EFFECT OF OSCILLATING TURBULENT FLOW ON A LONG MARINE RISER

by

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ABSTRACT

The deep seabed is one of the potentially most rewarding frontiers that challenges mankind in its quest for material achievement. Deep-ocean mining promises to make an enormous contribution to the world's resource base once its potential is fully realized. Of particular interest are manganese nodules and cobalt rich crust which are deposited over and beneath the ocean floor at depths of 800 ~ 6,000-m (Chung and Tsurusaki, 1994). Manganese nodules are known to contain high quantities of cobalt, copper, manganese, and nickel and possibly molybdenum, titanium and vanadium (Chung and Tsurusaki, 1994).

Mining either manganese nodules or ocean crust requires a long, slender pipe to transport the slurry from the miner on the sea floor to the ship on the ocean surface. In order to aid in the design and control of the nodule slurry hoisting system, the pipe dynamics of the string must be properly analyzed. Investigation into the torsional effect on a 3-D nonlinear, coupled axial, bending and torsional responses on a 4,000-ft pipe system was recently investigated by Chung, Cheng, and Huttelmaier (1994). The pipe system was subjected to current induced periodic horizontal and vertical motions and their analysis resulted in the determination of external shear forces resulting from hydrodynamic forces which is believed to be the principal design constraints.

The focus of the present research is to determine the magnitude of the shear stress which will affect the responses of a 18,000-ft vertical pipe containing a turbulent periodic internal pipe flow. The effect of mean flow and external axial shear stress on the responses of the pipe system was previously studied with a nonlinear finite element method (FEM) code (Chung, Cheng, and Huttelmaier, 1994) for the oscillatory frequency of the pipe. The present analysis utilizes the eddy viscosity approach in modeling the turbulent flow for oscillating shear stress.

The present analysis clearly illustrates that (i) oscillating velocity is slightly influenced by the magnitude of the mean velocity, (ii) both oscillating velocity and shear stress are related to the oscillation period and (iii) oscillating velocity is directly related to oscillating shear stress.

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GLOSSARY

A,B	oscillating pressure constants
c	isentropic speed of sound
C_j , D_j	boundary condition constants in region j
g	acceleration due to gravity
h_L	headloss in a pipe
I_0	modified Bessel function of the first kind of zero order
I_1	modified Bessel function of the first kind of first order
j	designates either a region boundary or region area
k_j	slope factor for the modified viscosity models in region j
K	constant used for solving oscillating shear stress
KU_j	constant used for solving oscillating velocity
K_0	modified Bessel function of the second kind of zero order
K_1	modified Bessel function of the second kind of first order
L	signifies Laplace transform
m_j	intercept for the modified viscosity model in region j
m	subscript denotes mean
n	signifies the number of regions in the model

os	subscript denotes oscillating component
p	pressure
Q^*	criterion function representing the square of the error
r	radial direction, distance away from pipe center
r_j	region j boundary
R	inside pipe radius
R*	dimensionless R used in modified viscosity model
Re	Reynolds number
s	parameter of Laplace transformation
st	subscript denotes steady component
t	time
T	period
u	velocity in the axial direction
u^*	friction velocity
v	velocity in the radial direction
w	velocity in the θ direction
$\bar{u}_{m,st}$	mean-steady upward velocity
y	distance measured away from the pipe wall
y^*	dimensionless y distance used in modified viscosity model

z	axial direction
$z_j(r)$	function used for transformation of variables, applicable to region j
ξ	eddy viscosity
λ	friction factor
ρ	fluid density
τ	shear stress
τ_R	Reynolds stress
μ	dynamic viscosity
ν	kinematic viscosity
ν_E	modified eddy viscosity
γ	wave propagation constant
γ_R	real component of the wave propagation constant
γ_I	imaginary component of the wave propagation constant
γ_{sw}	fluid specific weight
ω	angular frequency
$\bar{}$	denotes sort-time average value
$'$	denotes fluctuating component

CHAPTER 1

INTRODUCTION

1.1 Background

The ocean floor possesses massive deposits of cobalt, manganese and nickel in the form of nodules and from the crust surface itself. These minerals are deposited over and beneath the ocean floor from depths of 800~6,000-m (Chung and Tsurusaki, 1994). The potential for profitable mining of these minerals from the sea floor involves the coordination or integration of five distinct systems of operations (Chung and Tsurusaki, 1994):

(1) Exploration survey

(2) Mineral collection (nodule and crust) from the seafloor

(3) Hoisting the aggregate to the mining ship

(4) Transportation to land

(5) Processing onshore or in the ocean

This paper focuses only on the third stage, which is analyzed using both fluid mechanics and pipe dynamics. The pipe is subjected to numerous external forces including external hydrodynamic forces, large-amplitude vertical and horizontal motions, vertically varying current flows and a periodic internal flow. The top of the pipe is

pinned to a ship, incorporating a heave compensator to allow for vertical motion of the pipe. The bottom of the pipe is attached to a buffer and other additional equipment. The bottom end is considered free and independent of the self-propelled seafloor miner. The internal pipe flow for the hydraulic hoist consists of a mixture of nodules or solids, soil particles and seawater whose mixture is controlled by a buffer connected to the bottom portion of the pipe. In the present example, the flow is considered to be oscillating at the axially oscillating frequency of the pipe. The slurry flow is fully turbulent with Reynolds numbers in the order of 1.5×10^5. This oscillating, turbulent pipe flow induces an oscillating shear stress on the inside surface of the pipe which can change the dynamic response of a marine riser or pipe.

1.2 Previous Work

The majority of the early research conducted on oscillating flow consisted of analytical studies of periodic laminar flow conducted by Sexl and Szymanski (1930), Uchida (1956) and Rosenhead (1963). Information from these studies were often used for early comparative studies of transitional and turbulent flows.

Many of the studies on unsteady transitional flows have been reported mainly by bio-fluid mechanics because of its practical application to bio-fluid problems. Because of the difficult nature of transitional flow experiments, several studies led to conflicting

results and did not contribute to the fundamental understanding of unsteady flow (Gerrard, 1971).

Several researchers have studied periodic turbulent flow experimentally as well as analytically. Some of the earliest studies in this area were conducted by in 1940 by Schultz-Grunow (1940) with more recent research conducted by Gerrard (1971), Mizushina, Maruyama, and Hirasawa (1975), Ohmi and Usui (1976) and Ramaprian and Tu (1982). In all these cases, fully developed turbulent pipe flow was induced by superimposing a periodic pulsation on a steady mean flow. The majority of the studies to date have been aimed at determining the effect pulsation has on time dependent values such as velocity, pressure loss and wall shear stress.

Direct measurements of the above noted properties have been conducted by numerous researchers, most notably Laufer (1953) and Nunner (1956). Typically measurements are taken using hot-wire anemometers, cine-photography or electrochemical methods. Analytical models most commonly used to account for the presence of turbulence include quasi-turbulent models such as the k- ε model which is based on energy and the eddy viscosity model which attempts to account for Reynolds stresses.

1.3 Purpose

Potential deep-ocean mining operations require accurate modeling of slurry transport from the miner at the seafloor to the ship on the ocean surface in order to account for its influence on shear stress. To date, analysis of the coupled axial, bending and torsional responses have been investigated by Cheng, Chung, and Huttelmaier (1994) with the pipe subject to current induced periodic horizontal and vertical motions and their analysis resulted in the determination of external shear forces resulting from hydrodynamic forces which is believed to be the principle design constraints.

The focus of the present analysis is to determine the magnitude of the shear stress which will affect the responses of a 18,000-ft vertical pipe containing a turbulent periodic internal pipe flow. This paper presents the effect of average velocity on the oscillating velocity profile and the effect of periodic flow on the shear stress.

CHAPTER 2

MODELING OF VISCOSITY

2.1 Laminar and Turbulent Flow Theory

Laminar flow is characterized by fluid particles that flow in parallel layers with agitation only in a molecular nature. Particles remain in parallel paths due to the dominance of viscous forces. With no mixing between layers, the flow may be considered one dimensional in the direction of the pipe axis. The majority of the real world applications involve turbulent flow which is characterized by the chaotic, random movement of fluid parcels between layers. For these types of flows, inertia forces dominate which result in variations in velocity, pressure, shear stress and temperature. As figure (2.1) indicates, turbulent velocity has v and w components, normal to the pipe axis, with the dominant flow velocity in the z direction (pipe axis).

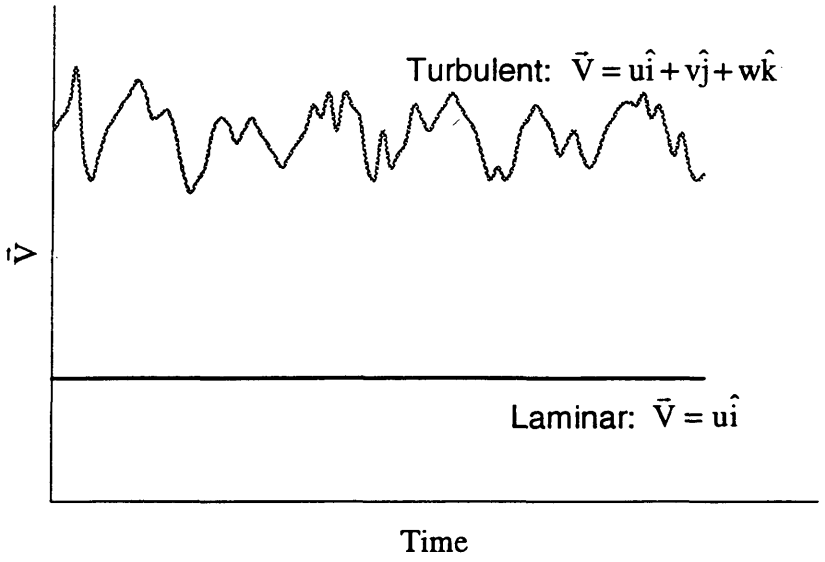


Figure 2.1 Comparison of Laminar and Turbulent Velocity

2.2 Laminar and Turbulent Shear Stress

Particles in laminar flows are constrained to parallel paths by viscosity. Shearing stress for laminar flow is thus a function of viscosity and is expressed in terms of viscosity and the velocity gradient:

$$\tau = \mu \frac{\partial u}{\partial y} \tag{2.1}$$

Equation (2.1) is only applicable to Newtonian fluids. Newtonian fluids exhibit a linear relationship between the shear stress and strain.

Turbulent flow is typically described in terms of mean values on which fluctuations are superimposed. For instance, turbulent velocity and pressure components are expressed as:

$$u = \bar{u} + u'$$
$$v = \bar{v} + v'$$
$$w = \bar{w} + w'$$
$$p = \bar{p} + p' \qquad (2.2)$$

The overscore denotes the short-time average and the prime denotes the fluctuating component.

Several experimental and theoretical studies have shown that utilizing $\bar{u}$ in the expression for turbulent shear stress (equation 2.1) is not valid (Rouse, 1959). This is due to the presence of eddies which transport fluid particles between layers resulting in additional shear stress. For example, the fluctuating velocities, u' and v', which occur in the axial (z) and radial (r) directions, cause effective shear stress in turbulent flows as Reynolds stresses:

$$\tau_R = -\rho\overline{u'v'} \qquad (2.3)$$

Reynolds stress can also be expressed in terms of $\overline{v'w'}$ and $\overline{u'w'}$. Hence for turbulent flows, the total shear stress may be expressed as the sum of the laminar and Reynolds shear stresses:

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} \tag{2.4}$$

It has been found that Reynolds stress term, $-\rho\overline{u'v'}$, is positive (Yih, 1969) which indicates that shear stress for turbulent flow is greater than that for laminar flow.

2.3 Eddy Viscosity

The turbulent shear expression in equation (2.4) is frequently expressed as:

$$\tau = \xi \frac{\partial \bar{u}}{\partial y} \tag{2.5}$$

where ξ is the eddy viscosity, introduced by Boussinessq (1877). Determining the values for eddy viscosity is very difficult, as it is a function of both the flow condition and the fluid. Hence, eddy viscosity can change from one turbulent flow to another and can also vary based on location within the flow.

Laufer (1954) and Nunner (1956) and others have measured eddy viscosity across pipe and duct for various fluids at different flow conditions. From these data, several theories, many of which are empirical, have been proposed to approximate values of ξ. Two eddy viscosity distribution models referenced frequently are the Taylor-Prandtl and von Kàrmàn models. Hinze (1959) calculated the eddy distribution profiles from data collected from both Laufer and Nunner for steady pipe flow.

In a paper dealing with pulsating turbulent pipe flow, Ohmi and Usui (1976) implement a modified eddy viscosity, ν_E, which is defined as the sum of the eddy and kinematic viscosities.

$$\nu_E = \xi + \nu \tag{2.6}$$

Their model utilizes constant eddy viscosity in the core region and linear approximations of the eddy viscosity data taken by Laufer (1954) in the vicinity of the wall. Their assumption in the core region agrees well with Hinze's (1959) study. The core region is defined as $0 \leq r/R \leq .825$, where R is the internal pipe radius, and possesses a modified eddy viscosity equal to $0.07\, u^* R$ were u^* is the friction velocity:

$$u^* = \bar{u}_{m,st}\sqrt{\lambda/8} \tag{2.7}$$

λ is the friction factor and $\overline{u}_{m,st}$ signifies the mean-steady flow. Eddy viscosity near the wall is modeled by using straight line approximations of Laufer's measurements. The distance y is defined as the distance measured from the pipe wall towards the pipe center (i.e. y = R-r). Figure 2.2 shows the straight line approximations used by Ohmi and Usui (1976) to approximate Laufer's modified eddy viscosity measurements.

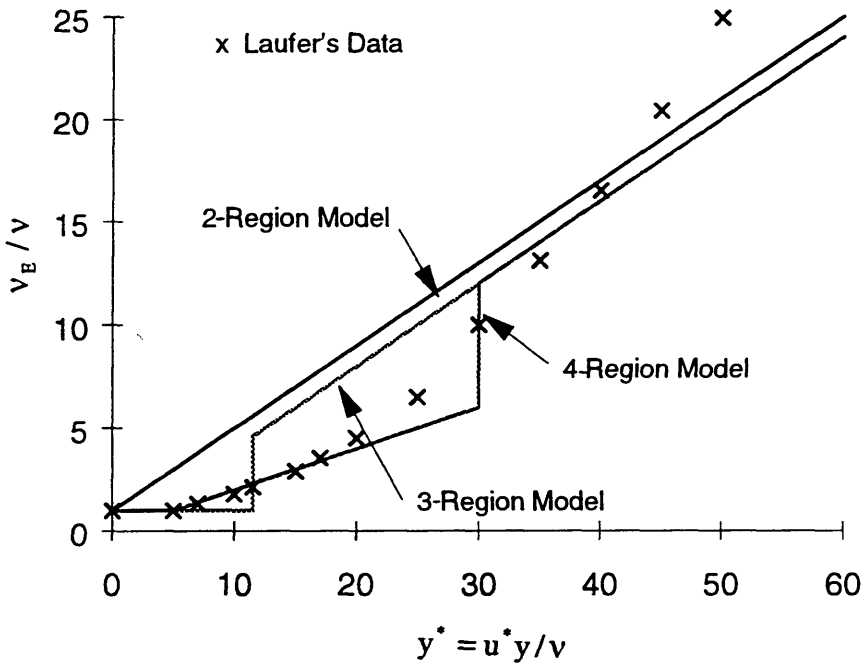


Figure 2.2 Comparison of Laufer's Data to Existing Eddy Viscosity Models

Ohmi and Usui (1976) used 2, 3, 4 and 13-region eddy viscosity distribution models, but concluded that the 4-region model was satisfactory which is why the 13-

region model is not included in figure 2.2. Note the number of regions shown in figure 2.2 is one less than the total number used in the approximation since the core region counts as one region. For every model, modified eddy viscosity is 0.07 u^* R from $0 \leq r/R \leq .825$ which corresponds $0 \leq y^* \leq 0.175$ R* were R* is defined as:

$$R^* = \frac{u^* R}{\nu} \tag{2.8}$$

The y^* term is the dimensionless distance from the wall and is related to the pipe radius and the boundary radii by:

$$\frac{r_j}{R} = 1 - \frac{y_j^*}{R^*} \tag{2.9}$$

where r_j and y_j^* are the radius and y^* boundary values between the j^{th} region and the $j+1^{th}$ region.

The equation used by Ohmi and Usui (1976) for the straight-line approximation of the modified eddy viscosity in the wall region is given by:

$$\frac{\nu_{E,j}}{\nu} = k_j y_j^* + m_j \tag{2.10}$$

where the subscript j denotes the quantity in the j^{th} region and the values of k_j, m_j and y_j^* for the 2, 3 and 4-region models are given in tables 2.1~2.3 (Ohmi and Usui, 1976).

