GENERALIZING EMERGENT GAUGE THEORIES TO NON-ABELIAN SETTINGS AND EXPLORING EMERGENCE IN GEOMETRICAL UNDERPINNINGS USING FIBER BUNDLES

by

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A thesis submitted to the Faculty and the Board of Trustees of the Colorado School of Mines in partial fulfillment of the requirements for the degree of Master of Science (Applied Physics).

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ABSTRACT

Attempts to devise a quantum theory of gravity consistent with the standard model have so far been unsuccessful; however, it is widely agreed that gravity can be described as a non-abelian gauge theory, and that the particle mediating the gravitational force (the graviton) would be a spin-2 boson. Under certain circumstances it is possible for gauge symmetries to emerge from theories containing only physical symmetries. It can also be shown that the gauge theories characteristic of the standard model are low energy approximations emerging from more fundamental principles [2]. While it is true that the Weinberg-Witten theorem excludes a large class of emergent theories from describing gravitation [9], it has been shown that non-abelian theories can be formulated such that the Weinberg-Witten theorem does not apply. Currently, this result has only been shown using a functional formalism. The standard model has been described using both functional and a geometrical interpretations. The geometrical interpretation involves describing gauge transformations as connections on principal bundles [11]. The purpose of this thesis is to form a geometrical description of non-abelian emergent gauge theories using a fiber bundle formalism.

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LIST OF SYMBOLS

four-gradient, Laplacian . . . . ∂^μ, $\Box$

covariant derivative, exterior derivative . . . . D, d

wedge product . . . . $\wedge$

gauge potential, field strength tensor . . . . A_μ, $F_{\mu\nu}$

Noether current, Noether charge . . . . J^μ, Q

Lagrangian, spinor field . . . . $\mathcal{L}$, ψ

Dirac matrices, Gell-Mann matrices . . . . γ^μ, λ^a

gauge transformation matrix, scalar field, structure constant . . . . U, χ, f^{abc}

total space, total space (principal) . . . . T, P

base space, fiber, structure group . . . . B, F, G

fiber bundle, principal bundle . . . . $\pi : T \to B$, $\pi : P \to B$

local neighborhood of points, local trivialization . . . . U_i, $\phi_i : U_i \times F \to \pi^{-1}(U_i)$

diffeomorphism . . . . ϕ_i

section, transition function . . . . σ, g_{ij}

right action, left action, Lie algebra . . . . R, L,$\mathcal{G}$

connection one-form, local connection one-form . . . . ω, $\mathcal{A}$

connection two-form, local connection two-form . . . . Ω, $\mathcal{F}$

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ACKNOWLEDGEMENTS

I would like to thank my parents for their undying love and support as well as my advisor, Alex Flournoy, for his patience and guidance over the last two years.

CHAPTER 1
INTRODUCTION

General relativity as it stands is an incomplete theory because it exhibits singular behavior at the center of black holes. This breakdown occurs because general relativity fails to accommodate quantum mechanics in its predictions. In other words, general relativity is not renormalizable, which has led to great difficulty in defining a theory that is "ultraviolet complete" i.e. predictive at all possible high energy and small distance scales. [1] Attempts to remedy this have included string theory, loop quantum gravity, supergravity, and many more theories which have, as of yet, been unproven [1] including emergent gravity, the methods of which will be the focus of this thesis.

Emergent gravity is based on the observation that it is possible to have non-trivial phenomena emerge from a theory originally containing none, when subject to some constraint. A good example of this is the standard model, which is an example of a theory containing gauge symmetries that emerge in a low energy limit [2]. This thesis will analyze a specific class of emergent phenomena referred to as "emergent gauge theories", which is based upon the idea that a theory containing only gauge symmetries can "emerge" from a different theory containing only physical symmetries when subject to some constraint. General relativity can be formulated as a non-abelian gauge theory exhibiting invariance under the Poincare group. As a result, it might be possible that gravity is not by itself fundamental, but that it is an effective theory that emerges in some low energy limit from a different fundamental theory. It has been thought that the Weinberg-Witten theorem excludes emergent theories from describing a spin-2 boson such as the graviton; however, this is generally only the case for abelian emergent theories, and there are indeed ways to formulate non-abelian emergent theories that do not violate this theorem. The Weinberg Witten theorem places limits on the maximum spin of massless particles charged under a Lorentz covariant current, but currents defined in non-abelian theories, like the currents associated with the charges of gluons, are often not Lorentz covariant, which renders this theory irrelevant. The non-abelian theory explored in this thesis possesses Noether currents that are not Lorentz covariant, and is unscathed by the Weinberg-Witten theorem [12].

In order to create a systematic way to search for and define emergent gauge theories it is first necessary to be able to mathematically distinguish between a gauge symmetry and a physical symmetry. It is well known that gauge symmetries represent mathematical redundancies in our description of physics, often interpreted as extraneous degrees of freedom. In other words, gauge symmetries represent transformations that leave the Lagrangian of our theory invariant, but do not produce physically distinguishable states, while physical symmetries leave the Lagrangian invariant, but can still produce states that are physically distinguishable. This heuristic description is well known, but Barcelo et al. provide a novel method to analyze the forms of the Noether currents generated by symmetry transformations of the Lagrangian in order to determine if they are gauge transformations or physical transformations. This method will be utilized in a non-abelian setting to establish that it is possible to formulate non-abelian emergent gauge theories. Furthermore, the standard model has two main interpretations. One interpretation is purely functional and is what most physicists are used to dealing with. This approach deals with Lagrangians, Noether currents, gauge transformations etc. The second interpretation is a geometrical interpretation. This interpretation can be used to describe Yang-Mills theories (as well as electromagnetism) in terms of topological objects called fiber bundles. It is possible to define fields as well as gauge transformations as sections over principal fiber bundles and retrieve the full standard model in a purely geometrical interpretation. This duality suggests that it might also be possible to translate non-abelian emergence into a geometrical regime as well. This thesis will attempt to find a geometric underpinning for emergent gauge theories in terms of fiber bundles such that the topological properties of such theories can be explored. Such a geometrical description could be very helpful if the methods explored in this thesis were to ever be used to formulate a theory of gravity as an emergent gauge theory.

CHAPTER 2

BACKGROUND

2.1 Gauge Symmetries and Physical Symmetries

2.1.1 Definition of Emergence and Noether's Theorem

Before we begin it is important to define what exactly the term emergence means in the context of this project. For the remainder of this paper the term emergence will refer to physical symmetries transforming into gauge symmetries when a theory is subject to certain constraints. To explain how this happens we must first find a precise functional way to differentiate between gauge symmetries and physical symmetries. A gauge symmetry is simply a mathematical redundancy in our description of physics [2] while a physical symmetry defines a transformation invariance that produces a physically distinguishable result. Noether currents can be defined for both gauge and physical symmetries; however, since gauge transformations are intrinsically non-physical they will produce Noether currents that are trivially conserved when evaluated on-shell [3]. First, lets consider the action for a relativistic field theory.

$$S = \int \mathcal{L}(\varphi_i(x^\mu), \partial_\mu \varphi_i(x^\mu)) d^4x \quad [i = 1, ..., n] \tag{2.1}$$

For simplicity, we will drop the i index and consider a system with one field. By varying the action with respect to arbitrary field deformations and requiring it to be stationary we get equations of motion via Hamilton's Principle. This yields an Euler-Lagrange equation describing the evolution of each field in the theory:

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = \frac{\delta S}{\delta \varphi} = 0. \tag{2.2}$$

Noether's Theorem follows a similar analysis, except instead of arbitrary field deformations we are considering a specific symmetry transformation of the field. Say

$$\varphi \to \varphi + \delta\varphi \tag{2.3}$$

is an infinitesimal transformation corresponding to a symmetry of the Lagrangian up to a four gradient. This means

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}, \quad \delta\mathcal{L} = \partial^\mu(B_\mu). \tag{2.4}$$

Using the multivariate chain rule, we can write the infinitesimal change in the Lagrangian density as

$$\delta\mathcal{L} = \partial^\mu(B_\mu) = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\partial_\mu\varphi. \tag{2.5}$$

Substituting in Eq.2.2 we get:

$$\partial^\mu(B_\mu) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right) + \frac{\partial S}{\partial\varphi}\delta\varphi. \tag{2.6}$$

By requiring the solutions to be on shell, meaning they satisfy equations of motion (Eq.2.2), the second term on the right hand side of Eq.2.6 goes to zero, and what's remaining is the statement of a conserved quantity defined as the Noether current [4]:

$$0 = \partial_\mu(J^\mu), \quad \text{with} \quad J^\mu \equiv \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi - \partial^\mu(B_\mu)\right). \tag{2.7}$$

Note that the Lagrangian density doesn't need to be totally invariant, as long as it is invariant up to a total four gradient then we are still able to define a conserved quantity. The zeroth component of the Noether current is defined as the charge density, while the rest of it specifies components of the current density vector. A Noether charge can then be written as [4]:

$$Q = \int J^0 d^3x. \tag{2.8}$$

These quantities can be used to systematically distinguish gauge and physical symmetries. Noether charges are non-trivial when corresponding to a physical symmetry [3]. When corresponding to a gauge symmetry, the Noether charges are trivial, and the associated Noether current takes a special superpotential form:

$$J^\mu = W^\mu + \partial_\nu N^{\nu\mu}. \tag{2.9}$$

Here, W^μ is taken to be zero on shell and $N^{\nu\mu}$ represents a second rank tensor with anti-symmetric spacetime indices. From this, we can say that the Noether current for a gauge symmetry is trivially conserved by index structure when evaluated on shell [5].

