RENORMALIZATION GROUP FLOW EQUATIONS FOR CHIRAL NUCLEAR MODELS

by

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ABSTRACT

The renormalization group (RG) is a tool for the qualitative and quantitative nonperturbative understanding of physical systems. Historically, perturbation theory has provided the basis for the understanding of a variety of phenomena. But, though it has seen many remarkable successes, it has at times proved to be more of a crutch than a panacea. There are many examples of physical systems that defy any perturbative approach, *e.g.* strongly correlated statistical systems and strongly coupled quantum field theories. The currently accepted theory of the strong interactions, Quantum Chromodynamics (QCD), is an example of the latter. Unlike the case of its gauge theory counterpart, Quantum Electrodynamics (QED), many consequences of QCD cannot be computed using perturbation theory. Instead, closed form perturbative solutions of QCD are possible only for a limited subset of phenomena such as high momentum-transfer scattering processes. These solutions afford little insight into the most ubiquitous and experimentally accessible consequences of QCD: the bound states of the theory, *e.g.* nucleons and nuclei. Clearly, approaches to field theories which are nonperturbative from the outset and which incorporate many of the important symmetries of QCD are in demand.

In this thesis we present a nonperturbative solution of the σ-model which was originally proposed in the late 50s as a phenomenological description of the dynamics of nucleons and mesons. In our version of the model the fermions are interpreted as quarks which interact via the sigma and pi mesons. The model exhibits an approximate $SU(2)\times SU(2)$ chiral symmetry which is understood as a low energy consequence of QCD. We use the Renormalization Group to study the behavior of the model as we evolve from a high to a low momentum scale and as chiral symmetry is both

spontaneously and explicitly broken. The results show a marked improvement over the perturbative calculation and are consistent with experiment and other nonperturbative calculations such as chiral perturbation theory and lattice gauge theory.

By way of introduction we first give a résumé of the salient features of gauge theories and QCD with an emphasis on the QCD origin of the approximate chiral symmetries of nuclear interactions. We review some of the many approaches to QCD including lattice gauge theory, chiral perturbation theory and Bag Models. Chiral symmetry is then introduced in the context of the linear sigma model.

We next review the Renormalization Group idea first with a heuristic example drawing from the contrast between the *hydrodynamic* and the *statistical* continuum limit. For physical systems in which the microscopic behavior *does not* sufficiently decouple from the macroscopic behavior, the *de facto* use of the hydrodynamic limit fails both qualitatively and quantitatively. These systems require the use of the RG for an understanding beyond that provided by the hydrodynamic continuum limit or mean field theory.

The historical development of the RG idea is reviewed with sufficient depth to put the present work into context. The seeds of the RG are found in the work of Dyson and others on the renormalization theory of the first viable quantum field theory: Quantum Electrodynamics (QED). This is traced through the work of Gell-Mann and Low in the early 50s up to Kadanoff's spin blocking in statistical systems in the late 60s. The ensuing revolution in both field theory and statistical physics inspired by Wilson's work in the early 70s is discussed. Recent work on the RG is reviewed with an emphasis on the development of approximation schemes to the highly complicated exact RG equations in order to extract useful results.

Next we discuss the derivation of the exact RG equations and the differential

flow equations under various approximations for model field theories containing both bosonic and fermionic degrees of freedom. The RG flow equations for the boson and generalized Yukawa effective potentials to leading order (LO) in the derivative expansion (DE) are derived in detail and compared with previously published results. The derivation of the σ-model flow equations is outlined and the results, which are quite lengthy, are catalogued in an appendix.

We present the numerical solution of the LO flow equations for the Yukawa coupled fermions and the σ-model treating the field variables and the momentum scale as independent continuous variables. The results for the flow of the boson and fermion Yukawa couplings are in agreement with those previously published. The results for the σ-model include the calculation of $\pi\pi$ scattering lengths which are an improvement on the old perturbative calculation and essentially in agreement with experiment.

Conventions and technical derivations of many results used in the text are presented in the appendices.