Table 2.1

Data for 2-Region Model

y*	Region number	k_j	m_j
0			
	1	0.4	1.0
0.175R*			
	2	0.0	0.07R*+1

Table 2.2

Data for 3-Region Model

y*	Region number	k_j	m_j
0			
	1	0.0	1.0
11.5			
	2	0.4	0.0
0.175R*			
	3	0.0	0.07R*
R*			

Table 2.3

Data for 4-Region Model

y*	Region number	k_j	m_j
0			
	1	0.0	1.0
5.0			
	2	0.2	0.0
30.0			
	3	0.4	0.0
0.175R*			
	4	0.0	0.07R*
R*			

2.4 Least Squares Method

Inspection of figure 2.2 indicates that a better approximation to Laufer's data can be determined. In order to accomplish this task, the method of least squares was implemented to determine the best fit line representing the modified eddy viscosity data. Let X and Y be arbitrary variables with n representing the total number of data points the approximation is accounting for:

$$X = \{x_1, x_2, \ldots, x_n\}$$

$$Y = \{y_1, y_2, \ldots, y_n\} \qquad (2.11)$$

Assuming the best fit equation has a quadratic form, the approximation can be expressed as:

$$y_i = ax_i^2 + bx_i + c \qquad (i=1,2,...,n) \tag{2.12}$$

The criterion function represents the square of the error which is the difference between the true value of the function and its quadratic approximation.

$$Q^* = \sum_{i=1}^{n} \left[y_i - (ax_i^2 + bx_i + c) \right]^2 \tag{2.13}$$

Minimum error occurs at the optimum values of the parameters a, b and c. Taking the derivative of the criterion function with respect to these parameters yields:

$$\frac{\partial Q^*}{\partial a} = 2x_i^2 \left[y_i - (ax_i^2 + bx_i + c) \right] = 0$$

$$\frac{\partial Q^*}{\partial b} = 2x_i \left[y_i - (ax_i^2 + bx_i + c) \right] = 0$$

$$\frac{\partial Q^*}{\partial c} = 2\left[y_i - (ax_i^2 + bx_i + c) \right] = 0 \tag{2.14}$$

Equations (2.14) can then be expressed as:

$$\left(\sum_{i=1}^{n} y_i x_i^2\right) - \left(\sum_{i=1}^{n} x_i^4\right)a - \left(\sum_{i=1}^{n} x_i^3\right)b - \left(\sum_{i=1}^{n} x_i^2\right)c = 0$$

$$\left(\sum_{i=1}^{n} y_i x_i\right) - \left(\sum_{i=1}^{n} x_i^3\right)a - \left(\sum_{i=1}^{n} x_i^2\right)b - \left(\sum_{i=1}^{n} x_i\right)c = 0$$

$$\left(\sum_{i=1}^{n} y_i\right) - \left(\sum_{i=1}^{n} x_i^2\right)a - \left(\sum_{i=1}^{n} x_i\right)b - \left(\sum_{i=1}^{n} 1\right)c = 0 \qquad (2.15)$$

Substituting values in for x_i and y_i from Laufer's data points, the above equations were solved for a, b and c. Knowing these values, it was determined that the best-fit line equation was:

$$\frac{\nu_E}{\nu} = 0.01y^{*2} + 0.009y^{*} + 0.768 \qquad (2.16)$$

2.5 Modified Eddy Viscosity Model

Using the best-fit equation of Laufer's experimental data, a new model was developed by dividing the wall region into several sections. For the present model, the wall region was divided into 7 straight-line approximations taking the form of equation

(2.10). The slope of these lines was obtained by taking the derivative of equation (2.16), allowing the determination of slope at any point along the curve to be expressed as:

$$\frac{d\frac{\nu_E}{\nu}}{dy^*} = 0.02y^* + 0.009 \qquad (2.17)$$

Table 2.4 summarizes the k_j (slope) and m_j (intercept) terms for the 8-region model (recall $0 \leq y^* \leq 50$ is broken into 6 regions with the two remaining regions making up the remainder of the pipe radius). Note that k_j and m_j values were adjusted to ensure continuity of viscosity at the region intersection points.

Table 2.4

Data for 8-Region Model

y*	Region number	k_j	m_j
0			
	1	0.0000	1.0000
5.0			
	2	0.1606	0.2364
10.0			
	3	0.3054	-1.2914
20.0			
	4	0.5090	-5.3153
30.0			
	5	0.7090	-11.3150
40.0			
	6	0.9090	-19.3150
50.0			
	7	0.4000	0.0000
0.175R*			
	8	0.0000	0.07R*
R*			

Figure 2.3 illustrates that the 8-region model approximates the Laufer's measurements better than the 2, 3 or 4-region models used by Ohmi and Usui, 1976.

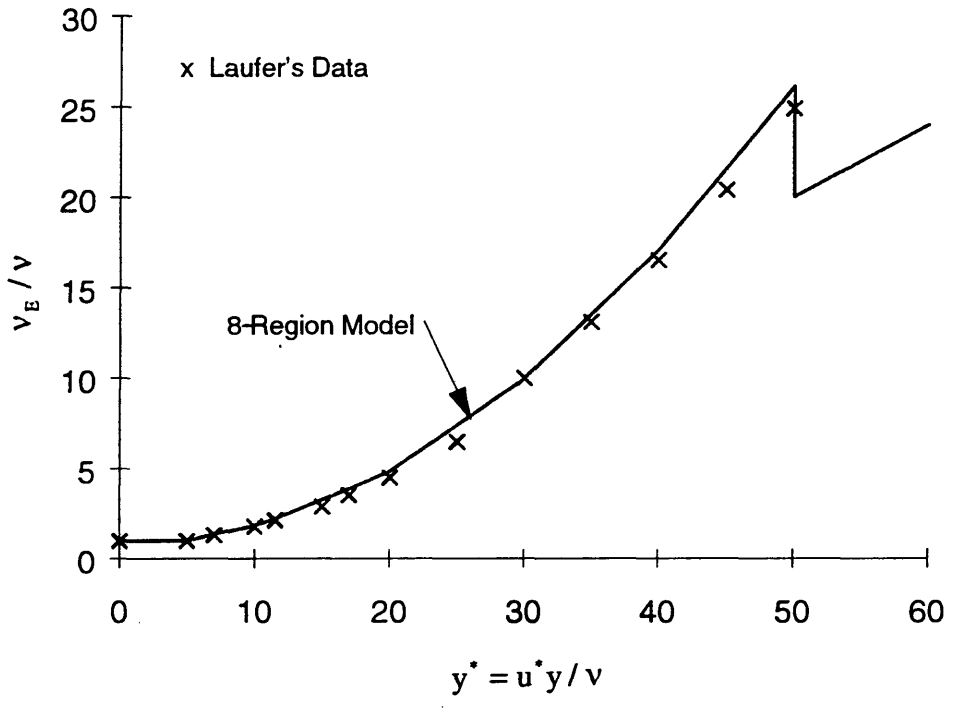


Figure 2.3 Comparison of Laufer's Data to the 8-Region Model

The discontinuity in Figure 2.3 at y*=50 is attributed to the point at $y^* = 0.175R^*$ being fixed which must tie into the region 6 boundary at a fixed point.

CHAPTER 3

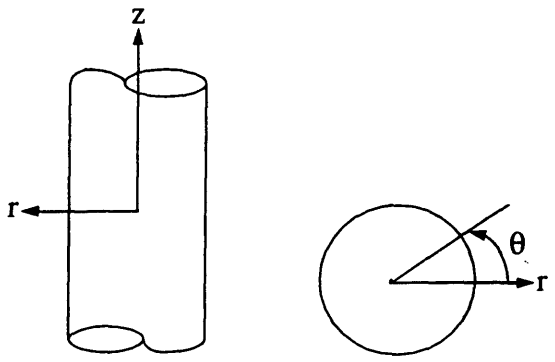
NAVIER-STOKES EQUATION

3.1 General Formulation for Laminar Flow

The Navier-Stokes equation and the continuity equation provide a complete mathematical description of the flow of fluids (Yih, 1969). The Navier-Stokes equations are considered to be the governing differential equations of fluid flow. The simplest manner to express the Navier-Stokes equations is that, for each element of fluid, the inertia forces are balanced by the sum of the pressure force, body force and viscous force. The general expression for the Navier-Stokes equation of a Newtonian fluid is:

$$\rho\frac{DU_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu\nabla^2 u_i \qquad (i=1,2,3) \qquad (3.1)$$

Because of the cylindrical geometry of a pipe, it is convenient to express both the Navier-Stokes and continuity equations in cylindrical coordinates. Figure 3.1 illustrates the pipe geometry and flow designations used throughout this thesis.



$x_1 = r$
$x_2 = \theta$
$x_3 = z$
$u_1 = u_r = v$
$u_2 = u_\theta = w$
$u_3 = u_z = u$

Figure 3.1 Pipe Geometry and Flow Directions

Based on the above notation, equation (3.1) can then be expressed in the radius, θ, and z directions as (Yih, 1969):

$$\rho\left(\frac{Dv}{Dt}-\frac{w^2}{r}\right)=-\frac{\partial p}{\partial r}+\rho g_r+\mu\left(\nabla^2 v-\frac{v}{r^2}-\frac{2}{r^2}\frac{\partial w}{\partial \theta}\right)$$

$$\rho\left(\frac{Dw}{Dt}+\frac{vw}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho g_\theta+\mu\left(\nabla^2 w+\frac{2}{r^2}\frac{\partial v}{\partial r}-\frac{w}{r^2}\right)$$

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial z}+\rho g_z+\mu\nabla^2 u \qquad (3.2)$$

where:

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+v\frac{\partial}{\partial r}+\frac{w}{r}\frac{\partial}{\partial \theta}+u\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$g = g_r = g_\theta = g_z \tag{3.3}$$

The continuity equation for an incompressible fluid in cylindrical coordinates is expressed as:

$$\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial u}{\partial z} = 0 \tag{3.4}$$

Flow in the radial and θ directions is negligible implying that the flow can be considered in the axial direction for this application. Expanding equation (3.2) for the axial velocity component, by the use of equation (3.3), yields:

$$\rho\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial r} + \frac{w}{r}\frac{\partial u}{\partial \theta} + u\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{3.5}$$

Equation (3.5) can not be solved directly without several assumptions and mathematical manipulations. The present assumptions are as follows:

(1) Flow is fully developed along the axis which implies that $\frac{\partial u}{\partial z} \rightarrow 0$

(2) There is no swirling in the flow, implying that $\frac{\partial u}{\partial \theta} \rightarrow 0$

(3) Flow in the axial (u) direction dominates, implying that $v \rightarrow 0$

Based on these assumptions, equation (3.5) reduces to:

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial z} + \rho g + \mu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \tag{3.6}$$

Dividing equation (3.6) by ρ yields:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + g + \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \tag{3.7}$$

Equation (3.7) is a commonly used expression for vertical laminar pipe flow and agrees with equations from several researchers including Hinze (1959) and Gerrard (1971). Mathematical transformation and the eventual model will be discussed in later sections.

3.2 Formulation for Turbulent Flow

Turbulent flow is characterized by the presence of eddies which transport fluid particles between layers causing fluctuations in velocity and pressure. Equation (2.2)

indicates that turbulent velocity and pressure is expressed as the sum of the short-time averaged component and the fluctuating term. Equation (3.7) is applicable to laminar flow and turbulent flows but needs to be modified in order to illustrate the effect turbulent motions have on mean motion. Hence, the velocity and pressure terms in equation (3.7) need to be replaced with the turbulent (instantaneous) velocity and pressure expressions from equation (2.2) an then incorporated into an averaging technique.

Hinze (1959) indicated that if the turbulent flow is quasi-steady, then averaging with respect to time would be applicable and, if the turbulent flow is homogeneous, then averaging with respect to space would be appropriate. However, in this application the turbulent flow is neither homogenous or steady, so utilizing either method would not be correct. Hinze (1959) implements an ensemble-mean method which involves taking the average over a large number of experiments that have the same initial and boundary conditions (Hinze, 1959). He utilizes the following rules based on $A = \overline{A} + a$ and $B = \overline{B} + b$ where the overscore denotes the average and the lower case letters represent fluctuating components:

Rule 1: $\overline{a} = 0, \overline{b} = 0$

Rule 2: $\overline{A} = \overline{\overline{A} + a} = \overline{\overline{A}} + \overline{a}$

Rule 3: $\overline{\overline{A}\,\overline{B}} = \overline{\overline{A}\,\overline{B}} = \overline{A}\,\overline{B}$

Rule 4: $\overline{AB} = \overline{(\overline{A} + a)(\overline{B} + b)} = \overline{A}\,\overline{B} + \overline{ab}$

Examining figure 3.2, it is apparent that the distributions are equally distributed on either side of the average which aids in proving the first of Hinze's rules.

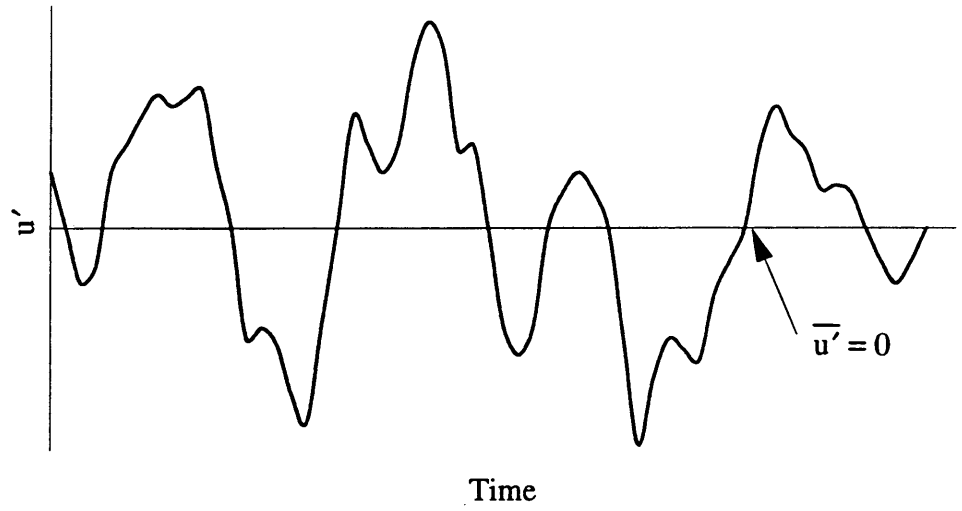


Figure 3.2 Graph of Fluctuating Velocity

Now lets apply Hinze's rules for ensemble-averaging to come up with an expression for the Navier-Stokes equation for turbulent flow. Starting with the Navier-Stokes equation (3.2) for flow in the z direction:

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial z} + \rho g_z + \mu\nabla^2 u \tag{3.8}$$

The $\rho\frac{Du}{Dt}$ term represents the inertia force where $\frac{Du}{Dt}$ is a substantial derivative of the velocity consisting of local and convective terms. The inertia force is expressed as:

$$\rho\frac{Du}{Dt}=\rho\left(\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial r}+\frac{w}{r}\frac{\partial u}{\partial \theta}+u\frac{\partial u}{\partial z}\right) \tag{3.9}$$

Using the viscous force term from equation (3.6) along with equations (3.8) and (3.9) the Navier-Stokes equation for the flow in the z direction can be expressed as:

$$\rho\left(\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial r}+\frac{w}{r}\frac{\partial u}{\partial \theta}+u\frac{\partial u}{\partial z}\right)=\frac{-\partial p}{\partial z}+\rho g+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{3.10}$$

Lets now apply Hinze's ensemble-mean averaging rules to the left hand side of equation (3.10). Substituting instantaneous values of velocities in terms of their short-time averages and fluctuating components (equation 2.2) as well as implementing Hinze's rule 2 yields:

$$\overline{\rho\frac{Du}{Dt}}=\overline{(\bar{\rho}+\rho')\frac{\partial(\bar{u}+u')}{\partial t}}+\overline{(\bar{\rho}+\rho')(\bar{v}+v')\frac{\partial(\bar{u}+u')}{\partial r}}$$

$$+\overline{(\bar{\rho}+\rho')\frac{(\bar{w}+w')}{r}\frac{\partial(\bar{u}+u')}{\partial\theta}}+\overline{(\bar{\rho}+\rho')(\bar{u}+u')\frac{\partial(\bar{u}+u')}{\partial z}} \tag{3.11}$$

Applying Hinze's rules 1 ~ 4 and rearranging the equation yields:

$$\begin{aligned}\overline{\rho\frac{Du}{Dt}} &= \bar{\rho}\left(\frac{\partial\bar{u}}{\partial t}+\bar{v}\frac{\partial\bar{u}}{\partial r}+\frac{\bar{w}}{r}\frac{\partial\bar{u}}{\partial\theta}+\bar{u}\frac{\partial\bar{u}}{\partial z}\right)+\bar{\rho}\left(\overline{v'\frac{\partial u'}{\partial r}}+\overline{\frac{w'}{r}\frac{\partial u'}{\partial\theta}}+\overline{u'\frac{\partial u'}{\partial z}}\right)\\ &+\overline{\rho'\frac{\partial u'}{\partial t}}+\overline{\rho'v'}\frac{\partial\bar{u}}{\partial r}+\bar{v}\overline{\rho'\frac{\partial u'}{\partial r}}+\overline{\rho'v'\frac{\partial u'}{\partial r}}+\overline{\rho'\frac{w'}{r}}\frac{\partial\bar{u}}{\partial\theta}+\frac{\bar{w}}{r}\overline{\rho'\frac{\partial u'}{\partial\theta}}\\ &+\overline{\rho'\frac{w'}{r}\frac{\partial u'}{\partial\theta}}+\overline{\rho'u'}\frac{\partial\bar{u}}{\partial z}+\bar{u}\overline{\rho'\frac{\partial u'}{\partial z}}+\overline{\rho'u'\frac{\partial u'}{\partial z}}\end{aligned} \tag{3.12}$$

