# A MODEL OF WAVE MOTION IN THREE-DIMENSIONAL HETEROGENEOUS MEDIA

by

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A report submitted to the Faculty and Board of Trustees of the Colorado School of Mines in partial fulfillment of the requirements for the degree of Master of Engineering (Engineering Systems).

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Date Aug. 9, 1999

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# ABSTRACT

A model of seismic wave motion in a half-space with heterogeneous layers, generated by a buried seismic dislocation source, is developed. Using the first-order perturbation approach, it is shown that the total wave field may be obtained as a superposition of the mean and the scattered wave fields. The mean wave field is obtainable as a response solution for a layered half-space without heterogeneity subjected to a buried seismic dislocation source. The scattered wave field is obtained as a response solution for the same layered half-space as used in the mean wave field, but is subjected to the equivalent virtual distributed body forces that mathematically replace the heterogeneity. These virtual body forces have the same effects as the existence of heterogeneity, and can be evaluated as a function of the heterogeneity parameters and the mean wave field. The explicit expressions for the responses in both the mean and the scattered wave fields are derived with the aid of the integral transform approach and wave propagation analysis.

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## ACKNOWLEDGEMENTS

*"The Lord is my shepherd, I shall not be in want. He makes me lie down in green pastures, he leads me beside quite waters, he restores my soul. He guides me in paths of righteousness for his name's sake." (Psalm 23: 1-3)*

My God has been with me, helping me to study at the Colorado School of Mines and enabling me to meet the following wonderful people.

First, I would like to thank my advisor, Dr. Ray Zhang for his patience and guidance. I will be thankful all of my life for his kind understanding of my situation and the invaluable research I have learned from him. I would also like to thank my committee members, Dr. Jin Chung and Dr. Ron Knaushaug, for their encouragement and time devoted in reviewing the thesis work. Dr. Jung has been particularly supportive through this project.

This work was supported by National Science Foundation through the Trainee Grant entitled "Integrated systems approach to Civil Infrastructure Education" (GER-9554559) with Joan Gosink as PI, and Colorado Advanced Software Institute through the grant entitled "Model-based wavelet processing for seismic survey" with Ray Zhang as PI. I would like to take this time to express my sincere thanks to them.

Last, but by no means least, I wish to thank to my lovely wife, Hannah, and my son, Hayoung.

# CHAPTER 1
# INTRODUCTION

The nature of near-field motion is probably best investigated by using theoretical models of earthquake wave motion to simulate 3D propagating waves in an earth medium (e.g., Bolt, 1999), since not nearly enough near-field motion records have been collected and thus used for qualitative or quantitative analysis. Such theoretical models usually consist of three major components: a seismic source mechanism, an earth profile, and an analysis of wave propagation and scattering in the medium. Their engineering applications could be found in, for example, Heaton et al. (1995) and Shinozuka et al. (1999), which examine near-field, low-frequency pulse-like peak motion as well as its effects on high-rise buildings. Such models could also be used to investigate the nature of spatial variations of near-field strong motion since the amount of near-field motion data is even less sufficient for quantitative analysis of spatial variations than is peak motion.

Results of studies with such models indicate that spatial variations could be caused by a combination of some or all of the following major factors: wave interference of the seismic signals from different locations on a fault (i.e., temporal-spatial effects of source), multiple reflection and refraction of waves at the Earth's surface and subsurface interfaces (i.e., site/wave-path effects), various surface waves and wave dispersion (i.e., wave-type/wave-passage effects), and wave scattering in a random-heterogeneity (RH) medium (i.e., RH effects, which could also be regarded as part of site/wave-path effects).

3D weak-random heterogeneity is probably a ubiquitous property of the Earth. Since dimension of spatial variations of heterogeneity is comparable to that of elongated structures, wave scattering and interference due to the RH could influence the spatial variations of near-field motion over separation distances comparable to the dimension of elongated structures. As a matter of fact, a recent study with a relatively simplistic earth model (O'Connell, 1999) indicated that the incorporation of 3D RH into a seismic wave motion model can simulate near-field peak motion that is more consistent with observation data than is that obtained with the same approach with a homogeneous medium. O'Connell's results also show that ignoring heterogeneity yields overestimates of the near-field peak motion. Inspired by this and other pertinent studies (e.g., Frankel and Clayton, 1986, Zhang et al., 1997a,b), it is reasonable to believe that 3D RH plays an important role in spatial variations of near-field motion as well as its implications for damage of elongated structures. In addition, the 3D RH model could also facilitate an explanation of the nature of near-field motion (in terms of peak motion, spatial variations, time duration, dominant frequency, etc.) in ways that complement and are more viable than influences of nonlinear media, basin edge, and other factors.

From an engineering viewpoint, reliability of elongated structures designed to resist seismic waves is strongly dependent upon the appropriate and precise description of the spatial variations, in addition to that of peak value, dominant frequency and time duration. Unfortunately, current engineering models of spatial variation used in most structural analysis and design are primarily limited to being site-specific and earthquake-specific (e.g., models established on the basis of statistical analysis of specific or multiple similar earthquake records at particular areas/locations). Consequently, uncertainty for the seismic capacity of elongated structures will increase if such models of spatial variation are simply used in other locations (particularly in the near-field) without any modification or justification.

To this end, an integrated seismological and engineering approach becomes natural and essential for comprehensive understanding of the spatial variations as well as the consequent quantitative implications for structural damage. In particular, theoretical wave-motion models that consider 3D RH, together with other factors (e.g., a seismologically-consistent dislocation source and a realistic geological profile including sedimentary basins and irregular topography), could be employed to examine the qualitative nature of spatial variations of near-field motion and to quantify its implications for damage to elongated structures.

For the Earth medium with weak heterogeneity, wave scattering may be easily described with the aid of the first-order perturbation technique (see, e.g., Chow, 1972, 1975; Aki and Richards, 1980). Kennett (1972) first used the Born-type expansion in the wave number domain to solve a two-dimensional wave scattering problem in the Earth medium with both lateral and vertical heterogeneity. The Born approximation approach was also applied by Sato (1984, 1989) to deal with a three-dimensional wave scattering problem in a full-space with a randomly heterogeneous lithosphere. Associated developments on the subject were also carried out by Chu et al. (1981), Wu (1982), Weaver (1984), Wu and Aki (1985), Farra and Madariaga (1987), and Bataille and Flatte (1988), among others. However, their analyses are restricted in either two-dimensional wave scattering in such modeled Earth medium as a layered half-space or three-dimensional wave scattering in the Earth medium with heterogeneity existing in a finite area. In practice, the Earth medium is probably best described by a layered half-space with top layer or layers (close to ground surface) being of weak heterogeneity (see Askar and Cakmak, 1988). Stauber (1985) investigated a three-dimensional wave scattering problem in a layered half-space with laterally heterogeneous layers. However, only acoustic waves are involved in his analysis, which may not be directly applied to seismic waves. In particular, when seismic waves such as P and S waves hit the heterogeneous medium, coupling between different kind of waves such as P-SV and SH waves occurs (Zhang et al. 1997), which can be found neither for seismic waves in a two-dimensional heterogeneous medium nor for the acoustic waves in a three-dimensional heterogeneous medium.

In this study, seismic waves in a layered half-space with three dimensional heterogeneity, generated by a buried seismic dislocation source, are investigated. The problem is first formulated and a corresponding approach is then provided. The explicit expressions for responses are derived using the integral transform approach and wave propagation analysis, in which P-SV and SH waves are decoupled and analyzed separately.

# CHAPTER 2
# OUTLINE OF PROBLEM AND ITS APPROACH

As shown in Figure 1, the Earth is modeled as a layered half-space. Each layer is assumed to be homogeneous, isotropic, and linearly elastic, except for layer $j$ that is heterogeneous. The assumption of one heterogeneous layer is simply for demonstration. It is straightforward to be applied to any layers. A seismic dislocation source is buried in layer $k$ where the seismic waves are originally generated and then propagated through the earth medium. The seismic wave motion can be obtained by solving the following governing equation:

$$L\{\vec{u}(x,y,z,t)\} = \vec{p}(x,y,z,t)\,. \tag{1}$$

where $\vec{u} = \{u_x, u_y, u_z\}^T$ denotes the displacement vector, $\vec{p} = \{p_x, p_y, p_z\}^T$ represents the body force vectors which can be found in terms of the dislocation of a seismic source, and $L$ is a linear operator, characterizing wave motion in the layered half-space with heterogeneity and being a function of such physical parameters as wave speeds of each layer. In equation (1), body force $\vec{p}$ is equivalent to the effects of the dislocation at the broken seismic fault and can be found by

$$\vec{p}(x,y,z,t) = \begin{cases} \Im\{\Delta\vec{u}(x,y,z,t)\} & \textit{if } (x,y,z) \in \textit{seismic source} \\ 0 & \textit{otherwise} \end{cases} \tag{2}$$

or

$$\vec{p}(x,y,z,t)=\begin{cases}\Re\{\Delta\vec{\tau}(x,y,z,t)\} & if\ (x,y,z)\in seismic\,source\\ 0 & otherwise\end{cases} \tag{3}$$

where $\Delta\vec{u}(x,y,z,t)$ and $\Delta\vec{\tau}(x,y,z,t)$ represent, respectively, the seismic displacement and stress discontinuity sources at a seismic source, and $\Im$ and $\Re$ are known functions (e.g., Aki and Richards, 1980).

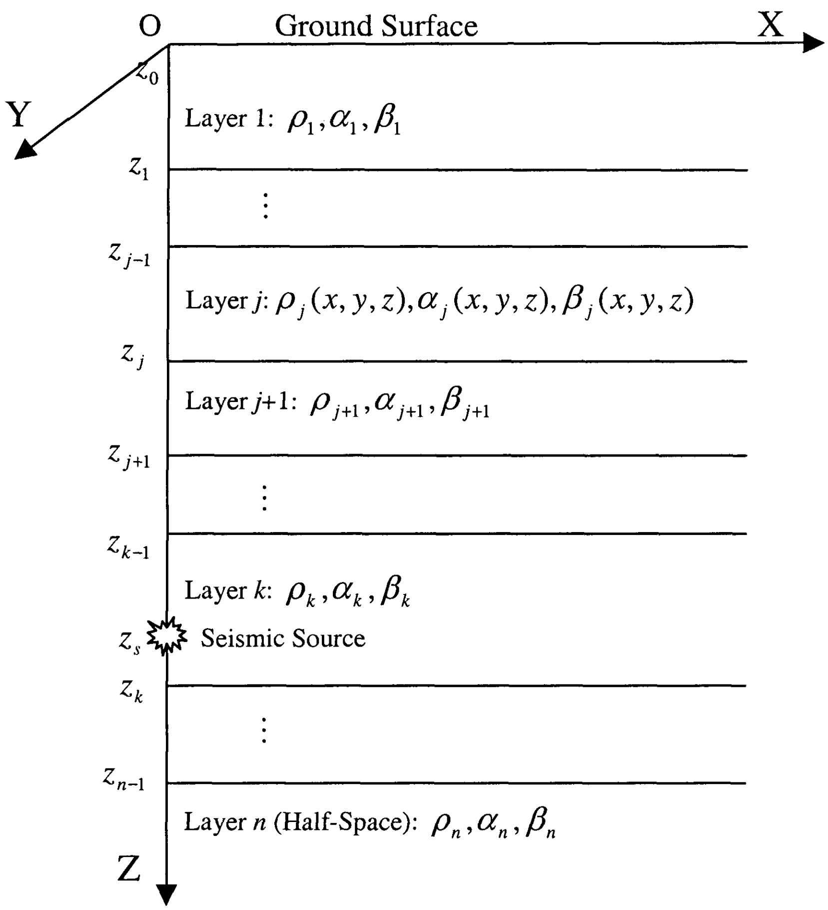


**Figure 1**. A layered half-space with one heterogeneous layer subjected to a seismic dislocation source ( $\rho$ =density, $\alpha$ =P wave speed, and $\beta$ =S wave speed).

Here, the stress vector $\vec{\tau}(x,y,z,t)$ consists of 9 components, i.e., $\vec{\tau} = \{\tau_{xx}, \tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yy}, \tau_{yz}, \tau_{zx}, \tau_{zy}, \tau_{zz}\}^T$. Based on mechanics, $\Delta\vec{u}$ and $\Delta\vec{\tau}$ in equations (2) and (3) do not simultaneously exist in a seismic source area.

The heterogeneity in layer $j$ is described by P and S waves speeds ($\alpha_j$ and $\beta_j$) and density ($\rho_j$) or equivalently by Lame's constants ($\lambda_j$ and $\mu_j$) and density ($\rho_j$) as functions of ($x$, $y$, $z$), i.e.,

$$\zeta_j(x,y,z) = \zeta_j^{\,m} + \varepsilon\zeta_j^{\,s}(x,y,z)\,;\ \zeta = \alpha, \beta \text{ or } \rho \text{ (or equivalently } \zeta = \lambda, \mu \text{ or } \rho) \tag{4}$$

where the first part is the average value, independent of ($x$, $y$, $z$), and the second part is the deviation from the average value, assumed to be of order of small quantity ε. For convenience, the superscript $m$ is used to denote the corresponding mean part, while the superscript $s$ represents the corresponding perturbed part due to the heterogeneity.

Correspondingly, the displacement field may also be decomposed into two parts as

$$\vec{u}(x,y,z,t) = \vec{u}^{\,m}(x,y,z,t) + \varepsilon\vec{u}^{\,s}(x,y,z,t) \tag{5}$$

where $\vec{u}^m$ is referred as to the mean wave field and $\varepsilon\vec{u}^s$ to the scattered wave field. Since the linear operator $L$ in equation (1) is a function of physical parameters $\zeta_j(x,y,z)$, it may also be expanded as a Taylor series with respect to the corresponding mean parts, i.e.,

$$L = L^m + \varepsilon L^s(1 + \varepsilon\eta_2 + \varepsilon^2\eta_3 + \cdots) \tag{6}$$

where $L^m$ denotes the mean linear operator, characterizing the layered half-space with homogeneous media; $\varepsilon L^s$ denotes the first-order scattered linear operator due to the heterogeneity; and $\varepsilon^i \eta_i L^s$ $(i = 2,3,\cdots)$ corresponds to the $i^{\text{th}}$-order scattered linear operators. If the second- and higher-order terms associated with $\eta_i (i = 2,3,\cdots)$ are neglected, equation (6) represents a first-order perturbation scheme, which was used previously by Kennett (1972), Ostoja-Starzewski (1987) and others in solving similar problems. Otherwise, equation (6) takes into account the effects of higher-order perturbation, and was used by Liu (1991) in investigating the wave scattering problem. In this study, the first-order perturbation approach is applied, which is considered adequate to reveal the fundamental characteristics of the ground motion with the weak heterogeneity. For a comprehensive survey of perturbation methods applied to wave scattering problems, readers are referred to the article by Chow (1972, 1975).

By substituting equations (5) and (6) into equation (1) and neglecting the second- and higher-order terms of small qunatity $\varepsilon$, one may obtain two sets of equations for the original problem, i.e.,

$$L^m \left\{ \vec{u}^m (x, y, z, t) \right\} = \vec{p}(x, y, z, t)\,, \tag{7}$$

$$L^m \left\{ \vec{u}^s (x, y, z, t) \right\} = \vec{p}^s (x, y, z, t)\,, \tag{8}$$

where

$$\vec{p}^s (x, y, z, t) = \begin{cases} -L^s \left\{ \vec{u}^m (x, y, z, t) \right\} & \textit{if } (x, y, z) \in \textit{layer } j \\ 0 & \textit{otherwise} \end{cases} \tag{9}$$

The physical meanings of equations (7) and (8) can be explained as follows. Both mean and scattered wave fields share the same linear operator $L^m$, indicating that they exist in the same layered half-space with homogeneous media. The mean wave field is obtained first from equation (7) with body forces $\vec{p}$ generated by a seismic discontinuity source (see equations (2) or (3)). The scattered wave field can then be obtained from equation

(8) in which the distributed virtual forces $\vec{p}^s$ are computed in terms of the mean wave field and the quantities of the heterogeneity involved in the operator $L^s$ (see equation (9)). The total wave field $\vec{u}$ in a layered half-space with one heterogeneous layer subjected to a buried seismic discontinuity source may finally be obtained as a superposition of these two wave fields via equation (5).