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Chapter 1

INTRODUCTION AND LITERATURE REVIEW

It is still unclear whether or not the strong interactions are comprehensible to the human mind. The question persists mainly because of the complicated nature of the currently accepted theory of the strong interactions, Quantum Chromodynamics (QCD) which to a large extent resists calculation. Why is it accepted? There are a number of reasons. Where calculations are possible there is excellent agreement with experiment; in addition, the theory brings together a number a previously disparate phenomena and is a cousin, a close blood cousin, to the most successful theory in the history of science, Quantum Electrodynamics (QED). So whence the original assertion, the question of the intelligibility of the strong interactions? Many physicists would take issue, pointing to the successes where the theory is tractable and argue that the understanding of the strong interaction is at hand—we know the underlying dynamics don't we?—what remains is merely mop-up work, getting the numbers out. There is, however, an alternative point of view that invests this "mop-up" work, in light of many examples from the history of science, with an urgency not experienced by most janitors, and which supplies the motivation for the research done for this thesis. The fact is, until the numbers are out for a convincing array of phenomena, we don't know if we have the right theory. Ptolemists used this argument to refute Copernican astronomy and largely won until they all died out. It took a century before computation with the heliocentric model could compete and surpass those of the intricate and arcane system of Ptolemy. More sophisticated attempts at calculation with a theory always either further the evidence for its adoption or

succeed in finding some chink in the armor from which to hang a new outlook. The resistance of the perihelion motion of Mercury to explanation under Newton's theory (so successful in most every other respect) is a conspicuous recent example. No matter how convinced some might be of the success of a physical theory it never hurts to keep calculating and even though the situation in QCD is, perhaps, less severe than these historic examples, still its resistance to over two decades of concerted effort to extract analytic results begs the question: do we really understand the strong interactions?

The calculations presented in this thesis may be useful in addressing this problem. The crux of the difficulty hinges on the fact that the most experimentally accessible manifestation of the strong interactions, nuclear phenomena, are nonperturbative in origin. Historically, perturbation theory provided the first venue for understanding and computing the consequences of quantum field theory. In effect, physicists are still getting over this initial brush with near perfection in the double headed success of theory and experiment in QED.[1] In QED the bound states are essentially nonrelativistic. Relativistic effects can be computed as perturbations on the nonrelativistic bound states. Thus it was not necessary to know about the full interacting field theory to quantitatively understand most aspects of atomic and solid state physics. For the nuclear interaction, the picture is quite the opposite. The experimental entities that we see, nucleons and nuclei for example, are fully relativistic bound states of QCD, thus from the outset all the machinery of quantum field theory is necessary. As discussed in the succeeding section, perturbation theory can be used for the strong interaction but only at high momentum transfer where the strong coupling becomes small. And, alas, these high energies are far from nuclear bound states. Nonrela-

[1]Theory and experiment are in agreement to 11 decimal places in QED [1]. Feynman [2] has compared this level of accuracy to knowing the distance between New York and Los Angeles to within the width of a human hair (50 microns). Also see Ref.[3] p.348.

tivistic bound states do exist in QCD but only for the "heavy-flavored" quarks which sadly play no role in nuclear phenomena. So understanding the nucleon and nuclei in terms of QCD involves understanding QCD where it is hardest to get analytic results. What ensues then is an industry similar to many parts of condensed matter theory where the underlying dynamics are assumed to be known and the resulting emergent phenomena are pursued by a mix of model building and first principles techniques.

The difficulties encountered in understanding the strong interaction are not unique and the technique employed in this thesis has many and varied applications. On a very general level the *Renormalization Group* (RG) is a way of understanding physical theories independent of dynamics on the basis of symmetries and scaling. This states accurately both its power and its limitations. For instance, the RG is useful for computing the critical exponents of statistical systems near second order phase transitions. But although one can calculate the values of these exponents very accurately using RG techniques, this says nothing about the underlying dynamics. Indeed it is another success of the approach, since *Universality*, the property that all systems that exhibit second order phase transitions look precisely the same at the critical point, is well known and well tested experimentally. Since the experimenters and theorists are both interested in the determination of the critical exponents and not the dynamics, this does not amount to much of an issue for the understanding of phase transitions. In the strong interactions, however, both dynamics and symmetry play a role to an extent which remains to be unentangled. The RG is useful in identifying the role of the symmetries in the emergence of nuclear phenomena, but it has little to say about dynamics. The dynamics of QCD are completely determined (as we'll discuss in detail in the next section) by the principle of *gauge invariance*. But in the nuclear domain it is another symmetry of QCD that is paramount and that

provides one focus for this thesis. Unlike gauge invariance which has its origins, perhaps, in the deep geometrical properties of spacetime, the *chiral symmetry* of QCD is *accidental* in the sense that it arises (as presently understood) from the sizes (relative to a characteristic energy scale associated with QCD) of the quark masses which are *inputs* to the theory. In this thesis we present the RG solution of a model field theory (the σ-model) which manifests chiral symmetry. This aspect of the strong interactions has been exploited since before QCD but here, using the RG, we track for the first time the flow of the various parameters in the σ-model as the theory evolves from high to low momentum and as chiral symmetry is broken both explicitly and spontaneously. Since this closely mimics the behavior of chiral symmetry breaking in QCD it is an important test case regardless of the specific dynamics. As a bonus (as will be discussed in detail later) the model also closely resembles the dynamics of the low energy degrees of freedom of QCD. Also, being an RG approach, the results transcend perturbation theory and as such are valid where many other approaches can give only tenuous extrapolations.

