Establishing Design Continuity in Pareto Fronts with Active Subspaces

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EARTH • ENERGY • ENVIRONMENT

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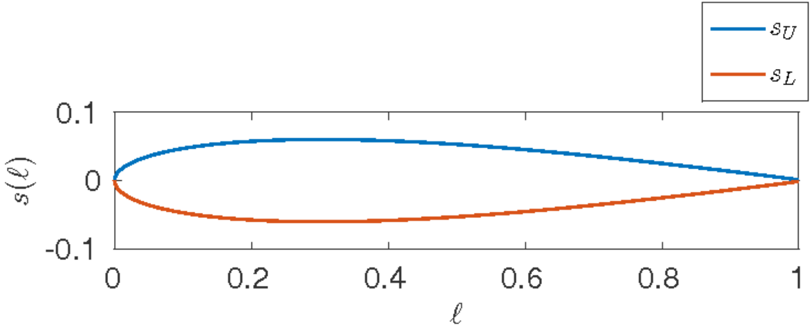
Transonic Flows

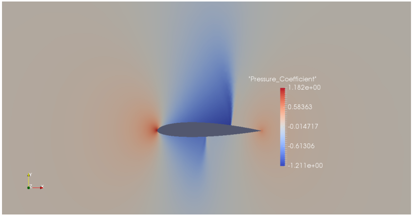
Image Credit: Flickr User, Marcos Vasconcelos Photography

Airfoil Shape (Rotated Teardrop Manifold)

$$\mathcal{A} \subset \mathbb{R}^2$$



Transonic Airfoil Problem



Optimal lift given drag req.

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbb{R}^m}{\text{minimize}} \quad & f_1(\boldsymbol{x}) := -C_\ell(\boldsymbol{x}) \\ \text{subject to} \quad & \|\boldsymbol{x}\|_\infty \leq 1 \\ & C_d(\boldsymbol{x}) - D \leq 0 \end{aligned}$$

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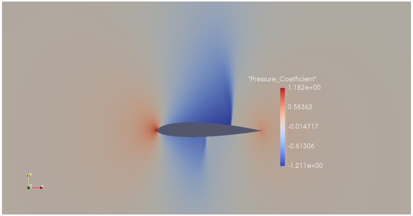

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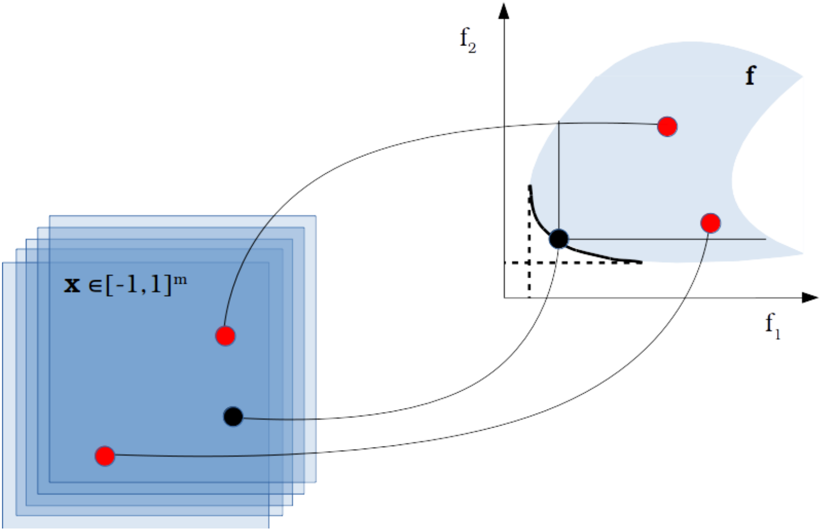
$$\boxed{\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \gamma f_1(\boldsymbol{x}) + (1-\gamma) f_2(\boldsymbol{x}) \\ \text{subject to} \quad & \boldsymbol{x} \in [-1,1]^m \end{aligned} \quad \text{for all } \gamma \in [0,1]}$$

Multi-Objective Concepts (Pareto)