# CHAPTER 3
# DECOMPOSITION OF SEISMIC WAVE FIELDS

For ease in describing the problem at hand, the following displacement and stress vectors are defined

$$\vec{u}(x, y, z, t) = u_x \vec{e}_x + u_y \vec{e}_y + u_z \vec{e}_z \tag{10}$$

$$\vec{t}(x, y, z, t) = \tau_{xz} \vec{e}_x + \tau_{yz} \vec{e}_y + \tau_{zz} \vec{e}_z \tag{11}$$

where $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ are, respectively, the orthogonal unit vectors in the $x$, $y$ and $z$ directions. Here stress vector, $\vec{t}(x, y, z, t)$ is a derivative with respect to $z$ in $\vec{\tau}(x, y, z, t)$, which appears only on the components of the traction across a horizontal plane. The total wave field in a layered half-space with a heterogeneous layer subjected to a seismic discontinuity source can be found by solving the following equations in conjunction with proper conditions.

*Governing equations*

The 3-D wave motion in each isotropic and linear elastic layer is governed by three partial differential equations.

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + p_x = \rho \frac{\partial^2 u_x}{\partial t^2} \tag{12}$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + p_y = \rho \frac{\partial^2 u_y}{\partial t^2} \tag{13}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + p_z = \rho \frac{\partial^2 u_z}{\partial t^2} \tag{14}$$

Here, body forces ($p_x$, $p_y$ and $p_z$) exist only in the source layer (see equations (2) and (3)).

*Constitutive relationship*

The constitutive relationship between displacements and stresses is expressed as:

$$\tau_{xx} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu \frac{\partial u_x}{\partial x} \tag{15}$$

$$\tau_{yy} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu \frac{\partial u_y}{\partial y} \tag{16}$$

$$\tau_{zz} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu \frac{\partial u_z}{\partial z} \tag{17}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{18}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \tag{19}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \tag{20}$$

in which $\lambda$ and $\mu$ are the Lame's elastic constants and can be found in terms of P and S wave speeds and density as follows:

$$\lambda = \rho(\alpha^2 - 2\beta^2) \tag{21}$$

$$\mu = \rho\beta^2 \tag{22}$$

where $\alpha = ((\lambda + 2\mu)/\rho)^{1/2}$ and $\beta = (\mu/\rho)^{1/2}$.

*Continuity conditions at each interface*

The continuity condition at each interface requires that $\vec{u}$ and $\vec{t}$ be continuous across $z = z_i$:

$$\vec{u}(x,y,z,t)\,|_{z=z_i^-} = \vec{u}(x,y,z,t)\,|_{z=z_i^+}\,, \tag{23}$$

$$\vec{t}(x,y,z,t)\,|_{z=z_i^-} = \vec{t}(x,y,z,t)\,|_{z=z_i^+}\,, \tag{24}$$

where $z_i^-$ and $z_i^+$ represent respectively the upper and lower sides of interface $z = z_i$.

*Boundary condition at ground surface*

The boundary condition at the free ground surface is

$$\vec{t}(x,y,z,t)\,|_{z=0} = 0 \tag{25}$$

*Radiation condition*

The radiation condition means that no propagating waves come from the place where $(x, y, z)$ are infinity.

*Seismic discontinuity source*

The seismic discontinuity source can be written in the form:

$$\vec{u}(x_s^+, y_s^+, z_s^+, t) - \vec{u}(x_s^-, y_s^-, z_s^-, t) = \Delta\vec{u}(x_s, y_s, z_s, t), \tag{26}$$

or

$$\vec{t}(x_s^+, y_s^+, z_s^+, t) - \vec{t}(x_s^-, y_s^-, z_s^-, t) = \Delta\vec{t}(x_s, y_s, z_s, t), \tag{27}$$

where $(x_s, y_s, z_s) \in$ source area. Displacement field $\vec{u}$ and stress field $\vec{t}$ are, of course, continuous outside the source area. With the knowledge of $\Delta\vec{u}$ or $\Delta\vec{t}$, the equivalent body force can be found from equations (2) and (3). More specifically, the equivalent body forces $\vec{p}$ in equations (2) and (3) can be represented by seismic moment components, which are related to the seismic dislocation quantities (Aki and Richard, 1980). For example, for a point seismic source equivalent to a dislocation $\Delta\vec{u}$ on a fault area centered at location $(x_s, y_s, z_s)$ and activated at time $t_s$, the double couple of body forces can be found as (Aki and Richards, 1980):

$$p_x(x, y, z, t) = \sum_{\varsigma = x, y, z} \frac{\partial[M_{x\varsigma}(t - t_s)\delta(x - x_s)\delta(y - y_s)\delta(z - z_s)]}{\partial\varsigma} \tag{28}$$

$$p_y(x, y, z, t) = \sum_{\varsigma = x, y, z} \frac{\partial[M_{y\varsigma}(t - t_s)\delta(x - x_s)\delta(y - y_s)\delta(z - z_s)]}{\partial\varsigma} \tag{29}$$

$$p_z(x,y,z,t) = \sum_{\varsigma=x,y,z} \frac{\partial[M_{z\varsigma}(t-t_s)\delta(x-x_s)\delta(y-y_s)\delta(z-z_s)]}{\partial\varsigma} \tag{30}$$

where $M_{\varsigma_1,\varsigma_2}(\varsigma_1,\varsigma_2 = x,y,z)$ are the seismic moment components and can be found by

$$M_{xy}(t-t_s) = M_o \sin(\delta)\cos(\lambda)F(t-t_s) \tag{31}$$

$$M_{xz}(t-t_s) = -M_o \cos(\delta)\cos(\lambda)F(t-t_s) \tag{32}$$

$$M_{yy}(t-t_s) = -M_o \sin(2\delta)\sin(\lambda)F(t-t_s) \tag{33}$$

$$M_{yz}(t-t_s) = M_o \cos(2\delta)\sin(\lambda)F(t-t_s) \tag{34}$$

$$M_{zz}(t-t_s) = M_o \sin(2\delta)\sin(\lambda)F(t-t_s) \tag{35}$$

in which dip $\delta$ and rake $\lambda$ are defined in Figure 2 for fault area $A$, and $F$ is the slip function with a unit final slip and the magnitude of the seismic moment $M_o$ is

$$M_o = \mu_k A\overline{\Delta u_F} \tag{36}$$

where $\mu_k$ is the shear rigidity of layer $k$, and $\overline{\Delta u_F}$ is the average of final slip on the fault, and can be found

$$\overline{\Delta u_F} = \left[\sqrt{(\Delta u_x)^2 + (\Delta u_y)^2 + (\Delta u_z)^2}\right]_{t=\infty} \tag{37}$$

*Description of heterogeneous media*

The properties in heterogeneous layer $j$, as seen in equation (4), may be expressed by

$$\alpha_j(x,y,z) = \alpha_j^m + \varepsilon\alpha_j^s(x,y,z) \quad (38)$$

$$\beta_j(x,y,z) = \beta_j^m + \varepsilon\beta_j^s(x,y,z) \quad (39)$$

$$\rho_j(x,y,z) = \rho_j^m + \varepsilon\rho_j^s(x,y,z) \quad (40)$$

or

$$\mu_j = \mu_j^m + \varepsilon\mu_j^s(x,y,z) \quad (41)$$

$$\lambda_j = \lambda_j^m + \varepsilon\lambda_j^s(x,y,z) \quad (42)$$

$$\rho_j = \rho_j^m + \varepsilon\rho_j^s(x,y,z) \quad (43)$$

With the aid of equations (21) and (22), the relationship between two sets of parameters can be found as

$$\mu_j^m = \rho_j^m \beta_j^{m^2} \quad (44)$$

$$\mu_j^s = 2\rho_j^m \beta_j^m \beta_j^s + \rho_j^s \beta_j^{m^2} \quad (45)$$

$$\lambda_j^m = \rho_j^m \left(\alpha_j^{m^2} - 2\beta_j^{m^2}\right) \quad (46)$$

$$\lambda_j^s = 2\rho_j^m \left(\alpha_j^m \alpha_j^s - 2\beta_j^m \beta_j^s\right) + \rho_j^s \left(\alpha_j^{m^2} - 2\beta_j^{m^2}\right) \quad (47)$$

Decomposition of wave field

Since the displacements in equation (5) can be decomposed into two parts as

$$u_x = u_x^m + \varepsilon u_x^s, \quad (48)$$

$$u_y = u_y^m + \varepsilon u_y^s, \quad (49)$$

$$u_z = u_z^m + \varepsilon u_z^s, \quad (50)$$

the stresses in equations (15) through (20) may also be decomposed accordingly. For example, stress component $\tau_{xx}$ is presented in detail in Appendix A. It is interesting to note that from equation (A1) through (A19), the total stresses are decomposed into three parts. The first is mean stress, which is a function of mean displacements and mean Lame's constants; the second is scattered stress, which relates to scattered displacements and mean Lame's constants; and the third part is equivalent stress which is due to mean displacements and scattered Lame's constants.

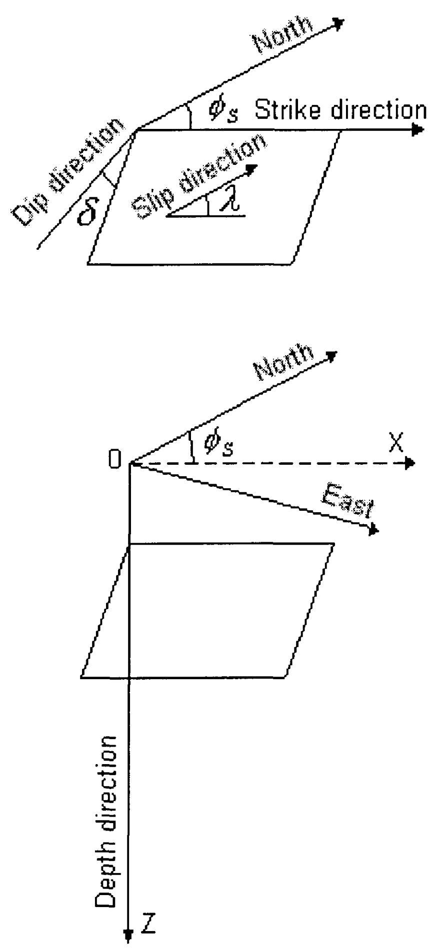


**Figure 2**. Fault orientation and related terminology.

All the decomposed stress fields presented in Appendix A can be expressed in a concise form as

$$\tau_{xx} = \tau_{xx}^{m} + \varepsilon\tau_{xx}^{s} + \varepsilon\tau_{xx}^{e} \tag{51}$$

$$\tau_{yy} = \tau_{yy}^{m} + \varepsilon\tau_{yy}^{s} + \varepsilon\tau_{yy}^{e} \tag{52}$$

$$\tau_{zz} = \tau_{zz}^{m} + \varepsilon\tau_{zz}^{s} + \varepsilon\tau_{zz}^{e} \tag{53}$$

$$\tau_{xy} = \tau_{yx} = \tau_{xy}^{m} + \varepsilon\tau_{xy}^{s} + \varepsilon\tau_{xy}^{e} \tag{54}$$

$$\tau_{yz} = \tau_{zy} = \tau_{yz}^{m} + \varepsilon\tau_{yz}^{s} + \varepsilon\tau_{yz}^{e} \tag{55}$$

$$\tau_{xz} = \tau_{zx} = \tau_{xz}^{m} + \varepsilon\tau_{xz}^{s} + \varepsilon\tau_{xz}^{e} \tag{56}$$

The governing equations (12) – (14) may be decomposed into two sets, by using three decomposed parameters in equations (41) – (43), displacements in equations (48) – (50) and stresses in equations (51) - (56) into the governing equations. The governing equation (12), for example, can be rewritten as, by using equations (43), (48), (51), (54) and (56),

$$\begin{aligned} &\frac{\partial}{\partial x}\left[\tau_{xx}^{m} + \varepsilon\tau_{xx}^{s} + \varepsilon\tau_{xx}^{e}\right] + \frac{\partial}{\partial y}\left[\tau_{xy}^{m} + \varepsilon\tau_{xy}^{s} + \varepsilon\tau_{xy}^{e}\right] + \frac{\partial}{\partial z}\left[\tau_{xz}^{m} + \varepsilon\tau_{xz}^{s} + \varepsilon\tau_{xz}^{e}\right] + p_x \\ &\quad = \left(\rho_j^{m} + \varepsilon\rho_j^{s}\right)\frac{\partial^2}{\partial t^2}\left(u_x^{m} + \varepsilon u_x^{s}\right) \end{aligned} \tag{57}$$

Similarly, governing equations (13) and (14) can also be rewritten by substituting equations (43), (49), (52), (54) and (55) into equation (13), and equations (43), (50), (53), (55) and (56) into equation (14). After collecting the terms in mean and scattered wave fields, the governing equations (12), (13) and (14) can be expressed in two sets of equations as

$$\frac{\partial \tau_{xx}^{m}}{\partial x}+\frac{\partial \tau_{xy}^{m}}{\partial y}+\frac{\partial \tau_{xz}^{m}}{\partial z}+p_{x}=\rho_{j}^{m}\frac{\partial^{2} u_{x}^{m}}{\partial t^{2}} \tag{58}$$

$$\frac{\partial \tau_{yx}^{m}}{\partial x}+\frac{\partial \tau_{yy}^{m}}{\partial y}+\frac{\partial \tau_{yz}^{m}}{\partial z}+p_{y}=\rho_{j}^{m}\frac{\partial^{2} u_{y}^{m}}{\partial t^{2}} \tag{59}$$

$$\frac{\partial \tau_{zx}^{m}}{\partial x}+\frac{\partial \tau_{zy}^{m}}{\partial y}+\frac{\partial \tau_{zz}^{m}}{\partial z}+p_{z}=\rho_{j}^{m}\frac{\partial^{2} u_{z}^{m}}{\partial t^{2}} \tag{60}$$

and

$$\frac{\partial \tau_{xx}^{s}}{\partial x}+\frac{\partial \tau_{xy}^{s}}{\partial y}+\frac{\partial \tau_{xz}^{s}}{\partial z}+p_{x}^{s}=\rho_{j}^{m}\frac{\partial^{2} u_{x}^{s}}{\partial t^{2}} \tag{61}$$

$$\frac{\partial \tau_{yx}^{s}}{\partial x}+\frac{\partial \tau_{yy}^{s}}{\partial y}+\frac{\partial \tau_{yz}^{s}}{\partial z}+p_{y}^{s}=\rho_{j}^{m}\frac{\partial^{2} u_{y}^{s}}{\partial t^{2}} \tag{62}$$

$$\frac{\partial \tau_{zx}^{s}}{\partial x}+\frac{\partial \tau_{zy}^{s}}{\partial y}+\frac{\partial \tau_{zz}^{s}}{\partial z}+p_{z}^{s}=\rho_{j}^{m}\frac{\partial^{2} u_{z}^{s}}{\partial t^{2}} \tag{63}$$

where $p_x^s$, $p_y^s$ and $p_z^s$ are the equivalent body forces, and can be found as

$$p_{x}^{s}=\frac{\partial \tau_{xx}^{e}}{\partial x}+\frac{\partial \tau_{xy}^{e}}{\partial y}+\frac{\partial \tau_{xz}^{e}}{\partial z}-\rho_{j}^{s}\frac{\partial^{2} u_{x}^{m}}{\partial t^{2}} \tag{64}$$

$$p_{y}^{s}=\frac{\partial \tau_{yx}^{e}}{\partial x}+\frac{\partial \tau_{yy}^{e}}{\partial y}+\frac{\partial \tau_{yz}^{e}}{\partial z}-\rho_{j}^{s}\frac{\partial^{2} u_{y}^{m}}{\partial t^{2}} \tag{65}$$

$$p_{z}^{s}=\frac{\partial \tau_{zx}^{e}}{\partial x}+\frac{\partial \tau_{zy}^{e}}{\partial y}+\frac{\partial \tau_{zz}^{e}}{\partial z}-\rho_{j}^{s}\frac{\partial^{2} u_{z}^{m}}{\partial t^{2}} \tag{66}$$

It can be seen clearly that, using the above approach, the constitutive law is still valid for the mean and the scattered waves, i.e., $\tau^m - u^m$ relationship in equations (58) - (60) and $\tau^s - u^s$ relationship in equations (61) - (63).