In this chapter we begin with a brief summary of QCD and gauge theories in general. Then some various approaches to understanding QCD are discussed including the first principles *Lattice Gauge Theory* and other more phenomenological models. Chiral symmetry is introduced as an approximate symmetry of QCD. Finally, we discuss the σ-model and the concepts of spontaneous and explicit symmetry breaking arise naturally in this context. Contact is made between the pion mass parameter in the σ-model and the u and d quark masses in QCD. Throughout the discussion of QCD and the σ-model, references to the literature are provided and the presentation is meant as both a pedagogical introduction and a literature review. Finally, the last section of the chapter introduces the RG approach to field theories in a general

context. An autonomous literature review of the RG is provided separate from the previous discussion of strong interaction models.

In the second chapter are collected all the derivations following a discussion of the Leading Order (LO) or Local Potential Approximation (LPA) to the RG equations. The general case is considered first; this gives a general equation that can be used to compute leading order RG flow equations for any model by the specification of the action. The flow equations are derived for scalar (ϕ^4), scalar plus Yukawa coupled fermion, and σ-models. In the case of the σ-model we obtain in limiting cases the results for the other models obtained earlier in the chapter.

In the third chapter we discus the numerical methods used to solve the flow equations. We present results for each of the models treated in the previous chapter with an emphasis on the σ-model. Further discussion of the results for the σ-model and possible ramifications for QCD-inspired models in nuclear physics are discussed. Future extensions to the calculations are discussed as well. Seven appendices contain a list of the conventions used in this thesis and derive various results used in the text.

1.1 A Résumé of QCD

Quantum Chromodynamics or QCD is the theory of interacting quarks and gluons believed to underlie all hadronic and nuclear phenomena. It is truly the fundamental theory of matter;—you, the reader, are comprised of over 99% hadrons! There is a strong family resemblance between the *gauge theories* QCD and QED. In fact, one easy tack for understanding the basics of gauge theories is to consider the more familiar case of QED first. In the following, we present a cursory treatment of gauge theories and outline the basic ingredients of QCD to motivate the use of chiral models in the nonperturbative solution of QCD.

Consider the free Dirac Lagrangian for the electron:

$$\mathcal{L}_{Dirac} = \overline{\psi}(i\not{\partial} - m)\psi, \tag{1.1}$$

where $\not{\partial} = \gamma^\mu \partial_\mu$, γ^μ = Dirac gamma matrices (see Appendix A), and m is the electron mass. From Eq.(1.1) one uses standard techniques to derive the equations of motion for the relativistic electron (the Dirac Equation) and to derive corrections to nonrelativistic quantum mechanics such as the hyperfine splitting of the levels of the hydrogen atom.[2] The free particle solutions can also be viewed as predictions of the existence of the positron, the antiparticles of electrons. But there is nothing in Eq.(1.1) referring to photons. The gauge construction is an elegant way to get from $\mathcal{L}_{Dirac}$ to an interacting theory of electrons and photons or QED.

Consider the effect of changing the phase of the Dirac field at each point in space,

$$\psi(x) \longrightarrow \mathrm{e}^{-i\theta(x)}\psi(x), \tag{1.2}$$

on $\mathcal{L}_{Dirac}$,

$$\mathcal{L}_{Dirac} \longrightarrow \mathcal{L}'_{Dirac} = \overline{\psi}(i\not{\partial} + \not{\partial}\theta - m)\psi. \tag{1.3}$$

Notice $\mathcal{L}'_{Dirac} \neq \mathcal{L}_{Dirac}$: the invariance has been spoiled by the spacetime dependence of the phase $\theta(x)$. To reconcile this we can define a new derivative, a covariant derivative, $D_\mu = \partial_\mu - ieA_\mu$ where the new field transforms as,

$$A_\mu \longrightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\theta. \tag{1.4}$$

Notice that D_μ is invariant with respect to the transformation Eq.(1.2); e is an ar-