For an incompressible flow, such as seawater in the present case, density is considered to be constant which implies that the fluctuating component, ρ', goes to zero. For now, density will simply be referred to as $\bar{\rho}$. With $\rho' = 0$, this eliminates the last ten terms from equation (3.12):

$$\overline{\rho\frac{Du}{Dt}} = \bar{\rho}\left(\frac{\partial\bar{u}}{\partial t}+\bar{v}\frac{\partial\bar{u}}{\partial r}+\frac{\bar{w}}{r}\frac{\partial\bar{u}}{\partial\theta}+\bar{u}\frac{\partial\bar{u}}{\partial z}\right)+\bar{\rho}\left(\overline{v'\frac{\partial u'}{\partial r}}+\overline{\frac{w'}{r}\frac{\partial u'}{\partial\theta}}+\overline{u'\frac{\partial u'}{\partial z}}\right) \tag{3.13}$$

Now lets apply Hinze's averaging rules to the right hand side of equation (3.10) and express instantaneous pressure and velocity as the sum of the average and fluctuating components as before. After considerable arrangement and using the fact that $\overline{u'} = 0$ and $\overline{p'} = 0$ from rule 1, the equation simplifies to:

$$\overline{-\frac{\partial(\bar{p}+p')}{\partial z}+(\bar{\rho}+\rho')g+\mu\left(\frac{\partial^2(\bar{u}+u')}{\partial r^2}+\frac{1}{r}\frac{\partial(\bar{u}+u')}{\partial r}\right)}$$

$$=-\frac{\partial\bar{p}}{\partial z}+\bar{\rho}g+\mu\left(\frac{\partial^2\bar{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{u}}{\partial r}\right) \tag{3.14}$$

The Navier-Stokes equation can now be expressed as:

$$-\bar{\rho}\left(\frac{\partial\bar{u}}{\partial t}+\bar{v}\frac{\partial\bar{u}}{\partial r}+\frac{\bar{w}}{r}\frac{\partial\bar{u}}{\partial\theta}+\bar{u}\frac{\partial\bar{u}}{\partial z}\right)$$

$$=-\frac{\partial\bar{p}}{\partial z}+\bar{\rho}g+\mu\left(\frac{\partial^2\bar{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{u}}{\partial r}\right)-\bar{\rho}\left(\overline{v'\frac{\partial u'}{\partial r}}+\overline{\frac{w'}{r}\frac{\partial u'}{\partial\theta}}+\overline{u'\frac{\partial u'}{\partial z}}\right) \tag{3.15}$$

Equation (3.4) can be expressed as:

$$u\left(\frac{\partial v}{\partial r}+\frac{v}{r}+\frac{1}{r}\frac{\partial w}{\partial \theta}+\frac{\partial u}{\partial z}\right)=0 \tag{3.16}$$

or

$$u\frac{\partial v}{\partial r}+\frac{uv}{r}+\frac{u}{r}\frac{\partial w}{\partial \theta}+u\frac{\partial u}{\partial z}=0 \tag{3.17}$$

Substitution of instantaneous values of velocity and pressure into equation (3.17) and application of Hinze's averaging rules yields:

$$\overline{u\frac{\partial v}{\partial r}+\frac{uv}{r}+\frac{u}{r}\frac{\partial w}{\partial \theta}+u\frac{\partial u}{\partial z}}=\overline{(\bar{u}+u')\frac{\partial(\bar{v}+v')}{\partial r}}+\overline{\frac{(\bar{u}+u')(\bar{v}+v')}{r}}$$
$$+\overline{\frac{(\bar{u}+u')}{r}\frac{\partial(\bar{w}+w')}{\partial \theta}}+\overline{(\bar{u}+u')\frac{\partial(\bar{u}+u')}{\partial z}}=0 \tag{3.18}$$

Applying Hinze's rule 3 to equation (3.18) results in:

$$\bar{u}\frac{\partial \bar{v}}{\partial r}+\overline{u'\frac{\partial v'}{\partial r}}+\frac{\bar{u}\bar{v}}{r}+\frac{\overline{u'v'}}{r}+\frac{\bar{u}}{r}\frac{\partial \bar{w}}{\partial \theta}+\overline{\frac{u'}{r}\frac{\partial w'}{\partial \theta}}+\bar{u}\frac{\partial \bar{u}}{\partial z}+\overline{u'\frac{\partial u'}{\partial z}}=0 \tag{3.19}$$

Rouse (1959) expressed equation (3.16) as:

$$\bar{u}\left(\frac{\partial \bar{v}}{\partial r}+\frac{\bar{v}}{r}+\frac{1}{r}\frac{\partial \bar{w}}{\partial \theta}+\frac{\partial \bar{u}}{\partial z}\right)=0$$

which implies that the remaining portion of equation (3.19) is:

$$\overline{u'\frac{\partial v'}{\partial r}}+\frac{\overline{u'v'}}{r}+\overline{\frac{u'}{r}\frac{\partial w'}{\partial \theta}}+\overline{u'\frac{\partial u'}{\partial z}}=0$$

or

$$\bar{\rho}\left(\overline{u'\frac{\partial v'}{\partial r}}+\frac{\overline{u'v'}}{r}+\overline{\frac{u'}{r}\frac{\partial w'}{\partial \theta}}+\overline{u'\frac{\partial u'}{\partial z}}\right)=0 \qquad (3.20)$$

Now subtracting equation (3.20) from the right hand side of equation (3.15):

$$\bar{\rho}\left(\frac{\partial \bar{u}}{\partial t}+\bar{v}\frac{\partial \bar{u}}{\partial r}+\frac{\bar{w}}{r}\frac{\partial \bar{u}}{\partial \theta}+\bar{u}\frac{\partial \bar{u}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial z}+\bar{\rho}g+\mu\left(\frac{\partial^2 \bar{u}}{\partial r^2}+\frac{1}{r}\frac{\partial \bar{u}}{\partial r}\right)$$

$$-\bar{\rho}\left(\overline{v'\frac{\partial u'}{\partial r}}+\overline{\frac{w'}{r}\frac{\partial u'}{\partial \theta}}+\overline{u'\frac{\partial u'}{\partial z}}+\overline{u'\frac{\partial v'}{\partial r}}+\frac{\overline{u'v'}}{r}+\overline{\frac{u'}{r}\frac{\partial w'}{\partial \theta}}+\overline{u'\frac{\partial u'}{\partial z}}\right) \qquad (3.21)$$

Rouse (1958) also indicated that one can apply the product rule to the turbulent terms. In other words:

$$\frac{\partial}{\partial r}\overline{u'v'} = \overline{u'\frac{\partial v'}{\partial r}} + \overline{v'\frac{\partial u'}{\partial r}} \quad \text{or} \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{u'v'}\right) = \overline{u'\frac{\partial v'}{\partial r}} + \overline{v'\frac{\partial u'}{\partial r}} + \frac{\overline{u'v'}}{r}$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\overline{u'w'} = \overline{\frac{u'}{r}\frac{\partial w'}{\partial \theta}} + \overline{\frac{w'}{r}\frac{\partial u'}{\partial \theta}}$$

$$\frac{\partial}{\partial z}\overline{u'^2} = \frac{\partial}{\partial z}\overline{u'u'} = \overline{u'\frac{\partial u'}{\partial z}} + \overline{u'\frac{\partial u'}{\partial z}} \tag{3.22}$$

Substitution of equation (3.22) into equation (3.21) gives us:

$$-\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{v}\frac{\partial \bar{u}}{\partial r} + \frac{\bar{w}}{r}\frac{\partial \bar{u}}{\partial \theta} + \bar{u}\frac{\partial \bar{u}}{\partial z}\right) = -\frac{\partial \bar{p}}{\partial z} + \bar{\rho}g + \mu\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r}\right)$$

$$-\rho\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{u'v'}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\overline{u'w'}\right) + \frac{\partial}{\partial z}\overline{u'^2}\right] \tag{3.23}$$

The $\overline{u'v'}$, $\overline{u'w'}$, and $\overline{u'u'}$ terms are Reynolds stresses which are a result of the stress created in turbulent flow by the mixing of fluid particles between layers. Equation (3.23) matches the equation used by Hinze. Under the same assumptions as listed in section 3.1 for laminar flow:

$$\frac{\partial \bar{u}}{\partial z} \to 0, \quad \frac{\partial \bar{u}}{\partial \theta} \to 0, \quad \bar{v} \to 0$$

Equation (3.23) can be expressed as:

$$\bar{\rho}\frac{\partial \bar{u}}{\partial t} = -\frac{\partial \bar{p}}{\partial z} + \bar{\rho}g + \mu\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r}\right) - \bar{\rho}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{u'v'}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\overline{u'w'}\right) + \frac{\partial}{\partial z}\overline{u'^2}\right] \quad (3.24)$$

As noted in section 3.1, swirl is assumed to be negligible in the present analysis which implies that $\bar{w} = 0$ and $w' = 0$. Also, for the present analysis it is assumed that the flow was fully developed so the $\frac{\partial}{\partial z}\overline{u'^2}$ term approaches zero. Equation (3.24) can then be expressed as:

$$\bar{\rho}\frac{\partial \bar{u}}{\partial t} = -\frac{\partial \bar{p}}{\partial z} + \bar{\rho}g + \mu\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r}\right) - \bar{\rho}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{u'v'}\right)\right] \quad (3.25)$$

Dividing equation (3.25) by $\bar{\rho}$ yields our final expression for the Navier-Stokes equation for vertical turbulent flow:

$$\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} + g + \nu\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{u'v'}\right) \quad (3.26)$$

Equation (3.26) matches the expression used by Ohmi and Usui (1976) except for the body force term which they did not include because their pipe has horizontal. Equation (3.26) is the basis of further modeling of oscillating velocity and shear stress.

CHAPTER 4

MODELING OSCILLATING VELOCITY

4.1 Expression for Oscillating Velocity

In chapter 2, eddy stress was introduced in order to relate turbulent shear stress to the velocity gradient (equation 2.5):

$$\tau = \xi \frac{\partial \bar{u}}{\partial y}$$

In cylindrical coordinates the above equation is expressed as:

$$\tau = \xi \frac{\partial \bar{u}}{\partial r} \tag{4.1}$$

Likewise, the concept of modified eddy viscosity, as shown by Ohmi and Usui, 1976, was also introduced:

$$\nu_E = \xi + \nu \tag{4.2}$$

ξ is actually the steady eddy viscosity component. The oscillating eddy viscosity term, $\Delta\xi$, was not used by Ohmi and Usui (1976) because they concluded that the oscillating term is insignificant at high frequencies due to the inertia dominance.

The present model is very closely based on the model used by Ohmi and Usui (1976) which expresses the Reynolds stress in terms of eddy viscosity:

$$\overline{\rho u'v'} = -\rho\xi\frac{\partial \bar{u}}{\partial r} \tag{4.3}$$

Using ρ instead of $\bar{\rho}$ from this point on and substituting equation (4.3) into the Navier-Stokes expression at the conclusion of chapter 3 (equation 3.26) gives:

$$\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} + g + \nu\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r}\right) + \xi\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{\partial \xi}{\partial r}\frac{\partial \bar{u}}{\partial r} + \frac{\xi}{r}\frac{\partial \bar{u}}{\partial r}$$

or

$$\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} + g + \frac{\partial^2 \bar{u}}{\partial r^2}(\nu+\xi) + \frac{\partial \bar{u}}{\partial r}\left(\frac{\partial \xi}{\partial r} + \frac{(\nu+\xi)}{r}\right) \tag{4.4}$$

Assuming kinematic viscosity, ν, is homogeneous throughout the flow and that modified viscosity, ν_E, is independent of z, substitution of equation (4.2) into equation (4.4) yields:

$$\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} + g + \frac{\partial^2 \bar{u}}{\partial r^2}\nu_E + \frac{\partial \bar{u}}{\partial r}\left(\frac{d\nu_E}{dr} + \frac{\nu_E}{r}\right) \tag{4.5}$$

Short-time average values of $\bar{p}$ and $\bar{u}$ have a long-time and an oscillating component due to pulsation. Gravity, g, has only a long-time component. Long-time average values are denoted using a subscript ta and oscillating components are shown with an os subscript as follows:

$$\bar{p} = p_{ta} + p_{os}$$
$$\bar{u} = u_{ta} + u_{os} \tag{4.6}$$

where:

p_{ta} = long-time average pressure

p_{os} = oscillating pressure

u_{ta} = long-time average velocity in the axial (z) direction

u_{os} = oscillating velocity in the axial (z) direction

Substitution of equations (4.6) into equation (4.5) yields:

$$\frac{\partial u_{ta}}{\partial t}+\frac{\partial u_{os}}{\partial t}=-\frac{1}{\rho}\frac{\partial p_{ta}}{\partial z}+g_{ta}+\frac{\partial^2 u_{ta}}{\partial r^2}\nu_E+\frac{\partial u_{ta}}{\partial r}\left(\frac{d\nu_E}{dr}+\frac{\nu_E}{r}\right)$$

$$-\frac{1}{\rho}\frac{\partial p_{os}}{\partial z}+\frac{\partial^2 u_{os}}{\partial r^2}\nu_E+\frac{\partial u_{os}}{\partial r}\left(\frac{d\nu_E}{dr}+\frac{\nu_E}{r}\right) \tag{4.7}$$

The long-time average components of equation (4.7) were verified in equation (3.7) except for the $\frac{1}{r}\frac{\partial}{\partial r}\left(\overline{ru'v'}\right)$ term which was introduced through Hinze's averaging procedure which is not applicable to long-time average flow. Hence, the oscillating components must hold true:

$$\frac{\partial u_{os}}{\partial t}=-\frac{1}{\rho}\frac{\partial p_{os}}{\partial z}+\frac{\partial^2 u_{os}}{\partial r^2}\nu_E+\frac{\partial u_{os}}{\partial r}\left(\frac{d\nu_E}{dr}+\frac{\nu_E}{r}\right) \tag{4.8}$$

Equation (4.8) is a second-order differential equation which can not be solved directly. Hence, we must transform the equation into a form that can be evaluated. First let us apply a Laplace transformation to equation (4.8). We start by defining L(f), the Laplace transform of f(t), as (Boas, 1983):

$$L(f) = \int_0^\infty f(t)e^{-st}dt \tag{4.9}$$

Likewise, the Laplace transformation of a first-order differential equation $y'(t)$ is expressed as:

$$L(y') = \int_0^\infty y'(t)e^{-st}dt \tag{4.10}$$

Integrating equation (4.10), using integration by parts with $dv = y'(t)dt$ and $u = e^{-st}$, yields:

$$L(y') = y(\infty)e^{-\infty} - y(0)e^{0} - (-s)\int_0^\infty y(t)e^{-st}dt$$

$$L(y') = sL(y) - y(0) \tag{4.11}$$

Hence, the Laplace transformation of the velocity component with respect to time with zero initial values yields:

$$L\left(\frac{\partial \bar{u}}{\partial t}\right) = \hat{u}_{os}s \tag{4.12}$$

where s is the parameter of the Laplace transformation and the ^ signifies the transform (Ohmi and Usui, 1976). Since the remaining terms of equation (4.8) are independent of time, their Laplace transforms are expressed as:

$$L\left(\frac{\partial \bar{p}}{\partial z}\right)=\frac{\partial \hat{p}_{os}}{\partial z} \quad , \quad L\left(\frac{\partial \bar{u}}{\partial r}\right)=\frac{\partial \hat{u}_{os}}{\partial r} \quad , \quad L\left(\frac{\partial^2 \bar{u}}{\partial r^2}\right)=\frac{\partial^2 \hat{u}_{os}}{\partial r^2} \tag{4.13}$$

Equation (4.8) can now be expressed as:

$$\hat{u}_{os}s=-\frac{1}{\rho}\frac{\partial \hat{p}_{os}}{\partial z}+\nu_E\frac{\partial^2 \hat{u}_{os}}{\partial r^2}+\frac{\partial \hat{u}_{os}}{\partial r}\left(\frac{d\nu_E}{dr}+\frac{\nu_E}{r}\right)$$

or

$$\hat{u}_{os}+\frac{1}{\rho s}\frac{\partial \hat{p}_{os}}{\partial z}=\frac{1}{s}\left[\nu_E\frac{\partial^2 \hat{u}_{os}}{\partial r^2}+\frac{\partial \hat{u}_{os}}{\partial r}\left(\frac{d\nu_E}{dr}+\frac{\nu_E}{r}\right)\right] \tag{4.14}$$

Lets now designate the right hand side of equation (4.14) as V, the equation can be expressed as:

$$\hat{u}_{os}=V-\frac{1}{\rho s}\frac{\partial \hat{p}_{os}}{\partial z} \tag{4.15}$$

where:

$$V = \frac{1}{s}\left[\nu_E \frac{\partial^2 \hat{u}_{os}}{\partial r^2} + \frac{\partial \hat{u}_{os}}{\partial r}\left(\frac{d\nu_E}{dr} + \frac{\nu_E}{r}\right)\right]$$

Assuming the pressure gradient is constant and independent of the radius, the first and second derivatives of equation (4.15) are:

$$\frac{\partial \hat{u}_{os}}{\partial r} = \frac{\partial V}{\partial r} \quad \text{and} \quad \frac{\partial^2 \hat{u}_{os}}{\partial r^2} = \frac{\partial^2 V}{\partial r^2} \tag{4.16}$$