The equivalent body forces in equations (64) – (66) can be rewritten more elaborately using equations (41) - (43) and (A2) - (A19). By substituting equations (A4), (A13), (A19) and (41) - (43) into equation (64), equivalent body force $p_x^s$ can be expressed as

$$\begin{aligned} p_x^s &= \frac{\partial}{\partial x}\left[\lambda_j^s\left(\frac{\partial u_x^m}{\partial x}+\frac{\partial u_y^m}{\partial y}+\frac{\partial u_z^m}{\partial z}\right)+2\mu_j^s\frac{\partial u_x^m}{\partial x}\right]+\frac{\partial}{\partial y}\left[\mu_j^s\left(\frac{\partial u_x^m}{\partial y}+\frac{\partial u_y^m}{\partial x}\right)\right] \\ &\quad +\frac{\partial}{\partial z}\left[\mu_j^s\left(\frac{\partial u_x^m}{\partial z}+\frac{\partial u_z^m}{\partial x}\right)\right]-\rho_j^s\frac{\partial^2 u_x^m}{\partial t^2} \\ &= \lambda_{j,x}^s\left(\frac{\partial u_x^m}{\partial x}+\frac{\partial u_y^m}{\partial y}+\frac{\partial u_z^m}{\partial z}\right)+2\mu_{j,x}^s\frac{\partial u_x^m}{\partial x}+\lambda_j^s\left(\frac{\partial^2 u_x^m}{\partial x^2}+\frac{\partial^2 u_y^m}{\partial x\partial y}+\frac{\partial^2 u_z^m}{\partial x\partial z}\right) \\ &\quad +2\mu_j^s\frac{\partial^2 u_x^m}{\partial x^2}+\mu_{j,y}^s\left(\frac{\partial u_x^m}{\partial y}+\frac{\partial u_y^m}{\partial x}\right)+\mu_j^s\left(\frac{\partial^2 u_x^m}{\partial y^2}+\frac{\partial^2 u_y^m}{\partial x\partial y}\right) \\ &\quad +\mu_j^s\left(\frac{\partial^2 u_x^m}{\partial z^2}+\frac{\partial^2 u_z^m}{\partial x\partial z}\right)++\mu_{j,z}^s\left(\frac{\partial u_x^m}{\partial z}+\frac{\partial u_z^m}{\partial x}\right)-\rho_j^s\frac{\partial^2 u_x^m}{\partial t^2} \end{aligned} \tag{67}$$

Similarly, by substituting the equations (A7), (A13), (A16) and (41) – (43) into the equation (65); and equations (A10), (A16), (A19) and (41) – (43) into the equation (66), the other equivalent body forces ( $p_y^s$ and $p_z^s$ ) can be expressed as

$$p_y^s = \mu_{j,x}^s \left( \frac{\partial u_x^m}{\partial y} + \frac{\partial u_y^m}{\partial x} \right) + \mu_j^s \left( \frac{\partial^2 u_x^m}{\partial x \partial y} + \frac{\partial^2 u_y^m}{\partial x^2} \right) + \lambda_{j,y}^s \left( \frac{\partial u_x^m}{\partial x} + \frac{\partial u_y^m}{\partial y} + \frac{\partial u_z^m}{\partial z} \right)$$
$$+ 2\mu_{j,y}^s \frac{\partial u_y^m}{\partial y} + \lambda_j^s \left( \frac{\partial^2 u_x^m}{\partial x \partial y} + \frac{\partial^2 u_y^m}{\partial y^2} + \frac{\partial^2 u_z^m}{\partial y \partial z} \right) + 2\mu_j^s \frac{\partial^2 u_y^m}{\partial y^2} \tag{68}$$
$$+ \mu_j^s \left( \frac{\partial^2 u_y^m}{\partial z^2} + \frac{\partial^2 u_z^m}{\partial y \partial z} \right) + \mu_{j,z}^s \left( \frac{\partial u_y^m}{\partial z} + \frac{\partial u_z^m}{\partial y} \right) - \rho_j^s \frac{\partial^2 u_y^m}{\partial t^2}$$

$$p_z^s = \mu_{j,x}^s \left( \frac{\partial u_x^m}{\partial z} + \frac{\partial u_z^m}{\partial x} \right) + \mu_j^s \left( \frac{\partial^2 u_x^m}{\partial x \partial z} + \frac{\partial^2 u_z^m}{\partial x^2} \right) + \mu_{j,y}^s \left( \frac{\partial u_y^m}{\partial z} + \frac{\partial u_z^m}{\partial y} \right)$$
$$+ \mu_j^s \left( \frac{\partial^2 u_y^m}{\partial y \partial z} + \frac{\partial^2 u_z^m}{\partial y^2} \right) + \lambda_{j,z}^s \left( \frac{\partial u_x^m}{\partial x} + \frac{\partial u_y^m}{\partial y} + \frac{\partial u_z^m}{\partial z} \right) + \lambda_j^s \left( \frac{\partial^2 u_x^m}{\partial x \partial z} + \frac{\partial^2 u_y^m}{\partial y \partial z} + \frac{\partial^2 u_z^m}{\partial z^2} \right) \tag{69}$$
$$+ 2\mu_{j,z}^s \frac{\partial u_z^m}{\partial z} + 2\mu_j^s \frac{\partial^2 u_z^m}{\partial z^2} - \rho_j^s \frac{\partial^2 u_z^m}{\partial t^2}$$

Therefore, equivalent body forces ( $p_x^s$, $p_y^s$ and $p_z^s$) can be found from equations (67) – (69), if the deviation parts of the Lame's constants and density as well as mean displacement fields are known.

In summary, it is found using the above approach that both the mean and the scattered wave fields have to satisfy the same equations of motion for the layered half-space with heterogeneous media, the same constitutive relationships, the same continuity conditions at each and every interface, and the same boundary conditions at the free surface, and the same radiation conditions. Since the seismic discontinuity source stated in equations (26) and (27) is supposed to be accounted for in determining the mean wave field, no discontinuity needs to be provided at the source location for the scattered wave field. In other words, continuities will be found in the seismic source area for the scattered wave field.

The total wave field in a layered half-space with heterogeneous media subjected to a buried seismic source can be obtained as superposition of two wave fields, without regard to the presence of heterogeneity. One is the mean wave field obtained with a seismic source layer, while the other is the scattered wave field obtained with a virtual force equivalent to the effects of heterogeneity.

# CHAPTER 4
# SOLUTIONS FOR WAVE RESPONSES

The total wave fields have been decomposed into mean and scattered wave fields in the previous chapter. The response solution for the total wave field can be found if the responses of both the mean and the scattered wave fields are obtained. The latter can be solved with the aid of the reflectivity method, which combines a finite integral transform approach and the wave propagation technique. In this chapter, explicit expressions for the responses in both the mean and the scattered wave fields at any level $z$, especially at ground surface $z = z_0$, are derived.

In particular, the $d$-direction ($d = x, y, z$) displacement may have the finite Fourier representation as

$$u_d(x, y, z, t) = \frac{\pi}{t_t}\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\tilde{u}_d(k_x, k_y, z, \omega)\exp\left[ik_x x + ik_y y + i\omega t\right] \tag{70}$$

where $\omega$ denotes the frequency, $k_x$ and $k_y$ the wavenumbers in $x$ and $y$ directions, and $t_t$, $x_t$ and $y_t$ the truncated time and distances in the $x$ and $y$ directions, respectively. Revoking equation (5) in Chapter 2, the total wave response can be expressed by

$$u_d(x, y, z, t) = u_d^m(x, y, z, t) + \varepsilon u_d^s(x, y, z, t) \tag{71}$$

Based on equation (70), the counterparts in the frequency-wavenumber domain for equation (71) can be found as

$$\tilde{u}_d(k_x,k_y,z,\omega)=\tilde{u}_d^m(k_x,k_y,z,\omega)+\varepsilon\tilde{u}_d^s(k_x,k_y,z,\omega) \tag{72}$$

$\tilde{u}_d$ and $u_d$ (or $\tilde{u}_d^m$, $\tilde{u}_d^s$ and $u_d^m$, $u_d^s$) will be solved in the following sub-sections.

### 4.1. Response in the Mean Wave Field

To obtain the responses for the mean wave field with a seismic discontinuity source, the finite integral transform procedure is applied. Specifically, triple finite Fourier transforms are used between time-space domain $(x,y,z,t)$ and frequency-wavenumber domain $(k_x,k_y,z,\omega)$. It can be shown (e.g., Zhang, 1992) that the displacement and stress vectors can be represented in a concise form with the aid of orthogonal vector harmonics, i.e.,

$$\vec{u}^m(x,y,z,t)=\frac{\pi}{t_t}\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(W_R^m\vec{e}_R+W_S^m\vec{e}_S+W_T^m\vec{e}_T)e^{i\omega t},$$

$$|t|\le t_t,\ |x|\le x_t \text{ and } |y|\le y_t \tag{73}$$

$$\vec{t}^m(x,y,z,t)=\frac{\pi}{t_t}\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\omega(F_R^m\vec{e}_R+F_S^m\vec{e}_S+F_T^m\vec{e}_T)e^{i\omega t},$$

$$|t|\le t_t,\ |x|\le x_t \text{ and } |y|\le y_t \tag{74}$$

Orthogonal vector harmonics $(\vec{e}_R,\vec{e}_S,\vec{e}_T)$, first introduced by Takeuchi and Saito (1972) in dealing with surface waves in cylindrical coordinates, are defined as

$$\vec{e}_R = e^{i(k_x x + k_y y)}\vec{e}_z \tag{75}$$

$$\vec{e}_S = \left(\frac{ik_x}{k_r}\vec{e}_x + \frac{ik_y}{k_r}\vec{e}_y\right)e^{i(k_x x + k_y y)} \tag{76}$$

$$\vec{e}_T = \left(\frac{ik_y}{k_r}\vec{e}_x - \frac{ik_x}{k_r}\vec{e}_y\right)e^{i(k_x x + k_y y)} \tag{77}$$

in which $k_x = m\pi / x_t$ and $k_y = n\pi / y_t$ are respectively wave numbers in $x$ and $y$ directions, $\omega = l\pi / t_t$ is the frequency, and $k_r = \sqrt{k_x^{\ 2} + k_y^{\ 2}}$ is the wave number in radial direction.

Conversely, variables $W_X^m$ and $F_X^m (X = R, S, orT)$ in equations (73) and (74) may be expressed in terms of displacement vector $\vec{u}^m$ and stress vector $\vec{t}^m$ as

$$W_X^m(k_x, k_y, z, \omega) = \frac{1}{(2\pi)^3}\int_{-t_t}^{t_t} dte^{-i\omega t}\int_{-x_t}^{x_t} dx\int_{-y_t}^{y_t} dy\{\vec{u}^m \cdot [\vec{e}_X]^*\}, \tag{78}$$

$$F_X^m(k_x, k_y, z, \omega) = \frac{1}{\omega(2\pi)^3}\int_{-t_t}^{t_t} dte^{-i\omega t}\int_{-x_t}^{x_t} dx\int_{-y_t}^{y_t} dy\{\vec{t}^m \cdot [\vec{e}_X]^*\}, \tag{79}$$

where the asterisk denotes the complex conjugate and the dot between two vectors represents a dot product.

As a limit case, summations in equations (73) and (74) will become integrals when the truncated parameters of $t_t$, $x_t$ and $y_t$ approach infinity, indicating the conventional integral transform approach. Equations (73) and (74) imply that the 3 D wave motion in the Cartesian coordinates $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ in the time-space domain can be projected into three new orthogonal harmonic directions $(\vec{e}_R, \vec{e}_S, \vec{e}_T)$ in the transformed domain. The 3 D wave motion in the Cartesian coordinates $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ can also be projected in the Cartesian coordinates $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ themselves in the transformed domain. For example, equation (73) may alternatively be written in the following explicit form:

$$\vec{u}^m(x,y,z,t) = \frac{\pi}{t_t}\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(\tilde{u}_x^m\vec{e}_x + \tilde{u}_y^m\vec{e}_y + \tilde{u}_z^m\vec{e}_z)e^{i(k_x x + k_y y + \omega t)},$$

$$|t| \le t_t,\ |x| \le x_t \text{ and } |y| \le y_t \qquad (80)$$

where the relations between displacement components in the frequency-wavenumber domain $\tilde{u}_d^m (d = x, y, z)$ and variables $W_X^m$ ($X$ = $R$, $S$ or $T$) can be easily found from equations (73) – (77) and (80) as

$$\tilde{u}_x^m = \frac{ik_x}{k_r}W_S^m + \frac{ik_y}{k_r}W_T^m \qquad (81)$$

$$\tilde{u}_y^m = \frac{ik_y}{k_r}W_S^m - \frac{ik_x}{k_r}W_T^m \qquad (82)$$

$$\tilde{u}_z^m = W_R^m \qquad (83)$$

The stress vector may also be written in a way similar to equations (80) through (83). It can be shown (Kennett, 1983) that the motions of $W_R^m$, $W_S^m$, $F_R^m$ and $F_S^m$ are associated with P-SV waves, while the motions of $W_T^m$ and $F_T^m$ are with SH waves. Therefore, equations (81) – (83) indicate that the displacement responses in $x$ and $y$ directions are composed of the motion of both P-SV and SH waves, while the displacement response in the $z$ direction is only contributed by the motion of the P-SV wave. After performing the transformation in equations (78) – (79), the governing equations of wave motion (see equations (58) – (60)) in the time-space domain $(x, y, z, t)$ in each layer, except for the source layer, may be rearranged in a general form in the transformed domain $(k_x, k_y, z, \omega)$, i.e.,

$$\frac{d}{dz}\begin{Bmatrix} \vec{w}^m \\ \vec{f}^m \end{Bmatrix} = [A(i)]\begin{Bmatrix} \vec{w}^m \\ \vec{f}^m \end{Bmatrix}, \quad i = 1, 2, \ldots, n; \; i \neq j \tag{84}$$

where $[A(i)]$ is the coefficient matrix and a function of the physical parameters of layer $i$: $\vec{w}^m = \vec{w}^m_{P-SV} \equiv \{W_R^m, W_S^m\}^T$ and $\vec{f}^m = \vec{f}^m_{P-SV} \equiv \{F_R^m, F_S^m\}^T$ for the P-SV waves, while $\vec{w}^m = \vec{w}^m_{SH} \equiv W_T^m$ and $\vec{f}^m = \vec{f}^m_{SH} \equiv F_T^m$ for the SH waves. Here, $\equiv$ denotes "by definition." Coefficient matrix $[A(i)]$ in equation (84) for the P-SV waves is

$$[A(i)] \equiv [A(i)_{P-SV}] = \begin{bmatrix} 0 & k_r\left(1 - \dfrac{2\beta_i^2}{\alpha_i^2}\right) & \dfrac{\omega}{\rho_i \alpha_i^2} & 0 \\ -k_r & 0 & 0 & \dfrac{\omega}{\rho_i \beta_i^2} \\ -\rho_i \omega_i & 0 & 0 & k_r \\ 0 & \rho_i \omega\left(\dfrac{\gamma_i k_r^{\,2}}{\omega^2} - 1\right) & -k_r\left(1 - \dfrac{2\beta_i^2}{\alpha_i^2}\right) & 0 \end{bmatrix} \tag{85}$$

and for the SH waves

$$[A(i)] \equiv [A(i)_{SH}] = \begin{bmatrix} 0 & \dfrac{\omega}{\rho_i \beta_i^2} \\ \rho_i \omega\left(\dfrac{\beta_i^2 k_r^2}{\omega^2} - 1\right) & 0 \end{bmatrix} \tag{86}$$

where $\gamma_i = 4\beta_i^2(1-\beta_i^2/\alpha_i^2)$. Correspondingly, the continuity conditions at each interface (see equations (23) and (24)) become

$$\begin{Bmatrix} \vec{w}^m \\ \vec{f}^m \end{Bmatrix}_{z=z_i^+} = \begin{Bmatrix} \vec{w}^m \\ \vec{f}^m \end{Bmatrix}_{z=z_i^-} \tag{87}$$

and the boundary condition at free surface, as defined in equation (25) for the total wave fields, is

$$\vec{f}^m \,|_{z=0} = 0\,. \tag{88}$$

The discontinuity source (e.g., equations (26) and (27) in Chapter 3) becomes

$$\begin{Bmatrix} \vec{w}^m \\ \vec{f}^m \end{Bmatrix}_{z=z_i^+} - \begin{Bmatrix} \vec{w}^m \\ \vec{f}^m \end{Bmatrix}_{z=z_i^-} = \begin{Bmatrix} \Delta\vec{w}^m \\ \Delta\vec{f}^m \end{Bmatrix}. \tag{89}$$

With the use of equations (28) – (30), $\Delta\vec{w}^m$ and $\Delta\vec{f}^m$ can be found as follows:

$$\Delta W_R = \frac{1}{(2\pi)^2} e^{(-ik_x x_s - ik_y y_s)} \frac{\tilde{M}_{zz}}{\rho_k \alpha_k^2} \tag{90}$$

$$\Delta W_S = \frac{1}{(2\pi)^2} e^{(-ik_x x_s - ik_y y_s)} \left( -\frac{ik_x}{k_y}\tilde{M}_{xz} - \frac{ik_y}{k_y}\tilde{M}_{yz} \right) \frac{1}{\rho_k \beta_k^2} \tag{91}$$

$$\Delta F_R = \frac{1}{(2\pi)^2} e^{(-ik_x x_s - ik_y y_s)} \frac{1}{\omega} \left[ ik_x (\tilde{M}_{zx} - \tilde{M}_{xz}) \right] + ik_y \left[ \tilde{M}_{zy} - \tilde{M}_{yz} \right] \tag{92}$$

$$\Delta F_S = \frac{1}{(2\pi)^2} e^{(-ik_x x_s - ik_y y_s)}$$
$$\frac{1}{\omega}\left[\frac{1}{k_y}(k_x^2 \tilde{M}_{xx} + k_y^2 \tilde{M}_{yy}) + \frac{k_x k_y}{k_y}(\tilde{M}_{xy} + \tilde{M}_{yx}) - k_y\left(1 - \frac{2\beta_k^2}{\alpha_k^2}\right)\tilde{M}_{zz}\right] \tag{93}$$

$$\Delta W_T = \frac{1}{(2\pi)^2} e^{(-ik_x x_s - ik_y y_s)}\left(-\frac{ik_y}{k_y}\tilde{M}_{xz} + \frac{ik_x}{k_y}\tilde{M}_{yz}\right)\frac{1}{\rho_k \beta_k^2} \tag{94}$$

$$\Delta F_T = \frac{1}{(2\pi)^2} e^{(-ik_x x_s - ik_y y_s)} \frac{1}{\omega}\left[\frac{k_x k_y}{k_y}\left(\tilde{M}_{xx} - \tilde{M}_{yy}\right) + \frac{1}{k_y}\left(k_y^2 \tilde{M}_{xy} + k_x^2 \tilde{M}_{yx}\right)\right] \tag{95}$$

where $\tilde{M}_{\varsigma_1,\varsigma_2}(\varsigma_1, \varsigma_2 = x, y, z)$ are the Fourier transformed counterpart of equations (31) – (35) in Chapter 3 and the subscript $k$ stands for the layer number.