[2]see, for example, Ref[4]

bitrary parameter associated with the coupling of the Dirac field to the new vector field A_μ. Thus the modified Dirac Lagrangian is

$$\mathcal{L}_{Dirac}^{mod} = \overline{\psi}(i\not{D} - m)\psi \tag{1.5}$$

and is invariant with respect to Eq.(1.2). $\mathcal{L}_{Dirac}^{mod}$ cannot be the total Lagrange density for a physically interesting theory, however, since we've included no dynamics for the vector field A_μ. The simplest choice that is still invariant with respect to Eq.(1.2) would be a term proportional to $F_{\mu\nu}^2$ where[3]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.6}$$

Once we interpret A_μ as the photon field, the proportionality constant can be chosen to get the conventional normalization of the Maxwell equations. Then we have the Lagrangian for QED:

$$\mathcal{L}_{QED} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}^2. \tag{1.7}$$

So, beginning with a theory of free electrons and following the path of the symmetry transformation Eq.(1.2) we've arrived at an interacting theory of electrons and photons. The transformation Eq.(1.2), called a *gauge transformation*, has guided us towards this enlargement of the theory and the specification of the dynamics. Notice that Eq.(1.4) is just the expression of the fact that the vector potential is arbitrary up to shifts by the gradient of a scalar function, as is familiar from classical electrodynamics.

[3]There are other gauge invariant but nonrenormalizable terms that can be added to Eq.(1.5), such as a Pauli term. The dual of $F_{\mu\nu}$ is gauge invariant and renormalizable but violates parity (P) and time reversal (T) symmetry. The photon mass term, $\frac{1}{2}m_\gamma(A_\mu)^2$, is not gauge invariant.

This construction was seen as an obscure though admittedly elegant arcanum until the early 70s when it became the centerpiece of what is now called the Standard Model of elementary particle interactions. The turning point was the proof by 't Hooft [5] that a certain class of gauge theories are indeed renormalizable. More specifically QCD can be viewed as a generalization of the above procedure. In general, we can consider transformations such as,

$$\mathbf{\Psi}(x) \longrightarrow \mathrm{e}^{-iT^a\theta^a(x)}\mathbf{\Psi}(x) \tag{1.8}$$

(with a sum on a assumed) where the $\{T^a\}$ are the generators of some continuous symmetry group parametrized by the set of $n \times n$ unitary matrices, $a = 1, \ldots, n^2 - 1$ and $\mathbf{\Psi}$ is now an n-column vector. Eq.(1.2) is special case of Eq.(1.8) where $n = 1$ and the symmetry group is $U(1)$, the phase transformations. Under the transformation Eq.(1.8) we have the covariant derivative,

$$\mathbf{D}_\mu = \partial_\mu \mathbf{1} - i\ g\ T^a A^a_\mu, \tag{1.9}$$

and

$$A^a_\mu \longrightarrow A^{a'}_\mu - \frac{1}{g}\partial_\mu \theta^a + h^{abc}\theta^b A^c_\mu. \tag{1.10}$$

The new penultimate term involving h^{abc}, called the *structure constants* of the symmetry group, arises since the generators $\{T^a\}$ don't commute,

$$[T^a, T^b] = i\ h^{abc}\ T^c, \tag{1.11}$$

and as a consequence the elements of the symmetry group don't commute and the group is called *nonabelian.* This is in contrast to the *abelian* $U(1)$ transformations in

QED. Just as in QED we need to add a kinetic term for the gauge field that respects Eq.(1.8), $(F^a_{\mu\nu})^2$ where now,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i\ g\ h^{abc} A^b_\mu A^c_\nu. \tag{1.12}$$

The Lagrange density invariant with respect to Eq.(1.8) is then,

$$\mathcal{L}_{Yang-Mills} = \overline{\mathbf{\Psi}}(i\not{\mathbf{D}} - \mathbf{m})\mathbf{\Psi} - \frac{1}{4}(F^a_{\mu\nu})^2. \tag{1.13}$$

In their original paper [6] Yang and Mills considered the $SU(2)$ gauge transformations where $\mathbf{T}^a = \tau^a/2$ and τ^a are the Pauli matrices and $h^{abc} = \epsilon^{abc}$, the Levi-Civita tensor. Here $\mathcal{L}_{Yang-Mills}$ is invariant with respect to $SU(n)$ gauge transformations.