Active Subspaces

Given: $f : \mathbb{R}^m \to \mathbb{R}, \quad \boldsymbol{x} \in [-1, 1]^m,$

$$C = \int \nabla f(\boldsymbol{x}) \nabla f(\boldsymbol{x})^T \rho(\boldsymbol{x}) d\boldsymbol{x} = W \Lambda W^T,$$

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Approximate: $f(\boldsymbol{x}) \approx g\left(W_1^T \boldsymbol{x}\right)$ where

$$W = \left[W_1 \;\; W_2\right], \quad W_1 \in \mathbb{R}^{m \times n}, \quad n < m,$$

and the input parameters can be written

$$\boxed{\boldsymbol{x} = W_1 \boldsymbol{y} + W_2 \boldsymbol{z}.}$$

Design Procedure

1. Parameterize an expansion
(e.g., CST: $\boldsymbol{x}_0 = (\boldsymbol{b}_U; \boldsymbol{b}_L) \in \mathbb{R}^{10}$, $k = 5$)

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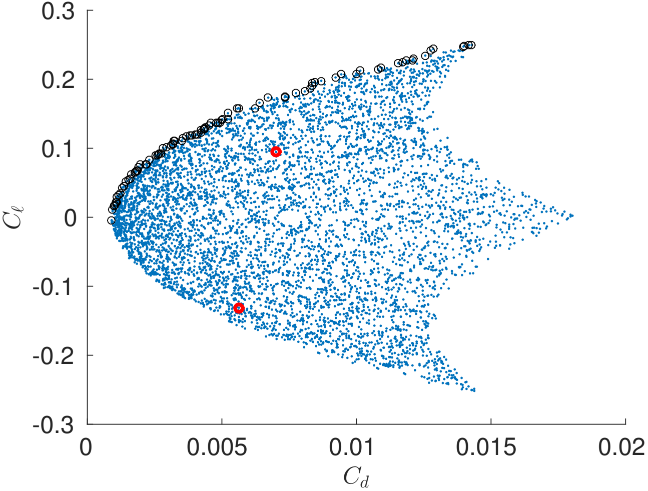
$$\boldsymbol{x} \longmapsto s_U, s_L \longmapsto f_1, f_2$$

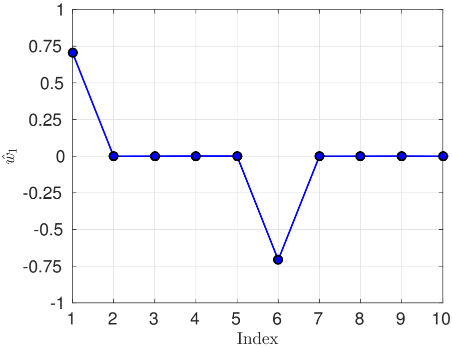
Multi-Objective Space

$$\texttt{X} \in \mathbb{R}^{6500\times 10} \implies \texttt{CL.F} \in \mathbb{R}^{6500\times 1}, \quad \texttt{CD.F} \in \mathbb{R}^{6500\times 1}.$$