2.1.2 Physical Symmetries

To cement this concept, we will look at well known examples of physical symmetries and gauge symmetries. The first example we will look at is a physical symmetry. The example we will use is a Lagrangian with a single generalized coordinate that displays translational symmetry. This is a well known example that leads to momentum conservation under the right circumstances that we will go over in detail. First, lets start with a Lagrangian that is only dependent upon a generalized spatial coordinate, $q(t)$, and its time derivative, $\dot{q}(t)$ i.e.

$$S = \int \mathcal{L}(q, \dot{q}) dt. \tag{2.10}$$

Now, lets transform our field by applying a simple spatial transformation parameterized by some constant β.

$$q(t) \rightarrow q'(t) = q(t) + \beta \tag{2.11}$$

To get our Noether current, all we must do is apply equation 2.7. To do this, we must modify equation 2.7 slightly to accommodate our simplified system whose action depends upon a single spatial coordinate.

$$J = \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q = \frac{\partial \mathcal{L}}{\partial \dot{q}} \frac{\partial q}{\partial \beta} \tag{2.12}$$

We must be careful here, because going back through Noether's theorem for this simplified case would reveal that this transformation is not a symmetry of the Lagrangian and therefore has no conserved current if the Lagrangian has any explicit q dependence. This is not an issue in this example, because we will take our Lagrangian to be a simple kinetic energy term associated with our coordinate, q.

$$\mathcal{L} = \frac{1}{2} m \dot{q}^2 \tag{2.13}$$

We finally see that we get conservation of momentum as our conservation law, which is to say that our Noether current is

$$J = m\dot{q}, \tag{2.14}$$

and our Noether charge is

$$Q = \int J dt \tag{2.15}$$

evaluated at the boundary of all time. Clearly this charge is not trivial. To demonstrate further that we indeed have a physical symmetry, we will evaluate evaluate our Noether current on-shell and show that it does not reduce to the super-potential form. The equation of motion for this system is found by applying the Euler-Lagrange equation as follows:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} \rightarrow m\ddot{q} = 0. \tag{2.16}$$

Applying this to our Noether current only enforces the condition that our velocity, $\dot{q}$, is constant, in other words,

$$J = mk \neq 0 \tag{2.17}$$

for some constant k. As a result, we can say that our current is not of the superpotential form and that we indeed have a physical symmetry.

To summarize, we can say that for a simple system that is moving at a constant velocity, the Lagrangian of this system has translational symmetry, that is to say that a constant translation of the system renders the action invariant, but produces a physically distinguishable state. This symmetry is physical because the Noether current associated with this transformation is not of the superpotential form when evaluated on-shell.

2.1.3 Gauge Symmetries

Next, we will apply the same analysis to a well known gauge theory, U(1). We will begin by writing out the full Lagrangian describing the electromagnetic interaction.

$$\mathcal{L} = \mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^{\mu}(i\partial_{\mu} - A_{\mu})\psi, \quad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{2.18}$$

Just like the previous example, we will now apply equation 2.7. Since we now have a gauge field, spinor field, and adjoint spinor field to take into consideration, we generalize equation 2.7 as follows:

$$J^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}A^{a}_{\nu})}\frac{\partial(A^{a}_{\nu})'}{\partial\beta} + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\psi)}\frac{\partial\psi'}{\partial\beta} + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\bar{\psi})}\frac{\partial\bar{\psi}'}{\partial\beta}. \tag{2.19}$$

Now we will define our local transformations for these fields. We will say $\psi \rightarrow U\psi$ and $\bar{\psi} \rightarrow \bar{\psi}\bar{U}$ as well as $A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\chi$. In this case χ is a scalar field with spacetime dependence, $U = e^{-i\chi}$, and $\bar{U} = e^{i\chi}$. To first order, we can parameterize these transformations by β as follows:

$$A_{\mu} \rightarrow (A_{\mu})' = A_{\mu} + \beta\partial_{\mu}(\chi) \tag{2.20}$$

$$\psi \rightarrow \psi' = (1 - i\beta\chi)\psi \tag{2.21}$$

$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}(1 + i\beta\chi). \tag{2.22}$$

Finally we will plug in these transformation rules along with our Lagrangian into equation 2.19 to get our Noether current.

$$J^\mu = -\partial_\nu (F^{\mu\nu}\chi) + \chi(\partial_\nu F^{\mu\nu} + j^\mu) \tag{2.23}$$

Here we have made the substitution $j^\mu = \bar{\psi}\gamma^\mu\psi$. Once more, we will use the Euler-Lagrange equation to find the equation of motion for this Lagrangian so that the Noether current can be evaluated on-shell. The resulting equation of motion is

$$j^\mu = \partial_\nu F^{\nu\mu}. \tag{2.24}$$

Plugging this in to our Noether current and taking advantage of the fact that $F^{\mu\nu}$ is anti-symmetric in its indices we get

$$J^\mu|_{shell} = -\partial_\nu F^{\mu\nu}. \tag{2.25}$$

Which is trivially conserved due to its index structure. To summarize, our Lagrangian (which is just electromagnetism) possesses local symmetries that are indeed gauge symmetries. These transformations do not produce new, physically distinguishable states. We have verified that these symmetries are gauge symmetries by showing that the Noether current associated with these transformations is a superpotential when evaluated on-shell, that is to say all terms are either rendered trivial or are trivially conserved due to their index structure.

2.2 Abelian Emergence

Now that we have a concrete idea of how physical symmetries can be distinguished from gauge symmetries, we will look at an example presented by C. Barcelo et al. [3] In this example, we will take a Lagrangian that is identical to the Lagrangian for electromagnetism plus an additional term. We will show how this additional term changes the system such that the Lagrangian possesses physical symmetries. Finally, we will enforce a projection on this theory that allows gauge symmetries to emerge where before there were only physical symmetries. The Lagrangian we will be using is as follows:

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_\zeta = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^\mu(i\partial_\mu - A_\mu)\psi + \frac{\zeta}{2}(\partial^\mu A_\mu)(\partial^\nu A_\nu) \tag{2.26}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This Lagrangian exhibits invariance under the following local transformations:

$$\psi \rightarrow \psi' = e^{-i\chi}\psi, \quad \bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}e^{i\chi} \tag{2.27}$$

$$A^\mu \rightarrow (A^\mu)' = A^\mu + \partial^\mu \chi, \quad \text{with} \quad \Box\chi = 0. \tag{2.28}$$

We will now apply the exact same process as in the two previous examples. First, we will find the Noether current associated with these transformations by linearizing them as we did in the electromagnetism example in equations 2.21 and 2.22. To find the Noether current, we will plug our new Lagrangian and our linearized transformations into equation 2.19 to get the following:

$$J^\mu = -\partial_\nu(F^{\mu\nu}\chi) + \zeta\partial_\nu A^\nu \partial^\mu \chi + \chi(\partial_\nu F^{\mu\nu} + j^\mu). \tag{2.29}$$

Where $j^\mu = \bar{\psi}\gamma^\mu\psi$ as in the electromagnetism example. Next, we need to evaluate this current on-shell, so we will need to find the equation of motion. Using the Euler-Lagrange equation as in the previous examples gives us the following field equation:

$$j^\mu = \partial_\nu F^{\nu\mu} - \zeta\partial^\mu\partial_\nu A^\nu. \tag{2.30}$$

Finally, when we apply equation 2.30 to equation 2.29 we get the following:

$$J^\mu|_{shell} = -\partial_\nu(F^{\mu\nu}\chi) + \zeta(\partial_\nu A^\nu\partial^\mu\chi - \chi\partial^\mu\partial_\nu A^\nu). \tag{2.31}$$

Now, before we go on to analyze this on-shell current it is enlightening to backtrack and analyze our equation of motion given by the Euler Lagrange equation. Without the ζ-term it is clear to see that our equation of motion gives us two of Maxwell's equations, namely Gauss's law, and Ampere's law:

$$\nabla \cdot \vec{E} = \rho \tag{2.32}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}. \tag{2.33}$$

Where $j^0 = \rho$ and $j^i = \vec{J}$ (here a letter superscript indicates a 3-vector i.e. spatial components of the associated 4-vector). However, with the ζ-term included, we end up with an extra piece of information in these two equations. Now we get

$$\nabla \cdot \vec{E} + \frac{\partial}{\partial t}(\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A}) = \rho \tag{2.34}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} + \nabla(\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A}) = \vec{J}. \tag{2.35}$$

This is an interesting piece of information that will come back later, as it suggests that taking the Lorenz gauge reduces these equations back to Gauss's law and Ampere's law, respectively. It is also important to note that the remaining two Maxwell equations result from $F^{\mu\nu}$ satisfying a Bianchi identity, which in components is [13]

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0. \tag{2.36}$$

Calculating this out explicitly yields

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{2.37}$$

$$\nabla \cdot \vec{B} = 0. \tag{2.38}$$

These equations arise purely from $F^{\mu\nu}$ and are not altered by the addition of the ζ-term.

Referring back to equation 2.31, this on-shell current is identical to the current we got for electromagnetism with the exception of an additional ζ-term which does not vanish, nor is it trivially conserved. This tells us that the local transformations generating this Noether current are indeed physical. Next, we will follow the process of Barcelo et al. to see how we can project out a form of this theory that only contains gauge symmetries.