It turns out that the relevant case for the strong interactions is the $SU(3)$ gauge transformations. The Lagrangian for QCD then looks just like $\mathcal{L}_{Yang-Mills}$ with all the group theory specialized to $SU(3)$, *i.e.* $T^a = \lambda^a/2$, $a = 1, \ldots, 8$ (with $\lambda^a =$ the generators of $SU(3)$, analogues of the Pauli matrices, the *Gell-Mann* matrices) and $h^{abc} = f^{abc}$ the structure constants of $SU(3)$. The gauge field-strength tensor is then,

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i\ g\ f^{abc} A^b_\mu A^c_\nu, \tag{1.14}$$

and the covariant derivative is,

$$\mathbf{D}_\mu = \partial_\mu \mathbf{1} - i\ g\ \frac{\lambda^a}{2} A^a_\mu. \tag{1.15}$$

Also we can generalize to n_f different kinds of fermions or *flavors* with the index f;

then,

$$\mathcal{L}_{QCD} = \sum_{f=1}^{n_f} \overline{\Psi}_f (i\not{D} - \mathbf{m}_f)\Psi_f - \frac{1}{4}(G^a_{\mu\nu})^2. \tag{1.16}$$

In QCD the fermions are interpreted as *quarks*,[4] elementary particles that carry fractional electric charge and a new tertiary internal quantum number, induced by the $SU(3)$ gauge transformations, called *color*. The quarks interact via their color "charge" mediated by the eight gauge bosons A^a_μ called *gluons*. In contrast to QED where the photons are neutral, the gluons carry color.

On the face of it the two Lagrangians, $\mathcal{L}_{QED}$ and $\mathcal{L}_{QCD}$ are deceptively similiar seeing as their consequences are so disparate. The difference turns on the similarity, the gauge structure of the two theories. Since $U(1)$ is an abelian group, the gauge boson A_μ carries no charge and thus there is no direct coupling of photons to each other in QED. As we saw, however, the $SU(3)$ gauge bosons A^a_μ carry color and this induces a nonlinearity in QCD, not present in QED, through the gluon self-interactions. Thus the behavior of quarks and gluons is very different from electrons and photons. Indeed were it not for the gluonic self-interactions, QCD would be a mere copy of QED with extra $SU(3)$ group theory factors and the visible world would be quite different.

It is this nonabelian structure that makes QCD a far richer and more complicated theory than QED. Consider the dielectric properties of the vacua of each of the theories. It is well known that the QED vacuum *screens* electric charge through the creation of a virtual cloud of electron-positron pairs around the electron. This is vacuum polarization. As a result the electromagnetic coupling constant, α, grows as one probes closer to the charge. It turns out that, as a consequence of gluonic self-interactions just the opposite happens in QCD where the vacuum *antiscreens*

[4]see Ref.[7] p.383.

color-charge. The coupling of colored quarks *decreases* as they approach closer to each other *i.e.* the strong coupling constant α_s decreases as one probes closer to the quark. This property is called *asymptotic freedom* [8] and serves as the basis for the understanding of deep inelastic scattering in terms of QCD.[5] So for high momentum transfer the QCD coupling shrinks and the theory becomes amenable to a perturbative treatment. On the other hand the coupling grows for small momentum transfers and this leads to the presumed exact *confinement* of quarks and gluons into color singlets, and thus color is forever buried in the bound states of the theory. This is in contrast to QED where electric charge is, of course, observable. Confinement has still not been derived as a rigorous consequence of QCD, but there are lattice calculations that indicate that, for strong enough gauge couplings, confinement arises as an inevitable consequence of nonabelian interactions [12]. Because of the gluonic self-interaction then there is an energy scale associated with confinement in QCD called Λ_{QCD}. This is the energy scale at which quark-gluon interactions become nonperturbatively strong. Roughly $\Lambda_{QCD} \simeq 2 - 300\text{MeV}$. The bound states of quarks and their properties, which presumably underlie all aspects of nuclear physics, are emergent phenomena of strong coupling QCD and so to understand these systems we must know how to do nonperturbative calculations in quantum field theory.

1.2 How To Solve QCD?

The discussion of the previous section revealed that understanding the strong interactions involves the quantitative nonperturbative solution of a quantum field theory. Whereas perturbation theory provided more than an adequate paradigm

[5] Deep Inelastic Scattering (DIS) was first understood in terms of the quark-parton model [9] and later corrections were computed using QCD [11].

for the solution of QED and its bearing out with experiment, QCD and the strong interaction is quite another story. Indeed the fundamental degrees of freedom of QCD are in principle not directly observable: we see no asymptotic color charge nor fermions with fractional electric charge. The familiar nuclear phenomena are an *emergent* property of QCD much as, say liquidity is an emergent property of the molecular dynamics of H_2O *en masse.*