Substitution of equations (4.16) into the right hand side of equation (4.14) gives:

$$V = \frac{1}{s}\left[\nu_E \frac{\partial^2 V}{\partial r^2} + \frac{\partial V}{\partial r}\left(\frac{d\nu_E}{dr} + \frac{\nu_E}{r}\right)\right]$$

or

$$\nu_E \frac{\partial^2 V}{\partial r^2} + \frac{\partial V}{\partial r}\left(\frac{d\nu_E}{dr} + \frac{\nu_E}{r}\right) - sV = 0 \tag{4.17}$$

To further simplify equation (4.17), a transformation variables, z_j (r), is introduced. For the modified eddy viscosity regions with $k_j = 0$, (refer to tables 2.1~2.4 in chapter 2) the function $z_j(r)$ is expressed as (Ohmi and Usui, 1976):

$$z_j(r) = r\sqrt{\frac{s}{\nu_{E,j}}} \tag{4.18}$$

The subscript j denotes the quantities in the j^{th} region. For regions where $k_j \neq 0$:

$$z_j(r) = \frac{2}{k_j u^*}\sqrt{s\nu_{E,j}} \tag{4.19}$$

The term u^* is the friction velocity (Ohmi and Usui, 1976):

$$u^* = \bar{u}_{m,st}\sqrt{\frac{\lambda}{8}} \tag{4.20}$$

$\bar{u}_{m,st}$ signifies mean-steady velocity and λ the friction factor. Using equations (2.8) ~ (2.10), the transformation is conducted as follows:

1. $k_j \neq 0$

$$\frac{dz_j(r)}{dr} = \frac{dz_j(r)}{d\nu_{E,j}}\frac{d\nu_{E,j}}{dr} = \frac{\sqrt{s}}{\sqrt{\nu_{E,j}}k_j u^*}\left(\frac{-\nu k_j R^*}{R}\right) = -\sqrt{\frac{s}{\nu_{E,j}}}\left(\frac{\nu R^*}{u^* R}\right) \tag{4.21}$$

Ohmi and Usui (1976) indicate $\left(\frac{\nu R^*}{u^* R}\right) = 1$, so equation (4.21) simplifies to:

$$\frac{dz_j(r)}{dr} = -\sqrt{\frac{s}{\nu_{E,j}}} \tag{4.22}$$

The second derivative of $z_j(r)$ with respect to r is:

$$\frac{d^2 z_j(r)}{dr^2} = \frac{d}{dr}\left[\frac{dz_j(r)}{dr}\right] = \frac{d}{dr}\left(-\sqrt{\frac{s}{\nu_{E,j}}}\right) = \frac{0 + \sqrt{s}\left[\frac{1}{2}\nu_{E,j}^{-\frac{1}{2}}\left(\frac{d\nu_{E,j}}{dr}\right)\right]}{\nu_{E,j}} \tag{4.23}$$

Using equations (2.8)~ (2.10), modified eddy viscosity in region j is expressed as:

$$\nu_{E,j} = k_j u^* R - k_j u^* r + m_j \nu \tag{4.24}$$

or

$$\frac{d\nu_{E,j}}{dr} = -k_j u^* \tag{4.25}$$

Substitution of equation (4.25) into equation (4.23) yields:

$$\frac{d^2 z_j(r)}{dr^2} = \frac{-\sqrt{s} k_j u^*}{2(\nu_{E,j})^{\frac{3}{2}}} \tag{4.26}$$

Continuing with the transformation using equations (4.22) and (4.26):

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial z_j(r)} \frac{dz_j(r)}{dr} = -\frac{\partial V}{\partial z_j(r)} \sqrt{\frac{s}{\nu_{E,j}}} \tag{4.27}$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial z_j(r)} \frac{dz_j(r)}{dr}\right) = \frac{\partial V}{\partial z_j(r)} \frac{d^2 z_j(r)}{dr^2} + \frac{\partial^2 V}{\partial z_j{}^2(r)} \left(\frac{dz_j(r)}{dr}\right)^2$$

$$\frac{\partial^2 V}{\partial r^2} = -\frac{\partial V}{\partial z_j(r)} \left[\frac{\sqrt{s} k_j u^*}{2(\nu_{E,j})^{\frac{3}{2}}}\right] + \frac{\partial^2 V}{\partial z_j{}^2(r)} \left(\frac{s}{\nu_{E,j}}\right) \tag{4.28}$$

Looking at the modified eddy viscosity derivative:

$$z_j{}^2(r) k_j{}^2 u^{*2} = 4 s \nu_{E,j}$$

$$\frac{d}{dr}\left(z_j{}^2(r) k_j{}^2 u^{*2}\right) = \frac{d}{dr}\left(4 s \nu_{E,j}\right)$$

$$2 z_j(r) \frac{dz_j(r)}{dr} k_j{}^2 u^{*2} = 4 s \frac{d\nu_{E,j}}{dr}$$

$$-2k_j{}^2u^{*2}\left(\frac{2}{k_ju^*}\sqrt{s\nu_{E,j}}\right)\sqrt{\frac{s}{\nu_{E,j}}}=4s\frac{d\nu_{E,j}}{dr}$$

$$\frac{-4k_ju^*s}{4s}=\frac{d\nu_{E,j}}{dr}$$

$$\frac{d\nu_{E,j}}{dr}=-k_ju^* \qquad (4.29)$$

Substitution of equations (4.27)~ (4.29) into equation (4.17) for $k_j \neq 0$ yields:

$$\nu_E\frac{\partial^2 V}{\partial z_j(r)^2}\left(\frac{s}{\nu_{E,j}}\right)-\nu_E\frac{\partial V}{\partial z_j(r)}\left[\frac{\sqrt{s}k_ju^*}{2\left(\nu_{E,j}\right)^{\frac{3}{2}}}\right]-\frac{\partial V}{\partial z_j(r)}\sqrt{\frac{s}{\nu_{E,j}}}\left(-k_ju^*+\frac{\nu_E}{r}\right)-sV=0$$

$$s\frac{\partial^2 V}{\partial z_j(r)^2}-\frac{\partial V}{\partial z_j(r)}\left[\frac{\sqrt{s}k_ju^*}{2\sqrt{\nu_{E,j}}}\right]-\frac{\partial V}{\partial z_j(r)}\sqrt{\frac{s}{\nu_{E,j}}}\left[-k_ju^*+\frac{\left(\nu k_jy^*+m_j\nu\right)}{r}\right]-sV=0$$

$$s\frac{\partial^2 V}{\partial z_j(r)^2}-\frac{\partial V}{\partial z_j(r)}\left[\frac{\sqrt{s}k_ju^*}{2\sqrt{\nu_{E,j}}}\right]+\frac{\partial V}{\partial z_j(r)}\sqrt{\frac{s}{\nu_{E,j}}}\left[2k_ju^*-\frac{\left(k_ju^*R+m_j\nu\right)}{r}\right]-sV=0$$

In their paper, Ohmi and Usui (1976) indicate that the term $\left[2k_ju^*-\left(\frac{k_ju^*R+m_j\nu}{r}\right)\right]$ can be approximated by k_ju^*. This simplifies the above equation to:

$$s\frac{\partial^2 V}{\partial z_j(r)^2}+\frac{\partial V}{\partial z_j(r)}\left(\frac{-\sqrt{s}k_j u^*}{2\sqrt{\nu_{E,j}}}+\frac{\sqrt{s}k_j u^*}{\sqrt{\nu_{E,j}}}\right)-sV=0 \qquad (4.30)$$

The transformation term, $z_j(r)$, for $k_j \neq 0$ in equation (4.19) is;

$$z_j(r)=\frac{2}{k_j u^*}\sqrt{s\nu_{E,j}} \quad \text{or} \quad k_j u^*=\frac{2\sqrt{s\nu_{E,j}}}{z_j(r)}$$

Equation (4.30) simplifies to:

$$s\frac{\partial^2 V}{\partial z_j(r)^2}+\frac{\partial V}{\partial z_j(r)}\left[\frac{-s}{z_j(r)}+\frac{2s}{z_j(r)}\right]-sV=0$$

$$s\frac{\partial^2 V}{\partial z_j(r)^2}+\frac{s}{z_j(r)}\frac{\partial V}{\partial z_j(r)}-sV=0$$

$$\frac{\partial^2 V}{\partial z_j(r)^2}+\frac{1}{z_j(r)}\frac{\partial V}{\partial z_j(r)}-V=0 \qquad (4.31)$$

2. $k_j = 0$

Modified eddy viscosity in this region is constant and is expressed using equation (2.10) as:

$$\nu_{E,j} = m_j \nu \tag{4.32}$$

so

$$\frac{d\nu_{E,j}}{dr} = 0 \tag{4.33}$$

Using equations (4.18) and (4.33), the transformation is conducted as:

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial z_j(r)} \frac{dz_j(r)}{dr} = \frac{\partial V}{\partial z_j(r)} \sqrt{\frac{s}{\nu_{E,j}}} \tag{4.34}$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial z_j(r)} \frac{dz_j(r)}{dr}\right) = \frac{\partial V}{\partial z_j(r)} \frac{d^2 z_j(r)}{dr^2} + \frac{\partial^2 V}{\partial z_j{}^2(r)} \left(\frac{dz_j(r)}{dr}\right)^2$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{\partial^2 V}{\partial z_j{}^2(r)} \left(\frac{s}{\nu_{E,j}}\right) \tag{4.35}$$

Looking at the modified eddy viscosity derivative:

$$z_j{}^2(r)\nu_{E,j} = r^2 s$$

$$\frac{d}{dr}\left(z_j{}^2(r)\nu_{E,j}\right) = \frac{d}{dr}\left(r^2 s\right)$$

$$\frac{d\nu_{E,j}}{dr} z_j{}^2(r) + 2z_j(r)\nu_{E,j} \frac{dz_j(r)}{dr} = 2rs$$

$$\frac{d\nu_{E,j}}{dr} z_j^{\,2}(r) = 2rs - 2\nu_{E,j} r\left(\frac{s}{\nu_{E,j}}\right) = 0$$

$$\frac{d\nu_{E,j}}{dr} = 0 \tag{4.36}$$

Substitution of equations (4.34)~ (4.36) into equation (4.17) for $k_j = 0$ yields:

$$\nu_{E,j}\frac{\partial^2 V}{\partial z_j(r)^2}\left(\frac{s}{\nu_{E,j}}\right) + \frac{\partial V}{\partial z_j(r)}\frac{z_j(r)}{r}\left(0 + \frac{\nu_E}{r}\right) - sV = 0$$

$$\nu_{E,j}\frac{\partial^2 V}{\partial z_j(r)^2}\left(\frac{s}{\nu_{E,j}}\right) + \frac{\partial V}{\partial z_j(r)}\frac{z_j(r)}{r}\left(\frac{rs}{z_j(r)^2}\right) - sV = 0$$

$$\frac{\partial^2 V}{\partial z_j(r)^2}s + \frac{\partial V}{\partial z_j(r)}\frac{s}{z_j(r)} - sV = 0$$

$$\frac{\partial^2 V}{\partial z_j(r)^2} + \frac{1}{z_j(r)}\frac{\partial V}{\partial z_j(r)} - V = 0 \tag{4.37}$$

Equation (4.37) is identical to the equation used by Ohmi and Usui (1976) and is equal to equation (4.31) for $k_j \neq 0$.

Equation (4.37) is a homogeneous differential equation with a solution involving a modified Bessel function of the first and second kinds of zero order (Boas, 1983). The

general homogenous solution for V is a linear combination of these modified Bessel functions (Kreyszig, 1988) and is expressed as:

$$V = C_j I_0\left[z_j(r)\right] + D_j K_0\left[z_j(r)\right] \tag{4.38}$$

where I_0 is a modified Bessel function of the first kind of zero order, K_0 is a modified Bessel function of the second kind of zero order, and C_j and D_j are integration constants.

Assuming a fully developed flow with a constant pressure gradient, substitution of equation (4.38) into equation (4.15) yields a solution for the oscillating velocity in terms of the oscillating pressure gradient:

$$\hat{u}_{j,os} = \left\{C_j I_0\left[z_j(r)\right] + D_j K_0\left[z_j(r)\right] - 1\right\}\frac{1}{\rho s}\frac{d\hat{p}_{j,os}}{dz} \tag{4.39}$$

4.2 Boundary Conditions

Solving equation (4.39) for oscillating velocity in region j requires solving for the constants C_j and D_j. These constants are solved using 4 boundary conditions. Figure 4.1 below is a schematic of a 2-region model, which will aid in expressing the boundary conditions.

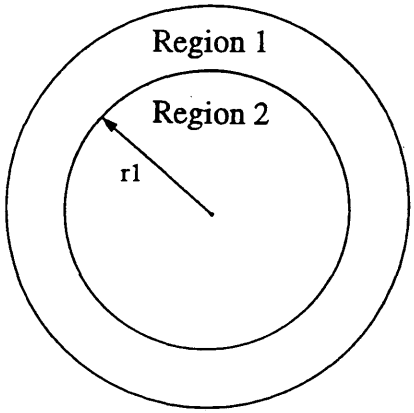


Figure 4.1 Schematic of 2-Region Model

The first boundary condition is no slip at the wall, implying that the oscillating velocity is assumed to be zero. Setting equation (4.39) equal to zero yields:

$$u_{j,os} = \left\{C_j I_0\left[z_j(r)\right] + D_j K_0\left[z_j(r)\right] - 1\right\}\frac{1}{\rho s}\frac{d\hat{p}_{j,os}}{dz} = 0 \tag{4.40}$$

The oscillating pressure gradient, $\frac{d\hat{p}_{j,os}}{dz}$, is constant. Hence, for equation (4.40) to be equal to zero, the following must equal zero:

$$C_j I_0\left[z_j(r)\right] + D_j K_0\left[z_j(r)\right] - 1 = 0 \tag{4.41}$$

The transformation variable, z_j , is a function of the radial location. At the wall j=1 and the radius is r_1 =R, so equation (4.41) can be expressed as:

$$C_1 I_0\left[z_j(R)\right] + D_1 K_0\left[z_j(R)\right] - 1 = 0 \tag{4.42}$$

The second boundary condition is that the velocity must be continuous (equal) at the region boundaries. Referring to figure 4.1, the velocity at the boundary r_1 relative to region 1 must equal the velocity at boundary r_1 relative to region 2. Assuming the pressure gradient terms are also equal at the region boundaries, the second boundary condition can also be expressed as:

$$u_j\left(r_j\right) = u_{j+1}\left(r_j\right)$$

$$C_j I_0\left[z_j\left(r_j\right)\right] + D_j K_0\left[z_j\left(r_j\right)\right] = C_{j+1} I_0\left[z_{j+1}\left(r_j\right)\right] + D_{j+1} K_0\left[z_{j+1}\left(r_j\right)\right] \tag{4.43}$$

The third boundary condition is that the shear stress at the boundary points also be continuous. For Newtonian fluids such as water in the present case, the shear stress is generally expressed as:

$$\tau = \mu \frac{\partial u}{\partial r} \tag{4.44}$$

For the oscillating shear stress in region j, equation (4.44) is expressed as:

$$\tau_{j,os} = \mu \frac{\partial \hat{u}_{j,os}}{\partial r} = \rho \nu_{E,j} \frac{\partial \hat{u}_{j,os}}{\partial r} \tag{4.45}$$

Taking the derivative of the oscillating velocity with respect to the radius requires knowing the derivatives of the modified Bessel functions of the first and second kinds of zero order. Using the technique outlined by Boas (1983) and the chain rule, the derivatives were determined to be:

$$\frac{d}{dx}\left[I_0(x)\right] = I_1(x) \qquad \text{and} \qquad \frac{d}{dx}\left[K_0(x)\right] = -K_1(x) \tag{4.46}$$

where I_1 is a modified Bessel function of the first kind of first order and K_1 is a modified Bessel function of the second kind of first order.

Using equations (4.46), the derivative of the oscillating velocity with respect to the radius is:

$$\frac{\partial \hat{u}_{j,os}}{\partial r} = \left\{C_j I_0\left[z_j(r)\right] + D_j K_0\left[z_j(r)\right] - 1\right\} \frac{1}{\rho s} \frac{\partial}{\partial r}\left(\frac{d\hat{p}_{j,os}}{dz}\right)$$

$$+\left\{C_j I_1\left[z_j(r)\right]\frac{dz_j(r)}{dr} - D_j K_1\left[z_j(r)\right]\frac{dz_j(r)}{dr}\right\}\frac{1}{\rho s}\frac{d\hat{p}_{j,os}}{dz} \qquad (4.47)$$

As noted earlier, the pressure gradient is assumed to be constant, independent of the radius. Because it is constant, the subscript j will not be included from this point on.