CST - Eigenvectors

```
plot(CL.W1);
```

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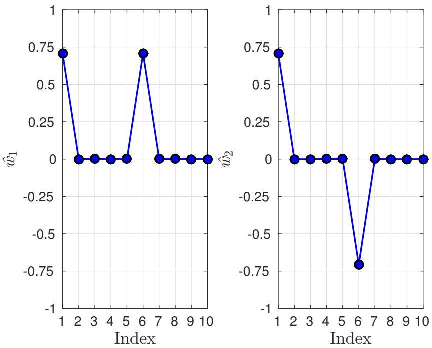
plot(CD.W1);
```



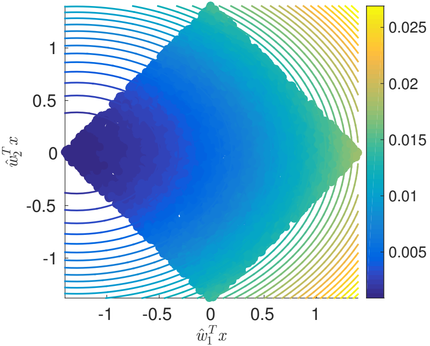
CST - Optimization Implications

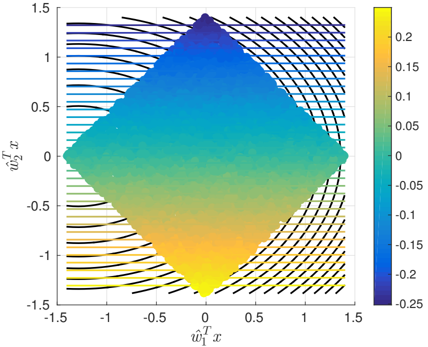
Take the drag active subspace approximations such that

$$\hat{W}_1 = [\hat{\boldsymbol{w}}_1 \;\; \hat{\boldsymbol{w}}_2] \in \mathbb{R}^{10\times 2}, \quad \boldsymbol{y} = \hat{W}_1^T \boldsymbol{x} \in \mathbb{R}^2,$$

and fit least squares approximations:

$$C_d(\boldsymbol{x}) \approx g_{quad}(\hat{W}_1^T \boldsymbol{x}) \qquad\qquad C_\ell(\boldsymbol{x}) \approx g_{lin}(\hat{W}_1^T \boldsymbol{x})$$

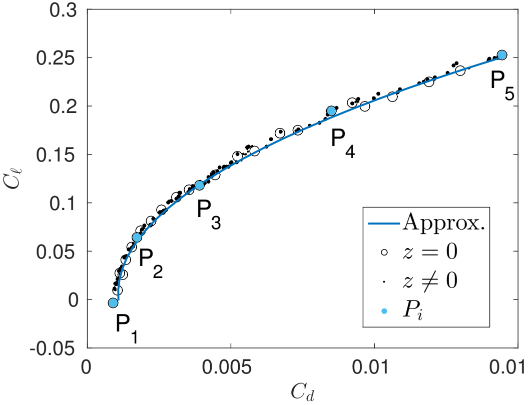


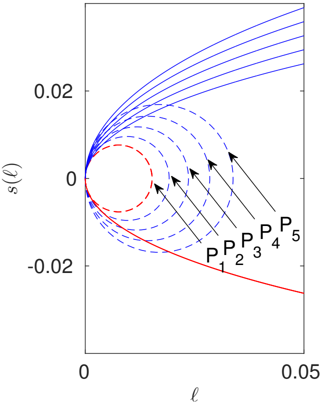


CST - Approximate Pareto Front

$$\text{Define :} \quad \mathcal{P} = \left\{ \boldsymbol{x} \in [-1,1]^{10} \;\middle|\; \boldsymbol{x} = \hat{W}_1 \boldsymbol{y}^* + \hat{W}_2 \boldsymbol{z}^* \right\}$$

$$\text{Approximate :} \quad \boldsymbol{f} : \{\boldsymbol{x} \in \mathcal{P}\} \to \left\{ P \in \mathbb{R}^2 \right\}$$





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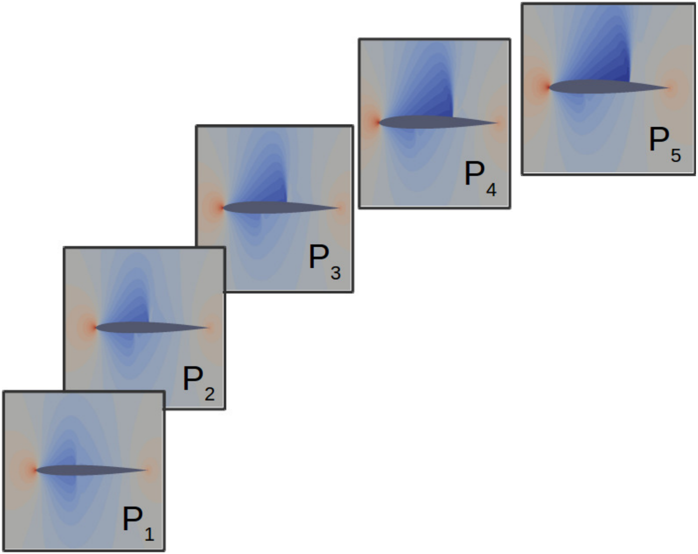


Summary

- We have found low-dimension active subspaces that help study lift and drag in transonic flow regimes
- The subspaces relate to linear changes in circular curvature of the leading edge
- Shape subsets of interest are consistent with first principle aerodynamics
- The two-dimensional active subspace enables visualization that leads to a continuous approximation of the Pareto front

Thank you.

Mean-squared directional derivatives

Lemma

The mean-squared directional derivative of f with respect to the eigenvector $\boldsymbol{w}_i$ is equal to the corresponding eigenvalue,

$$\int \left(\nabla f(\boldsymbol{x})^T \boldsymbol{w}_i\right)^2 \rho(\boldsymbol{x})\, d\boldsymbol{x} = \lambda_i, \quad i = 1, \ldots, m.$$

Where $\boldsymbol{w}_i$ is the i^{th} column of W corresponding to the i^{th} largest eigenvalue λ_i.

Global Quadratic Approximation