To pursue this analogy further, in condensed matter physics it is not the microscopic dynamics, nonrelativistic quantum mechanics, nor is it the postulates of statistical mechanics that are in question in scientific investigation, it is, rather how these postulated underlying principles give way to the emergent macroscopic phenomena. In principle, the only necessary inputs for any condensed matter physics ought to be microscopic parameters such as atomic weights, ionic charges and so on. From these all the phenomena are derivable. Indeed this is many times possible but not always practical; but even in the cases where it is not practical to think in terms of first principles, it is still necessary to establish to some level of satisfaction that the emergent properties are not *inconsistent* with the tenets of quantum and statistical mechanics. To do otherwise would not be scientific. So the use of semi-quantitative phenomenological models ensues. These models are many times more useful in the sense that they more readily succumb to computation. It is pleasing when the models also embody certain important aspects of the underlying theory.

The situation in strong interaction physics, at present, is analogous to the foregoing discussion of condensed matter theory. There are strong indications from, *e.g.* hadron spectroscopy and the deep inelastic scattering of electrons off of the nucleon, that the fundamental degrees of freedom that give rise to all strong interaction phenomena are quarks interacting through $SU(3)$ gauge fields. But there is still a lacuna

Table 1.1. The quark masses and electric charges.

Quark Flavor	Mass	Electric Charge (e)
u	$2-8$MeV	$2/3$
d	$5-15$MeV	$-1/3$
s	$100-200$MeV	$-1/3$
c	$1-1.6$GeV	$2/3$
b	$4.1-4.5$GeV	$-1/3$
b	$170-200$GeV	$2/3$

of first principles understanding of most hadronic phenomena, particularly in the nuclear domain.

Empirically there appear to be six flavors of quarks in order of their relative masses: u (up), d (down), s (strange), c (charm), b (bottom), and t (top) [10]. (See Table 1.1.) Evidence for the heaviest, the top quark was just found recently [10]. Because of their differing masses relative the QCD scale $\Lambda_{QCD} \approx$ 2-300MeV), only the u, d and s quarks form bound states that appear in nuclear phenomena. States involving the heavy quarks, the c, b, and t (which all have masses much larger than Λ_{QCD}) have been created in accelerators but are so short lived as to play no significant role in nuclear processes. So in this thesis we will focus on the light sector of QCD and in particular the two lightest quarks, the u and d.

A whole host of models have grown up around QCD since the early 70s as well as a systematic first principles approach called Lattice Gauge Theory. Each model emphasizes certain parts of QCD at the expense of ignoring others. We first discuss the *ab initio* approach.

1.2.1 Lattice Gauge Theory

Lattice Gauge Theory was founded by Wilson [12] in 1974 and has developed into a field in it's own right.[6] The strategy is to put the points of spacetime on a lattice (almost always simple cubic). The quark flavor and color quantum numbers live on these points and the gauge fields live on the links between them. QCD is thus reduced to a statistical mechanics problem with a large but finite number of degrees of freedom. Various expectation values, which allow one to compute different aspects of the theory, can be related to the partition function which can be calculated with Monte Carlo techniques. The lattice constant hardwires in the momentum cutoff $\Lambda_0 \sim 1/a$ (where a is the lattice constant) so the theory is UV finite. This means that calculations are unencumbered by the apparatus of perturbative renormalization, familiar in QED. Lattice calculations are nonperturbative from the outset. The rub is in taking the continuum limit after the lattice simulation. This is still a technically complicated and nontrivial process. To give a recent example, the results for the glueball mass obtained by Weingarten and the UKQCD collaboration differed by a couple hundred MeV [14]. This difference (substantial in meson spectroscopy) can mostly be ascribed to the different techniques used in each case to take the continuum limit. Lattice QCD is promising and in principle, with enough computer firepower, will eventually solve many if not all pertinent aspects of QCD. But even in this eventuality it never hurts to have a few simple models to draw from. In many cases one can compute things on the lattice to fix free parameters in these models; thus the fitting is to *ab initio* calculations and not experiment.

[6] For an introduction see Ref. [13, 15].

1.2.2 Chiral Symmetry and QCD

So besides and in conjunction with lattice calculations it is useful to have several models in mind when considering the QCD basis of nuclear phenomena. As in the case with lattice gauge theory many of these QCD-inspired models have become industries. At the heart most of these models is the issue of how the quark-gluon degrees of freedom turn into the familiar baryon-meson degrees of freedom. It is still an open question what role the fundamental QCD degrees of freedom play in nuclear physics. At very low energies ($\tilde{<}$ 100 MeV) it is clear that the only relevant degrees of freedom in the strong interaction are nucleons and pions. This is the energy region where nonperturbative confinement is "done" and only confined degrees of freedom are present. At very high energies ($\tilde{>}$ a few GeV) we have a weakly interacting system of gluons and quarks. It is in the intermediate energy range (between roughly 100MeV and a GeV) where there is a question as to which degrees of freedom are most relevant. Before considering some models in detail we digress on the important issue of chiral symmetry.