The goal in this step is to try to find a condition that can be applied to our our fields such that our physical symmetries become gauge. It is also important that this condition does not trivialize our fields. In essence, we are subjecting our fields to a constraint that "picks out" a subset of solutions for our fields that are physically unchanged by our transformations. Looking at equation 2.31 we can see that setting $\partial_\nu A^\nu = 0$ renders the entire ζ-term trivial and reduces our on-shell Noether current to the same current displayed by electromagnetism i.e. trivially conserved by index structure. This means that under our constraint our theory exhibits gauge symmetry, and without our constraint our theory exhibits physical symmetry. It is also worth noting that the analysis by Barcelo et al. goes on to prove that this "emerged" theory not only exhibits the same gauge invariance as electromagnetism in the Lorenz gauge, but also contains the same number of degrees of freedom as electromagnetism – that is to say that this theory is equivalent to electromagnetism when subject to our constraint, but not otherwise.

The results of this abelian analysis are promising and give an excellent example of how a well-understood gauge theory can be formulated as emerging from a more complicated theory whose fields are subject to a natural constraint [3]. This result also sets the stage for attempts to describe other gauge theories as emergent phenomena. Of course, other theories that would be of interest are non-abelian e.g. electro-weak theory,quantum chromodynamics, and general relativity. While it is interesting to ponder whether or not an emergent theory of gravity could be formulated to fit in with the standard model, this analysis alone was done in a strictly abelian regime and does not justify such claims alone. In order to show that emergence could be a potential area of interest for such efforts it will be necessary to first show that this sort of analysis can be generalized to non-abelian theories. This was the focus of the first part of this thesis and will be the subject of the next chapter

CHAPTER 3
NON-ABELIAN GENERALIZATION

3.1 Non-Abelian Theory and Gauge Transformations

Now that we have a method to distinguish between physical and gauge symmetries, we will introduce a Lagrangian in a non-abelian setting and define local transformations for this Lagrangian. Barcelo et al. used the following Lagrangian to show emergence in an abelian setting.

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_{\zeta} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^{\mu}(i\partial_{\mu} - A_{\mu})\psi + \frac{\zeta}{2}(\partial^{\mu}A_{\mu})(\partial^{\nu}A_{\nu}) \tag{3.1}$$
$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$$

The transformations they consider are similar to those of electromagnetism:

$$\psi \to \psi' = e^{-i\chi}\psi, \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{i\chi} \tag{3.2}$$
$$A^{\mu} \to (A^{\mu})' = A^{\mu} + \partial^{\mu}\chi, \quad \text{with} \quad \Box\chi = 0. \tag{3.3}$$

We would like to apply their analysis to a non-abelian setting, so the Lagrangian we will analyze is as follows:

$$\mathcal{L} = \overbrace{-\frac{1}{4}F^{a\,\mu\nu}F^{a}_{\mu\nu} + \bar{\psi}\gamma^{\mu}(i\partial_{\mu} - \lambda^{a}A^{a}_{\mu})\psi}^{\mathcal{L}_{YM}} + \overbrace{\frac{\zeta}{2}(\partial^{\mu}A^{a}_{\mu})^{2}}^{\mathcal{L}_{\zeta}} \tag{3.4}$$
$$F^{a}_{\mu\nu} \equiv \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} + f^{abc}A^{b}_{\mu}A^{c}_{\nu}.$$

Where we now have a Lagrangian that looks like a standard Yang-Mills Lagrangian plus an additional "ζ-term". The transformation rules used in this analysis are as follows:

$$(\lambda^{a}A^{a}_{\mu}) \to (\lambda^{a}A^{a}_{\mu})' = U\lambda^{a}A^{a}_{\mu}\bar{U} + i\partial_{\mu}(U)\bar{U} \tag{3.5}$$

$$\psi \to \psi' = U\psi \tag{3.6}$$

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}\bar{U}. \tag{3.7}$$

with $U \equiv e^{-i\lambda^{a}\chi^{a}}$, and $\bar{U} \equiv e^{i\lambda^{a}\chi^{a}}$. As is implied in Eq.3.1 this Lagrangian is simply the Lagrangian for electromagnetism plus an additional ζ term, that will ultimately be the source of emergence in this theory.

The non-abelian transformation rules are simply the transformation rules for a standard Yang-Mills theory. Since we have an additional ζ term in our theory it is important to check to make sure that this theory is invariant under our non-abelian local transformations as we have defined them. It is also important to see if there are any conditions that must be satisfied by χ in order for local invariance to hold. To do this we will first linearize our transformations as follows:

$$(\lambda^a A^a_\mu)' = (1 - i\lambda^a\chi^a)\lambda^b A^b_\mu(1 + i\lambda^c\chi^c) + i(1 - i\lambda^a\chi^a)\partial_\mu(-i\lambda^b\chi^b)(1 + i\lambda^c\chi^c)$$
$$\implies (A^a_\mu)' = A^a_\mu + \partial_\mu(\chi^a) - f^{bca}A^b_\mu\chi^c. \tag{3.8}$$

Using this linearization we find that our Lagrangian is indeed invariant to first order in χ^a satisfying a similar condition as in the non-abelian case:

$$\Box\chi^a = f^{bca}\partial^\nu(A^b_\nu\chi^c). \tag{3.9}$$

This is just a non-homogeneous wave equation, whereas in the abelian case our condition was a homogeneous wave equation. To summarize, we have shown that the Lagrangian given by Barcelo et al. is indeed invariant under our local transformations in a non-abelian regime as well as found the analogous condition under which this is true.

3.2 Noether currents, Symmetries, and Emergence

Now that we have a theory that is invariant under our local transformations we will find the Noether currents associated with these local transformations and deduce whether the symmetries of this theory are physical or gauge. Since we ultimately need to evaluate our Noether currents on-shell the first thing we will find is the equation of motion.

3.2.1 Equation of Motion

To find the equation of motion we simply need to apply the Euler-Lagrange equation to our Lagrangian.

$$\frac{\partial\mathcal{L}}{\partial(A^a_\mu)} = \partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A^a_\mu)}\Big) \tag{3.10}$$

First we will find the right hand side of Eq.3.10.

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^h_\beta)} = \frac{\partial}{\partial(\partial_\alpha A^h_\beta)}\Big(-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\,\nu} - \partial^\nu A^{a\,\mu}) \tag{3.11}$$

$$-\frac{1}{2}f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^{b\,\nu}A^{c\,\nu} + \frac{\zeta}{2}(\partial^\mu A^a_\mu)^2\Big) \tag{3.12}$$

$$= -\frac{1}{2}F^{h\,\alpha\beta} + \frac{1}{2}F^{h\,\beta\alpha} + \zeta\eta^{\alpha\beta}\partial^\mu A^h_\mu \tag{3.13}$$

We can now relabel some indices and get the following:

$$\partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A^a_\mu)}\Big) = -\frac{1}{2}\partial_\nu(F^{a\,\nu\mu}) + \frac{1}{2}\partial_\nu(F^{a\,\mu\nu}) + \zeta\partial^\mu(\partial^\nu A^a_\nu). \tag{3.14}$$

We will now introduce an additional constraint in order to simplify Eq.3.14, and ultimately the left hand side of the equation of motion when we derive it. We will assume we are working with a semi-simple compact group i.e. SU(N). This will also allow us to ultimately compare our results with results from well-known theories, specifically QCD, in the limit where ζ vanishes. This gives us the following relationships between our structure constants:

$$f^{abc} = f^{bca} = f^{cab} = -f^{cba} = -f^{bac} = -f^{acb}. \tag{3.15}$$

This now allows us to simplify Eq.3.14 as follows:

$$\partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A^a_\mu)}\Big) = \partial_\nu F^{a\,\mu\nu} + \zeta\partial^\mu(\partial^\nu A^a_\nu). \tag{3.16}$$

Now we will find the left hand side of Eq.3.10.

$$\frac{\partial\mathcal{L}}{\partial(A^h_\gamma)} = \frac{\partial}{\partial(A^h_\gamma)}\Big(-\frac{1}{2}f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)\eta^{\mu\alpha}\eta^{\nu\beta}A^b_\alpha A^c_\beta - \tag{3.17}$$

$$\frac{1}{4}f^{abc}f^{ade}\eta^{\mu\alpha}\eta^{\nu\beta}A^b_\mu A^c_\nu A^d_\alpha A^e_\beta - \bar{\psi}\gamma^\mu(i\partial_\mu - \lambda^a A^a_\mu)\psi\Big) \tag{3.18}$$

$$= -\frac{1}{2}f^{ahc}A^c_\mu F^{a\,\gamma\mu} - \frac{1}{2}f^{abh}A^b_\mu F^{a\,\mu\gamma} - \bar{\psi}\gamma^\gamma\lambda^h\psi \tag{3.19}$$

With a relabeling of indices, we get the LHS side of Eq.3.10:

$$\frac{\partial\mathcal{L}}{\partial(A^a_\mu)} = -\frac{1}{2}f^{cab}A^b_\nu F^{c\,\mu\nu} - \frac{1}{2}f^{cba}A^b_\nu F^{c\,\nu\mu} - \bar{\psi}\gamma^\mu\lambda^a\psi. \tag{3.20}$$

It is important to note that the above result is not trivial and requires one to expand the relation completely using the product rule and cleverly relabel indices in order to combine and eliminate the resulting terms to get a simpler result. Next, we will apply our structure constant relations detailed in Eq.3.15 and obtain the left hand side of Eq.3.10 in its simplest form.

$$\frac{\partial \mathcal{L}}{\partial (A^a_\mu)} = -f^{abc} A^b_\nu F^{c\,\mu\nu} - j^{a\,\mu}, \quad \text{with} \quad j^{a\,\mu} \equiv \bar{\psi}\gamma^\mu \lambda^a \psi \tag{3.21}$$