If no gradient information is available, suppose

$$f(\boldsymbol{x}) \approx \frac{1}{2}\boldsymbol{x}^T H\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{v} + c, \quad H = H^T \in \mathbb{R}^{m\times m}, \quad \boldsymbol{v} \in \mathbb{R}^m, \quad c \in \mathbb{R}.$$

Also assume the gradient is well approximated by

$$\nabla f(\boldsymbol{x}) \approx H\boldsymbol{x} + \boldsymbol{v}.$$

Given uniform density, ρ,

$$C \approx \frac{1}{3}H^2 + \boldsymbol{v}\boldsymbol{v}^T = \hat{W}\hat{\Lambda}\hat{W}^T.$$

Convergence

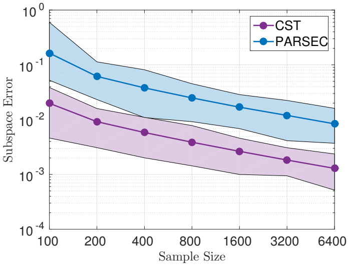
CST Note

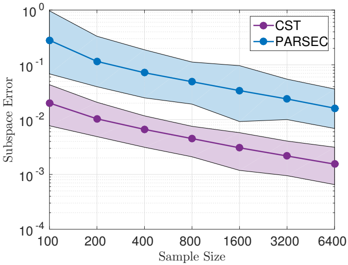
Choose $\boldsymbol{x} \in \mathbb{R}^{10}$ and $\pm 20\%$ bounds of coefficient values for least squares fit of NACA 0012, $\boldsymbol{x}_0$, such that

$$0.8\boldsymbol{x}_0 \leq \boldsymbol{x}_0 \leq 1.2\boldsymbol{x}_0 \implies \boldsymbol{x} \in [-1, 1]^{10}.$$

Given the scaled and centered parameters, sample and compute

$$\left\{ X \in \mathbb{R}^{N \times 10},\ F \in \mathbb{R}^{N} \right\},\ N \leq 6500 \longmapsto ||W_1^T \hat{W}_2||_2.$$





Round Nose Airfoil Representations

Parameterization Types

- Karhunen-Loève
- Parametric splines
- Conformal maps
- Design expansions

e.g., Design expansions

- PARSEC
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PARSEC:

$$s(\ell; \boldsymbol{a}) = \sum_{j=1}^{6} a_j \ell^{j-1/2}, \quad \{\boldsymbol{a} \in \mathbb{R}^6 \mid M\boldsymbol{a} = \boldsymbol{p}\}$$

CST:

$$s(\ell; \boldsymbol{b}) = C(\ell)S(\ell; \boldsymbol{b}) = \sqrt{\ell}(1-\ell)\sum_{j=0}^{k-1} b_j \ell^j, \quad \boldsymbol{b} \in \mathbb{R}^k$$

The Intermediate Scale

$$t = \sqrt{\ell} \implies s(t) = \sum_{j=1}^{k} a_j t^{2j-1}$$

Consider: $0 < \epsilon \ll 1$,

$$t \in (0, \epsilon] \implies s(t) \approx a_1 t.$$

Find a_1 that approximates a circle of radius and center ϵ,

$$s_c^2 + (\ell - \epsilon)^2 = \epsilon^2 \implies s_c(t) = t\sqrt{2\epsilon - t^2}.$$

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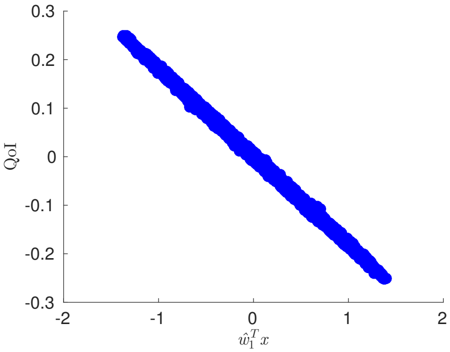
Answer: $t \neq 0 \implies a_1 = \sqrt{2\epsilon}.$

CST - Shadow Plots