Chiral symmetry was realized as an important aspect of nuclear phenomena even before the advent of QCD. As the name implies chiral symmetry involves the discrete symmetry of space reflection. More specifically, when we project out the components of a Dirac spinor of definite chirality, defined as

$$\psi_{L,R} = \frac{1}{2}(1 \mp \gamma^5)\psi, \tag{1.17}$$

a field theory is said to be chirally symmetric when the transformations

$$\psi_{L,R} \longrightarrow U_{L,R}\psi_{L,R} \tag{1.18}$$

leave the Lagrangian unchanged. That is, the theory is invariant with respect to separate global symmetry transformations on the left- and right-handed spinors. The spinors of different chirality in a chirally symmetric theory don't talk to each other in the sense that there is no interaction that changes (for example) a right-handed spinor to a left-handed one. Accordingly, their will be two conserved Noether currents associated with each of these symmetries,

$$j^{\mu a}_{L,R} = \overline{\psi}_{L,R}\gamma^{\mu}T^{a}\psi_{L,R} \tag{1.19}$$

where $\{T^a\}$ are again the generators of $SU(n)$. A Lagrangian with these symmetries is termed symmetric under $SU_L(n) \times SU_R(n)$ transformations and the corresponding Noether currents are conserved,

$$\partial_{\mu} j^{\mu a}_{L,R} = 0 \tag{1.20}$$

The obvious example of a chiral fermion is the neutrino which is always left-handed. There is no evidence for right-handed neutrinos. Neutrinos are also massless and it turns out that this is essential for chiral invariance. Consider a mass term for a Dirac fermion in terms of the left- and right-handed components:

$$\overline{\psi}m\psi = \overline{\psi}_L m\psi_L + \overline{\psi}_R m\psi_R + \overline{\psi}_R m\psi_L + \overline{\psi}_L m\psi_R. \tag{1.21}$$

Since we can write

$$\begin{aligned} \psi_L &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \phi_L \\ \psi_R &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \phi_R \end{aligned} \tag{1.22}$$

where $\phi_{L,R}$ are the two component Weyl spinors and $\overline{\psi}_{L,R} = \psi^{\dagger}_{L,R}\gamma^0$, the two symmetric terms vanish,

$$\overline{\psi}_L m\psi_L = \overline{\psi}_R m\psi_R = 0; \tag{1.23}$$

so the mass term becomes,

$$\overline{\psi}m\psi = \overline{\psi}_R m\psi_L + \overline{\psi}_L m\psi_R. \tag{1.24}$$

This shows that a generic fermion mass term mixes states of definite helicity in with each other. Thus massive fermions must always come in both left- and right-handed varieties. Another way of saying this is that in chirally symmetric theories the number of left- and right-handed particles are separately conserved as expressed in Eq.(1.20).

What has this to do with QCD? From Eq.(1.16) it is obvious that the theory is not chirally symmetric since there is a mass term for the quarks. But as we have said the three quarks pertinent to nuclear phenomena are light with respect to the QCD scale. This is an empirical observation; the masses presumably come from the Higgs mechanism, but even at this level they are inputs in the form of Yukawa couplings of the Higgs boson to the quarks;[7] we will take them as given. Since the u and d quarks are light, this sector of QCD is *approximately* chirally symmetric. One can begin with the massless theory and add the masses as a perturbation.

Consider the u-d part of $\mathcal{L}_{QCD}$ (Eq.(1.16)), which can be written as,

$$\mathcal{L}^{ud} = \overline{u}i\partial\!\!\!/ u + \overline{d}i\partial\!\!\!/ d - \mathcal{L}_{\chi B} \tag{1.25}$$

$$\begin{aligned} \mathcal{L}_{\chi B} &= \overline{u}m_u u + \overline{d}m_d d \\ &= \frac{1}{2}(m_u + m_d)(\overline{u}u + \overline{d}d) + \frac{1}{2}(m_d - m_u)(\overline{d}d - \overline{u}u). \end{aligned} \tag{1.26}$$

[7] Discussion of the Higgs mechanism can be found in many texts. A lucid account is in Ref. [16].