Since $\frac{\partial}{\partial r}\left(\frac{d\hat{p}_{j,os}}{dz}\right) = 0$, equation (4.47) can be expressed as:

$$\frac{\partial \hat{u}_{j,os}}{\partial r} = \left\{C_j I_1\left[z_j(r)\right]\frac{dz_j(r)}{dr} - D_j K_1\left[z_j(r)\right]\frac{dz_j(r)}{dr}\right\}\frac{1}{\rho s}\frac{d\hat{p}_{os}}{dz} \qquad (4.48)$$

Equations (4.45) and (4.48) allows us to express oscillating shear as:

$$\tau_{j,os} = \frac{\nu_{E,j}}{s}\left\{C_j I_1\left[z_j(r)\right]\frac{dz_j(r)}{dr} - D_j K_1\left[z_j(r)\right]\frac{dz_j(r)}{dr}\right\}\frac{d\hat{p}_{os}}{dz} \qquad (4.49)$$

For regions where the modified eddy viscosity is constant (k_j=0):

$$z_j(r_j) = r_j\sqrt{\frac{s}{\nu_{E,j}}} \qquad \text{or} \qquad \frac{dz_j(r_j)}{dr} = \sqrt{\frac{s}{\nu_{E,j}}}$$

For regions where the modified eddy viscosity is not constant ($k_j \neq 0$):

$$z_j(r_j) = \frac{2}{k_j u^*}\sqrt{s\nu_{E,j}} \quad \text{or} \qquad \frac{dz_j(r_j)}{dr} = -\sqrt{\frac{s}{\nu_{E,j}}}$$

Hence, for regions where $k_j=0$, the oscillating shear in region j can be expressed as:

$$\tau_{j,os} = \sqrt{\frac{\nu_{E,j}}{s}}\left\{C_j I_1[z_j(r)] - D_j K_1[z_j(r)]\right\}\frac{d\hat{p}_{os}}{dz} \tag{4.50}$$

For regions where $k_j \neq 0$, the oscillating shear in region j is expressed as:

$$\tau_{j,os} = -\sqrt{\frac{\nu_{E,j}}{s}}\left\{C_j I_1[z_j(r)] - D_j K_1[z_j(r)]\right\}\frac{d\hat{p}_{os}}{dz} \tag{4.51}$$

The third boundary condition is expressed as:

$$\tau_j(r_j) = \tau_{j+1}(r_j) \tag{4.52}$$

For the cases in which $k_j \neq 0$ or $k_j = 0$ in both the j^{th} and $j+1^{th}$ regions, equation (4.52) is expressed as:

$$\sqrt{\nu_{E,j}}\left\{C_j I_1\left[z_j(r_j)\right] - D_j K_1\left[z_j(r_j)\right]\right\} = \sqrt{\nu_{E,j+1}}\left\{C_{j+1} I_1\left[z_{j+1}(r_j)\right] - D_{j+1} K_1\left[z_{j+1}(r_j)\right]\right\} \tag{4.53}$$

For cases in which $k_j=0$ in the j^{th} region and with $k_j\neq 0$ in the $j+1^{th}$ region, equation (4.52) is expressed as:

$$\sqrt{\nu_{E,j}}\left\{C_j I_1\left[z_j(r_j)\right] - D_j K_1\left[z_j(r_j)\right]\right\} = -\sqrt{\nu_{E,j+1}}\left\{C_{j+1} I_1\left[z_{j+1}(r_j)\right] - D_{j+1} K_1\left[z_{j+1}(r_j)\right]\right\} \tag{4.54}$$

If the j^{th} region has $k_j\neq 0$ and the $j+1^{th}$ region has $k_j=0$, the boundary condition would be similar to equation (4.54) but with the negative sign on the left hand side instead of the right hand side.

Referring to figure 2.3, modified eddy viscosity is continuous at all the boundary points, y^*, except for at the boundary between region 6 and region 7 ($y^* = 50$). For this point, $\sqrt{\nu_{E,6}(r_6)} \neq \sqrt{\nu_{E,7}(r_6)}$, so these terms can not be eliminated from equation (4.53). Equations (4.53) and (4.54) match the equation used by Ohmi and Usui (1976) except the authors use a plus/minus sign in front of the right hand side equation. It is noted that the positive sign is taken for j=2, ...,n-2 for the 4 and 13-region models. This is simply a shorthand notation and agrees with the present expressions.

The fourth boundary condition is that the velocity must be finite at the centerline of the pipe. The boundary condition is used to ensure that the problem remains bounded. The second and third boundary conditions (continuity of velocity and shear stress) are applicable to the region boundaries and thus have n-1 equations each where n = number of regions. Along with the first boundary condition, there are a total of 2n-1 equations. For the problem to be bounded, there must be 2n-1 variables and known terms. This implies that the variables must be C_j (j =1,2,...,n) and D_j (j=1,2,...n-1) or restated:

$$D_n = 0 \tag{4.55}$$

CHAPTER 5

DETERMINATION OF OSCILLATING SHEAR STRESS

5.1 Expression for Oscillating Pressure

Equation (4.45) expressed oscillating shear stress as:

$$\tau_{j,os} = \mu \frac{\partial \hat{u}_{j,os}}{\partial r} = \rho \nu_{E,j} \frac{\partial \hat{u}_{j,os}}{\partial r}$$

Where the oscillating velocity in region j was determined to be:

$$\hat{u}_{j,os} = \left[C_j I_0(z_j) + D_j K_0(z_j) - 1 \right] \frac{1}{\rho s} \frac{d\hat{p}_{os}}{dz}$$

Hence, in order to determine the oscillating shear stress, an expression for the oscillating pressure gradient is needed.

The equation of continuity for a compressible fluid in the cylindrical coordinates (Rouse, 1959) is:

$$\frac{D\rho}{Dt} + \mathrm{div}(u_\alpha \rho) = 0 \tag{5.1}$$

where:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial r} + \frac{w}{r}\frac{\partial \rho}{\partial \theta} + u\frac{\partial \rho}{\partial z}$$

$$\mathrm{div}(u_\alpha \rho) = \frac{1}{r}\frac{\partial(\rho v r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho w)}{\partial \theta} + \frac{\partial(\rho u)}{\partial z} \tag{5.2}$$

For an incompressible fluid, ρ is considered constant with regards to r, z and θ. Allowing density to remain as a function of time, the equation of continuity can then be expressed as:

$$\frac{\partial \rho}{\partial t} + \rho\left[\frac{1}{r}\frac{\partial(vr)}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial u}{\partial z}\right] = 0$$

or simply

$$\frac{\partial \rho}{\partial t} + \rho\left[\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial u}{\partial z}\right] = 0 \tag{5.3}$$

Assuming the flow in the θ direction to be negligible, the equation of continuity can be expressed as:

$$\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial u}{\partial z}\right) = 0 \tag{5.4}$$

The isentropic speed of sound, c, is expressed as (Ohmi and Usui, 1976):

$$\frac{\partial p}{\partial \rho} = c^2 \tag{5.5}$$

Integration of equation (5.5) yields:

$$p = c^2 \rho \tag{5.6}$$

So the derivative of density with respect to time yields:

$$\frac{\partial \rho}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} \tag{5.7}$$

substitution of equation (5.7) into equation (5.4) results in:

$$\frac{\partial p}{\partial t} = -c^2 \rho \left(\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial u}{\partial z} \right) \tag{5.8}$$

In Chapter 3, instantaneous values of velocity and pressure components were expressed as the sum of the short-time average component and the turbulent fluctuating components:

$$\begin{aligned} p &= \bar{p} + p' \\ u &= \bar{u} + u' \\ v &= \bar{v} + v' \end{aligned} \tag{5.9}$$

Substitution of equations (5.9) into equation (5.8) and using Hinze's ensemble averaging rule 3 (Chapter 3) yields:

$$\begin{aligned} \overline{\frac{\partial(\bar{p}+p')}{\partial t}} &= -\overline{c^2 \rho \frac{\partial(\bar{v}+v')}{\partial r}} - \overline{c^2 \rho \frac{(\bar{v}+v')}{r}} - \overline{c^2 \rho \frac{\partial(\bar{u}+u')}{\partial z}} \\ \frac{\partial(\bar{\bar{p}}+\bar{p'})}{\partial t} &= -c^2 \rho \frac{\partial(\bar{\bar{v}}+\bar{v'})}{\partial r} - c^2 \rho \frac{(\bar{\bar{v}}+\bar{v'})}{r} - c^2 \rho \frac{\partial(\bar{\bar{u}}+\bar{u'})}{\partial z} \end{aligned} \tag{5.10}$$

Hinze's first rule (Chapter 3) indicates that $\overline{p'}, \overline{v'}, \overline{u'} \rightarrow 0$ and rule 3 indicates that $\overline{\overline{p}} = \overline{p}$ (etc.) so equation (5.10) becomes:

$$\frac{\partial \overline{p}}{\partial t} = -c^2 \rho \left(\frac{\partial \overline{v}}{\partial r} + \frac{\overline{v}}{r} + \frac{\partial \overline{u}}{\partial z} \right) \tag{5.11}$$

Average values of $\overline{p}, \overline{v}$ and $\overline{u}$ have both a long-time and oscillating component due to the pulsation (Ohmi and Usui, 1976). As with Chapter 4, subscripts ta and os are used to designate these components:

$$\begin{aligned} \overline{p} &= p_{ta} + p_{os} \\ \overline{v} &= v_{ta} + v_{os} \\ \overline{u} &= u_{ta} + u_{os} \end{aligned} \tag{5.12}$$

Substitution of equations (5.12) into equation (5.11) yields:

$$\frac{\partial p_{ta}}{\partial t} + \frac{\partial p_{os}}{\partial t} = -c^2 \rho \left(\frac{\partial v_{ta}}{\partial r} + \frac{v_{ta}}{r} + \frac{\partial u_{ta}}{\partial z} \right) - c^2 \rho \left(\frac{\partial v_{os}}{\partial r} + \frac{v_{os}}{r} + \frac{\partial u_{os}}{\partial z} \right) \tag{5.13}$$

The long-time average components of the above equation were verified in equation (5.8) which indicates that the oscillating components hold true:

$$\frac{\partial p_{os}}{\partial t} = -c^2 \rho \left(\frac{\partial v_{os}}{\partial r} + \frac{v_{os}}{r} + \frac{\partial u_{os}}{\partial z} \right) \tag{5.14}$$

Now taking the cross-sectional average of the right hand side of equation (5.14):

$$\text{Ave(RHS)} = \frac{-c^2 \rho}{\pi R^2} \int_0^R \left(\frac{\partial v_{os}}{\partial r} + \frac{v_{os}}{r} + \frac{\partial u_{os}}{\partial z} \right) 2\pi r dr \tag{5.15}$$

The above integral can be broken up into 3 separate integrals:

$$\frac{-2c^2 \rho}{R^2} \int_0^R \frac{\partial v_{os}}{\partial r} r dr - \frac{-2c^2 \rho}{R^2} \int_0^R v_{os} dr - \frac{c^2 \rho}{\pi R^2} \int_0^R \frac{\partial u_{os}}{\partial z} 2\pi r dr \tag{5.16}$$

Integrating the first integral, using integration by parts with $dv = \frac{\partial v_{os}}{\partial r}$ and u = r, yields:

$$\frac{-2c^2 \rho}{R^2} \left[R v_{os}(R) - 0 - \int_0^R v_{os} dr \right]$$

Assuming oscillating velocity is zero at the wall, this integral simplifies to:

$$\frac{2c^2\rho}{R^2}\int_0^R v_{os}dr$$

which cancels with the second integral. Hence, the cross-sectional average of the expression is the third integral:

$$-\frac{c^2\rho}{\pi R^2}\int_0^R \frac{\partial u_{os}}{\partial z}2\pi rdr = -c^2\rho\frac{\partial}{\partial z}\left(\frac{1}{\pi R^2}\int_0^R u_{os}2\pi rdr\right)$$

The $\left(\frac{1}{\pi R^2}\int_0^R u_{os}2\pi rdr\right)$ term is by definition the cross-sectional average of u_{os} which will denoted as $u_{m,os}$. Hence the cross-sectional average is:

$$\frac{\partial p_{os}}{\partial t} = -c^2\rho\frac{du_{m,os}}{dz} \tag{5.17}$$

Application of a Laplace transform of the above equation yields:

$$\hat{p}_{os} = -\frac{c^2 \rho}{s} \frac{d\hat{u}_{m,os}}{dz} \tag{5.18}$$

In Chapter 3, the Navier-Stokes equation for turbulent flow was expressed as:

$$\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\rho}\frac{d\bar{p}}{dz} + \nu\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{u'v'}\right)$$

or

$$\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\rho}\frac{d\bar{p}}{dz} + \nu\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r}\right) - \frac{\partial \overline{u'v'}}{\partial r} - \frac{\overline{u'v'}}{r} \tag{5.19}$$

Using equation (5.12), the $\bar{u}$ and $\bar{p}$ terms can be expressed in terms of long-time and oscillating components. As with equation (4.7), the long-time average components are verified in equation (3.7), except for the Reynolds stress terms which don't apply to long-time average flow which implies:

$$\frac{\partial u_{os}}{\partial t} = -\frac{1}{\rho}\frac{dp_{os}}{dz} + \nu\left(\frac{\partial^2 u_{os}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{os}}{\partial r}\right) - \frac{\partial \overline{u'v'}}{\partial r} - \frac{\overline{u'v'}}{r} \tag{5.20}$$

At the wall $k_j = 0$ and $\nu_E = \nu$. Using equations (4.2) and (4.3) and assuming a homogeneous kinematic viscosity, equation (5.20) reduces to:

$$\frac{\partial u_{os}}{\partial t} = -\frac{1}{\rho}\frac{dp_{os}}{dz} + \nu\frac{\partial^2 u_{os}}{\partial r^2} + \frac{\nu}{r}\frac{\partial u_{os}}{\partial r} \tag{5.21}$$

Taking the cross-sectional average of the right hand side of equation (5.21):

$$\text{Ave(RHS)} = \frac{1}{\pi R^2}\int_0^R \left(-\frac{1}{\rho}\frac{dp_{os}}{dz} + \nu\frac{\partial^2 u_{os}}{\partial r^2} + \frac{\nu}{r}\frac{\partial u_{os}}{\partial r} \right) 2\pi r dr \tag{5.22}$$

The above integral can be broken up into 3 separate integrals:

$$\frac{-1}{\rho\pi R^2}\int_0^R \frac{dp_{os}}{dz} 2\pi r dr + \frac{2\nu}{R^2}\int_0^R \frac{\partial^2 u_{os}}{\partial r^2} r dr + \frac{2\nu}{R^2}\int_0^R \frac{\partial u_{os}}{\partial z} dr \tag{5.23}$$

Integrating the second integral, using integration by parts with $dv = \frac{\partial^2 u_{os}}{\partial r}$ and u=r, yields:

$$\frac{2\nu}{R^2}\left[R\left(\frac{\partial u_{os}}{\partial r}\right)_{r=R} - 0 - \int_0^R \frac{\partial u_{os}}{\partial r} dr \right]$$

Simplifying and combining with the third integral yields:

$$\frac{2\nu}{R}\left(\frac{\partial u_{os}}{\partial r}\right)_{r=R} - \frac{2\nu}{R^2}\int_0^R \frac{\partial u_{os}}{\partial z}dr + \frac{2\nu}{R^2}\int_0^R \frac{\partial u_{os}}{\partial z}dr = \frac{2\nu}{R}\left(\frac{\partial u_{os}}{\partial r}\right)_{r=R} \tag{5.24}$$

The first integral can be expressed as:

$$\frac{-1}{\rho\pi R^2}\int_0^R \frac{dp_{os}}{dz}2\pi r dr = \frac{-1}{\rho}\left(\frac{1}{\pi R^2}\int_0^R \frac{dp_{os}}{dz}2\pi r dr\right)$$

The $\left(\frac{1}{\pi R^2}\int_0^R \frac{dp_{os}}{dz}2\pi r dr\right)$ term is by definition the cross-sectional average of $\frac{dp_{os}}{dz}$.