```
plot(X*CL.W1,CL.F);
```

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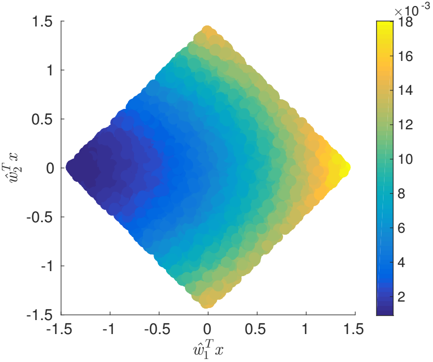


CST - Approximate Pareto Front

By inspection of shadow plots

$$\mathcal{P}_{\boldsymbol{y}} = \left\{ \boldsymbol{y} \in \mathbb{R}^2 \mid \boldsymbol{y} = \gamma \begin{bmatrix} y_{1,\text{min}} \\ 0 \end{bmatrix} + (1-\gamma) \begin{bmatrix} 0 \\ y_{2,\text{min}} \end{bmatrix}, \ \gamma \in [0,1] \right\},$$

where

$$y_{1,\text{min}} = \underset{\boldsymbol{x} \in [-1,1]^{10}}{\text{minimum}} \ \hat{\boldsymbol{w}}_1^T \boldsymbol{x}, \quad y_{2,\text{min}} = \underset{\boldsymbol{x} \in [-1,1]^{10}}{\text{minimum}} \ \hat{\boldsymbol{w}}_2^T \boldsymbol{x}.$$

Then, for $\boldsymbol{y}^* \in \mathcal{P}_{\boldsymbol{y}}$ define

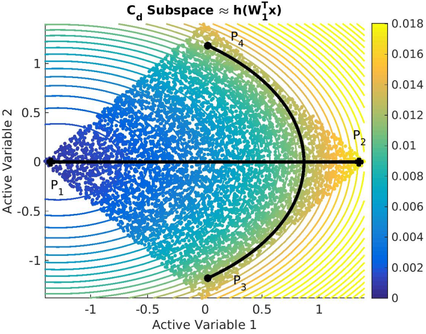
$$\mathcal{P}_z = \left\{ \boldsymbol{z} \in \mathbb{R}^8 \mid \ -\boldsymbol{1} - \hat{W}_1 \boldsymbol{y}^* \le \hat{W}_2 \boldsymbol{z} \le \boldsymbol{1} - \hat{W}_1 \boldsymbol{y}^* \right\}.$$