Finally, we will combine Eq.3.21 and Eq.3.16 to get our equation of motion.

$$j^{a\,\mu} = -\partial_\nu F^{a\,\mu\nu} - f^{abc} A^b_\nu F^{c\,\mu\nu} - \zeta \partial^\mu (\partial^\nu A^a_\nu) \tag{3.22}$$

3.2.2 Noether Current

The next step we will take is finding the Noether current and evaluating it on-shell. Using our definition of the Noether current as shown in Eq.2.7 we get the following relation:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\nu)} \frac{\partial (A^a_\nu)'}{\partial \beta} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \frac{\partial \psi'}{\partial \beta} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \frac{\partial \bar{\psi}'}{\partial \beta}. \tag{3.23}$$

The third term in Eq.3.23 vanishes because our Lagrangian does not depend on the four-gradient of the adjoint spinor field. In this case, β will be some constant that parameterizes our local transformations as follows:

$$A^a_\mu \to (A^a_\mu)' = A^a_\mu + \beta \partial_\mu (\chi^a) - \beta f^{bca} A^b_\mu \chi^c \tag{3.24}$$

$$\psi \to \psi' = (1 - i\beta \lambda^a \chi^a)\psi \tag{3.25}$$

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}(1 + i\beta \lambda^a \chi^a). \tag{3.26}$$

Finding the variations of the fields with respect to the variational parameter as well as finding the variation of the Lagrangian with respect to $\partial_\mu \psi$ is straightforward and the results are shown below.

$$\frac{\partial (A^a_\nu)'}{\partial \beta} = \frac{\partial}{\partial \beta}(A^a_\nu + \beta \partial_\nu (\chi^a) - \beta f^{bca} A^b_\nu \chi^c) = \partial_\nu (\chi^a) - f^{bca} A^b_\nu \chi^c \tag{3.27}$$

$$\frac{\partial \psi'}{\partial \beta} = \frac{\partial}{\partial \beta}((1 - i\beta \lambda^a \chi^a)\psi) = -i\lambda^a \chi^a \psi \tag{3.28}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{\partial}{\partial(\partial_\mu \psi)}(\bar{\psi}\gamma^\mu (i\partial_\mu - \lambda^a A^a_\mu)\psi) = i\bar{\psi}\gamma^\mu. \tag{3.29}$$

We have already found the variation of the Lagrangian with respect to the four-gradient of the gauge fields when the equation of motion was derived. Coupling that result with these equations we can see that the Noether current is

$$
\begin{aligned}
J^{\mu} &= (F^{a\,\nu\mu} + \xi\eta^{\nu\mu}(\partial^{\gamma}A^{a}_{\gamma}))(\partial_{\nu}(\chi^{a}) - f^{bca}A^{b}_{\nu}\chi^{c}) + (i\bar{\psi}\gamma^{\mu})(-i\lambda^{a}\chi^{a}\psi) \\
&= -F^{a\,\mu\nu}\partial_{\mu}(\chi^{a}) + f^{bca}F^{a\,\mu\nu}A^{b}_{\nu}\chi^{c} + \zeta(\partial^{\mu}A^{a}_{\mu})\partial^{\nu}(\chi^{a}) - f^{bca}\zeta(\partial^{\nu}A^{a}_{\nu})A^{b\,\mu}\chi^{c} + j^{a\,\mu}\chi^{a}. \quad (3.30)
\end{aligned}
$$

We can evaluate this on-shell by plugging in the value of $j^{a\mu}\chi^{a}$ as given by our equation of motion in Eq.3.22. After simplifying and applying Eq.3.15 as we did with our equation of motion, we get

$$
\begin{aligned}
J^{\mu}\big|_{\text{shell}} &= -\partial_{\nu}(\chi^{a}F^{a\,\mu\nu}) + \zeta\bigg(\varphi^{a}\partial^{\mu}(\chi^{a}) - \partial^{\mu}(\varphi^{a})\chi^{a} - f^{abc}\varphi^{a}A^{b\,\mu}\chi^{c}\bigg) \quad (3.31) \\
\varphi^{a} &\equiv \partial^{\gamma}A^{a}_{\gamma}.
\end{aligned}
$$

With this form of the Noether current we can determine that the symmetries corresponding to our local transformations are physical symmetries. We can determine this by looking at the two terms of Eq.3.31. The first term is indeed trivially conserved on-shell by its index structure, as $F^{a\mu\nu}$ is anti-symmetric in its spacetime indices and $\partial_{\mu}\partial_{\nu}$ is symmetric in its spacetime indices. The ζ term does not vanish when evaluated on shell, and it is clearly not trivially conserved by index structure, so we can conclude that we have physical symmetries.

3.2.3 Emergence

In order to get emergence, we need to find a constraint that will turn our physical symmetries into gauge symmetries. This requires a constraint that will make our Noether current trivially conserved on-shell. With the Noether current in the form given in Eq.3.31 we can see that every term with a ζ coefficient has a factor of ϕ^{a}. As a result, if we impose the condition $\phi^{a} = 0$ then the ζ term will disappear entirely and our Noether current will now be trivially conserved on-shell by its index structure. It is also important to make sure that this constraint does not trivialize our fields. We can show this by first taking a Helmholtz decomposition of our fields.

$$
A^{a}_{\mu} = \bar{A}^{a}_{\mu} + \partial_{\mu}\xi^{a}, \quad \partial^{\mu}\bar{A}^{a}_{\mu} = 0 \quad (3.32)
$$

This is equivalent to splitting the field into spin-1 and spin-0 contributions. This is easier to see by writing ξ using a Green function.

$$
\varphi^{a} = \partial^{\mu}(A^{a}_{\mu}) = \partial^{\mu}(\bar{A}^{a}_{\mu} + \partial_{\mu}\xi^{a}) = \Box\xi^{a} \quad (3.33)
$$

$$\implies \xi = \int d^4x' G(x,x')\varphi(x') \tag{3.34}$$

This means that as $\varphi^a \to 0$, $A^a_\mu \to \bar{A}^a_\mu$ with $\partial^\mu(\bar{A}^a_\mu) = 0$. This is indeed a non-trivial result, which tells us that this constraint allows gauge symmetries to emerge without trivializing the fields. The last thing we need to check is that this new theory "stays emerged". To check this we will apply a gauge transformation to our new fields and ensure that they retain invariance.

$$\varphi^a \to (\varphi^a)' = \partial^\mu(\bar{A}^a_\mu + \partial_\mu(\chi^a) - f^{bca}A^b_\mu\chi^c) = \Box\chi^a - f^{bca}\partial^\mu(A^b_\mu\chi^c) = 0 \tag{3.35}$$

Recall that the final step in the above transformation involves the application of the condition on χ^a shown in Eq.3.9. This confirmation concludes the process used for developing a consistent theory in which physical symmetries can be transformed to gauge symmetries in a non-abelian setting without trivializing any of the fields.

CHAPTER 4
FIBER BUNDLES AND DIFFERENTIAL GEOMETRY

In order to thoroughly explain the goals of this project it is necessary to first dedicate some time to clearly define fiber bundles and different types of fiber bundles. A fiber bundle is typically specified with four components: (T,B,π,F). T, B, and F are all smooth manifolds and will also be referred to as the total space, base space, and fiber, respectively. The last component, π, is the surjective map $\pi : T \to B$. The total space locally looks like $B \times F$. In other words, if we have a set of open coverings (also referred to as local neighborhoods) $\{U_i\}$ of B then T looks like $B \times F$ near these coverings. In the case that $T = B \times F$ we have what is called a trivial bundle [13]. Another important component that is not in the list of four mentioned above is a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ which we will call a local trivialization. This also allows us to define individual fibers as $f_i : \pi^{-1}(x) \to F, x \in B$. This function gives us a corresponding fiber for every point in the base space. It is important to remember that the fiber is a differentiable manifold, so π^{-1} is not an inverse operation of π. This is demonstrated in Figure 4.1.

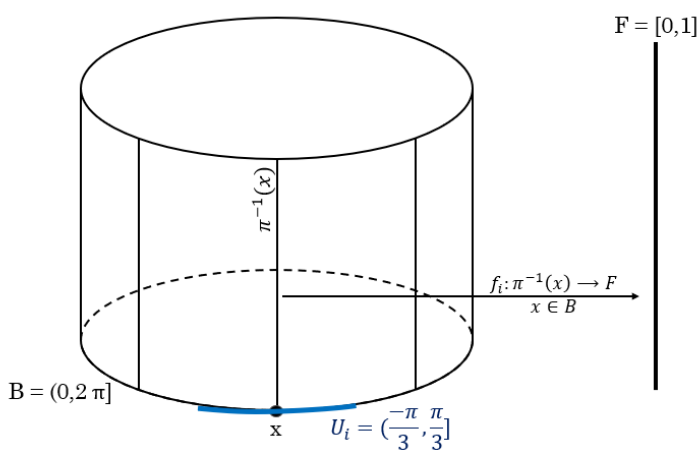

Figure 4.1: This is an example of a fiber bundle as defined in this section. This simple example is just a cylinder. In this example F is simple a linear interval from 0 to 1, B is a circle (interval from 0 to 2π) and $T = B \times F$, so we have a trivial bundle.

A fiber bundle also has a Lie group, G, that acts on the left of F and is called the structure group. With this information, we can introduce and define transition functions. Transition functions are necessary to map between fibers and will become more important later on when we define connections on bundles. We will denote the fiber at a point as $\phi_{i,p} = \phi(p, f)$, that is to say $\pi \circ \phi_{i,p} = \pi(\phi(p, f)) = p$. To map between fibers we can consider $g_{ij} \equiv \phi_{i,p}^{-1} \circ \phi_{j,p}$.