Because the oscillating pressure gradient is not a function of r, Ohmi and Usui (1976) designate the left hand side of equation (5.21) in terms of $u_{m,os}$. Hence the cross-sectional average is:

$$\frac{\partial u_{m,os}}{\partial t} = -\frac{1}{\rho s}\frac{dp_{os}}{dz} + \frac{2\nu}{R}\left(\frac{\partial u_{os}}{\partial r}\right)_{r=R} \tag{5.25}$$

Taking the Laplace transform of the above equation using zero initial values results in:

$$\hat{u}_{os}s = -\frac{1}{\rho s}\frac{d\hat{p}_{os}}{dz} + \frac{2\nu}{R}\left(\frac{\partial \hat{u}_{os}}{\partial r}\right)_{r=R}$$

or

$$\hat{u}_{os} = -\frac{1}{\rho s}\frac{d\hat{p}_{os}}{dz} + \frac{2\nu}{Rs}\left(\frac{\partial \hat{u}_{os}}{\partial r}\right)_{r=R} \tag{5.26}$$

Cross-sectional averaging and differentiation of equation (5.26) with respect to z yields:

$$\frac{\partial \hat{u}_{m,os}}{\partial z} = -\frac{1}{\rho s}\frac{d^2\hat{p}_{os}}{dz^2} + \frac{2\nu}{Rs}\frac{d}{dz}\left(\frac{\partial \hat{u}_{os}}{\partial r}\right)_{r=R} \tag{5.27}$$

where $\hat{u}_{m,os}$ designates the mean oscillating velocity. From Chapter 4, the oscillating velocity in region j in terms of the pressure gradient was expressed as:

$$\hat{u}_{j,os} = \left\{C_j I_0\left[z_j(r)\right] + D_j K_0\left[z_j(r)\right] - 1\right\}\frac{1}{\rho s}\frac{d\hat{p}_{os}}{dz}$$

Taking the derivative of the above equation for the oscillating velocity with respect to the radius yields:

$$\frac{\partial \hat{u}_{j,os}}{\partial r} = \left\{C_j I_0\left[z_j(r)\right] + D_j K_0\left[z_j(r)\right] - 1\right\}\frac{1}{\rho s}\frac{\partial}{\partial r}\left(\frac{d\hat{p}_{os}}{dz}\right)$$

$$+\frac{1}{\rho s}\left\{C_j I_1\left[z_j(r)\right]\frac{dz_j(r)}{dr} - D_j K_1\left[z_j(r)\right]\frac{dz_j(r)}{dr}\right\}\frac{d\hat{p}_{os}}{dz} \tag{5.28}$$

Since the pressure gradient, $\frac{d\hat{p}_{os}}{dz}$, is constant with respect to the radius and $\nu_E = \nu$ at r = R, equation (5.28) can be expressed as:

$$\frac{\partial \hat{u}_{j,os}}{\partial r} = \frac{1}{\rho s}\left\{C_j I_1\left[z_j(r)\right]\frac{dz_j(r)}{dr} - D_j K_1\left[z_j(r)\right]\frac{dz_j(r)}{dr}\right\}\frac{d\hat{p}_{os}}{dz} \tag{5.29}$$

From Chapter 4, the value of $\frac{\partial z_j(r)}{\partial r}$ is $\pm\sqrt{\frac{s}{\nu_{E,j}}}$ depending on $k_j=0$ and $k_j\neq 0$, respectively. The 2-region model is the only model with $k_j\neq 0$ at the wall, so we will use the (+) sign for the derivative. Equation (5.29) can then be expressed as:

$$\frac{\partial \hat{u}_{j,os}}{\partial r} = \frac{1}{\rho s}\sqrt{\frac{s}{\nu}}\left\{C_j I_1\left[z_j(r)\right] - D_j K_1\left[z_j(r)\right]\right\}\frac{d\hat{p}_{os}}{dz} \tag{5.30}$$

Earlier in the chapter the oscillating pressure was determined to be:

$$\hat{p}_{os} = -\frac{c^2 \rho}{s} \frac{d\hat{u}_{m,os}}{dz}$$

or

$$\frac{d\hat{u}_{m,os}}{dz} = \frac{-\hat{p}_{os} s}{c^2 \rho} \tag{5.31}$$

Substitution of equations (5.30) and (5.31) into equation (5.27) yields:

$$\frac{-\hat{p}_{os} s}{c^2 \rho} = \frac{-1}{\rho s} \frac{d^2 \hat{p}_{os}}{dz^2} + \frac{1}{\rho s} \sqrt{\frac{s}{\nu}} \frac{2\nu}{Rs} \frac{d^2 \hat{p}_{os}}{dz^2} \left\{ C_j I_1 \left[z_j(R) \right] - D_j K_1 \left[z_j(R) \right] \right\}$$

or

$$\frac{1}{\rho s} \frac{d^2 \hat{p}_{os}}{dz^2} \left(1 - \sqrt{\frac{s}{\nu}} \frac{2\nu}{Rs} \left\{ C_j I_1 \left[z_j(R) \right] - D_j K_1 \left[z_j(R) \right] \right\} \right) - \frac{s}{c^2 \rho} \hat{p}_{os} = 0$$

or

$$\frac{d^2 \hat{p}_{os}}{dz^2} - \frac{\dfrac{s^2}{c^2}}{\left(1 - \sqrt{\dfrac{s}{\nu}} \dfrac{2\nu}{Rs} \left\{ C_j I_1 \left[z_j(R) \right] - D_j K_1 \left[z_j(R) \right] \right\} \right)} \hat{p}_{os} = 0 \tag{5.32}$$

Equation (5.32) can be expressed as:

$$\frac{d^2 \hat{p}_{os}}{dz^2} - \gamma^2 \hat{p}_{os} = 0 \tag{5.33}$$

where γ is the wave propagation constant (Ohmi and Usui, 1976):

$$\gamma = \frac{\frac{s}{c}}{\sqrt{1 - \frac{2}{R}\sqrt{\frac{\nu}{s}}\{C_1 I_1[z_1(R)] - D_1 K_1[z_1(R)]\}}} \tag{5.34}$$

For the 2-region model, equation (5.34) would be expressed as:

$$\gamma = \frac{\frac{s}{c}}{\sqrt{1 + \frac{2}{R}\sqrt{\frac{\nu}{s}}\{C_1 I_1[z_1(R)] - D_1 K_1[z_1(R)]\}}} \tag{5.35}$$

Equation (5.33) is a homogeneous differential equation with a solution of:

$$\hat{p}_{os} = Ae^{\gamma z} + Be^{-\gamma z} \tag{5.36}$$

where A and B are constants determined from pressure data.

5.2 Expression for Oscillating Shear Stress

The pressure gradient of equation (5.36) is:

$$\frac{d\hat{p}_{os}}{dz} = \gamma\left(Ae^{\gamma z} - Be^{-\gamma z}\right) \tag{5.37}$$

Substitution of equation (5.37) into equation (4.39) gives the following equation for oscillating velocity:

$$\hat{u}_{j,os} = \frac{\gamma}{\rho s}\left(Ae^{\gamma z} - Be^{-\gamma z}\right)\left\{C_j I_0\left[z_j(r)\right] + D_j K_0\left[z_j(r)\right] - 1\right\} \tag{5.38}$$

This equation agrees with the expression used by Ohmi and Usui (1976). Also note the difference between z and z_j. The z_j term represents a transformation of variable term for region j and is a function of r (refer to equations 4.18 and 4.19). The z term itself represents the axial direction since the marine riser for this analysis is considered to be in a vertical configuration.

Developing an expression for oscillating shear stress at the wall requires taking the derivative of oscillating velocity with respect to r (refer to equation 4.45). For the 3, 4 and 8-region models, the transformation variable, z_j , for k_j=0 (applicable to the wall region) is:

$$z_j(r) = r\sqrt{\frac{s}{\nu_{E,j}}} \qquad \text{so} \qquad \frac{\partial z_j}{\partial r} = \sqrt{\frac{s}{\nu_{E,j}}} \tag{5.39}$$

Using equations (5.38) and (5.39), the derivative of oscillating velocity is:

$$\frac{\partial \hat{u}_{j,os}}{\partial r} = \frac{\gamma}{\rho s}\left(Ae^{\gamma z} - Be^{-\gamma z}\right)\left\{C_j I_1\left[z_j(r)\right] - D_j K_1\left[z_j(r)\right]\right\}\sqrt{\frac{s}{\nu_{E,j}}} \tag{5.40}$$

Hence, the oscillating shear stress in region j is expressed as:

$$\tau_{j,os} = \rho\nu_{E,j}\frac{\partial \hat{u}_{j,os}}{\partial r} \tag{5.41}$$

$$\tau_{j,os} = \left(\rho\nu_{E,j}\right)\frac{\gamma}{\rho s}\sqrt{\frac{s}{\nu_{E,j}}}\left(Ae^{\gamma z} - Be^{-\gamma z}\right)\left\{C_j I_1\left[z_j(r)\right] - D_j K_1\left[z_j(r)\right]\right\}$$

$$\tau_{j,os} = \gamma\sqrt{\frac{\nu_{E,j}}{s}}\left(Ae^{\gamma z} - Be^{-\gamma z}\right)\left\{C_j I_1\left[z_j(r)\right] - D_j K_1\left[z_j(r)\right]\right\} \tag{5.42}$$

The wall shear stress (j=1) with $r_1 = R$ and $\nu_{E,1} = \nu$ from equation (5.42) is expressed as:

$$\tau_{w,os} = \gamma\sqrt{\frac{\nu}{s}}\left(Ae^{\gamma z} - Be^{-\gamma z}\right)\left[C_1 I_1\left(R\sqrt{\frac{s}{\nu}}\right) - D_1 K_1\left(R\sqrt{\frac{s}{\nu}}\right)\right] \tag{5.43}$$

CHAPTER 6

MODEL OF OSCILLATING VELOCITY AND SHEAR STRESS INSIDE A VERTICAL RISER

6.1 Application Background and Parameters

In deep-ocean mining applications, the manganese nodules or crust located at depths of 800–6,000-m, will be transported in a slurry flow from the miner on the seafloor to the ship on the ocean surface as illustrated in figure 6.1.

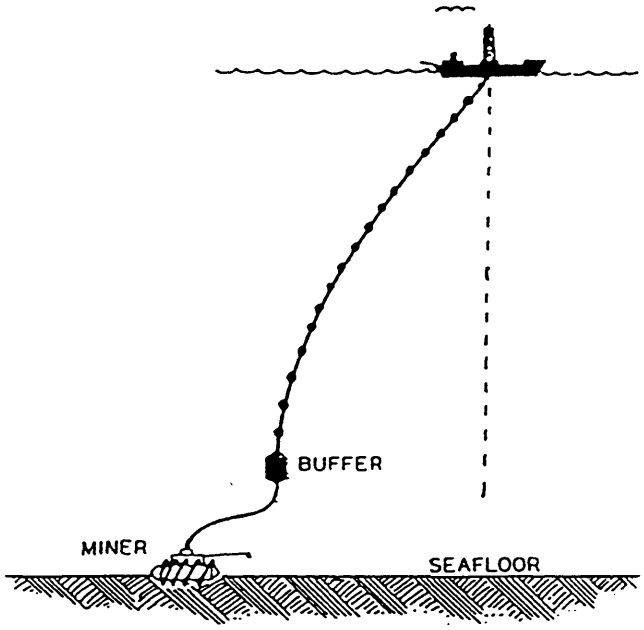


Figure 6.1 Schematic of the Hoisting System

For the present model, however, the internal flow is analyzed only as seawater with the intention of determining a simplified fundamental understanding of the effects of oscillating shear stress on the marine riser or pipe.

In the present application, seawater is transported through a 20 inch outer diameter (.6354 in thickness) commercial steel pipe with an upward mean-steady velocity of $\bar{u}_{m,st}$ = 10 ft/s (Cheng, Chung and Huttelmaier, 1994). Based on a kinematic viscosity of 1 x 10^{-5} ft^2/s, the flow has a Reynolds number (Re) of 1.56 x 10^6 and is considered to be fully turbulent. Based on this Reynolds number, the friction factor was determined to be λ=0.013. The top of the pipe is pinned to the ship, incorporating a heave compensator to compensate for the ship motions induced by the surface waves. The period of these oscillations can range from T = 3.1 to 11.90 s, which are common for Sea State 5 (Chung, 1993). The bottom of the pipe is attached to a buffer and is considered free and independent of the self-propelled seafloor miner.

6.2 Determination of C_j and D_j :

Solving equations (5.38) and (5.43) for oscillating velocity and shear stress in region j requires solving for constants C_j and D_j. Applying the boundary conditions outlined in section 4.2, the equations for the 8-region model are expressed as:

For r = R with boundary condition (4.42):

$$C_1 I_0[z_1(R)] + D_1 K_0[z_1(R)] = 1 \tag{6.1}$$

For $r = r_j$ ($j = 1,2,...,7$) with boundary condition (4.43):

$$C_j I_0[z_j(r_j)] + D_j K_0[z_j(r_j)] - C_{j+1} I_0[z_{j+1}(r_j)] - D_{j+1} K_0[z_{j+1}(r_j)] = 0 \tag{6.2}$$

For $r = r_1$, r_7 with boundary condition (4.54):

$$\sqrt{\nu_{E,j}(r_j)}\{C_j I_1[z_j(r_j)] - D_j K_1[z_j(r_j)]\} + \sqrt{\nu_{E,j+1}(r_j)}\{C_{j+1} I_1[z_{j+1}(r_j)] - D_{j+1} K_1[z_{j+1}(r_j)]\} = 0 \tag{6.3}$$

For $r = r_2$,..., r_6 with boundary condition (4.53):

$$\sqrt{\nu_{E,j}(r_j)}\{C_j I_1[z_j(r_j)] - D_j K_1[z_j(r_j)]\} - \sqrt{\nu_{E,j+1}(r_j)}\{C_{j+1} I_1[z_{j+1}(r_j)] - D_{j+1} K_1[z_{j+1}(r_j)]\} = 0 \tag{6.4}$$

For $r = 0$ with boundary condition (4.55):

$$D_8 = 0 \tag{6.5}$$

The viscosity portions of equations (6.3) and (6.4) are equal for every region boundary in the 8-region model except for at the boundary between region 6 and region 7 ($y^* = 50$) where the viscosity is not continuous (refer to figure 2.3). For the 8-region model, equations (6.1) ~ (6.4) can be expressed in matrix form as:

$$\begin{bmatrix} \text{Bessel} \\ \text{Functions} \\ \text{of } z_j \\ \text{terms} \end{bmatrix} * \begin{bmatrix} \text{Unknown} \\ C_j, D_j \\ \text{terms} \end{bmatrix} = \begin{bmatrix} \text{Known} \\ \text{RHS} \\ \text{values} \end{bmatrix} \tag{6.6}$$

where the "Bessel Function" matrix is 15 x 15 and both the "C_j ,D_j" and the "RHS values" matrices are 1 x 15 each. For the 8-region model, equation (6.6) can be expressed in detail as:

$$\begin{bmatrix} I_0[z_1(R)] & K_0[z_1(R)] & 0 & 0 & 0. & 0. & 0 \\ I_0[z_1(r_1)] & K_0[z_1(r_1)] & -I_0[z_2(r_1)] & -K_0[z_2(r_1)] & 0 & 0 & 0 \\ 0 & 0 & I_0[z_2(r_2)] & K_0[z_2(r_2)] & -I_0[z_3(r_2)] & . & 0 \\ 0 & 0 & 0 & 0 & I_0[z_3(r_3)] & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_1[z_1(r_1)] & -K_1[z_1(r_1)] & I_1[z_2(r_1)] & -K_1[z_2(r_1)] & 0 & 0. & 0 \\ 0 & 0 & I_1[z_2(r_2)] & -K_1[z_2(r_2)] & -I_1[z_3(r_2)] & . & 0 \\ 0 & 0 & 0 & 0 & I_1[z_3(r_3)] & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ D_1 \\ C_2 \\ D_2 \\ C_3 \\ D_3 \\ C_4 \\ D_4 \\ C_5 \\ D_5 \\ C_6 \\ D_6 \\ C_7 \\ D_7 \\ C_8 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.7}$$

The values of $z_j(r_\alpha)$, where $\alpha = j-1, j$ for $j > 1$ and $\alpha = j$ for $j=1$, are determined using equations (4.18) and (4.19) depending if $k_j = 0$ or $k_j \neq 0$ in region j. In either case, the value of $\nu_{E,j}$ was determined using equation (2.10) with values of k_j and m_j from the 8-region model listed in table 2.4. Equation (6.7) can be solved for C_j (j = 1,2,...,8) and D_j (j = 1,2,...,7) using either linear algebraic techniques or software packages. MathCad, is used for the present analysis.

6.3 Determination of A and B:

Constants A and B were determined using experimental pressure data from Ohmi, Usui, Tanaka and Toyama (1976). Their data consisted of oscillating pressure readings for both air and water at two stations separated by a distance L. Application of these pressure conditions to equation (5.36) yields:

$$\text{at } z = 0: \quad \hat{p}_{os}(z=0) = A + B$$

$$\text{at } z = L: \quad \hat{p}_{os}(z=L) = Ae^{\gamma L} + Be^{-\gamma L} \tag{6.8}$$

A and B can be expressed from (6.8) as:

$$A = \frac{\left(P_2 - P_1 e^{-\gamma L}\right)}{e^{\gamma L} - e^{-\gamma L}}$$

$$B = \frac{\left(P_1 e^{\gamma L} - P_2\right)}{e^{\gamma L} - e^{-\gamma L}} \tag{6.9}$$

where γ is determined using equation (5.34) and values of C_1 and D_1.

The experimental setup for water measurements used by the authors consisted of 79.0- mm I.D. horizontal brass pipe connected in series to pumps, head tanks and reservoirs. Their pressure measurement apparatus consisted of a strain-gauge-type

pressure transducer connected to a dynamic strain meter with the output voltage recorded on a photo oscillograph. This setup is identical to the setup used by Ohmi, Usui, Fukawa and Hirasaki in an earlier paper on laminar flow (1976). The distance used by the authors between both the air and water pressure readings was L=3 m and their pressure readings were recorded in mm Ag which has a density of 10,500 kg/m^3 (Lynch, 1974). In their paper they included the oscillating pressure graphs and a table of A and B values for several air tests but included only one graph of oscillating pressure for water.

Prior to determining values of A and B for the water test case, their values of A and B for the air tests were checked to ensure that the present calculation method and values for γ were correct. The present calculation of A and B for in kg/m^2 for air agreed with the authors' values, when their A and B values are multiplied by g=9.81 m/s^2. Their error involved an error in conversion between mass and force.

For their single turbulent water flow test, the pressure readings taken from the horizontal pipe setup at z = 0 was recorded as P_1 = 2375 mm Ag and at z = 3m the pressure was P_2 = 1000 mm Ag. The pressure difference is a measure of the headloss, h_L, due to friction. For the present application, it was necessary to modify the pressure readings to account for the body force associated with the vertical flow of a marine riser. Figure 6.2 illustrates both the experimental setup used by Ohmi, Usui, Tanaka and Toyama (1976) as well as the deep-ocean mining marine riser.