For simplicity, only a point source at $(x_s, y_s, z_s)$ is considered in the following solution. The response of the wave motion at a given depth in the transformed domain to a point source at $(x_s, y_s, z_s)$ can be obtained by solving equation (84) in conjunction with equations (87) - (89) and the boundary condition at infinite depth, i.e.,

$$\begin{Bmatrix} \vec{w}^m \\ \vec{f}^m \end{Bmatrix} = \begin{bmatrix} M_u(i) & M_d(i) \\ N_u(i) & N_d(i) \end{bmatrix} \begin{Bmatrix} I \\ R(z,0) \end{Bmatrix} [I - R(z, z_s^-)R(z,0)]^{-1} T(z_s^-, z)\vec{\mu}_u^m(z_s^-) \tag{96}$$
$$when \quad z_{i-1}^+ < z < z_i^- < z_s$$

and

$$\begin{Bmatrix} \vec{w}^m \\ \vec{f}^m \end{Bmatrix} = \begin{bmatrix} M_u(i) & M_d(i) \\ N_u(i) & N_d(i) \end{bmatrix} \begin{Bmatrix} R(z,\infty) \\ I \end{Bmatrix} [I - R(z_s^+, z)R(z,\infty)]^{-1} T(z_s^+, z)\vec{\mu}_d^m(z_s^+) \tag{97}$$
$$when \quad z_{i-1}^+ > z > z_i^- > z_s$$

where

$$\bar{\mu}_u^m(z_s^-) = [I - R(z_s^+,\infty)R(z_s^-,0)]^{-1} i\{[N_d^T(k) + R(z_s^+,\infty)N_u^T(k)]\Delta w_s \\ -[M_d^T(k) + R(z_s^+,\infty)M_u^T(k)]\Delta f_s\} \tag{98}$$

$$\bar{\mu}_d^m(z_s^+) = [I - R(z_s^+,\infty)R(z_s^-,0)]^{-1} i\{[R(z_s^-,0)N_d^T(k) + N_u^T(k)]\Delta w_s \\ -[R(z_s^-,0)M_d^T(k) + M_u^T(k)]\Delta f_s\} \tag{99}$$

In the above equations, coefficient $i$ denotes imaginary unit $\sqrt{-1}$ while $i$ in the subscripts and parentheses stands for layer number. Therefore, $z_{i-1}$ indicates the depth of the $i^{\text{th}}$ interface ($i$ = 1 for surface), its superscript of plus or minus corresponds the up or down side of the interface, $(\Delta w_s, \Delta f_s)$ are the discontinuities of $(w, f)$ in the source area which are equivalent to the effects of a seismic dislocation source and can be found in equations (90) – (95). In addition, columns in a matrix with sub-matrices $M(i)$ and $N(i)$ are the eigenvectors of matrix [$A$] in layer $i$, $R(z_i, z_j)$ and $T(z_i, z_j)$ are known as reflection and transmission matrices, respectively, which characterize properties of wave propagation in the earth medium between depths $z_i$ and $z_j$. In equations (96) – (99), matrix $I$ is the identity matrix, and matrices $M$, $N$, $R$ and $T$ are functions of the physical parameters of the earth medium, such as $P$ and $S$ wave speeds, etc., and the structure of the earth medium, such as depth of the interface. The explicit expressions for $M$, $N$, $R$ and $T$ matrices are provided in Appendix B.

Substituting equations (96) – (99) into equations (81) - (83), one may obtain the displacement response solution in the $x$, $y$ and $z$ directions in the transformed domain. The corresponding responses in the time-space domain can be obtained from equation (80) using the triple FFT algorithm.

### 4.2. Response in the Scattered Wave Field

The heterogeneity in layer *j* has been described in equation (4) in terms of P and S waves speeds as well as the density. Since the deviation part of the heterogeneity is function of coordinates (*x*, *y*, *z*), it can be expressed by the Fourier series as

$$\zeta_j^s(x,y,z) = \frac{\pi}{x_t'}\frac{\pi}{y_t'}\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\tilde{\zeta}_j^s(k_x',k_y',z)e^{i(k_x'x+ik_y'y)} \tag{100}$$

where $\zeta$ = $\alpha$, $\beta$ or $\rho$. More specifically, the deviations of each parameter of heterogeneous layer are expressed as

$$\alpha_j^s(x,y,z) = \frac{\pi}{x_t'}\frac{\pi}{y_t'}\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\tilde{\alpha}_j^s(k_x',k_y',z)e^{i(k_x'x+k_y'y)} \tag{101}$$

$$\beta_j^s(x,y,z) = \frac{\pi}{x_t'}\frac{\pi}{y_t'}\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\tilde{\beta}_j^s(k_x',k_y',z)e^{i(k_x'x+k_y'y)} \tag{102}$$

$$\rho_j^s(x,y,z) = \frac{\pi}{x_t'}\frac{\pi}{y_t'}\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\tilde{\rho}_j^s(k_x',k_y',z)e^{i(k_x'x+k_y'y)} \tag{103}$$

The heterogeneity will generate virtual body forces for the scattered wave responses, as seen equations (67) – (69) in Chapter 3. In order to find these equivalent body forces, partial derivatives on $u_d^m$ (or $\tilde{u}_d^m$) and $\zeta_j^s$ ($\zeta = \alpha, \beta$ *or* $\rho$) defined by equations (81) – (83) and (101) – (103) are needed, which is shown in detail in Appendix C. Substituting equations (73), (74), (81) - (83) and (101) - (103) into equations (67) –

(69), the equivalent forces in $x$, $y$ and $z$ direction for the scattered wave field can be found as

$$p_x^s = \frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} e^{i(k'_x x + k'_y y)} \frac{\pi}{t_t}\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{l''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\sum_{n''=-\infty}^{\infty} e^{i(k''_x x + k''_y y)} e^{i\omega t}$$

$$\left\{\begin{array}{l}
i(k'_x + k''_x)\tilde{\lambda}_j^s\left[-k''_r\dfrac{2\beta_j^{m^2}}{\alpha_j^{m^2}}W_S^m + \dfrac{\omega}{\rho_j^m\alpha_j^{m^2}}F_R^m\right] \\
+2i(k'_x + k''_x)\tilde{\mu}_j^s\left[-\dfrac{k_x''^2}{k''_r}W_S^m - \dfrac{k''_x k''_y}{k''_r}W_T^m\right] \\
+i(k'_y + k''_y)\tilde{\mu}_j^s\left[-2\dfrac{k''_x k''_y}{k''_r}W_S^m + \left(\dfrac{k_x''^2 - k_y''^2}{k''_r}\right)W_T^m\right] \\
+\tilde{\mu}_j^s\left[\begin{array}{l}
-ik''_x\left[k''_r\left(1-\dfrac{2\beta_j^{m^2}}{\alpha_j^{m^2}}\right) - \dfrac{1}{k''_r}\dfrac{\omega^2}{\beta_j^{m^2}}\left(\dfrac{\gamma k_r''^2}{\omega^2}-1\right)\right]W_S^m \\
-ik''_x\left[\dfrac{\omega}{\rho_j^m\alpha_j^{m^2}} + \dfrac{\omega}{\rho_j^m\beta_j^{m^2}}\left(1-\dfrac{2\beta_j^{m^2}}{\alpha_j^{m^2}}\right)\right]F_R^m \\
+\dfrac{ik''_y}{k''_r}\dfrac{\omega^2}{\beta_j^{m^2}}\left(\beta_j^{m^2}\dfrac{k_r''^2}{\omega^2}-1\right)W_T^m
\end{array}\right] \\
+\tilde{\mu}_{j,z}^s\left[\dfrac{ik''_x}{k''_r}\dfrac{\omega}{\rho^m\beta_j^{m^2}}F_S^m + \dfrac{ik''_y}{k''_r}\dfrac{\omega}{\rho^m\beta_j^{m^2}}F_T^m\right] \\
+\omega^2\left[\dfrac{ik''_x}{k''_r}W_S^m + \dfrac{ik''_y}{k''_r}W_T^m\right]\tilde{\rho}_j^s
\end{array}\right\} \tag{104}$$

$$p_y^s = \frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} e^{i(k'_x x + k'_y y)} \frac{\pi}{t_t}\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{l''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\sum_{n''=-\infty}^{\infty} e^{i(k''_x x + k''_y y)} e^{i\omega t}$$

$$\left\{\begin{array}{l}
i(k'_y + k''_y)\tilde{\lambda}_j^s\left(-k''_r \dfrac{2\beta_j^{m\,2}}{\alpha_j^{m\,2}} W_S^m + \dfrac{\omega}{\rho_j^m \alpha_j^{m\,2}} F_R^m\right) \\
+ 2i(k'_y + k''_y)\tilde{\mu}_j^s\left(-\dfrac{k_y''^2}{k''_r} W_S^m + \dfrac{k''_x k''_y}{k''_r} W_T^m\right) \\
+ i(k'_x + k''_x)\tilde{\mu}_j^s\left[-\dfrac{2k''_x k''_y}{k''_r} W_S^m + \left(\dfrac{k_x''^2 - k_y''^2}{k''_r}\right) W_T^m\right] \\
+ \tilde{\mu}_j^s\left[\begin{array}{l}
\dfrac{ik''_y}{k''_r}\left[-k_r''^2\left(1 - \dfrac{2\beta_j^{m\,2}}{\alpha_j^{m\,2}}\right) + \dfrac{\omega^2}{\beta_j^{m\,2}}\left(\dfrac{\gamma k_r''^2}{\omega^2} - 1\right)\right] W_S^m \\
- ik''_y \dfrac{\omega}{\rho_j^m}\left[\dfrac{1}{\alpha_j^{m\,2}} + \dfrac{1}{\beta_j^{m\,2}}\left(1 - \dfrac{2\beta_j^{m\,2}}{\alpha_j^{m\,2}}\right)\right] F_R^m \\
- \dfrac{ik''_x}{k''_r}\dfrac{\omega^2}{\beta_j^{m\,2}}\left(\beta_j^{m\,2}\dfrac{k_r''^2}{\omega^2} - 1\right) W_T^m
\end{array}\right] \\
+ \tilde{\mu}_{j,z}^s\left[\dfrac{ik''_y}{k''_r}\dfrac{\omega}{\rho_j^m \beta_j^{m\,2}} F_S^m - \dfrac{ik''_x}{k''_r}\dfrac{\omega}{\rho_j^m \beta_j^{m\,2}} F_T^m\right] \\
+ \omega^2\left[\dfrac{ik''_y}{k''_r} W_S^m - \dfrac{ik''_x}{k''_r} W_T^m\right]\tilde{\rho}_j^s
\end{array}\right\} \qquad (105)$$

$$p_z^s = \frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} e^{i(k_x' x + k_y' y)} \frac{\pi}{t_t}\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{l''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\sum_{n''=-\infty}^{\infty} e^{i(k_x'' x + k_y'' y)} e^{i\omega t}$$

$$\left\{\begin{aligned}
&\left(ik_x' + ik_x''\right)\tilde{\mu}_j^s\left(\frac{ik_x''}{k_r''}\frac{\omega}{\rho_j^m \beta_j^{m2}}F_S^m + \frac{ik_y''}{k_r''}\frac{\omega}{\rho_j^m \beta_j^{m2}}F_T^m\right)\\
&+\left(ik_y' + ik_y''\right)\tilde{\mu}_j^s\left(\frac{\omega}{\rho_j^m \beta_j^{m2}}F_R^m - \frac{ik_x''}{k_r''}\frac{\omega}{\rho_j^m \beta_j^{m2}}F_T^m\right)\\
&+\tilde{\lambda}_j^s\left[\begin{aligned}&-k_r''\frac{\omega}{\rho_j^m \beta_j^{m2}}F_S^m + k_r''^2\frac{2\beta_j^{m2}}{\alpha_j^{m2}}W_R^m\\ &+k_r''\left(1-\frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right)\frac{\omega}{\rho_j^m \beta_j^{m2}}F_S^m - \frac{\omega^2}{\alpha_j^{m2}}W_R^m + k_r''\frac{\omega}{\rho_j^m \alpha_j^{m2}}F_S^m\end{aligned}\right]\\
&+\tilde{\lambda}_{j,z}^s\left(-k_r''\frac{2\beta_j^{m2}}{\rho_j^m}W_S^m + \frac{\omega}{\rho_j^m \alpha_j^{m2}}F_R^m\right)\\
&+2\tilde{\mu}_j^s\left[\begin{aligned}&+k_r''\left(1-\frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right)\left(-k_r''W_R^m + \frac{\omega}{\rho_j^m \beta_j^{m2}}F_S^m\right)\\ &+\frac{\omega}{\rho_j^m \alpha_j^{m2}}\left(-\rho_j^m \omega W_R^m + k_r''F_S^m\right)\end{aligned}\right]\\
&+2\tilde{\mu}_{j,z}^s\left[k_r''\left(1-\frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right)W_S^m + \frac{\omega}{\rho_j^m \alpha_j^{m2}}F_R^m\right] + \omega^2\tilde{\rho}_j^s W_R^m
\end{aligned}\right\} \tag{106}$$

With the use of equation (74), equivalent force $\vec{p}^s = \{p_x^s, p_y^s, p_z^s\}^T$ can be written as

$$\vec{p}^s(x,y,z,t) = \frac{\pi}{t_t}\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{l=-\infty}^{\infty} e^{i\omega t}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\omega(P_R^s\vec{e}_R + P_S^s\vec{e}_S + P_T^s\vec{e}_T),$$

$$|t| \le t_t,\ |x| \le x_t \text{ and } |y| \le y_t \tag{107}$$

Correspondingly, $\left(P_R^s, P_S^s, P_T^s\right)$ can be found in terms of $\vec{p}^s$ as (see equation (79))

$$P_X^s(k_x,k_y,z,\omega)=\frac{1}{\omega(2\pi)^3}\int_{-t_t}^{t_t}dte^{-i\omega t}\int_{-x_t}^{x_t}dx\int_{-y_t}^{y_t}dy\left\{\vec{p}^s\cdot[\vec{e}_X]^*\right\};\quad X=R,S,T \tag{108}$$

Substituting equations (104) – (106) into equation (108), the relationship between $P_X^s$ and $W_X^m$ $(X = R,S\ or\ T)$ can be found as follows:

$$\omega\begin{Bmatrix} P_R^s\left(k_x,k_y,z,\omega\right) \\ P_S^s\left(k_x,k_y,z,\omega\right) \\ \cdots\cdots\cdots\cdots\cdots \\ P_T^s\left(k_x,k_y,z,\omega\right) \end{Bmatrix} =$$

$$\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\begin{bmatrix} a_{WR} & a_{WS} & a_{FR} & a_{FS} & \vdots & a_{WT} & a_{FT} \\ b_{WR} & b_{WS} & b_{FR} & b_{FS} & \vdots & b_{WT} & b_{FT} \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots \\ c_{WR} & c_{WS} & c_{FR} & c_{FS} & \vdots & c_{WT} & c_{FT} \end{bmatrix}\begin{Bmatrix} W_R^m\left(k_x'',k_y'',z,\omega\right) \\ W_S^m\left(k_x'',k_y'',z,\omega\right) \\ F_R^m\left(k_x'',k_y'',z,\omega\right) \\ F_S^m\left(k_x'',k_y'',z,\omega\right) \\ \cdots\cdots\cdots\cdots\cdots \\ W_T^m\left(k_x'',k_y'',z,\omega\right) \\ F_T^m\left(k_x'',k_y'',z,\omega\right) \end{Bmatrix} \tag{109}$$

where $k_x'' = k_x - k_x'$ and $k_y'' = k_y - k_y'$. The methodology to define the above relationship as well as coefficients $a_{ij}$ in the matrix are presented in detail in Appendix D. As mentioned in the previous section, the motions of $W_R$, $W_S$, $F_R$ and $F_S$ are associated with P-SV waves, while the motions of $W_T$ and $F_T$ are with SH waves. Therefore, $P_R^s$ and $P_S^s$ correspond to the forces related to P-SV scattered wave component, and $P_T^s$ does to the SH scattered wave component. Equation (109) also provides the apparent fact that each component (either P-SV or SH) of body forces for scattered wave field are generated by both the mean P-SV wave field and the mean SH mean wave.