Which maps $F \to F$. Using the structure group, which acts on the left of a fiber, f_i, we can conclude $gij : U_i \cap U_j \to G$. Now we can state the action of the transition function on a fiber as $f_j = g_{ij}(p)f_i$. This is an important point that will come in later in the paper.

4.1 Sections on Bundles

In order to define physical quantities in terms of fiber bundles we will need to work with objects called sections. A section is defined as a map that brings us from a point in the base space to a point in the total space: $\sigma : B \to T$. In addition, σ must satisfy $\pi(\sigma(x)) = x, x \in B$. Before it was mentioned that π^{-1} is not an inverse operation of π. It would be accurate to think of σ as an inverse operation to π. To summarize, π can take us from some point in the total space to a unique point in the base space, π^{-1} can take us from a point in the base space to the fiber associated with that point, and σ can take us from some point in the base space to a unique point in the total space.

When creating a geometrical interpretation of the standard model using fiber bundles, it is a primary concern to find some way to define spacetime scalar fields as well as gauge transformations using fiber bundles. By this point it should be clear that it does not make sense to try to define any of our three spaces as spacetime functions or gauge transformations due to the non-uniqueness inherent with a full bundle and its associated surjective map, π. However, we can define spacetime scalar fields as sections on bundles, and the same applies for gauge transformations [11]. Of course, this still leaves us with an important hole. More specifically, it is not yet clear where gauge fields will arise in this picture. We will find that in order to answer this, and ultimately in order to produce a working geometrical theory for the standard model, we will have to use two specific types of fiber bundles: principal bundles and associated bundles.

4.2 Principal Bundles, Associated Bundles, Dual Bundles, and Spinor Bundles

A principal bundle is a fiber bundle whose fiber is its own structure group i.e. $F = G$. When referencing principal bundles the fiber and structure group will both be denoted as G unless stated otherwise to avoid confusion, and the total space will be denoted as P instead of T to make it clear that we are referring to a principal bundle instead of a normal fiber bundle. Recall that the transition functions, g_{ij} defined the left action of G on F. Since G and F are now equivalent, G acts on itself and we can now also define a right action of G on itself. The left action of G on the fiber (the transition functions) was a map between fibers; the right action of G on the fiber can be seen as a map between points in the same fiber. We will denote this right action as $ug = \phi(p, u_i g)$ for $g \in G, u \in \pi^{-1}(U_i)$. Figure

4.2 demonstrates the structure of a principal bundle and the functions of the left and right actions.

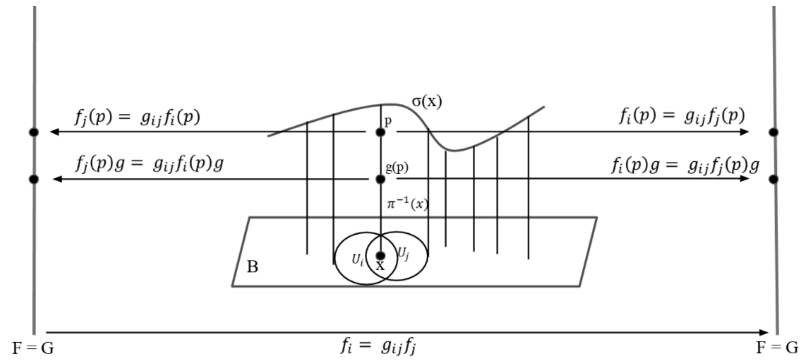

Figure 4.2: This is an example of a principal bundle as defined in this section. This demonstrates how the fiber and structure group are equivalent, and how the left and right action can be used in relation to fibers.

In addition to principal bundles, it is important to define associated bundles. Two bundles are associated bundles with respect to one another if they share the same base space, structure group, and transition functions.Note that they do not have to share the same fiber. Associated bundles are useful because they have been used to help make complete geometrical descriptions of theories in the standard model. Earlier it was mentioned that both spacetime scalar fields and gauge transformations can be described as sections on bundles, but this might raise some questions initially because it would not make sense to say they are different sections on the same bundle. As it turns out, the gauge transformations are sections on a principal bundle, while the scalar field is a section on the associated bundle of this principal bundle. This is demonstrated in Figure 4.3.

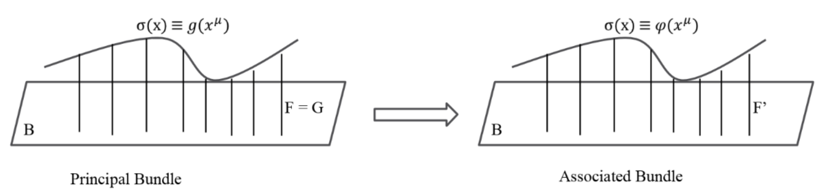

Figure 4.3: This is an example of a principal bundle and its associated bundle. This example shows the relationship between gauge transformations and scalar fields in the context of fiber bundles.

Dual bundles are important to define because they are required to differentiate between vectors and dual vectors in the context of fiber bundles. This will become important when we attempt to define Lagrangians in terms of fiber bundles because there will ultimately be terms in our Lagrangian that have both gauge vector fields and gauge dual vector fields, and we need a way to distinguish these objects if we are to define them in terms of fiber bundles.

For our purposes it makes the most sense to define a dual bundle with respect to another principal bundle, P. The dual bundle, P^*, is defined as the associated bundle of P whose fibers are the dual spaces to the fibers of P. The transition functions of P are g_{ij} while the transition functions of P^* are given by $g^*_{ij} = (g^T_{ij})^{-1}$

Spinor bundles are worth mentioning, but are not vital to our analysis of emergence, so we will not go into too much detail in defining them. Given some spin structure, one can define an associated spinor bundle. We can define spinor fields as sections over spinor bundles. This is mentioned primarily because it is important to see that all parts of a given Lagrangian can be defined in terms of geometric objects using fiber bundles, and even though spinor bundles have little relevance to this analysis, this puzzle would feel incomplete if spinor bundles were not at least meantioned.

4.3 The Pushforward and Pullback

It will soon be important to be able to clearly define an object called the connection one-form, which will allow us to define gauge fields in the context of fiber bundles. Before we can define the connection one-form; however, it is important that we understand operations referred to as the pushforward and the pullback.

Previously we have talked about several different maps that take us from points in one manifold to points in another manifold e.g. $\pi : T \to B$. However, in our analysis so far we have not been provided with any information regarding mappings between tangent spaces. This gives rise to the following question: What effect does a map between manifolds have, if any at all, on the tangent spaces to these manifolds? As it turns out, there is important information regarding maps between tangent spaces encoded into maps between manifolds, in other words, a map between two manifolds induces a map between their tangent spaces. Consider the map between the total space and the base space of a principal bundle $\pi : P \to B$. Also consider a map between the base space and the real numbers $h : B \to \mathbb{R}$ and a function γ such that $\gamma : \mathbb{R} \to P$ where $\gamma(0) = p \in P$ and $\dot{\gamma}(0) = x \in T_pP$. It is important to note a detail here that will come back several times throughout this paper. If we have a curve characterizing points in a manifold then differentiating this curve at a point will give us an element of the tangent space at that same point. Bearing this in mind, we will now define the pushforward of x by π as [14]

$$\pi_* x = \frac{d}{dt}\pi(\gamma(t))|_{t=0}. \tag{4.1}$$

This provides us with a map between the tangent space of P and the tangent space of B i.e. $\pi_* : T_pP \to T_{\pi(p)}B$. At this point it would be natural to ask if there is a well defined inverse operation that could take the pushforward of a vector in $T_{\pi(p)}B$ and give us back the original vector in T_pP. The answer to this is no; however, we can "pullback" a function. The pullback of a function will be defined as [14]

$$\pi^* hx = h(\pi_* x) = h(\frac{d}{dt}\pi(\gamma(t))|_{t=0}). \tag{4.2}$$

Essentially we are just looking at how h acts on vectors in B and using this to characterize a new function acting on vectors in $T_{\pi(p)}B$. It is also important to note how the definition of the pushforward is embedded in the definition of the pullback. In essence, the pullback is looking at how a function acts on the pushforward.

4.4 The Connection One-Form

When attempting to define Lagrangians in terms of fiber bundles it is important to be able to find a specific definition for every component of the Lagrangian using the fiber bundle language. Defining sections over bundles was the first step to doing this because sections allow us to define unique equations of motion as well as other continuous functions. The next big step in this process was to define associated bundles and dual bundles, which allowed us to relate scalar fields to gauge transformations and define general relationships between spaces and dual spaces using fiber bundles. The next step we will take is to define an object called the connection one-form, which will allow us to define gauge fields in the context of fiber bundles.

First lets consider the tangent space to a point in a principal bundle, T_uP where $u \in P$. We can divide the tangent space to our total space at a point u, T_uP, into two subspaces that we will call the horizontal subspace, H_uP and the vertical subspace, V_uP. In order to divide our space as stated the following three criteria must be met:

1. $T_uP = H_uP \oplus V_uP$,

2. A smooth vector field, x, on P is separated into smooth vector fields $x^H \in H_uP$ and $x^V \in V_uP$ with $x = x^H + x^V$,

3. $H_{ug}P = R_g^* H_uP$ for $u \in P$ and $g \in G$.

Figure 4.4 paints a picture for what this splitting of the tangent space looks like as well as how the right action of $g \in G$ changes this picture.