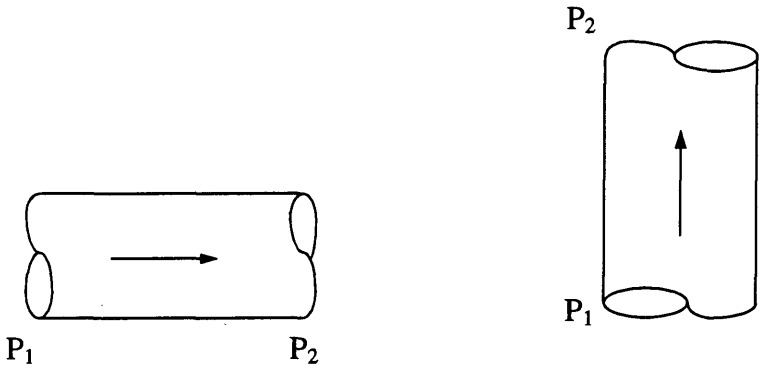


Figure 6.2 Pipe Orientation of Horizontal Pipe and Vertical Marine Riser

The pressure drop, ΔP, term reflects the effect of body force and headloss, h_L, due to friction, and is expressed as:

$$\Delta P = L\gamma_{sw} + h_L \tag{6.10}$$

where γ_{sw} designates the specific weight of seawater which is equal to 64.0 lb/ft^3 or 10,055.5 N/m^3. Hence, using L = 3m as the distance between pressure readings P_1 and P_2, the pressure drop due to body force is calculated as:

$$L\gamma_{sw} = (3\text{m}) * \left(10{,}055.5\frac{\text{N}}{\text{m}^3}\right) = 30{,}166.5\frac{\text{N}}{\text{m}^2} \tag{6.11}$$

Converting P_1 and P_2 from mm Ag into N/m^2 gives us:

$$P_1 = 2.375\text{m} * \left(\frac{10{,}500\text{kg}}{\text{m}^3}\right) * \left(\frac{9.81\text{m}}{\text{s}^2}\right) = 244{,}636.9\frac{\text{N}}{\text{m}^2}$$

$$P_2 = 1.00\text{m} * \left(\frac{10{,}500\text{kg}}{\text{m}^3}\right) * \left(\frac{9.81\text{m}}{\text{s}^2}\right) = 103{,}005.0\frac{\text{N}}{\text{m}^2} \qquad (6.12)$$

The difference between P_1 and P_2 is the headloss, h_L, which has a value of 141,631.9 $\frac{\text{N}}{\text{m}^2}$. Hence, the total pressure drop value from equation (6.10) is expressed as:

$$\Delta P = L\gamma_{sw} + h_L$$

or

$$\Delta P = 30{,}166.5\frac{\text{N}}{\text{m}^2} + 141{,}631.88\frac{\text{N}}{\text{m}^2} = 171{,}798.4\frac{\text{N}}{\text{m}^2}$$

Hence, the pressure reading at the top is $P_2 = 103{,}005.00\ \frac{\text{N}}{\text{m}^2}$ and $P_1 = 274{,}803.4\ \frac{\text{N}}{\text{m}^2}$ at the bottom.

Based on the parameters of Ohmi, Usui, Tanaka and Toyama (1976) (Re=74,100, ω=1.0 Hz), constants C_1 and D_1 were determined and used with equation (6.9) to solve

for A and B. Based on these values, A and B were determined as $|A| = 667{,}570.5$ N/m^2 (13,940.58 lb/ft^2) and $|B| = 667{,}864.8$ N/m^2 (13,946.73 lb/ft^2).

6.4 Modeling Oscillating Velocity due to Pipe Flow

Using the 8-region model boundary values from section 2.5, as well as the application parameters outlined in section 6.1, values of C_j and D_j were determined which were used in calculating the wave propagation constant, γ. Table 6.1 shows values of the propagation constant, γ, for periods within Sea State 5 criteria (Chung, 1993) at $\bar{u}_{m,st}$ = 10 ft/s:

Table 6.1

Values of γ for T =3~12 s at $\bar{u}_{m,st}$ = 10 ft/s

Period (s)	Wave Propagation Constant, γ, (ft^{-1})
3.0	$3.5023*10^{-6} + 4.3431*10^{-4}$ i
4.0	$3.4034*10^{-6} + 3.2647*10^{-4}$ i
5.0	$3.3352*10^{-6} + 2.6174*10^{-4}$ i
6.0	$3.2848*10^{-6} + 2.1859*10^{-4}$ i
7.0	$3.2458*10^{-6} + 1.8775*10^{-4}$ i
8.0	$3.2147*10^{-6} + 1.6462*10^{-4}$ i
9.0	$3.1892*10^{-6} + 1.4662*10^{-4}$ i
10.0	$3.1680*10^{-6} + 1.3222*10^{-4}$ i
11.0	$3.1500*10^{-6} + 1.2043*10^{-4}$ i
12.0	$3.1346*10^{-6} + 1.1060*10^{-4}$ i

As table 6.1 indicates, the γ term is complex (due to $s=i\omega$) and can be expressed as:

$$\gamma z = \gamma_R z + i\gamma_I z \tag{6.13}$$

where subscripts R and I signify the real and imaginary parts of γ and z is positive upward along the pipe. For the present study, z varies from 0-18,000 ft. Substitution of equation (6.13) into equation (5.38) results in:

$$\hat{u}_{j,os} = KU_j(\gamma_R + i\gamma_I)\left[Ae^{(\gamma_R+i\gamma_I)z} - Be^{-(\gamma_R+i\gamma_I)z}\right] \tag{6.14}$$

where KU_j is a constant within region j and equals:

$$KU_j = \frac{\left\{C_j I_0[z_j(r)] + D_j K_0[z_j(r)] - 1\right\}}{\rho s} \tag{6.15}$$

An expanded form of (6.14) is:

$$\hat{u}_{j,os} = KU_j(\gamma_R + i\gamma_I)\{Ae^{\gamma_R z}[\cos(\gamma_I z) + i\sin(\gamma_I z)]$$

$$-Be^{-\gamma_R z}\left[\cos(\gamma_I z) - i\sin(\gamma_I z)\right]\Big\} \tag{6.16}$$

The real part of (6.16) is of interest to the present application:

$$\hat{u}_{j,R,os} = KU_j\left[\gamma_R Ae^{\gamma_R z}\cos(\gamma_I z) - \gamma_I Ae^{\gamma_R z}\sin(\gamma_I z)\right.$$
$$\left.-\gamma_R Be^{-\gamma_R z}\cos(\gamma_I z) - \gamma_I Be^{-\gamma_R z}\sin(\gamma_I z)\right] \tag{6.17}$$

The oscillating velocity, $\hat{u}_{j,R,os}$, across a pipe for mean-steady velocities $\bar{u}_{m,st}$ = 1, 5, 10 and 20 ft/s calculated at periods T =5 and 6 s at z = 100 ft is shown in Figure 6.3. Examples of oscillating velocity for $\bar{u}_{m,st}$ = 1, 5, 10 and 20 ft/s as a function of z are shown in Figures 6.4–6.6. The figures illustrate that oscillating velocity is not related to the mean flow. In addition, Figures 6.3 and 6.4 show that the oscillating velocity is influenced by the oscillation period. The magnitude of the oscillating velocities at T = 5 s is larger than at T = 6 s. Absolute values of KU_j where used for the graphs.

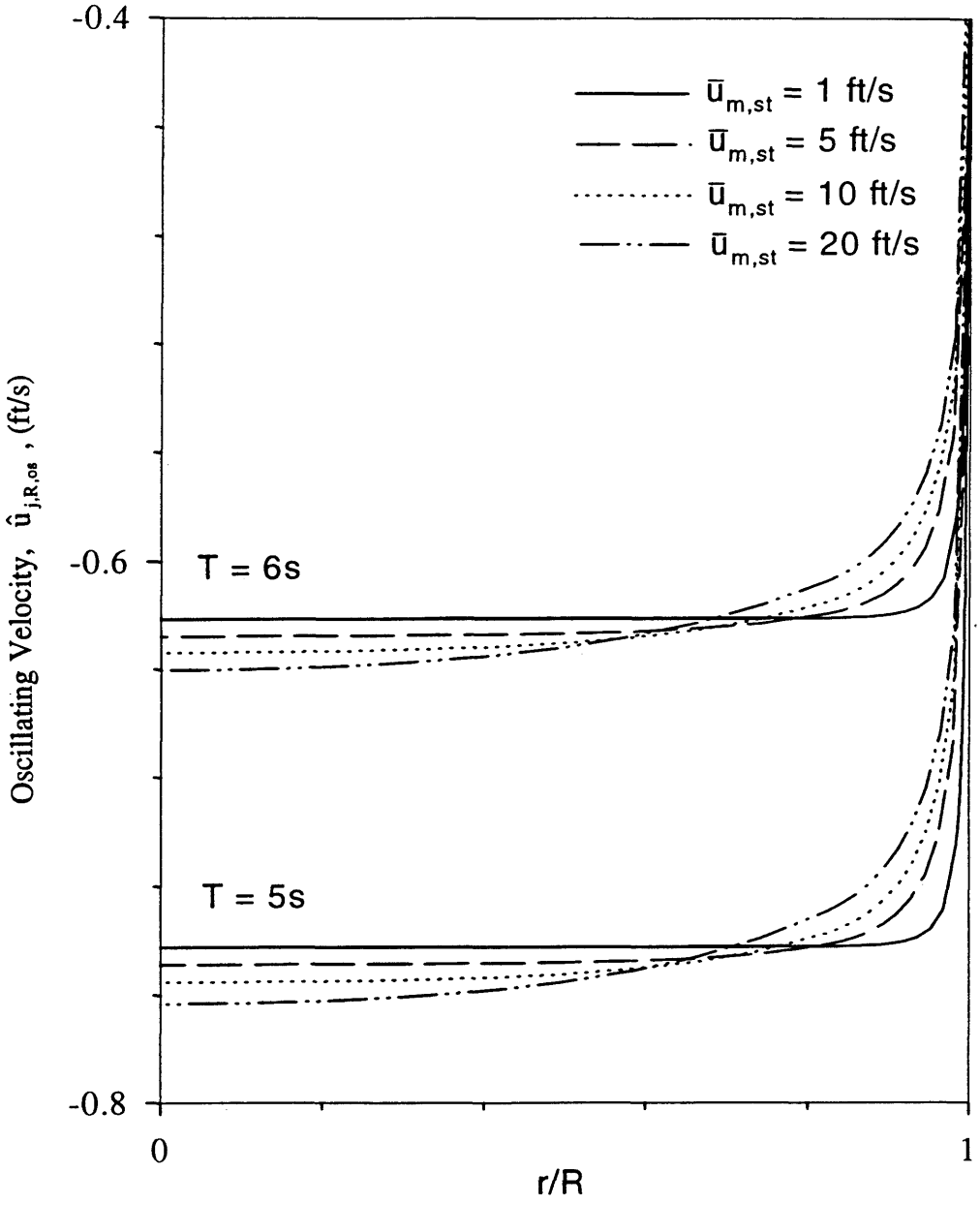


Figure 6.3 Oscillating Velocity Across a Pipe due to Oscillating Water Flow at $\bar{u}_{m,st}$ = 1, 5, 10 and 20 ft/s at T = 5 and 6 s.

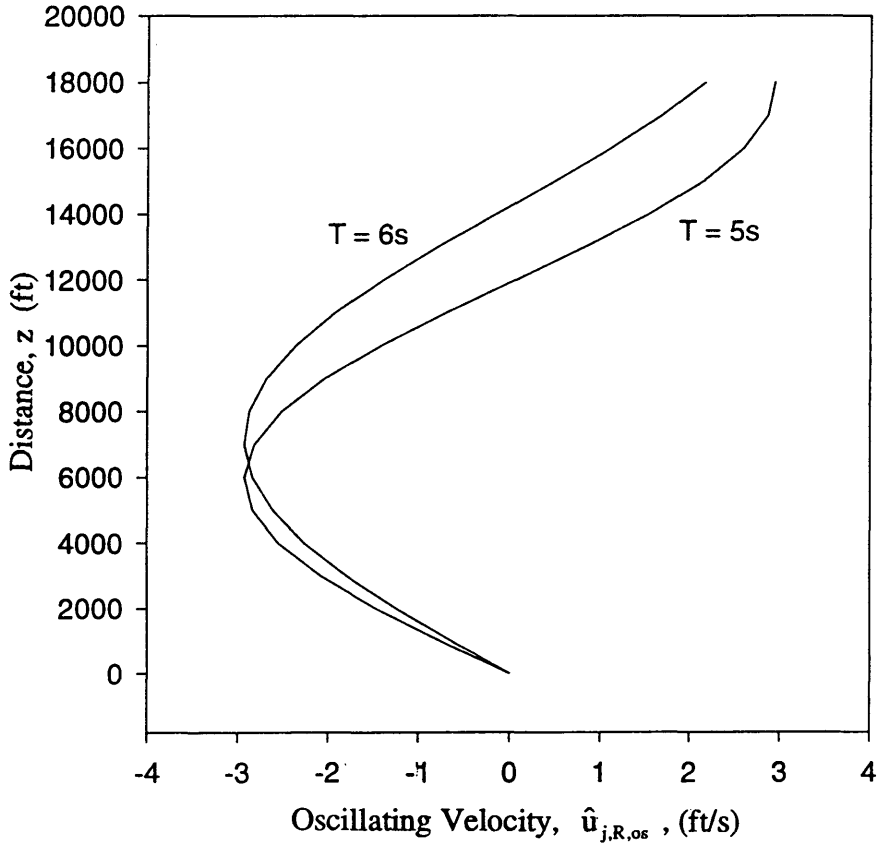


Figure 6.4 Oscillating Velocity as a Function of z at $\bar{u}_{m,st}$ = 10 ft/s at T =5 and 6 s

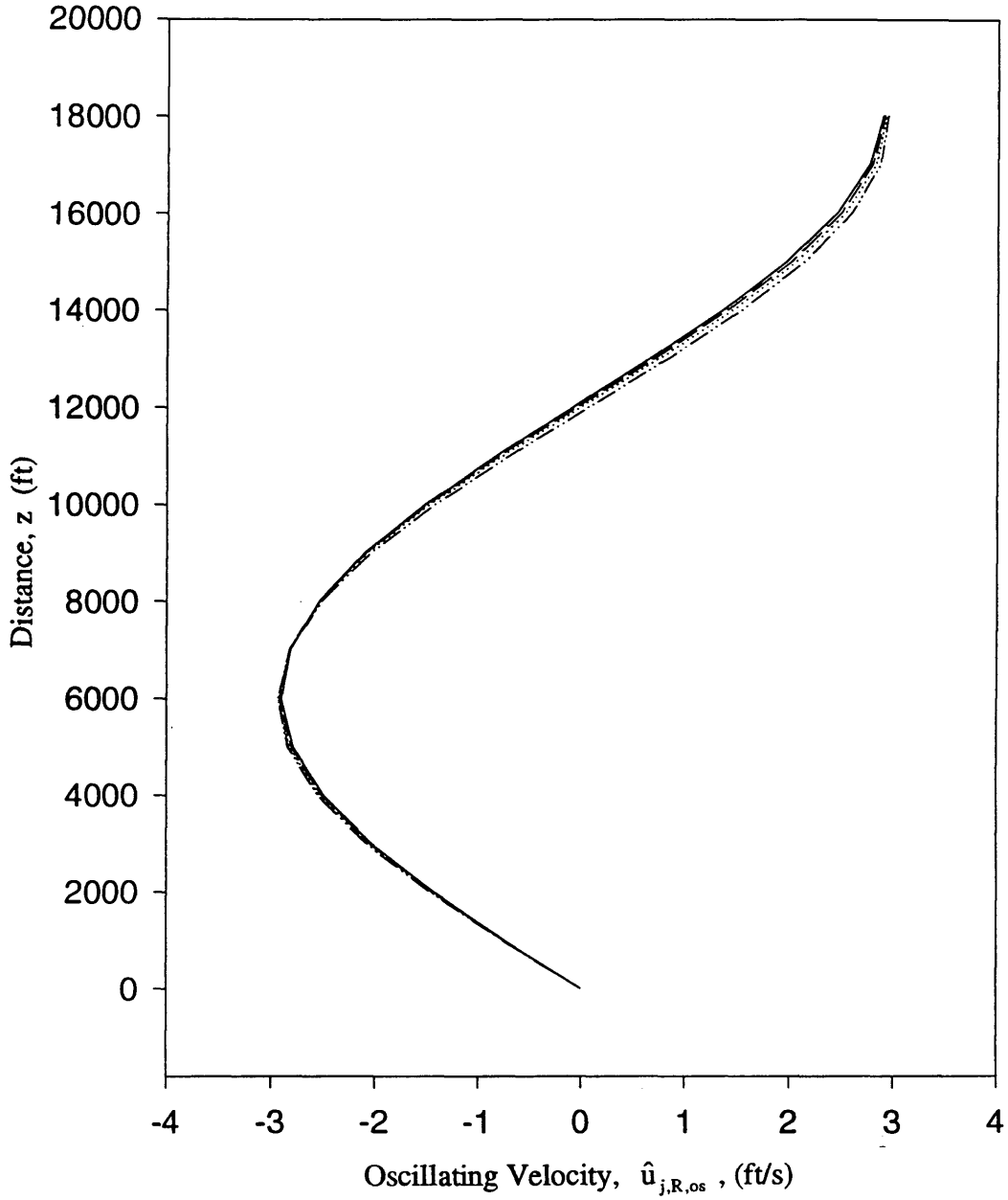


Figure 6.5 Oscillating Velocity as a Function of z at $\bar{u}_{m,st}$ = 1, 5, 10 and 20 ft/s at T = 5 s. Legend as in Figure 6.3.