Corresponding to equation (84) for mean wave field, the governing equations for the scattered wave field can be expressed in transformed domain as

$$\frac{d}{dz}\begin{Bmatrix} \vec{w}^s \\ \vec{f}^s \end{Bmatrix} = [A]\begin{Bmatrix} \vec{w}^s \\ \vec{f}^s \end{Bmatrix} - \begin{Bmatrix} 0 \\ \vec{P}^s \end{Bmatrix} \tag{110}$$

where the coefficient matrix [*A*] is the same as defined in equations (85) and (86). Here, the transformation is introduced as

$$\begin{Bmatrix} \vec{w}^s \\ \vec{f}^s \end{Bmatrix} = [D(j)]\begin{Bmatrix} \vec{\mu}_u^s \\ \vec{\mu}_d^s \end{Bmatrix} = \begin{bmatrix} M_u(j) & M_d(j) \\ N_u(j) & N_d(j) \end{bmatrix}\begin{Bmatrix} \vec{\mu}_u^s \\ \vec{\mu}_d^s \end{Bmatrix} \tag{111}$$

where columns in matrix [*D*(*j*)] are the eigenvectors of matrix [*A*] (see Appendix B for matrices *M* and *N*), $\vec{\mu}_u^s$ and $\vec{\mu}_d^s$ denote, respectively, an up- and down-going wave vectors as depicted in Figure 3. The superscripts of plus and minus correspond respectively to the down and upper sides of the interface. By substituting equation (111) into equation (110), the wave equation can be expressed as

$$\frac{d}{dz}\begin{Bmatrix} \vec{\mu}_u^s \\ \vec{\mu}_d^s \end{Bmatrix} = [D(j)]^{-1}[A][D(j)]\begin{Bmatrix} \vec{\mu}_u^s \\ \vec{\mu}_d^s \end{Bmatrix} - [D(j)]^{-1}\begin{Bmatrix} 0 \\ \vec{P}^s \end{Bmatrix} = \begin{bmatrix} -\lambda_j & 0 \\ 0 & \lambda_j \end{bmatrix}\begin{Bmatrix} \vec{\mu}_u^s \\ \vec{\mu}_d^s \end{Bmatrix} - \begin{Bmatrix} \vec{P}_u^s \\ \vec{P}_d^s \end{Bmatrix} \tag{112}$$

where the inverse of matrix [*D*(*j*)] is

$$[D(j)]^{-1} = i\begin{bmatrix} -N_d^T(j) & M_d^T(j) \\ N_u^T(j) & -M_u^T(j) \end{bmatrix} \tag{113}$$

Equation (112) can be separated into two parts as

$$\frac{d}{dz}\vec{\mu}_u^s = -\lambda_j \vec{\mu}_u^s - \vec{P}_u^s, \tag{114}$$

and

$$\frac{d}{dz}\vec{\mu}_d^s = \lambda_j \vec{\mu}_d^s - \vec{P}_d^s, \tag{115}$$

where $\lambda_j = i\omega q_{\alpha_j}$ or $i\omega q_{\beta_j}$. In Figure 3, $R(z_{j-1}^+, 0)$ (or $R(z_j^-, \infty)$) represents the reflection matrix, which characterizes properties of wave propagation in the earth medium between $z_{j-1}^+$ layer and the ground surface (or between $z_j^-$ layer and the infinite depth). Similarly, $T(z_j^-, z_{j-1}^+)$ represents the transmission matrix in a layer between $z_j^-$ and $z_{j-1}^+$ interfaces and expressed by the exponential, $e^{\lambda(z_j - z_{j-1})}$. Hence, the relationship of wave motions at any level can be expressed using both reflection and transmission matrices, as seen in Figure 3.

The solutions of equations (114) and (115), which are described in Appendix E, can be obtained as

$$\vec{\mu}_u^s(z_{j-1}^+) = e^{\lambda_j(z_j - z_{j-1})}\vec{\mu}_u^s(z_j^-) + \int_{z_{j-1}}^{z_j} e^{\lambda_j(s - z_{j-1})}\vec{P}_u^s(s)ds, \tag{116}$$

and

$$\vec{\mu}_d^s(z_j^-) = e^{\lambda\,(z_j - z_{j-1})}\vec{\mu}_d^s(z_{j-1}^+) - \int_{z_{j-1}}^{z_j} e^{\lambda_j(z_j - s)}\vec{P}_d^s(s)ds. \tag{117}$$

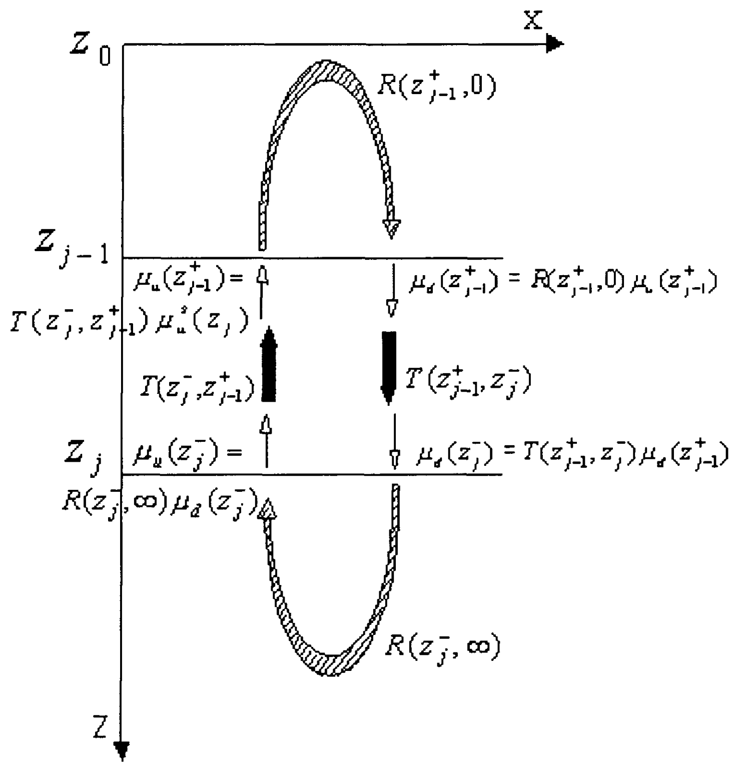


**Figure 3**. Up- and down-going waves from a layer.

Figure 4 depicts the seismic wave motions in the heterogeneous $j^{th}$ layer which are expressed in equations (116) and (117). The integration parts in Figure 4 represent the hetergeneities. As can be seen in Figure 4, the wave motions in $j^{th}$ layer consist of four unknowns (i.e., $\vec{\mu}_u^s(z_{j-1}^+)$, $\vec{\mu}_u^s(z_j^-)$, $\vec{\mu}_d^s(z_j^-)$ *and* $\vec{\mu}_d^s(z_{j-1}^+)$). They can be solved with the use of boundary conditions as well as the relationship between these wave vectors (see Appendix E for the detailed derivation) as follows:

$$\vec{\mu}_u^s(z_{j-1}^+) = \left[I - T(z_{j-1}^+, z_j^-)R(z_j^-,\infty)T(z_j^-, z_{j-1}^+)R(z_{j-1}^+,0)\right]^{-1}$$
$$\left[-\int_{z_{j-1}}^{z_j} T(s, z_{j-1}^+)\vec{P}_u^s(s)ds - T(z_{j-1}^+, z_j^-)R(z_j^-,\infty)\int_{z_{j-1}}^{z_j} T(s, z_j^-)\vec{P}_d^s(s)ds\right] \tag{118}$$

$$\vec{\mu}_d^s(z_j^-) = \left[I - T(z_j^-, z_{j-1}^+)R(z_{j-1}^+,0)T(z_{j-1}^+, z_j^-)R(z_j^-,\infty)\right]^{-1}$$
$$\left[-T(z_j^-, z_{j-1}^+)R(z_{j-1}^+,0)\int_{z_{j-1}}^{z_j} T(s, z_{j-1}^+)\vec{P}_u^s(s)ds - \int_{z_{j-1}}^{z_j} T(s, z_j^-)\vec{P}_d^s(s)ds\right] \tag{119}$$

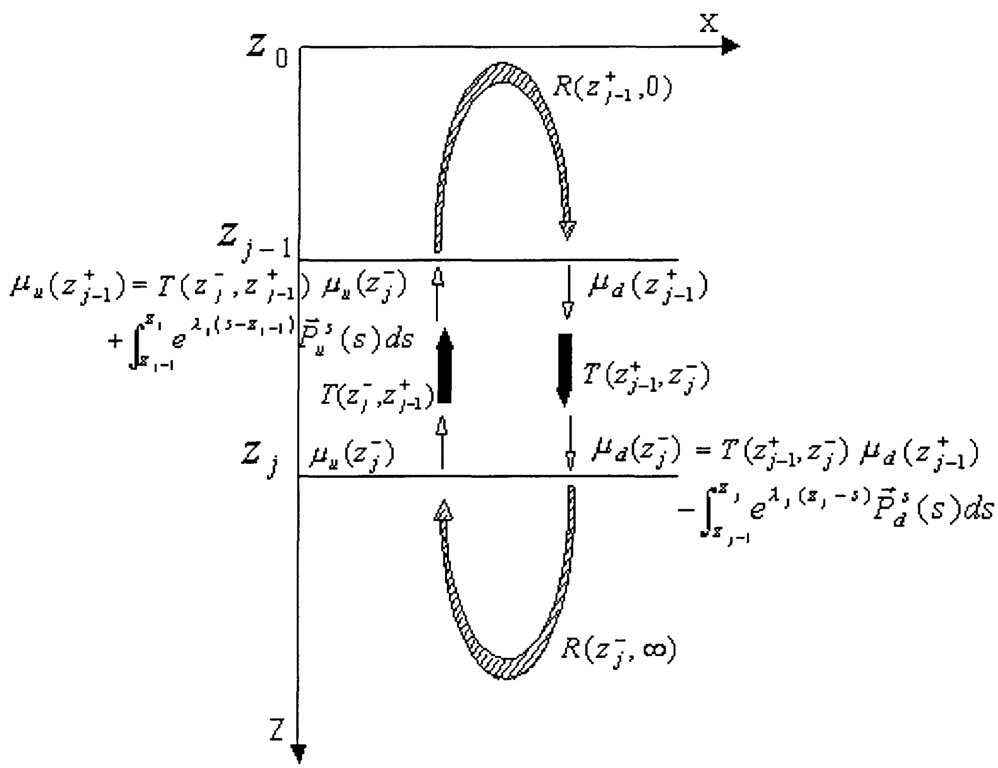


**Figure 4**. Seismic wave motions in $j^{th}$ layer.

Now, one can easily solve for the ground motion using equations (118) and (119). Substituting equations (118) and (119) into equation (96) and (97), except for the mean wave part, stress and displacement discontinuities can be calculated as

$$\begin{Bmatrix} \vec{w}^s \\ \vec{f}^s \end{Bmatrix} = \begin{bmatrix} M_u(i) & M_d(i) \\ N_u(i) & N_d(i) \end{bmatrix} \begin{Bmatrix} I \\ R(z,0) \end{Bmatrix} [I - R(z, z_s^-) R(z,0)]^{-1} T(z_s^-, z) \vec{\mu}_u^s (z_{j-1}^+), \qquad (120)$$

$$when \;\; z_{j-1}^+ < z < z_j^- < z_s$$

and

$$\begin{Bmatrix} \vec{w}^s \\ \vec{f}^s \end{Bmatrix} = \begin{bmatrix} M_u(i) & M_d(i) \\ N_u(i) & N_d(i) \end{bmatrix} \begin{Bmatrix} R(z,\infty) \\ I \end{Bmatrix} [I - R(z_s^+, z) R(z,\infty)]^{-1} T(z_s^+, z) \vec{\mu}_d^s (z_j^-) \qquad (121)$$

$$when \;\; z_{j-1}^+ > z > z_j^- > z_s$$

Finally, substituting equations (120) and (121) into following equations (cf. Equations (81) – (83) for the mean wave field), displacement of the scattered wave field in the frequency-wavenumber domain can be obtained,

$$\tilde{u}_x^s = \frac{ik_x}{k_r} W_S^s + \frac{ik_y}{k_r} W_T^s \qquad (122)$$

$$\tilde{u}_y^s = \frac{ik_y}{k_r} W_S^s - \frac{ik_x}{k_r} W_T^s \qquad (123)$$

$$\tilde{u}_z^s = W_R^s \qquad (124)$$

# CHAPTER 5
# CONCLUDING REMARKS

A model of three-dimensional wave scattering, generated by a seismic dislocation source in a layered half-space with 3 D heterogeneous media, is proposed. In this study, the explicit expressions for responses are derived using the finite integral transform approach and wave propagation analysis, in which P-SV and SH waves are decoupled and analyzed separately. The corresponding response is obtained as a superposition of the responses in both the mean and the scattered wave fields with the use of a first-order perturbation approach. Based on developed equations, it is shown theoretically that the heterogeneity causes the coupling between P-SV and SH waves in the total wave field. In particular, both the P-SV and SH mean waves give the contribution to the P-SV or SH scattered waves.

The numerical example of this model is recommended for a future work to see the extent of influences of the heterogeneous media on seismic wave motion in general and ground motion in particular.

