie in the definition of an equitable compromise. The displaced-ideal method uses a real analysis where each objective achieves the same proportion of optimality, or a formulation of minimizing the maximum of the deviations from optimality. The weighted-centroid method uses marginal analysis where a compromise is determined with respect to each objectives best and worst cases. The only worst cases for any objective function which are relevant to the analysis are where another objective reaches optimality. The weighted-centroid method assumes the the individual objectives have some "knowledge" as to their competitors' bargaining power. This bargaining power is assumed to be in terms of the relevant range on each objective in the analysis. This relevant range is in terms of each objective's value at each of the individual optimum corner points for all objective functions. Relative to the equitable compromise at the weighted-centroid, further optimization of all objectives is sought.

The weighted-centroid compromise programming algorithm was presented for the first time in this thesis. It is a new technique that needs much further testing and study. The weighted-centroid compromise programming algorithm can easily be applied to present linear and goal programming models when the decision maker identifies other criteria to be considered in the evaluation. In every application of the algorithm, a unique compromise solution was identified. The use of the algorithm is direct and easily implemented in conjunction with a standard linear programming package.

The weighted-centroid compromise programming method is not recommended for those problems which require integer solutions. Even if each of the linear objective functions is evaluated using an integer programming method, there is no guarantee that the compromise solution will yield integer values for the decision variables.

Application of the algorithm presents a potential problem when there are more objective functions in the model than there are decision variables. This relates to the fact the the number of decision variables corresponds to the number of dimensions defining the feasible region and the number of objectives corresponds to the number of possible directions in which the weighted-centroid is "pulled". Recall that the weighed-centroid is defined as that point in n-space, where n is the number of decision variables in the model, where equilibrium is reached when a force is exerted from each individual objective's optimum point, pulling the weighted-centroid toward it with a force proportional to the objective's user-assigned weight. This potential problem, which would be evident when there is no feasible solution to the goal reformulation model, warrants further investigation.

5.2 Conclusions

The weighted-centroid compromise programming algorithm offers a new marginal analysis approach to the solution of MCDM problems. Because the philosophy behind the algorithm was to more closely simulate the human decision making process while using popular modelling and solution methods from operations research, the algorithm has a natural intuitive appeal. It seems equally well-suited to all classical linear programming problems, with the exception of those that require integer solutions.

Besides a new solution method, the weighted-centroid compromise programming algorithm offers a new philosophical approach to the MCDM problem. Unlike other MCDM methods, the solution method here narrows the compromise area to the feasible region's relevant range with respect to the best and worst cases for each objective's value. This type of marginal analysis in determining an equitable compromise is philosophically consistent with the situation when the parties who have special interest in each objective's success have knowledge of each other's bargaining power. Zeleny's method is consistent with the situation where the parties do not have this knowledge and judge success based only on how good things could possible get without regard to the reasonable worst compromises they might be forced to make in the name of equity.

5.3 Recommendations for Further Research

This thesis represents the first efforts to apply the weighted-centroid compromise programming algorithm to small, but typical problems. Much further research, especially testing the method using larger, real-life problems is needed.

The effects of assigning different weights to the different problem objective functions were not explored in this thesis. All problems illustrated assumed that the objectives were each assigned equal weights. The possibility of assigning different weights arose as a means of compensating for the fact that the various objectives may have different levels of importance. The effect of weighting on the weighted-centroid is to pull the weighted-centroid proportionately more strongly in the direction of the optimum corner points with respect to the objective's relative weight. This additional refinement to the multiple objective/multiple goal model may be especially useful in identifying a solution which is truly equitable. The effect of weighting deserves much further exploration.

Other areas for further research include the potential problems that may arise when there are more problem objective functions than there are decision variables. The most important information as to the importance of the weighted-centroid compromise programming algorithm will come from its application to complicated, multiple-criteria problems encountered in business decision making.

REFERENCE LIST

Charnes, A. and W. W. Cooper. 1961. Management Models and Industrial Applications of Linear Programming. Vol. 1, New York: John Wiley and Sons.

_________., R. W. Clower, and K. O. Kortanek. 1967. "Effective Control Through Coherent Decentralization with Preemptive Goals." Econometrica. Vol. 35, No. 2.

Dantzig, G. B. 1948. "Programming in a Linear Structure." Comptroller, United States Air Force. Washington, D.C. February.

Evans, J. P. and R. E. Steuer. 1973. "Generating Efficient Extreme Points in Linear Multiple Objective Programming: Two Algorithms and Computing Experience." in Multiple Criteria Decision Making. Cochrane, J. L. and M. Zeleny, ed. Columbia: University of South Carolina Press. Pages 54-72.

Ignizio, J. P. 1976. Goal Programming and Extensions. Lexington: Lexington Books.

Ijiri, Y. 1965. Management Goals and Accounting for Control. Amsterdam: North Holland Publishing Company.

Koopmans, T. C. 1951. "Activity Analysis of Production and Allocation." Cowles Commission Monograph 13. New York: John Wiley and Sons.

Lee, S. M. 1972. Goal Programming for Decision Analysis. Philadelphia: Auerbach Publishers.

Pareto, V. 1906. Manuale di economia politica, con una introduzione ulla scienza sociale. Milan: Societa Editrice Libraria.

Saska. J. 1968. "Lineární Multiprogramováni." Ekonomicko-matematický Obzor. Vol. 4, No. 3.

Zeleny, M. 1974. "A Concept of Compromise Solutions and the Method of the Displaced Ideal." Computers and Operations Research. Vol. 1, No. 4.

________, ed. 1976. Multiple Criteria Decision Making: Kyoto 1975. New York: Springer-Verlag.

________. 1982. Multiple Criteria Decision Making. New York: McGraw-Hill.

Zionts, S. and J. Wallenius. 1983. "Interactive Multiple Objective Linear Programming." Management Science. Vol. 29, No. 5.