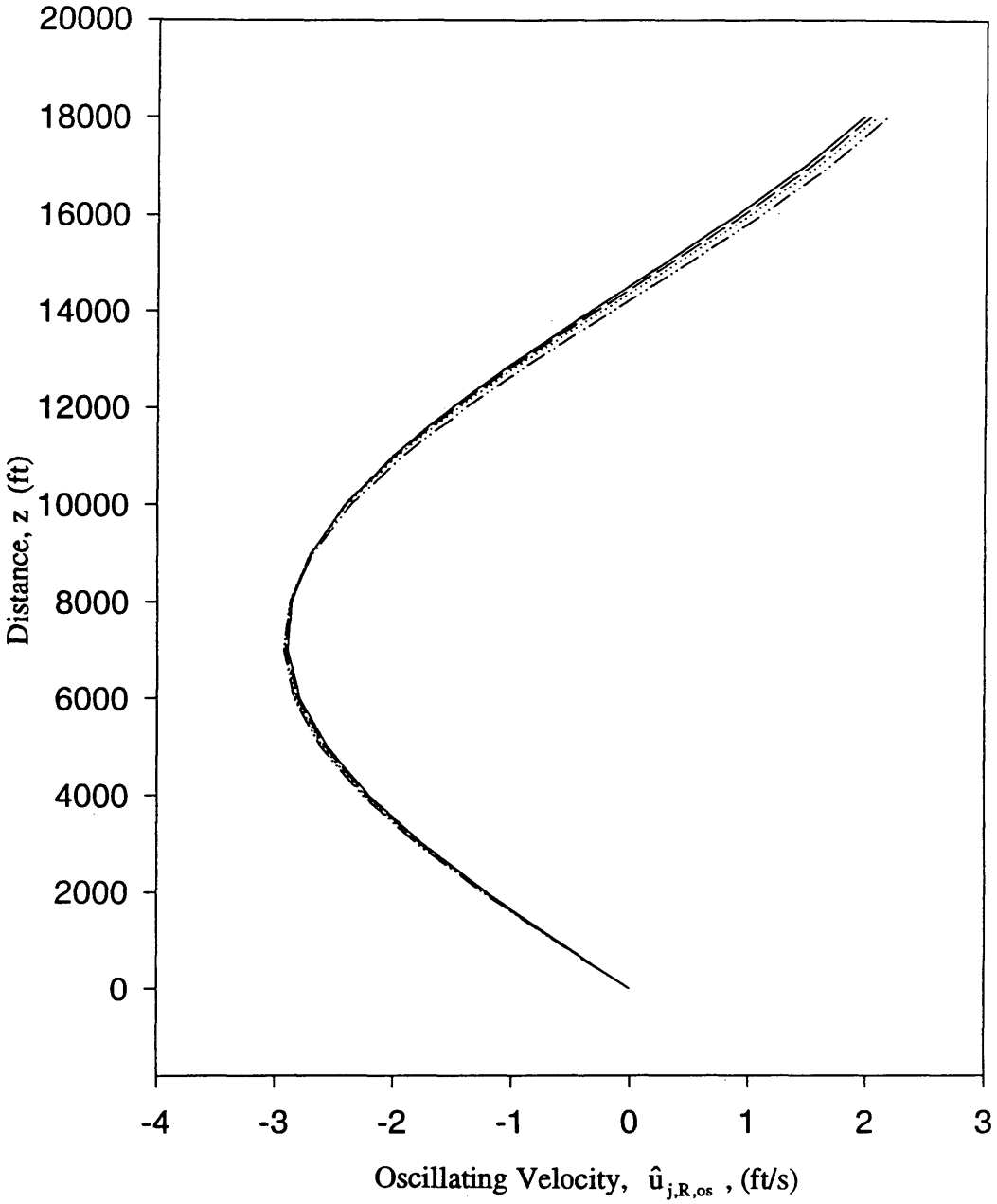


Figure 6.6 Oscillating Velocity as a Function of z at $\bar{u}_{m,st}$ = 1, 5, 10 and 20 ft/s at T = 6 s. Legend as in Figure 6.3.

6.5 Modeling Oscillating Shear Stress due to Pipe Flow

Using the values of $|A|$, $|B|$, and C_j , D_j and γ obtained from oscillating velocity calculations, equation (5.43) was solved for the oscillating shear stress. Using equation (6.13), the oscillating wall shear stress can be expressed as:

$$\tau_{w,os} = K(\gamma_R + i\gamma_I)\left\{Ae^{\gamma_R z}\left[\cos(\gamma_I z) + i\sin(\gamma_I z)\right]\right.$$
$$\left. -Be^{-\gamma_R z}\left[\cos(\gamma_I z) - i\sin(\gamma_I z)\right]\right\} \qquad (6.18)$$

The real part of equation (6.18) is expressed as:

$$\tau_{R,w,os} = K\left[\gamma_R Ae^{\gamma_R z}\cos(\gamma_I z) - \gamma_I Ae^{\gamma_R z}\sin(\gamma_I z)\right.$$
$$\left. -\gamma_R Be^{-\gamma_R z}\cos(\gamma_I z) - \gamma_I Be^{-\gamma_R z}\sin(\gamma_I z)\right] \qquad (6.19)$$

where K is a constant at mean-steady velocity and is a function of period:

$$K = \sqrt{\frac{\nu}{s}}\left[C_1 I_1\left(R\sqrt{\frac{s}{\nu}}\right) - D_1 K_1\left(R\sqrt{\frac{s}{\nu}}\right)\right] \qquad (6.20)$$

Table 6.2 shows values of the propagation constant, γ, for periods T =3~12 s at $\bar{u}_{m,st}$ = 10 ft/s:

Table 6.2

Values of K for T =3~12 s at $\bar{u}_{m,st}$ = 10 ft/s

Period (s)	K (ft)	|K| (ft)
3.0	0.00619 - 0.00619i	0.00876
4.0	0.00797 - 0.00797i	0.01127
5.0	0.00970 - 0.00970i	0.01372
6.0	0.01138 - 0.01138i	0.01610
7.0	0.01305 - 0.01305i	0.01845
8.0	0.01467 - 0.01467i	0.02075
9.0	0.01628 - 0.01628i	0.02302
10.0	0.01785 - 0.01785i	0.02525
11.0	0.01941 - 0.01941i	0.02745
12.0	0.02094 - 0.02094i	0.02962

Using values of γ_I and γ_R from Table 6.1, |K| values from Table 6.2 and values of |A| and |B| from section 6.3, equation (6.20) was plotted for periods T = 5 s and T = 6 s at mean-steady velocities $\bar{u}_{m,st}$ = 1, 5, 10 and 20 ft/s and is shown in Figures 6.7 and 6.8. These graphs illustrate that oscillating shear stress is larger at $\hat{T}$ = 5 s (ω = 1.2566 rad/s) than at T = 6 s (ω = 1.0472 rad/s). This was also true for oscillating velocity, so increased

oscillating velocity yields increasing shear stress. Absolute values of K were used for the graphs.

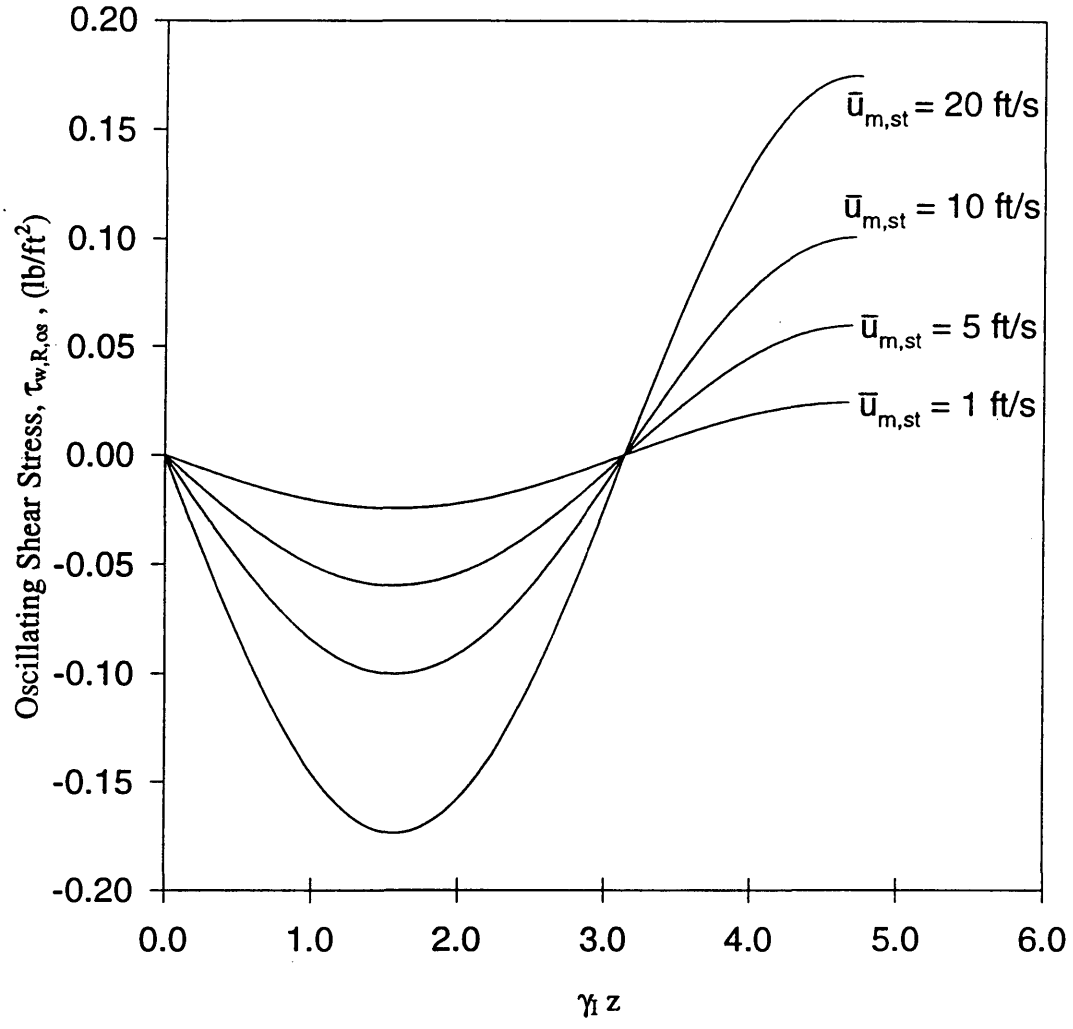


Figure 6.7 Oscillating Shear Stress as a Function of z at $\bar{u}_{m,st}$ = 1, 5, 10 and 20 ft/s at T = 5 s. Legend as in Figure 6.3.

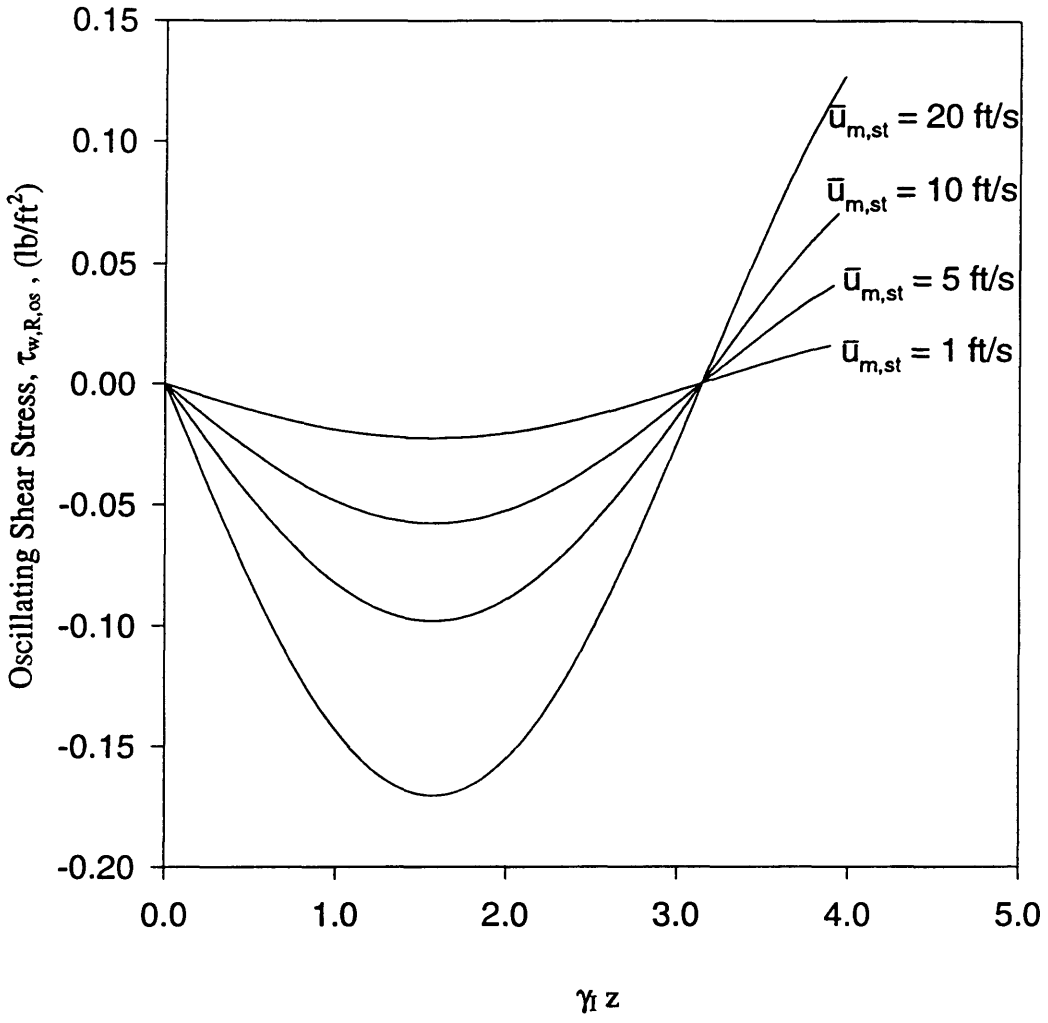


Figure 6.8 Oscillating Shear Stress as a Function of z at $\bar{u}_{m,st}$ = 1, 5, 10 and 20 ft/s at T = 6 s. Legend as in Figure 6.3.

6.6 Case Study

For a 20-inch outer diameter commercial steel pipe transporting oscillating turbulent seawater with a mean-steady vertical velocity of $\bar{u}_{m,st}$ = 10 ft/s at a period T = 5 s, the wave propagation constant, γ, equals:

$$\gamma = 3.34 * 10^{-6} + 2.62 * 10^{-4} i$$

Using the pressure constants A and B determined in section 6.3 and values of C_j and D_j from the boundary conditions, Figures 6.4 and 6.5 illustrate that the magnitude of the oscillating velocity at z = 18,000 ft is approximately 3 ft/s.

From Table 6.2, the magnitude of the shear stress constant is $|K| = 0.1372$. Using the imaginary component of the wave propagation constant as 2.62×10^{-4} and z = 18,000 ft, the $\gamma_I z$ value for Figure 6.7 is 4.72 which indicates an oscillating shear stress of 0.11 lb/ft^2.

Based on an 18.73-inch inside diameter, the internal perimeter is 4.90 ft. At a length of 18,000, the expected oscillating shear force is approximately 9709 lb. The pipe cross-sectional area is 0.27 ft^2, implying that the approximate shear (axial) stresses are 36,130 lb/ft^2 (251.1 psi).

6.7 Conclusions

Potential deep-ocean mining operations require accurate modeling of upward vertical slurry transport from the miner at the seafloor to the ship on the ocean surface in order to account for its influence on shear stress. The seawater in the present model is both periodic and turbulent, inducing shear stress on the inside surface of the pipe which can influence the response of the pipe.

The present analysis of upward vertical flows of water clearly illustrates that (i) oscillating velocity is slightly influenced by the magnitude of the mean velocity, (ii) both oscillating velocity and shear stress are related to the oscillation period and (iii) oscillating velocity is directly related to oscillating shear stress.

6.8 Recommendations

Analyzing the magnitude of the shear (axial) stress, one may interpret that the internal oscillating shear stress due to oscillating flow could be a minor factor in the design of a marine riser for deep-ocean mining applications. However, a comprehensive understanding of the internal oscillating shear stress effect can only be determined after the results of the shear force are incorporated into the nonlinear three-dimensional (3-D) finite element code, such as the one developed by Chung, Cheng and Huttelmaier (1994). Their code can provide a comprehensive analysis of the axial deformation of the oscillating pipe, when the analysis includes both axial shear forces resulting from external

hydrodynamic forces and the shear forces due to the present oscillating internal flow. Such an analysis is critical in order to accurately predict the responses of the marine riser for future deep-ocean operations.

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