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# APPENDIX

## A. Decomposition of Stress-Displacement Equations

The constitutive relationship between displacements and stresses expressed in equations (15) through (20) in Chapter 3 can be decomposed by using equations (48) - (50) and (21) - (22). Firstly, for example, the stress $\tau_{xx}$ is presented in detail below by substituting the equations (48) - (50) and (21) - (22) into equation (15).

$$\tau_{xx} = \lambda_j^m \left( \frac{\partial u_x^m}{\partial x} + \frac{\partial u_y^m}{\partial y} + \frac{\partial u_z^m}{\partial z} \right) + 2\mu_j^m \frac{\partial u_x^m}{\partial x} + \varepsilon \left[ \lambda_j^m \left( \frac{\partial u_x^s}{\partial x} + \frac{\partial u_y^s}{\partial y} + \frac{\partial u_z^s}{\partial z} \right) + 2\mu_j^m \frac{\partial u_x^s}{\partial x} \right] + \varepsilon \left[ \lambda_j^s \left( \frac{\partial u_x^m}{\partial x} + \frac{\partial u_y^m}{\partial y} + \frac{\partial u_z^m}{\partial z} \right) + 2\mu_j^m \frac{\partial u_x^m}{\partial x} \right] \tag{A1}$$

As can be seen, the equation (A1) is described in three parts

$$\tau_{xx}^m = \lambda_j^m \left( \frac{\partial u_x^m}{\partial x} + \frac{\partial u_y^m}{\partial y} + \frac{\partial u_z^m}{\partial z} \right) + 2\mu_j^m \frac{\partial u_x^m}{\partial x} \tag{A2}$$

$$\tau_{xx}^s = \lambda_j^m \left( \frac{\partial u_x^s}{\partial x} + \frac{\partial u_y^s}{\partial y} + \frac{\partial u_z^s}{\partial z} \right) + 2\mu_j^m \frac{\partial u_x^s}{\partial x} \tag{A3}$$

$$\tau_{xx}^e = \lambda_j^s \left( \frac{\partial u_x^m}{\partial x} + \frac{\partial u_y^m}{\partial y} + \frac{\partial u_z^m}{\partial z} \right) + 2\mu_j^s \frac{\partial u_x^m}{\partial x} \tag{A4}$$

Similarly, the other stress-displacement relationships can also be decomposed, by adopting the same methodology used above.

$$\tau_{yy}^{m} = \lambda_{j}^{m}\left(\frac{\partial u_{x}^{m}}{\partial x} + \frac{\partial u_{y}^{m}}{\partial y} + \frac{\partial u_{z}^{m}}{\partial z}\right) + 2\mu_{j}^{m}\frac{\partial u_{y}^{m}}{\partial y} \tag{A5}$$

$$\tau_{yy}^{s} = \lambda_{j}^{m}\left(\frac{\partial u_{x}^{s}}{\partial x} + \frac{\partial u_{y}^{s}}{\partial y} + \frac{\partial u_{z}^{s}}{\partial z}\right) + 2\mu_{j}^{m}\frac{\partial u_{y}^{s}}{\partial y} \tag{A6}$$

$$\tau_{yy}^{e} = \lambda_{j}^{s}\left(\frac{\partial u_{x}^{m}}{\partial x} + \frac{\partial u_{y}^{m}}{\partial y} + \frac{\partial u_{z}^{m}}{\partial z}\right) + 2\mu_{j}^{s}\frac{\partial u_{y}^{m}}{\partial y} \tag{A7}$$

$$\tau_{zz}^{m} = \lambda_{j}^{m}\left(\frac{\partial u_{x}^{m}}{\partial x} + \frac{\partial u_{y}^{m}}{\partial y} + \frac{\partial u_{z}^{m}}{\partial z}\right) + 2\mu_{j}^{m}\frac{\partial u_{z}^{m}}{\partial z} \tag{A8}$$

$$\tau_{zz}^{s} = \lambda_{j}^{m}\left(\frac{\partial u_{x}^{s}}{\partial x} + \frac{\partial u_{y}^{s}}{\partial y} + \frac{\partial u_{z}^{s}}{\partial z}\right) + 2\mu_{j}^{m}\frac{\partial u_{z}^{s}}{\partial z} \tag{A9}$$

$$\tau_{zz}^{e} = \lambda_{j}^{s}\left(\frac{\partial u_{x}^{m}}{\partial x} + \frac{\partial u_{y}^{m}}{\partial y} + \frac{\partial u_{z}^{m}}{\partial z}\right) + 2\mu_{j}^{s}\frac{\partial u_{z}^{m}}{\partial z} \tag{A10}$$

$$\tau_{xy}^{m} = \mu_{j}^{m}\left(\frac{\partial u_{x}^{m}}{\partial y} + \frac{\partial u_{y}^{m}}{\partial x}\right) \tag{A11}$$

$$\tau_{xy}^{s} = \mu_j^m \left( \frac{\partial u_x^s}{\partial y} + \frac{\partial u_y^s}{\partial x} \right) \tag{A12}$$

$$\tau_{xy}^{e} = \mu_j^s \left( \frac{\partial u_x^m}{\partial y} + \frac{\partial u_y^m}{\partial x} \right) \tag{A13}$$

$$\tau_{yz}^{m} = \mu_j^m \left( \frac{\partial u_y^m}{\partial z} + \frac{\partial u_z^m}{\partial y} \right) \tag{A14}$$

$$\tau_{yz}^{s} = \mu_j^m \left( \frac{\partial u_y^s}{\partial z} + \frac{\partial u_z^s}{\partial y} \right) \tag{A15}$$

$$\tau_{yz}^{e} = \mu_j^s \left( \frac{\partial u_y^m}{\partial z} + \frac{\partial u_z^m}{\partial y} \right) \tag{A16}$$

$$\tau_{xz}^{m} = \mu_j^m \left( \frac{\partial u_x^m}{\partial z} + \frac{\partial u_z^m}{\partial x} \right) \tag{A17}$$

$$\tau_{xz}^{s} = \mu_j^m \left( \frac{\partial u_x^s}{\partial z} + \frac{\partial u_z^s}{\partial x} \right) \tag{A18}$$

$$\tau_{xz}^{e} = \mu_j^s \left( \frac{\partial u_x^m}{\partial z} + \frac{\partial u_z^m}{\partial x} \right) \tag{A19}$$

## B. Matrices M, N, Q, R and T

The explicit expressions for M, N, Q, R and T matrices are provided as below:
For P-SV wave motion,

$$M_{u,d}(i) = \begin{bmatrix} \mp iq_{Pi}\varepsilon_{Pi} & \kappa_r \varepsilon_{Si}/\omega \\ \kappa_r \varepsilon_{Pi}/\omega & \mp iq_{Si}\varepsilon_{Si} \end{bmatrix} \tag{B1}$$

$$N_{u,d}(i) = \begin{bmatrix} \rho_i(2v_{Si}^2\kappa_r^2/\omega^2 - 1)\varepsilon_{Pi} & \mp 2i\rho_i v_{Si}^2 \kappa_r q_{Si}\varepsilon_{Si}/\omega \\ \mp 2i\rho_i v_{Si}\kappa_r q_{Pi}\varepsilon_{Pi}/\omega & \rho_i(2v_{Si}^2\kappa_r^2/\omega^2 - 1)\varepsilon_{Si} \end{bmatrix} \tag{B2}$$

and for SH wave motion,

$$M_{u,d}(i) = \varepsilon_{Si}/v_{Si} \tag{B3}$$

$$N_{u,d}(i) = \mp i\rho_i v_{Si} q_{Si}\varepsilon_{Si} \tag{B4}$$

where $v_{Pi}$, $v_{Si}$ and $\rho_i$ are respectively the P and S wave speeds and density in the $i^{\text{th}}$ layer, and

$$q_{Pi} = \sqrt{1/v_{Pi}^2 - \kappa_r^2/\omega^2}, \qquad q_{Si} = \sqrt{1/v_{Si}^2 - \kappa_r^2/\omega^2}, \tag{B5}$$

$$\varepsilon_{Pi} = (2\rho\ q_{Pi})^{-1/2}, \qquad \varepsilon_{Si} = (2\rho\ q_{Si})^{-1/2}. \tag{B6}$$

To account for damping, the real-valued speeds $v_{Pi}$ and $v_{Si}$ can be replaced by a pair of complex ones, i.e., by $v_{Pi}[1 + i\,\mathrm{sgn}(\omega)/(2Q_{Pi})]$ and $v_{Si}[1 + i\,\mathrm{sgn}(\omega)/(2Q_{Si})]$, where *sgn(ω)* denotes the sign of frequency $\omega$, $Q_{Pi}$ and $Q_{Si}$ are the attenuation factors for P and

S waves in layer *i*. The branch cuts for the radicals in the expressions for $q_{Pi}$ and $q_{Si}$ are taken to be

$$\mathrm{Im}(\omega q_{Pi}) \geq 0, \qquad \mathrm{Im}(\omega q_{Si}) \geq 0 \tag{B7}$$

The reflection and transmission matrices in layer *i*, which is bounded by depths $z_{i-1}^{+}$ and $z_i^{-}$, are

$$R(z_{i-1}^{+}, z_i^{-}) = R(z_i^{-}, z_{i-1}^{+}) = 0 \tag{B8}$$

$$T(z_{i-1}^{+}, z_i^{-}) = T(z_i^{-}, z_{i-1}^{+}) = \begin{cases} diag[e^{i\omega q_{Pi}\Delta z} \quad e^{i\omega q_{Si}\Delta z}] & , \ (P-SV) \\ e^{i\omega q_{Si}\Delta z} & , \ (SH) \end{cases} \tag{B9}$$

where *diag* indicates a diagonal matrix and $\Delta z = \left| z_i^{-} - z_{i-1}^{+} \right|$. The reflection and transmission matrices at the interface with depth $z_i$ bounded by $z_i^{-}$ and $z_i^{+}$ are

$$R(z_i^{-}, z_i^{+}) = Q_{12}Q_{22}^{-1}, \qquad R(z_i^{+}, z_i^{-}) = -Q_{22}^{-1}Q_{21} \tag{B10}$$

$$T(z_i^{-}, z_i^{+}) = Q_{22}^{-1}, \qquad T(z_i^{+}, z_i^{-}) = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} \tag{B11}$$

where

$$Q_{11} = -iN_d^T(i)M_u(i+1) + iM_d^T(i)N_u(i+1) \tag{B12}$$

$$Q_{12} = -iN_d^T(i)M_d(i+1) + iM_d^T(i)N_d(i+1) \tag{B13}$$

$$Q_{21} = iN_u^T(i)M_u(i+1) - iM_u^T(i)N_u(i+1) \tag{B14}$$

$$Q_{22} = iN_u^T(i)M_d(i+1) - iM_u^T(i)N_d(i+1) \tag{B15}$$

The reflection matrix at free surface is

$$R(0_i^+,0) = -[N_d(1)]^{-1} N_u(1) . \quad \text{(B16)}$$

Based on the fundamental reflection and transmission matrices for each uniform layer, each interface between each pair of neighboring layers and the ground surface (equations (B7) - (B16)), the corresponding reflection and transmission matrices between any two depths can be constructed using the following composite rule (Kennett, 1983)

$$R(z_i, z_k) = R(z_i, z_j) + T(z_j, z_i)R(z_j, z_k)[I - R(z_j, z_i)R(z_j, z_k)]^{-1} T(z_i, z_j) \quad \text{(B17)}$$

$$T(z_i, z_k) = T(z_j, z_k)[I - R(z_j, z_i)R(z_j, z_k)]^{-1} T(z_i, z_j) \quad \text{(B18)}$$

## C. Equivalent Body Forces for Scattered Wave Field

The equivalent body forces defined in equations (67) – (69) for the scattered wave field can be alternatively written, using equations (81) – (83) and (101) – (103). The first term in equation (67), for example, is presented in detail. Taking a partial derivative on $\lambda_j^s$ and $u_x^m$ over *x*, and $u_y^m$ over *y* in equation (67) are straightforward, but doing on $u_z^m$ over *z* is need to be paid attention because $W_R^m$ is also a function of *z*, that is,

$$\frac{\partial \tilde{u}_z^m}{\partial z} = \frac{\partial W_R^m}{\partial z} \tag{C1}$$

Hence, the partial derivatives on the displacement fields, $(\vec{u}^m(x,y,z,t))$, over *z* direction are defined, using the variables $W_X^m (X = R, S, orT)$ in equation (86),

$$\frac{\partial u_x^m}{\partial z} = \frac{ik_x''}{k_r''}\left( -k_r'' W_R^m + \frac{\omega}{\rho^m \beta_j^{m^2}} F_S^m \right) + \frac{ik_y''}{k_r''} \frac{\omega}{\rho^m \beta_j^{m^2}} F_T^m \tag{C2}$$

$$\frac{\partial u_y^m}{\partial z} = \frac{ik_y''}{k_r''}\left( -k_r'' W_R^m + \frac{\omega}{\rho^m \beta_j^{m^2}} F_S^m \right) - \frac{ik_x''}{k_r''} \frac{\omega}{\rho^m \beta_j^{m^2}} F_T^m \tag{C3}$$

$$\frac{\partial u_z^m}{\partial z} = k_r''\left( 1 - \frac{2\beta_j^{m^2}}{\alpha_j^{m^2}} \right) W_S^m + \frac{\omega}{\rho^m \alpha_j^{m^2}} F_R^m \tag{C4}$$

Therefore, the first term in the equation (67), using the partial derivatives of $u_x^m$ and $u_y^m$ in equations (81) and (82), and $u_z^m$ in equation (C4), can be expressed as

$$\lambda_{j,x}^{s}\left(\frac{\partial u_x^m}{\partial x}+\frac{\partial u_y^m}{\partial y}+\frac{\partial u_z^m}{\partial z}\right)=\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}e^{i(k_x'x+k_y'y)}\frac{\pi}{t_t}\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{l''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\sum_{n''=-\infty}^{\infty}e^{i(k_x''x+k_y''y)}e^{i\omega t}$$

$$\left\{ik_x'\tilde{\lambda}_j^s\left[\begin{array}{l}ik_x''\left(\frac{ik_x''}{k_r''}W_S^m+\frac{ik_y''}{k_r''}W_T^m\right)+ik_y''\left(\frac{ik_y''}{k_r''}W_S^m-\frac{ik_x''}{k_r''}W_T^m\right)\\+k_x''\left(1-\frac{2\beta_j^{m^2}}{\alpha_j^{m^2}}\right)W_S^m+\frac{\omega}{\rho_j^m\alpha_j^{m^2}}F_R^m\end{array}\right]\right\} \tag{C5}$$

Similarly to equation (C5), all components in equation (67) can be rewritten using equations (101) – (103) and (C2) – (C4). Thus, the equivalent body force in $x$ direction can be fully rewritten. The same methodology is also applied in equations (68) and (69) for the equivalent body forces in $y$ and $z$ directions, respectively. The equivalent body forces are elaborately expressed in the following equations (C6), (C7) and (C8) for $x$, $y$ and $z$ directions, respectively.

$$p_x^s = \frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} e^{i(k'_x x + k'_y y)} \frac{\pi}{t_t}\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{l''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\sum_{n''=-\infty}^{\infty} e^{i(k''_x x + k''_y y)} e^{i\omega t}$$

$$\left\{\begin{array}{l}
ik'_x\tilde{\lambda}_j^s\left[\begin{array}{l} ik''_x\left(\frac{ik''_x}{k''_r}W_S^m + \frac{ik''_y}{k''_r}W_T^m\right) + ik''_y\left(\frac{ik''_y}{k''_r}W_S^m - \frac{ik''_x}{k''_r}W_T^m\right) \\ + k''_r\left(1 - \frac{2\beta_j^{m\,2}}{\alpha_j^{m\,2}}\right)W_S^m + \frac{\omega}{\rho_j^m\alpha_j^{m\,2}}F_R^m \end{array}\right] \\
+ ik''_x\tilde{\lambda}_j^s\left[\begin{array}{l} ik''_x\left(\frac{ik''_x}{k''_r}W_S^m + \frac{ik''_y}{k''_r}W_T^m\right) + ik''_y\left(\frac{ik''_y}{k''_r}W_S^m - \frac{ik''_x}{k''_r}W_T^m\right) \\ + k''_r\left(1 - \frac{2\beta_j^{m\,2}}{\alpha_j^{m\,2}}\right)W_S^m + \frac{\omega}{\rho_j^m\alpha_j^{m\,2}}F_R^m \end{array}\right] \\
+ 2ik'_x\tilde{\mu}_j^s ik''_x\left(\frac{ik''_x}{k''_r}W_S^m + \frac{ik''_y}{k''_r}W_T^m\right) + 2\tilde{\mu}_j^s\left(ik''_x\right)^2\left(\frac{ik''_x}{k''_r}W_S^m + \frac{ik''_y}{k''_r}W_T^m\right) \\
+ ik'_y\tilde{\mu}_j^s\left[ik''_y\left(\frac{ik''_x}{k''_r}W_S^m + \frac{ik''_y}{k''_r}W_T^m\right) + ik''_x\left(\frac{ik''_y}{k''_r}W_S^m - \frac{ik''_x}{k''_r}W_T^m\right)\right] \\
+ ik''_y\tilde{\mu}_j^s\left[ik''_y\left(\frac{ik''_x}{k''_r}W_S^m + \frac{ik''_y}{k''_r}W_T^m\right) + ik''_x\left(\frac{ik''_y}{k''_r}W_S^m - \frac{ik''_x}{k''_r}W_T^m\right)\right] \\
+ \tilde{\mu}_j^s\left\{\begin{array}{l} -ik''_x\left[k''_r\left(1 - \frac{2\beta_j^{m\,2}}{\alpha_j^{m\,2}}\right)W_S^m + \frac{\omega}{\rho_j^m\alpha_j^{m\,2}}F_R^m\right] \\ + \frac{ik''_x}{k''_r}\frac{\omega}{\rho_j^m\beta_j^{m\,2}}\left[\rho_j^m\omega\left(\frac{\gamma k''^2_r}{\omega^2} - 1\right)W_S^m - k''_r\left(1 - \frac{2\beta_j^{m\,2}}{\alpha_j^{m\,2}}\right)F_R^m\right] \\ + \frac{ik''_y}{k''_r}\frac{\omega}{\rho_j^m\beta_j^{m\,2}}\left[\rho_j^m\omega\left(\beta_j^{m\,2}\frac{k''^2_r}{\omega^2} - 1\right)W_T^m\right] \end{array}\right\} \\
+ \tilde{\mu}_{j,z}^s\left\{\frac{ik''_x}{k''_r}\left(-k''_r W_R^m + \frac{\omega}{\rho^m\beta_j^{m\,2}}F_S^m\right) + \frac{ik''_y}{k''_r}\frac{\omega}{\rho^m\beta_j^{m\,2}}F_T^m + ik''_x W_R^m\right\} \\
- \tilde{\rho}_j^s\left(i\omega\right)^2\left(\frac{ik''_x}{k''_r}W_S^m + \frac{ik''_y}{k''_r}W_T^m\right)
\end{array}\right\} \qquad \text{(C6)}$$