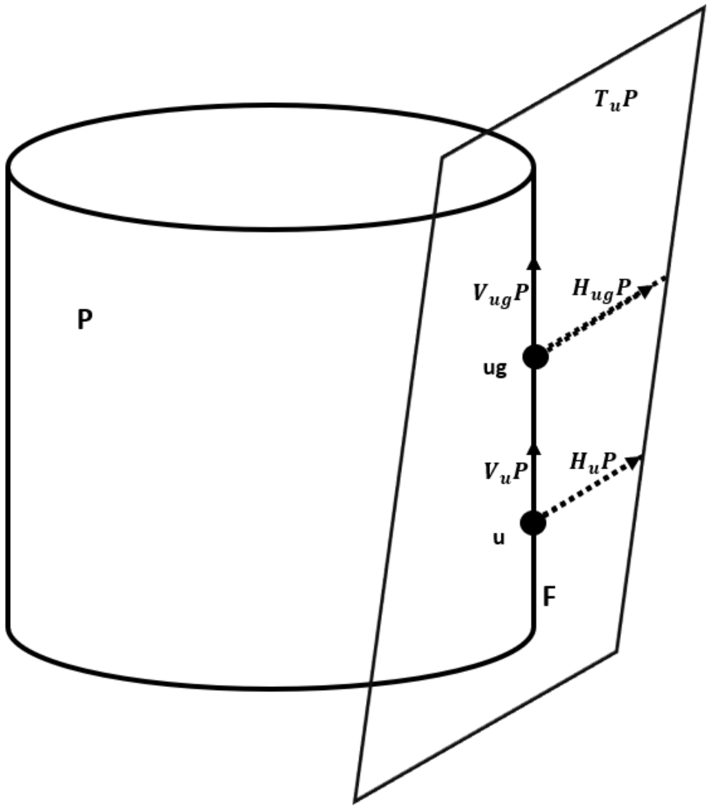

Figure 4.4: This is an example of what a splitting of the tangent space into vertical and horizontal components would look like for the cylindrical trivial bundle used in earlier sections at a point $u \in P$. This also demonstrates how the right action of $g \in G$ affects the horizontal subspace by "lifting" it along the fiber.

The vertical point of T_uP, v_u, will be defined as that which is parallel to our Lie group, G, at the point u. We can now act on points using a right action characterized by elements of the Lie algebra, $\mathcal{G}$.

$$R_t u = ue^{tA} \tag{4.3}$$

Where $A \in \mathcal{G}$ and t parameterizes a path through u. This path goes along a single fiber. In other words, the right action simply changes our position along a fiber, it does nothing to the base, nor does it bring us to a different fiber. It follows that

$$\pi(u) = \pi(ue^{tA}) = p \tag{4.4}$$

for some $p \in B$. Take some arbitrary function $f : P \to \mathbb{R}$. We now define vectors, $A^{\#}$ which are tangent to G. It is common and straightforward to define a tangent space at a point simply by differentiating at that point, so we will define $A^{\#}$ by

$$A^{\#}f(u) = \frac{d}{dt}f(ue^{tA})|_{t=0}. \tag{4.5}$$

This is an isomorphism which takes $A \to A^{\#}$, i.e. $\# : \mathcal{G} \to V_uP$. Bearing these things in mind, it is now possible to define a Lie algebra valued one-form, $\omega \in \mathcal{G} \otimes T^*P$, such that [13]

1. $\omega(A^{\#}) = A$,
2. $R_g^* = g^{-1}\omega_u(x)g$,
3. H_uP is now defined by the kernel of ω i.e. $H_uP = \{x \in T_uP|\omega(x) = 0\}$.

Now we have defined a connection one-form which satisfies the conditions outlined at the beginning of this section, but it is not clear with this information alone how this one-form relates to gauge fields or gauge transformations. As we have defined it, ω is a global quantity containing information about P. We need a way to promote this information to a local regime. To do this, first we will define the local pullback of the connection one form over a local neighboorhood of points, U_i, in the base space as

$$\mathcal{A}_i = \sigma_i^*\omega. \tag{4.6}$$

To get the proper gauge transformation rules on fiber bundles we need another important relationship. To define this relationship first consider a curve $\gamma : [0, 1] \to B$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = x$. Now, using the definition of the pushforward of a vector as well as the relationship governing local sections i.e. $\sigma_j(p) = \sigma_i(p)g_{ij}(p)$ we can show the following:

$$\begin{aligned}
\sigma_{j*}(x) &= \frac{d}{dt}\sigma_j(\gamma(t))|_{t=0} = \frac{d}{dt}\{\sigma_i(\gamma(t))g_{ij}(\gamma(t))\}|_{t=0} \\
&= \frac{d}{dt}\sigma_i(\gamma(t))g_{ij}(p) + \sigma_i(p)\frac{d}{dt}g_{ij}(\gamma(t))|_{t=0} \\
&= R_{g_{ij}*}(\sigma_{i*}x) + \sigma_j(p)g_{ij}(p)^{-1}\frac{d}{dt}g_{ij}(\gamma(t))|_{t=0}.
\end{aligned} \tag{4.7}$$

We now note that

$$g_{ij}(p)^{-1}dg_{ij}(x) = g_{ij}(p)^{-1}\frac{d}{dt}g_{ij}(\gamma(t))|_{t=0}, \tag{4.8}$$

where d denotes the exterior derivative. Since $g_{ij}(p)^{-1}gij(\gamma(t)) = e$ at $t = 0$ we can deduce that the second term in the last line of equation 4.7 represents $(g_{ij}^{-1}dg_{ij}(x))^{\#}$ evaluated at $\sigma_j(p)$ using equation 4.5. This gives us the following general relationship:

$$\sigma_{j*}x = R_{g_{ij}*}(\sigma_{i*}x) + (g_{ij}^{-1}dg_{ij}(x))^{\#} \tag{4.9}$$

Now, if we apply the connection one-form on equation 4.9 and use the definition of the pullback we can say

$$\begin{aligned} \sigma_j^*\omega(x) &= R_{g_{ij}*}\omega(\sigma_{i*}x) + (g_{ij}^{-1}dg_{ij}(x))^{\#} \\ &= g_{ij}^{-1}\omega(\sigma_{i*}x)g_{ij} + g_{ij}^{-1}dg_{ij}(x) \\ &\rightarrow \mathcal{A}_j = g_{ij}^{-1}\mathcal{A}_i g_{ij} + g_{ij}^{-1}dg_{ij} \end{aligned} \tag{4.10}$$

because this must hold for all $x \in T_pB$. The final line of equation 4.10 is the result we have been looking for. To see this more clearly, lets look at an example. If we let P be a U(1) bundle over B then our transition function can be given by $g_{ij} = e^{i\chi(p)}$ for some $\chi(p) \in \mathbb{R}$. U(1) is an abelian symmetry group, so equation 4.10 reduces quite nicely to $\mathcal{A}_i(p) + id\chi(p)$ which, in components, reduces further to $\mathcal{A}_{j\mu} = \mathcal{A}_{i\mu} + i\partial_\mu\chi$. It is worth noting that the local connection varies from the gauge potential by a Lie algebra factor, i.e. $\mathcal{A}_\mu = iA_\mu$ [13]. This result is our standard local gauge transformation rule for U(1). To summarize this whole section we can say the following: gauge fields can be represented by the pullback of the connection one-form to a local chart over the base space by a section of the principal bundle.

4.5 The Wedge Product and the Exterior Derivative

In the previous section the term "exterior derivative" was briefly mentioned and glossed over. The reason it was glossed over before was because the concept of the exterior derivative was not vitally important to the punchline of the previous section. However, moving forward it will be necessary that we understand what an exterior derivative is because such knowledge is necessary to define an object called the curvature two-form, which we will find can be used to represent the field strength tensor in the fiber bundle formalism.

In the field of differential geometry it is important to be able to define coordinate independent objects. It is also important to be able to do high dimensional calculus that is independent of coordinates. To do this, it is useful to define objects called differential forms. A rank p differential form is a rank (0,p) tensor that is completely anti-symmetric in its indices [7]. Based upon this it is clear that a scalar is simply a 0-form, and a dual vector is a 1-form. However, once we get into higher order forms things get non-trivial and we have to pay careful attention to how we construct these higher order differential forms. For example, a 2-form would have to obey the condition $F_{\mu\nu} = -F_{\nu\mu}$ and a 3-form would have to satisfy $F_{\mu\nu\lambda} = F_{\lambda\mu\nu}$ and so on for all even permutations of μ, ν, and λ as well as $F_{\mu\nu\lambda} = -F_{\nu\mu\lambda}$ and so on for all odd permutations of μ, ν, and λ. This same rule can be generalized to higher order differential forms as well and is, by definition, the same rule governing the components of a Levi-Civita symbol of arbitrary order.