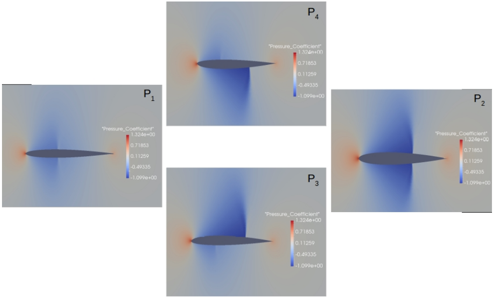
For $\boldsymbol{z}^* \in \mathcal{P}_{\boldsymbol{z}}$ the local Pareto support for a subset of shapes is in

$$\mathcal{P} = \left\{ \boldsymbol{x} \in [-1,1]^{10} \mid \ \boldsymbol{x} = \hat{W}_1 \boldsymbol{y}^* + \hat{W}_2 \boldsymbol{z}^* \right\}.$$

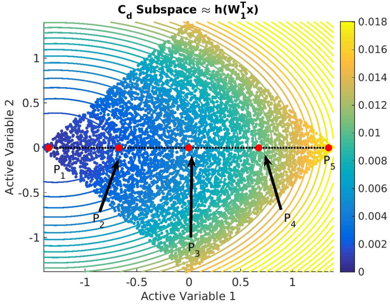
CST - Drag Shape Subsets

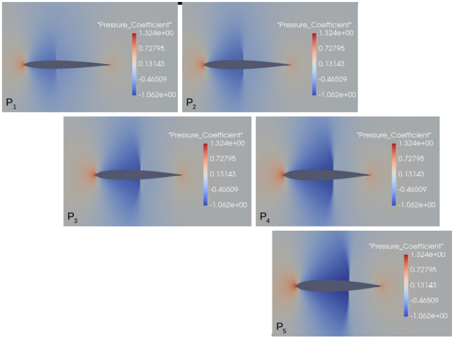
$$\boldsymbol{x}_k = W_1 \boldsymbol{y}_k \quad \text{for} \quad k = 1, \ldots, 4$$



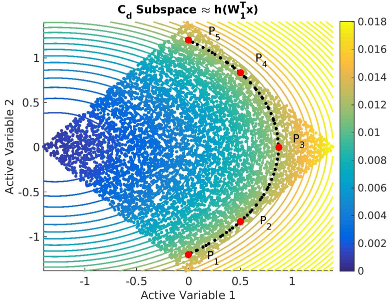


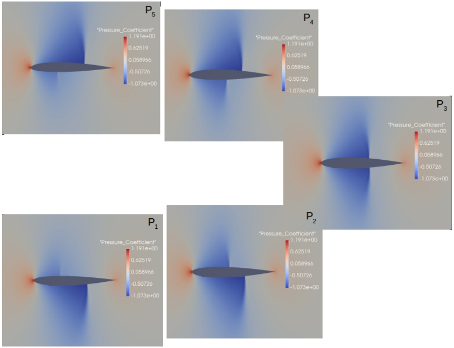
CST - Min/Max Drag Perturbations



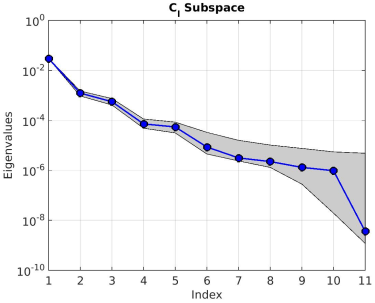


CST - Constant Drag Perturbations

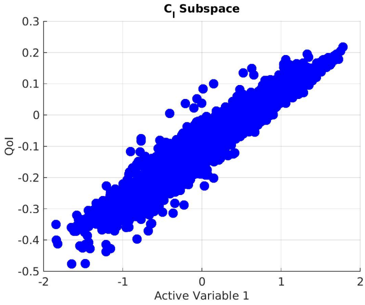


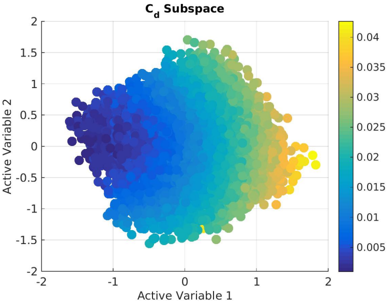


PARSEC - Eigenvalues



PARSEC - Summary Plots





PARSEC - Subspace Error