$$
\begin{aligned}
p_y^s = \frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} e^{i(k'_x x + k'_y y)} \frac{\pi}{t_t}\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{l''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\sum_{n''=-\infty}^{\infty} e^{i(k''_x x + k''_y y)} e^{i\omega t} \\
\left\{
\begin{aligned}
& ik'_y\tilde{\lambda}_j^s \left[\begin{aligned} & ik''_x\left(\frac{ik''_x}{k''_r}W_S^m + \frac{ik''_y}{k''_r}W_T^m\right) + ik''_y\left(\frac{ik''_y}{k''_r}W_S^m - \frac{ik''_x}{k''_r}W_T^m\right) \\ & + k''_r\left(1 - \frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right)W_S^m + \frac{\omega}{\rho_j^m\alpha_j^{m2}}F_R^m \end{aligned}\right] \\
& + ik''_y\tilde{\lambda}_j^s \left\{\begin{aligned} & ik''_x\left(\frac{ik''_x}{k''_r}W_S^m + \frac{ik''_y}{k''_r}W_T^m\right) + ik''_y\left(\frac{ik''_y}{k''_r}W_S^m - \frac{ik''_x}{k''_r}W_T^m\right) \\ & + k''_r\left(1 - \frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right)W_S^m + \frac{\omega}{\rho_j^m\alpha_j^{m2}}F_R^m \end{aligned}\right\} \\
& + 2ik'_y\tilde{\mu}_j^s ik''_y\left(\frac{ik''_y}{k''_r}W_S^m - \frac{ik''_x}{k''_r}W_T^m\right) + 2\tilde{\mu}_j^s\left(ik''_y\right)^2\left(\frac{ik''_y}{k''_r}W_S^m - \frac{ik''_x}{k''_r}W_T^m\right) \\
& + ik'_x\tilde{\mu}_j^s\left[ik''_y\left(\frac{ik''_x}{k''_r}W_S^m + \frac{ik''_y}{k''_r}W_T^m\right) + ik''_x\left(\frac{ik''_y}{k''_r}W_S^m - \frac{ik''_x}{k''_r}W_T^m\right)\right] \\
& + ik''_x\tilde{\mu}_j^s\left[ik''_y\left(\frac{ik''_x}{k''_r}W_S^m + \frac{ik''_y}{k''_r}W_T^m\right) + \left(ik''_x\right)^2\left(\frac{ik''_y}{k''_r}W_S^m - \frac{ik''_x}{k''_r}W_T^m\right)\right] \\
& + \tilde{\mu}_j^s\left\{\begin{aligned} & -ik''_y\left[k''_r\left(1 - \frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right)W_S^m + \frac{\omega}{\rho_j^m\alpha_j^{m2}}F_R^m\right] \\ & + \frac{ik''_y}{k''_r}\frac{\omega}{\rho_j^m\beta_j^{m2}}\left[\rho_j^m\omega\left(\frac{\gamma k_r''^2}{\omega^2} - 1\right)W_S^m - k''_r\left(1 - \frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right)F_R^m\right] \\ & - \frac{ik''_x}{k''_r}\frac{\omega}{\rho_j^m\beta_j^{m2}}\left[\rho_j^m\omega\left(\beta_j^{m2}\frac{k''_r}{\omega^2} - 1\right)W_T^m\right] \end{aligned}\right\} \\
& + \tilde{\mu}_{j,z}^s\left\{\frac{ik''_y}{k''_r}\left(-k''_r W_R^m + \frac{\omega}{\rho_j^m\beta_j^{m2}}F_S^m\right) - \frac{ik''_x}{k''_r}\frac{\omega}{\rho_j^m\beta_j^{m2}}F_T^m + ik''_y W_R^m\right\} \\
& - \tilde{\rho}_j^s\left(i\omega\right)^2\left(\frac{ik''_y}{k''_r}W_S^m - \frac{ik''_x}{k''_r}W_T^m\right)
\end{aligned}
\right\}
\end{aligned}
\tag{C7}
$$

$$
\begin{aligned}
p_z^s = {} & \frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} e^{i(k_x' x + k_y' y)} \frac{\pi}{t_t}\frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{l''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\sum_{n''=-\infty}^{\infty} e^{i(k_x'' x + k_y'' y)} e^{i\omega t} \\
& \left\{ \begin{aligned}
& ik_x'\tilde{\mu}_j^s\left[\frac{ik_x''}{k_r''}\left(-k_r''W_R^m + \frac{\omega}{\rho_j^m\beta_j^{m^2}}F_S^m\right) + \frac{ik_y''}{k_r''}\frac{\omega}{\rho_j^m\beta_j^{m^2}}F_T^m + ik_x''W_R^m\right] \\
& + ik_x''\tilde{\mu}_j^s\left[\frac{ik_x''}{k_r''}\left(-k_r''W_R^m + \frac{\omega}{\rho_j^m\beta_j^{m^2}}F_S^m\right) + \frac{ik_y''}{k_r''}\frac{\omega}{\rho_j^m\beta_j^{m^2}}F_T^m + ik_x''W_R^m\right] \\
& + \left(ik_y' + ik_y''\right)\tilde{\mu}_j^s\left[\frac{ik_y''}{k_r''}\left(-k_r''W_R^m + \frac{\omega}{\rho_j^m\beta_j^{m^2}}F_R^m\right) - \frac{ik_x''}{k_r''}\frac{\omega}{\rho_j^m\beta_j^{m^2}}F_T^m + ik_y''W_R^m\right] \\
& + \tilde{\lambda}_j^s \left\{ \begin{aligned}
& ik_x''\left[\frac{ik_x''}{k_r''}\left(-k_r''W_R^m + \frac{\omega}{\rho_j^m\beta_j^{m^2}}F_S^m\right) + \frac{ik_y''}{k_r''}\frac{\omega}{\rho_j^m\beta_j^{m^2}}F_T^m\right] \\
& + ik_y''\left[\frac{ik_y''}{k_r''}\left(-k_r''W_R^m + \frac{\omega}{\rho_j^m\beta_j^{m^2}}F_S^m\right) - \frac{ik_x''}{k_r''}\frac{\omega}{\rho_j^m\beta_j^{m^2}}F_T^m\right] \\
& + k_r''\left(1 - \frac{2\beta_j^{m^2}}{\alpha_j^{m^2}}\right)\left(-k_r''W_R^m + \frac{\omega}{\rho_j^m\beta_j^{m^2}}F_S^m\right) + \frac{\omega}{\rho_j^m\alpha_j^{m^2}}\left(-\rho_j^m\omega W_R^m + k_r''F_S^m\right)
\end{aligned} \right\} \\
& + 2\tilde{\mu}_j^s\left[k_r''\left(1 - \frac{2\beta_j^{m^2}}{\alpha_j^{m^2}}\right)\left(-k_r''W_R^m + \frac{\omega}{\rho_j^m\beta_j^{m^2}}F_S^m\right) + \frac{\omega}{\rho_j^m\alpha_j^{m^2}}\left(-\rho_j^m\omega W_R^m + k_r''F_S^m\right)\right] \\
& + \tilde{\lambda}_{j,z}^s \left[ \begin{aligned}
& ik_x''\left(\frac{ik_x''}{k_r''}W_S^m + \frac{ik_y''}{k_r''}W_T^m\right) + ik_y''\left(\frac{ik_y''}{k_r''}W_S^m - \frac{ik_x''}{k_r''}W_T^m\right) \\
& + k_r''\left(1 - \frac{2\beta_j^{m^2}}{\rho_j^m}\right)W_S^m + \frac{\omega}{\rho_j^m\alpha_j^{m^2}}F_R^m
\end{aligned} \right] \\
& + 2\tilde{\mu}_{j,z}^s\left[k_r''\left(1 - \frac{2\beta_j^{m^2}}{\alpha_j^{m^2}}\right)W_S^m + \frac{\omega}{\rho_j^m\alpha_j^{m^2}}F_R^m\right] \\
& - \tilde{\rho}_j^s(i\omega)^2\overline{W}_R^m
\end{aligned} \right\} \qquad \text{(C8)}
\end{aligned}
$$

**D. Relationship between $P_X^s$ and $W_X^m (X = R, S \ or \ T)$**

In equation (107), the scattered equivalent forces have been expressed in a concise form using the orthogonal vector harmonics $(\vec{e}_R, \vec{e}_S, \vec{e}_T)$. Correspondingly, the variables $P_X^s (X = R, S, or \ T)$ have been defined in equation (108). Each variable in equation (108) can be specifically expressed as

$$\omega P_R^s (k_x, k_y, z, \omega) = \frac{1}{(2\pi)^3} \int_{-t_t}^{t_t} dt e^{-i\omega t} \int_{-x_t}^{x_t} dx \int_{-y_t}^{y_t} dy \left( p_x^s \vec{e}_x + p_y^s \vec{e}_y + p_z^s \vec{e}_z \right) \cdot \vec{e}_z \quad \text{(D1)}$$

$$\omega P_S^s (k_x, k_y, z, \omega) = \frac{1}{(2\pi)^3} \int_{-t_t}^{t_t} dt e^{-i\omega t} \int_{-x_t}^{x_t} dx \int_{-y_t}^{y_t} dy \left( p_x^s \vec{e}_x + p_y^s \vec{e}_y + p_z^s \vec{e}_z \right) \cdot \left( -\frac{ik_x}{k_r} \vec{e}_x - \frac{ik_y}{k_r} \vec{e}_y \right) \quad \text{(D2)}$$

$$\omega P_T^s (k_x, k_y, z, \omega) = \frac{1}{(2\pi)^3} \int_{-t_t}^{t_t} dt e^{-i\omega t} \int_{-x_t}^{x_t} dx \int_{-y_t}^{y_t} dy \left( p_x^s \vec{e}_x + p_y^s \vec{e}_y + p_z^s \vec{e}_z \right) \cdot \left( -\frac{ik_y}{k_r} \vec{e}_x - \frac{ik_x}{k_r} \vec{e}_y \right) \quad \text{(D3)}$$

Knowing that the property of the orthogonal vectors, i.e.,

$$\begin{cases} \vec{e}_i \cdot \vec{e}_j = 1 & if \quad i = j \\ \vec{e}_i \cdot \vec{e}_j = 0 & otherwise \end{cases}, \quad \text{(D4)}$$

one can easily simplify equations (D1) – (D3) as

$$\omega P_R^s = \frac{\pi}{x_t} \frac{\pi}{y_t} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{p}_z^s \quad \text{(D5)}$$

$$\omega P_S^s = \frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left[-\frac{ik_x}{k_r}\tilde{p}_x^s - \frac{ik_y}{k_r}\tilde{p}_y^s\right] \tag{D6}$$

$$\omega P_T^s = \frac{\pi}{x_t}\frac{\pi}{y_t}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left[-\frac{ik_y}{k_r}\tilde{p}_x^s - \frac{ik_x}{k_r}\tilde{p}_y^s\right] \tag{D7}$$

where $\tilde{p}^s(\tilde{p}_x^s, \tilde{p}_y^s, \tilde{p}_z^s)$ represent the scattered equivalent body forces in the transformed domain $(k_x, k_y, z, \omega)$. Above three equations (D5) – (D7) can be combined in a matrix form as in equation (109).

The coefficients in the matrix in equation (109) can be obtained from equations (D5) – (D7). As can be seen from equation (D5), the first variable $P_R^s$ has a relationship with $\tilde{p}_z^s$ only. Therefore, the coefficients in first row in the matrix can be found by using equation (104) with the corresponding variables $W_X^s(X = R, S, orT)$, and they are expressed below.

$$\begin{aligned} a_{WR} &= \tilde{\lambda}_j^s\left(k_r^{n2}\frac{2\beta_j^{m2}}{\alpha_j^{m2}} - \frac{\omega^2}{\alpha_j^{m2}}\right) + 2\tilde{\mu}_j^s\left[-k_r^{n2}\left(1-\frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right) - \frac{\omega^2}{\alpha_j^{m2}}\right] + \omega^2\tilde{\rho}_j^s \\ &= \left(\tilde{\lambda}_j^s + 2\tilde{\mu}_j^s\right)\left(k_r^{n2}\frac{2\beta_j^{m2} - \omega^2}{\alpha_j^{m2}}\right) - 2k_r^{n2}\tilde{\mu}_j^s + \omega^2\tilde{\rho}_j^s \end{aligned} \tag{D8}$$

$$a_{WS} = -\tilde{\lambda}_{j,z}^s k_r^n\frac{2\beta_j^{m2}}{\rho_j^m} + 2\tilde{\mu}_{j,z}^s k_r^n\left(1-\frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right) \tag{D9}$$

$$a_{FR} = \left(ik_y' + ik_y''\right)\tilde{\mu}_j^s \frac{\omega}{\rho_j^m \beta_j^{m^2}} + \tilde{\lambda}_{j,z}^s \frac{\omega}{\rho_j^m \alpha_j^{m^2}} + 2\tilde{\mu}_{j,z}^s \frac{\omega}{\rho_j^m \alpha_j^{m^2}}$$
$$= ik_y \tilde{\mu}_j^s \frac{\omega}{\rho_j^m \beta_j^{m^2}} + \left(\tilde{\lambda}_{j,z}^s + 2\tilde{\mu}_{j,z}^s\right)\frac{\omega}{\rho_j^m \alpha_j^{m^2}} \quad \text{(D10)}$$

$$a_{FS} = \left(ik_x' + ik_x''\right)\tilde{\mu}_j^s \frac{ik_x''}{k_r''}\frac{\omega}{\rho_j^m \beta_j^{m^2}} + 2\tilde{\mu}_j^s\left[k_r''\left(1 - \frac{2\beta_j^{m^2}}{\alpha_j^{m^2}}\right)\frac{\omega}{\rho_j^m \beta_j^{m^2}} + k_r''\frac{\omega}{\rho_j^m \alpha_j^{m^2}}\right]$$
$$+ \tilde{\lambda}_j^s\left[-k_r''\frac{\omega}{\rho_j^m \beta_j^{m^2}} + k_r''\left(1 - \frac{2\beta_j^{m^2}}{\alpha_j^{m^2}}\right)\frac{\omega}{\rho_j^m \beta_j^{m^2}} + k_r''\frac{\omega}{\rho_j^m \alpha_j^{m^2}}\right] \quad \text{(D11)}$$
$$= \left(ik_x \frac{ik_x''}{k_r''}\tilde{\mu}_j^s - k_r''\tilde{\lambda}_j^s\right)\frac{\omega}{\rho_j^m \beta_j^{m^2}} + \left(\tilde{\lambda}_j^s + 2\tilde{\mu}_j^s\right)k_r''\frac{\omega\left(\alpha_j^{m^2} - \beta_j^{m^2}\right)}{\rho_j^m \alpha_j^{m^2}\beta_j^{m^2}}$$

$$a_{WT} = 0 \quad \text{(D12)}$$

$$a_{FT} = -\left(\frac{k_x k_y'' + k_y k_x''}{k_r''}\right)\frac{\omega}{\rho^m \beta_j^{m^2}}\tilde{\mu}_j^s \quad \text{(D13)}$$