The next step in understanding differential forms is understanding how to combine differential forms. Whenever we define new mathematical objects it is important to know how to combine them. An important condition for this is that combining two of the same mathematical object should yield an object of the same class as its constituents. For example, two scalar numbers can be combined via scalar multiplication to make another scalar number. With tensors we can obey Einstein summation notation to combine tensors e.g. $A^{\mu}B^{\nu} = C^{\mu\nu}$, $F^{\alpha}_{\mu\nu}G^{\gamma\beta}_{\alpha} = H^{\gamma\beta}_{\mu\nu}$ etc. The important thing to note here is that when we combine tensors in this way we get another tensor as a result. Likewise, it is important that combining two differential forms results in another differential form. Obviously, simply applying Einstein summation convention to the components will not suffice because there is no guarantee that the result will be anti-symmetric. To solve this problem we introduce what is called the wedge product. For some p-form and q-form the wedge product can be written as

$$A^{(p)} \wedge B^{(q)} = C^{(p+q)}. \tag{4.11}$$

Which, in components can be written as follows:

$$A_{\mu_1\mu_2\ldots\mu_p} \wedge B_{\nu_1\nu_2\ldots\nu_q} = \frac{(p+q)!}{p!q!}A_{[\mu_1\mu_2\ldots\mu_p}B_{\nu_1\nu_2\ldots\nu_p]}. \tag{4.12}$$

Where the brackets [] denote an anti-symmetrization of the components of A and B. The anti-symmetrization operation is very mechanical and can be accomplished as follows:

$$F_{[\mu_1\mu_2\ldots\mu_n]} = \frac{1}{n!}(\sum_{i=1}^{n!/2} F^{|i|}_{\mu_1+\mu_2+\ldots\mu_n} - \sum_{i=1}^{n!/2} F^{<i>}_{\mu_1+\mu_2+\ldots\mu_n}). \tag{4.13}$$

Where $F^{|i|}_{\mu_1+\mu_2+\ldots\mu_n}$ will represent all even permutations of the indices of F and $F^{<i>}_{\mu_1+\mu_2+\ldots\mu_n}$ will represent all odd permutations of the indices of F. As an example, we will assume A and B are both 1-forms. The wedge product of A and B would then look like

$$A_{\mu} \wedge B_{\nu} = A_{\mu}B_{\nu} - A_{\nu}B_{\mu}. \tag{4.14}$$

Now this example might seem a bit familiar. The form of this past example could be used to suggest that the electromagnetic field strength tensor $_{\mu\nu}$, is a wedge product of the gauge potential, A_{μ}, and the four gradient, ∂_{μ} due to the suggestive form of the following

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} = \partial_{[\mu}A_{\nu]}. \tag{4.15}$$

However, there is actually something slightly different at play here, and it has to do with the fact that we are trying to combine a 1-form with derivatives. Now earlier we learned of the importance of antisymmetrization to differential forms, and we used this rule to define an operation to combine differential forms. Now we will use that same argument to define differentiation of differential forms. In other words, we will define a new type of derivative that will act on a differential form and return to us another differential form. This new operation will be called the exterior derivative and will be defined as follows:

$$dA^p = B^{p+1}. \tag{4.16}$$

Which, in components, looks like

$$dA^p = \partial_{[\mu} A_{\nu_1 \nu_2 \dots \nu_n]}. \tag{4.17}$$

This is a very powerful operation because it allows us to classify how differential forms change in a way that is independent of coordinates and depends only on the topology of the space. The exterior derivative is of paramount importance to understanding the field strength tensor in terms of fiber bundles and the next section will go over this in detail. However, referring back to our example in equation 4.15, it is important to note that this is a specific example where the field strength tensor is equal to the exterior derivative of the gauge potential. As we will learn in the next section this is due to the fact that electromagnetism is described by U(1), which is an abelian symmetry group, and that for non-abelian symmetry groups we cannot relate these two objects with an exterior derivative alone.

4.6 The Curvature Two-form

In section 4.4 we discovered a way to express gauge fields as local pullbacks of connections on principal bundles. This was useful because it gave us a correspondence between fiber bundles and functional theory with regard to gauge fields, which is important when defining Lagrangians and classifying symmetries. However, there is another important object that is important to define in the context of fiber bundles: the electromagnetic field strength tensor, $F_{\mu\nu}$.

It is standard to define the electromagnetic field strength tensor (which will also be referred to as the electromagnetic two-form) in terms of the electromagnetic gauge potential, A^μ. More specifically, the electromagnetic two-form is the exterior derivative of the electromagnetic gauge potential [15].

$$F = dA \tag{4.18}$$

In components, this becomes the familiar expression:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.19}$$

This result is suggestive and we can use this result along with our understanding of the connection one-form to define an analogous two-form on a principal bundle called the curvature two-form.

Define the curvature two-form, Ω, as the covariant derivative of the connection one-form [13], i.e.

$$\Omega(X,Y) = D\omega(X,Y) = d_p\omega(X^H, Y^H). \tag{4.20}$$

Where the X and Y in parentheses simply denotes that a two-form will take two vectors X and Y as arguments and produce a scalar and X^H and Y^H are the horizontal projections of X and Y. The first step we will take is to specify a more useful form of equation 4.20 in terms of the exterior derivative and the wedge product. First, we will assert that for two vectors $X, Y \in T_uP$ that Ω and ω satisfy Cartan's structure theorem, that is to say

$$\Omega(X,Y) = d_p\omega(X,Y) + [\omega(X), \omega(Y)]. \tag{4.21}$$

To prove this assertion, it is sufficient to show that equation 4.21 is equivalent to equation 4.20 in the following three cases [13]: $(X, Y \in H_uP)$, $(X \in H_uP, Y \in V_uP)$, and $(X, Y \in V_uP)$.

1. $X, Y \in H_uP$: Since we have defined H_uP to be the kernel of ω, then we can say that $\omega(X) = \omega(Y) = 0$. The rest is simple, as equation 4.21 reduces to $\Omega(X,Y) = d_p\omega(X,Y)$, which is equal to equation 4.20 by definition since $X = X^H$ and $Y = Y^H$ in this case.

2. $X \in H_uP, Y \in V_uP$: Now, if $Y^H = 0$ then equation 4.20 reduces to $\Omega(X,Y) = d_p\omega(X^H, Y^H) = 0$, so we must show that equation 4.21 also reduces to zero. Since $\omega(X) = 0$ by definition we can see that the second term of equation 4.21 goes to zero, so we now only need to show that $d_p\omega(X,Y) = 0$. It can be shown that [13]

$$d_p\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]), \tag{4.22}$$

which reduces to $d_p\omega(X,Y) = X\omega(Y) - \omega([X,Y])$. Since $Y \in V_uP$ there exists some $A \in \mathcal{G}$ such that $Y = A^{\#}$, by definition of the # operation. Also recall one of the three properties of ω holds that $\omega(A^{\#}) = A$. Using this we can say $X\omega(Y) = X \cdot A = 0$. The last remaining step now is to show that $\omega([X,Y]) = 0$. It can be shown that [13] $[X,Y] \in H_uP$, so then $\omega([X,Y]) = 0$ and ultimately $d_p\omega(X,Y) = 0$.

3. $X, Y \in V_uP$: By definition, X and Y have no horizontal components, so we can say that $\Omega(X,Y) = D\omega(X,Y) = d_p\omega(X^H, Y^H) = 0$. So we must show that the right hand side of equation 4.21 goes to zero, that is

$$d_p\omega(X,Y) + [\omega(X), \omega(Y)] = X\omega(Y) - Y\omega(X) - \omega([X,Y]) + [\omega(X), \omega(Y)] = 0. \quad (4.23)$$

Using the same arguments as in the second case we can say that $X\omega(Y) = Y\omega(X) = 0$. Now all that remains is to show that $\omega([X,Y]) + [\omega(X), \omega(Y)] = 0$. For $X, Y \in V_uP$ it is true that $[X,Y] \in V_uP$ [13]. As a result, we can say that $[X,Y] = A^{\#} \in V_uP$ and that there exists some $A \in \mathcal{G}$ such that $\omega([X,Y]) = A$. We can also say that $X = B^{\#}$ and $Y = C^{\#}$ for some $B, C \in \mathcal{G}$. Then we can conclude that $[\omega(X), \omega(Y)] = [B, C] = A$, which tells us that $\omega([X,Y]) + [\omega(X), \omega(Y)] = A - A = 0$.

Now that we have shown that equation 4.20 can be written in the form of equation 4.21 we will simplify equation 4.21 further to get a result that is strictly in terms of an exterior derivatives and a wedge product. First define the commutator of a p-form, α, and a q-form, β, by [13]

$$[\alpha, \beta] = \alpha \wedge \beta - (-1)^{pq}\beta \wedge \alpha. \quad (4.24)$$

Now, consider the following expansion of $[\omega, \omega](X,Y)$ in terms of its basis vectors, $T_\mu \in \mathcal{G}$.

$$\begin{aligned} [\omega, \omega]\,(X,Y) &= [T_\mu, T_\nu]\omega^\mu \wedge \omega^\nu(X,Y) \\ &= [T_\mu, T_\nu][\omega^\mu(X), \omega^\nu(Y) - \omega^\nu(X)\omega^\nu(Y)] \\ &= [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)] = 2[\omega(X), \omega(Y)] \end{aligned} \quad (4.25)$$

Now, using equation 4.24 we can write equation 4.21 as

$$\Omega(X,Y) = (d_p\omega + \frac{1}{2}[\omega, \omega])(X,Y) = (d_p\omega + \omega \wedge \omega)(X,Y). \quad (4.26)$$

Now that we have a more amenable expression for the curvature two-form, we will take the local pullback of a section in the exact same way as we did for the connection one-form and define $\mathcal{F} = \sigma^*\Omega$. Plugging in our result from equation 4.26 we get that

$$\begin{aligned}\mathcal{F} &= \sigma^* d_p\omega + \sigma^*(\omega \wedge \omega) \\ &= d\sigma^*\omega + \sigma * \omega \wedge \sigma^*\omega \\ &= d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}.\end{aligned} \tag{4.27}$$

This is a very useful result because we can directly calculate the components of the local curvature two-form using the definitions of the exterior derivative and the wedge product in equations 4.17 and 4.12, respectively.

$$\mathcal{F} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + \mathcal{A}_\mu \mathcal{A}_\nu - \mathcal{A}_\mu \mathcal{A}_\nu = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{4.28}$$

Finally, we can expand the local connection one-form and curvature two-form in terms of the basis of $\mathcal{G}$ i.e. $\mathcal{A}_\mu = A^\alpha_\mu T_\alpha$ and $\mathcal{F}_{\mu\nu} = F^\alpha_{\mu\nu} T_\alpha$, and use the commutation relations [13] $[T_a, T_b] = f^c_{ab} T_c$ to obtain the expected expression for a Yang-Mills field strength tensor.

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu \tag{4.29}$$

It is also good to note that this result does indeed simplify appropriately to $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for electromagnetism, which as we have stated is an abelian symmetry group whose structure constants, f^c_{ab}, are all zero.

CHAPTER 5
RESULTS AND CONCLUSIONS

5.1 Geometrization of the Abelian Emergent Gauge Theory

The content of this chapter has been concerned primarily with familiarizing the reader with various types of fiber bundles, operations on these bundles, and geometrical quantities that can be defined on fiber bundles as well as physical interpretations of these objects in the context of particle physics. The physical interpretations are as follows:

1. Transformations can be described by sections over a principal fiber bundle with a specified structure group, G. An example of this would be a principal bundle whose fibers are U(1). Sections over this bundle can describe transformations that are symmetries of the Lagrangian describing electromagnetism.

2. A connection one-form can be defined whose purpose is to split the tangent space into horizontal and vertical subspaces. A gauge field can be described by a pullback of the connection one-form to a local chart over the base space by a section of the principal fiber bundle.

3. Associated spinor bundles can be defined for our principal bundle. These spinor bundles have transformation rules that ultimately depend on those described by the principal bundle due to its association. spinor fields can be described by local sections over spinor bundles.

4. The covariant derivatives of spinor functions used in the Lagrangian can be described by parallel propagation of elements of the associated spinor bundle along paths defined by local sections.

5. A field strength tensor (like $F^{\mu\nu}$) can be described by a pullback of the curvature two-form to a local chart over the base space by a section of the principal fiber bundle.

6. Dual bundles can be defined for the principal and spinor bundles. Local sections over these bundles give us quantities that we need to form invariants in our Lagrangian.

7. A Lagrangian is just a constraint specifying the allowed spaces of sections over the principal and spinor bundles. This Lagrangian may be symmetric under all transformations specified by the structure group of the principal bundle, some of them, or none of them.

If we take all these steps together we can arrive at an enlightening result. The geometrical quantities we define e.g. principal bundles, their respective fibers, relevant differential forms are defined irrespective of the Lagrangian we choose. The definitions of these quantities are also not dependent upon any choice of gauge fixing or restrictions on field solutions. Such restrictions only serve to narrow the space of local sections describing our theory. Emergence, as has been described earlier in this paper, comes from subjecting our gauge fields to the condition $\partial_\mu A^\mu = 0$. We can conclude that this condition does nothing to fundamentally redefine the bundle structure of our theory. Emergence is a phenomenon that depends primarily upon the form of the Lagrangian and the space of allowed field solutions. The Lagrangian is an object that itself does not change the underlying bundle structure of the theory, it is a constraint that can be built out of geometrical objects that arise from the bundle structure of the theory that it describes. To put it another way, earlier in this thesis it was noted that in the abelian emergence of electromagnetism two of Maxwell's equations come from the Bianchi identity that is satisfied by the theory. The Bianchi identity is ultimately just a statement about the curvature i.e. the geometry, of the theory. These geometrical equations of motion are not changed by modifying the Lagrangian. Changing the Lagrangian only serves to change the dynamical equations of motion governing the theory e.g. the remaining two Maxwell's equations for electromagnetism.

5.2 Geometrization of the Non-Abelian Gauge Theory

Up until this point our analysis has been primarily concerned with associating a geometric underpinning to abelian theories. The geometrical approach used prior to this section has been very general, and we will see that the general definitions and underlying geometry do not change significantly if at all when we go to a non-abelian setting. This section will explain how these ideas can be generalized to a non-abelian setting.

1. Transformations can be described by sections over a principal fiber bundle with structure group, G. This definition does not change, the only difference is that our structure group, G, is now taken to be a non-abelian group. Sections over this bundle can describe transformations that are symmetries of the Lagrangian just as before.

2. A connection one-form can be defined whose purpose is to split the tangent space into horizontal and vertical subspaces. The vertical subspace is spanned by a set of generators of G. A gauge field can be described by a pullback of the connection one-form to a local chart over the base space by a section of the principal fiber bundle. This definition does not change for a non-abelian theory; however, the form of these objects and their transformation rules will be altered by the fact that we are now dealing with

a non-abelian theory. Equation 4.10 demonstrates explicitly how the transformation rules of our gauge fields would change if the structure group, G, was specified to be non-abelian.

3. Associated spinor bundles can be defined for our principal bundle. These spinor bundles have transformation rules that ultimately depend on those described by the principal bundle due to its association. Spinor fields can be described by local sections over spinor bundles. This does not change in a non-abelian setting; however, the transformation rules governing these spinor bundles will now reflect the non-abelian structure group of the principal bundle.

4. The covariant derivatives of spinor functions used in the Lagrangian can be described by parallel propagation of elements of the associated spinor bundle along paths defined by local sections. This does not change in a non-abelian setting.

5. A field strength tensor (like $F^{\mu\nu}$) can be described by a pullback of the curvature two-form to a local chart over the base space by a section of the principal fiber bundle. The defining of the field strength tensor in terms of the curvature two-form does not change; however, these objects will take slightly different forms now. Recall that equations 4.27 through 4.29 simplify in the abelian case. In the non-abelian case these equations would take into account the non-commutative properties of a non-abelian theory and would take s lightly different form.

6. Dual bundles can be defined for the principal and spinor bundles. Local sections over these bundles give us quantities that we need to form invariants in our Lagrangian. This does not change in a non-abelian setting.

7. A Lagrangian is just a constraint specifying the allowed spaces of sections over the principal and spinor bundles. This Lagrangian may be symmetric under all transformations specified by the structure group of the principal bundle, some of them, or none of them. This does not change in a non-abelian setting.

The same basic conclusions made in the previous section can be applied here. As can be seen by reading through the list in this section, a theory being abelian or non-abelian does not change the process or definitions of the underlying geometric objects. The definitions of fiber bundles and their relevant differential forms are robust and account for non-commutative properties. Selecting a non-abelian theory can be boiled down to simply selecting a new non-abelian structure group to represent the theory in question. The only other notable changes this induces are changes in the forms of some quantities that acquire extra terms to account for the non-commutative properties of the theory. Emergence looks the same in this picture as it does in the abelian picture and it is still possible to fully define a geometric underpinning to the theory and its Lagrangian in terms of bundle structures and their differential forms.

5.3 Conclusions

Emergence can be a tricky phenomenon to clearly define. One way of doing so boils down to identifying conditions for which physical symmetries can transform into gauge symmetries. The start of this thesis introduced a method by Barcelo et al. that hinges on the fact that gauge symmetries of the Lagrangian generate Noether charges that are trivial on-shell while physical symmetries of the Lagrangian generate Noether charges that are not trivial on-shell. Barcelo et al. showed that emergence is possible in an abelian regime, and provided an explicit example of emergent electromagnetism. In this paper, their methods were generalized to a non-abelian theory, and it was shown that these same methods can be used to get an emergent Yang-Mills theory under the constraint $\partial_\mu A^\mu = 0$. This is an important result because it has been shown [12], [9] that non-abelian emergent theories (including the one analyzed in this thesis) can be constructed such that they do not violate the Weinberg-Witten theorem, which is an important step in defining a consistent theory of quantum gravity.

The ability to analyze the topology and geometry of a theory can be very valuable, especially when trying to formulate a theory that could have connections to a theory with clear geometric underpinnings, such as gravity. The standard model (and gauge theories in general) can be geometrically described using fiber bundles. The transformations that must be analyzed for a given Lagrangian are generated by the structure group, G, of the principal fiber bundle. Every object in the Lagrangian analyzed in this thesis can be described in terms of principal bundles, associated bundles, and differential forms on these bundles. Furthermore, whether or not a symmetry of the Lagrangian is physical or gauge depends on the Noether charges it generates, which ultimately hinges on the form of the Lagrangian. The Lagrangian is nothing more than a constraint on the allowed space of sections over the principal and associated spinor bundles, so regardless of whether or not the underlying symmetries are gauge or physical we must start with the same underlying bundle structure. In other words, when we start off with a modified Lagrangian to search for emergence, we have the same underlying bundle structure as we would have with an unmodified Lagrangian. The modification of the Lagrangian only serves to change the dynamical equations of motion as well as the Noether currents. Applying a constraint to the fields in order to get emergence restricts the space of sections that we can have on our principal and spinor bundles, which in the case of emergence causes transformations that carry spinor solutions into different spinor solutions to instead carry these solutions into the same solutions, i.e. we get a gauge redundancy. The fact that we are in a non-abelian setting does not change how we define our geometrical objects, it only changes the form of the differential forms that we define on our fiber bundles to accommodate the non-commutative properties of our structure group. This shows that emergence is well-defined in a non-abelian setting, amenable to ideas surrounding possible theories of quantum gravity, and does not fundamentally change the underlying bundle structure of a theory. Defining gravity as a gauge theory is a tricky undertaking with many nuances, and assigning a concrete bundle structure to such a theory adds to this difficulty [14], but using the methods in this thesis it may be possible in future attempts to take such a formulation of gravity and search for an emergent underpinning for quantum gravity.

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