Similarly, the coefficients in the second and third rows can be found from equations (D6) and (D7) as followed from equations (D14) to (D25).

$$b_{WR} = 0 \quad \text{(D14)}$$

$$b_{FR} = -\frac{ik_x}{k_r}\left[\left(ik_x' - ik_x''\right)\tilde{\lambda}_j^s \frac{\omega}{\rho_j^m \alpha_j^{m2}} - ik_x''\left(\frac{\omega}{\rho_j^m \alpha_j^{m2}} + \frac{\omega}{\rho_j^m \beta_j^{m2}}\left(1 - \frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right)\right)\tilde{\mu}_j^s\right]$$

$$-\frac{ik_y}{k_r}\left[\left(ik_y' - ik_y''\right)\tilde{\lambda}_j^s \frac{\omega}{\rho_j^m \alpha_j^{m2}} - ik_y''\left(\frac{\omega}{\rho_j^m \alpha_j^{m2}} + \frac{\omega}{\rho_j^m \beta_j^{m2}}\left(1 - \frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right)\right)\tilde{\mu}_j^s\right] \quad \text{(D15)}$$

$$= \left(\frac{k_x^{\ 2} - k_y^{\ 2}}{k_r}\right)\tilde{\lambda}_j^s \frac{\omega}{\rho_j^m \alpha_j^{m2}} - \left[\frac{\omega}{\rho_j^m \alpha_j^{m2}} + \frac{\omega}{\rho_j^m \beta_j^{m2}}\left(1 - \frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right)\right]\left(\frac{k_x''k_x + k_y''k_y}{k_r}\right)\tilde{\mu}_j^s$$

$$b_{FS} = \left(\frac{k_x''k_x + k_y''k_y}{k_r k_r''}\right)\frac{\omega}{\rho_j^m \beta_j^{m2}}\tilde{\mu}_{j,z}^s \quad \text{(D16)}$$

$$b_{WS} = -\frac{ik_x}{k_r}\left\{\begin{array}{l} -\left(ik_x' - ik_x''\right)\tilde{\lambda}_j^s k_r'' \dfrac{2\beta_j^{m2}}{\alpha_j^{m2}} - \left(ik_x' - ik_x''\right)2\tilde{\mu}_j^s \dfrac{k_x''^2}{k_r''} - \left(ik_y' - ik_y''\right)2\tilde{\mu}_j^s \dfrac{k_x''k_y''}{k_r''} \\ \tilde{\mu}_j^s\left[-ik_x''k_r''\left(1 - \dfrac{2\beta_j^{m2}}{\alpha_j^{m2}}\right) + \dfrac{ik_x''}{k_r''}\dfrac{\omega^2}{\beta_j^{m2}}\left(\dfrac{\gamma k_r''^2}{\omega^2} - 1\right)\right] + \omega^2\tilde{\rho}_j^s \dfrac{ik_x''}{k_r''} \end{array}\right\}$$

$$-\frac{ik_y}{k_r}\left\{\begin{array}{l} -\left(ik_y' - ik_y''\right)\tilde{\lambda}_j^s k_r'' \dfrac{2\beta_j^{m2}}{\alpha_j^{m2}} - \left(ik_y' - ik_y''\right)2\tilde{\mu}_j^s \dfrac{k_y''^2}{k_r''} - \left(ik_x' - ik_x''\right)2\tilde{\mu}_j^s \dfrac{k_x''k_y''}{k_r''} \\ \tilde{\mu}_j^s\left[-ik_y''k_r''\left(1 - \dfrac{2\beta_j^{m2}}{\alpha_j^{m2}}\right) + \dfrac{ik_y''}{k_r''}\dfrac{\omega^2}{\beta_j^{m2}}\left(\dfrac{\gamma k_r''^2}{\omega^2} - 1\right)\right] + \omega^2\tilde{\rho}_j^s \dfrac{ik_y''}{k_r''} \end{array}\right\}$$

$$= -\left(\frac{k_x^{\ 2} + k_y^{\ 2}}{k_r}\right)k_r''\frac{2\beta_j^{m2}}{\alpha_j^{m2}}\tilde{\lambda}_j^s - \left(\frac{k_x''^2 k_x^{\ 2} + k_y''^2 k_y^{\ 2} + 2\left(k_x''k_y''\right)\left(k_x k_y\right)}{k_r k_r''}\right)2\tilde{\mu}_j^s$$

$$-k_r''\left(\frac{k_x''k_x + k_y''k_y}{k_r}\right)\left(1 - \frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right)\tilde{\mu}_j^s - \left(\frac{k_x''k_x + k_y''k_y}{k_r k_r''}\right)\left(\frac{\omega^2}{\beta_j^{m2}}\left(\frac{\gamma k_r''^2}{\omega^2} - 1\right) + \omega^2\tilde{\rho}_j^s\right) \quad \text{(D17)}$$

$$b_{WT} = -\frac{ik_x}{k_r}\left\{\begin{array}{l} -\left(ik'_x + ik''_x\right)2\tilde{\mu}_j^s \dfrac{k''_x k''_y}{k''_r} + \left(ik'_y + ik''_y\right)\tilde{\mu}_j^s\left(\dfrac{k''^2_x - k''^2_y}{k''_r}\right) \\ + \tilde{\mu}_j^s \dfrac{ik''_y}{k''_r}\dfrac{\omega^2}{\beta_j^{m2}}\left(\beta_j^{m2}\dfrac{k''^2_r}{\omega^2} - 1\right) + \omega^2 \dfrac{ik''_y}{k''_r}\tilde{\rho}_j^s \end{array}\right\}$$

$$-\frac{ik_y}{k_r}\left\{\begin{array}{l} -\left(ik'_y + ik''_y\right)2\tilde{\mu}_j^s \dfrac{k''_x k''_y}{k''_r} + \left(ik'_x + ik''_x\right)\tilde{\mu}_j^s\left(\dfrac{k''^2_x - k''^2_y}{k''_r}\right) \\ - \tilde{\mu}_j^s \dfrac{ik''_x}{k''_r}\dfrac{\omega^2}{\beta_j^{m2}}\left(\beta_j^{m2}\dfrac{k''^2_r}{\omega^2} - 1\right) + \omega^2 \dfrac{ik''_y}{k''_r}\tilde{\rho}_j^s \end{array}\right\}$$

$$= 2\frac{k''_x k''_y}{k''_r}\left(\frac{-k_x{}^2 + k_y{}^2}{k_r}\right)\tilde{\mu}_j^s + \left(\frac{2k_x k_y}{k_r}\right)\left(\frac{k''^2_x - k''^2_y}{k''_r}\right)\tilde{\mu}_j^s$$

$$+\left(\frac{k_x k''_y - k''_x k_y}{k_r k''_r}\right)\left[\frac{\omega^2}{\beta_j^{m2}}\left(\beta_j^{m2}\frac{k''^2_r}{\omega^2} - 1\right)\tilde{\mu}_j^s + \omega^2\tilde{\rho}_j^s\right] \tag{D18}$$

$$b_{FT} = \left(\frac{k_x k''_y - k''_x k_y}{k_r k''_r}\right)\frac{\omega}{\rho_j^m \beta_j^{m2}}\tilde{\mu}_{j,z}^s \tag{D19}$$

$$c_{WR} = 0 \tag{D20}$$

$$c_{WS} = -\left(\frac{k_x k_y + k_x k_y}{k_r}\right)k''_r \frac{2\beta_j^{m2}}{\alpha_j^{m2}}\tilde{\lambda}_j^s - \left(\frac{k_x{}^2 + k_y{}^2}{k_r k''_r}\right)\left(k''_x k''_y\right)2\tilde{\mu}_j^s$$

$$-k''_r\left(\frac{k''_x k_x + k''_y k_y}{k_r}\right)\left(1 - \frac{2\beta_j^{m2}}{\alpha_j^{m2}}\right)\tilde{\mu}_j^s + \left(\frac{k''_x k_y + k_x k''_y}{k_r k''_r}\right)\left[\left(\frac{\gamma k''^2_r - \omega^2}{\beta_j^{m2}}\right) + \omega^2\tilde{\rho}_j^s\right] \tag{D21}$$

$$c_{FR} = \left(\frac{2k_x k_y}{k_r}\right)\tilde{\lambda}_j^s \frac{\omega}{\rho_j^m \alpha_j^{m\,2}} - \left(\frac{k_x'' k_y + k_x k_y''}{k_r}\right)\left[\frac{\omega}{\rho_j^m \alpha_j^{m\,2}} + \frac{\omega}{\rho_j^m \beta_j^{m\,2}}\left(1 - \frac{2\beta_j^{m\,2}}{\alpha_j^{m\,2}}\right)\right]\tilde{\mu}_j^s \quad \text{(D22)}$$

$$c_{FS} = \left(\frac{k_x'' k_y + k_x k_y''}{k_r k_r''}\right)\frac{\omega}{\rho_j^m \beta_j^{m\,2}}\tilde{\mu}_{j,z}^s \quad \text{(D23)}$$

$$c_{WT} = \left(\frac{k_x^{\,2} + k_y^{\,2}}{k_r}\right)\left(\frac{k_x''^2 - k_y''^2}{k_r''}\right)\tilde{\mu}_j^s + \left(\frac{k_y'' k_y - k_x'' k_x}{k_r k_r''}\right)\left[\frac{\omega^2}{\beta_j^{m\,2}}\left(\beta_j^{m\,2}\frac{k_r''^2}{\omega^2} - 1\right)\tilde{\mu}_j^s + \omega^2 \tilde{\rho}_j^s\right] \quad \text{(D24)}$$

$$c_{FT} = \left(\frac{k_y k_y'' - k_x'' k_x}{k_r k_r''}\right)\frac{\omega}{\rho_j^m \beta_j^{m\,2}}\tilde{\mu}_{j,z}^s \quad \text{(D25)}$$

**E. Solutions of the Ground Motion ($\vec{\mu}_u^s$ *and* $\vec{\mu}_d^s$) in j<sup>th</sup> Layer**

Wave equation is separated into two differential equations as

$$\frac{d}{dz}\vec{\mu}_u^s = -\lambda_j \vec{\mu}_u^s - \vec{P}_u^s \,, \tag{E1}$$

and

$$\frac{d}{dz}\vec{\mu}_d^s = \lambda_j \vec{\mu}_d^s - \vec{P}_d^s \,, \tag{E2}$$

where $\lambda_j = i\omega q_{\alpha_j}$ or $i\omega q_{\beta_j}$. The homogeneous solution for equation (E1) can be easily obtained as

$$\vec{\mu}_u^s(z) = e^{-\lambda_j(z-z_0)} f(z) \tag{E3}$$

By substituting equation (E3) into the equation (E1) and integrating both sides, $f(z)$ in equation (E3) becomes

$$f(z) = c - \int_{z_{j-1}}^{z} e^{\lambda_j(s-z_0)} \vec{P}_u^s(s)ds \,, \tag{E4}$$

where $c$ is a constant. Equation (E3) can rewritten using equation (E4) as

$$\vec{\mu}_u^s(z_j) = e^{-\lambda_j(z_j-z_0)}c - \int_{z_{j-1}}^{z_j} e^{\lambda_j(s-z_j)} \vec{P}_u^s(s)ds, \qquad for\ \ j^{th}\ \ layer \,, \tag{E5}$$

and

$$\vec{\mu}_u^s(z_{j-1}) = e^{-\lambda_j(z_{j-1}-z_0)}c, \qquad for\ \ j-1^{th}\ \ layer \,. \tag{E6}$$

Now, one can easily obtain the solution for equation (E1), by substituting equation (E6) into equation (E5) and canceling the constant *c*:

$$\begin{aligned}\vec{\mu}_u^s(z_j) &= e^{-\lambda_j(z-z_0)}\vec{\mu}_u^s(z_{j-1})e^{\lambda_j(z_{j-1}-z_0)} - \int_{z_{j-1}}^{z} e^{\lambda_j(s-z_j)}\vec{P}_u^s(s)ds \\ &= e^{\lambda_j(z_{j-1}-z_j)}\vec{\mu}_u^s(z_{j-1}) - \int_{z_{j-1}}^{z_j} e^{\lambda_j(s-z_j)}\vec{P}_u^s(s)ds\end{aligned} \tag{E7}$$

Similarly, by performing the same methodology, the solution for the equation (E2) can be defined as

$$\vec{\mu}_d^s(z_j) = e^{\lambda\,(z_j-z_{j-1})}\vec{\mu}_d^s(z_{j-1}) - \int_{z_{j-1}}^{z_j} e^{\lambda_j(z_j-s)}\vec{P}_d^s(s)ds \tag{E8}$$

The equation (E7) can be expressed for $z_{j-1}$ as

$$\vec{\mu}_u^s(z_{j-1}) = e^{\lambda_j(z_j-z_{j-1})}\vec{\mu}_u^s(z_j) + \int_{z_{j-1}}^{z_j} e^{\lambda_j(s-z_{j-1})}\vec{P}_u^s(s)ds \tag{E9}$$

As seen in Figure 4, the earth medium in $j^{th}$ layer can be represented by reflection matrices ($R(z_{j-1}^+,0)$ and $R(z_j^-,\infty)$) and the transmission matrices ($T(z_j^-,z_{j-1}^+)$ and $T(z_{j-1}^+,z_j^-)$). Hence, the wave equations in equations (E8) and (E9) can be expressed using both reflection and transmission matrices as

$$\vec{\mu}_u^s(z_{j-1}^+) = T(z_j^-,z_{j-1}^+)R(z_j^-,\infty)\vec{\mu}_d^s(z_j^-) + \int_{z_{j-1}}^{z_j} T(s,z_{j-1}^+)\vec{P}_u^s(s)ds \tag{E10}$$

$$\vec{\mu}_d^s(z_j^-) = T(z_{j-1}^+,z_j^-)R(z_{j-1}^+,0)\vec{\mu}_u^s(z_{j-1}^+) - \int_{z_{j-1}}^{z_j} T(s,z_j^-)\vec{P}_d^s(s)ds \tag{E11}$$

Multiplying equation (E10) with $T(z_{j-1}^{+}, z_j^{-})R(z_j^{-},\infty)$ and then adding it with the equation (E11) produce the combined equation as

$$\begin{aligned}\left[I - T(z_{j-1}^{+}, z_j^{-})R(z_j^{-},\infty)T(z_j^{-}, z_{j-1}^{+})R(z_{j-1}^{+},0)\right]\vec{\mu}_u^s(z_{j-1}^{+}) \\ = +\int_{z_{j-1}}^{z_j} T(s, z_{j-1}^{+})\vec{P}_u^s(s)ds - T(z_{j-1}^{+}, z_j^{-})R(z_j^{-},\infty)\int_{z_{j-1}}^{z_j} T(s, z_j^{-})\vec{P}_d^s(s)ds\end{aligned} \tag{E12}$$

Hence, $\vec{\mu}_u^s(z_{j-1}^{+})$ can be rearranged from equation (E12) as

$$\begin{aligned}\vec{\mu}_u^s(z_{j-1}^{+}) = \left[I - T(z_{j-1}^{+}, z_j^{-})R(z_j^{-},\infty)T(z_j^{-}, z_{j-1}^{+})R(z_{j-1}^{+},0)\right]^{-1} \\ \left[+\int_{z_{j-1}}^{z_j} T(s, z_{j-1}^{+})\vec{P}_u^s(s)ds - T(z_{j-1}^{+}, z_j^{-})R(z_j^{-},\infty)\int_{z_{j-1}}^{z_j} T(s, z_j^{-})\vec{P}_d^s(s)ds\right]\end{aligned} \tag{E13}$$

Similarly, $\vec{\mu}_d^s(z_j^{-})$ can be rewritten from (E11) by multiplying $T(z_j^{-}, z_{j-1}^{+})R(z_{j-1}^{+},0)$ into equation (E10), and then adding it again with equation (E11),

$$\begin{aligned}\vec{\mu}_d^s(z_j^{-}) = \left[I - T(z_j^{-}, z_{j-1}^{+})R(z_{j-1}^{+},0)T(z_{j-1}^{+}, z_j^{-})R(z_j^{-},\infty)\right]^{-1} \\ \left[-T(z_j^{-}, z_{j-1}^{+})R(z_{j-1}^{+},0)\int_{z_{j-1}}^{z_j} T(s, z_{j-1}^{+})\vec{P}_u^s(s)ds - \int_{z_{j-1}}^{z_j} T(s, z_j^{-})\vec{P}_d^s(s)ds\right]\end{aligned} \tag{E14}$$