If $\mathcal{L}_{\chi B} = 0$ then $\mathcal{L}^{ud}$ has the full $SU_L(2) \times SU_R(2)$ symmetry discussed above (as well as $U_L(1) \times U_R(1)$ symmetries). If $m_u = m_d = M \neq 0$ notice that,

$$\mathcal{L}_{\chi B} = \frac{1}{2}(m_u + m_d)(\overline{u}u + \overline{d}d) = M(\overline{u}u + \overline{d}d). \tag{1.27}$$

In this case the $SU_L(2) \times SU_R(2)$ symmetry is broken but Eq.(1.27) shows that $\mathcal{L}^{ud}$ is still symmetric under the interchange of the u and d quarks; this is just a symmetry under $SU(2)$ rotations of $\begin{pmatrix} u \\ d \end{pmatrix}$. The original motivation for using the $SU(2)$ isospin symmetry in the strong interaction was the apparent symmetry with respect to the interchange of the proton and the neutron. This u-d isospin symmetry in $\mathcal{L}^{ud}$ for $m_u = m_d$ is the modern understanding of this in terms of quarks. (Since the proton is a uud state and the neutron is a udd state it's clear that $u \leftrightarrow d \Longleftrightarrow p \leftrightarrow n$.) So even if m_u and m_d are equal but not zero there is a symmetry in $\mathcal{L}^{ud}$. How is this related to chiral $SU_L(2) \times SU_R(2)$? Notice that the Noether current for the u-d interchange symmetry is,

$$j^{\mu a} = \overline{\psi}\gamma^\mu \frac{\tau^a}{2}\psi = j_L^{\mu a} + j_R^{\mu a}, \tag{1.28}$$

where $j_{L,R}^{\mu a}$ are the $SU(2)$ versions of Eq.(1.19). We can also define the difference and write it as,

$$j^{\mu 5a} = j_R^{\mu a} - j_L^{\mu a} = \overline{\psi}\gamma^\mu\gamma^5 \frac{\tau^a}{2}\psi. \tag{1.29}$$

We term the symmetries represented by the conservation of $j^{\mu a}$ and $j^{\mu 5a}$ $SU_V(2)$ and $SU_A(2)$ respectively for "vector" and "axial vector". They correspond to the invariance of the Lagrangian with respect to,

$$\begin{aligned} \psi &\longrightarrow U_{V,A}\psi \\ U_V &= e^{-i\frac{\vec{\tau}\cdot\vec{\theta}}{2}} \simeq 1 - i\frac{\vec{\tau}\cdot\vec{\theta}}{2} \end{aligned}$$

$$U_A = e^{-i\gamma^5 \frac{\vec{\tau}\cdot\vec{\beta}_5}{2}} \simeq 1 - i\gamma^5 \frac{\vec{\tau}\cdot\vec{\theta}_5}{2}, \tag{1.30}$$

where $\vec{\theta}$ and $\vec{\theta}^5$ are the infinitesimal versions of $\vec{\beta}$ and $\vec{\beta}^5$. The $SU_L(2) \times SU_R(2)$ symmetry is equivalent to $SU_V(2) \times SU_A(2)$. This decomposition of the left- and right-handed currents into V and A pieces is useful since it lumps all the symmetry breaking into one current,

$$\begin{aligned} \partial_\mu j^{\mu a} &= 0 \\ \partial_\mu j^{\mu a 5} &\neq 0, \end{aligned} \tag{1.31}$$

whereas for $m_u = m_d = 0$ both currents are conserved. Since the u and d masses are both small and approximately equal in relation to the QCD scale ($m_u, m_d \tilde{<} 10$ MeV$\ll \Lambda_{QCD} \simeq$2-300MeV) we expect the strong interaction to be nearly symmetric under $SU_L(2) \times SU_R(2)$. The symmetry under $SU_V(2)$ is consistent with the experimental evidence of the multiplet pattern of the hadrons (*e.g.* (p, n), (π^+, π^0, π^-), etc.). Indeed the "Eightfold Way" [17] of the sixties was merely the extension of $SU_V(2)$ to $SU_V(3)$ to include states with strange quarks. But if $SU_A(2)$ is a good symmetry of the strong interactions then for each of the multiplets of $SU_V(2)$ there ought to be sisters with opposite parity. This is patently not the case. The smallness of the pion mass, however, indicates that if $SU_A(2)$ is broken, it is only broken weakly. (We'll show this in the next section.) Thus QCD has the approximate chiral symmetry expressed by the conservation of the vector current and the "Partial Conservation of the Axial Current" or PCAC.

Before QCD, PCAC was one linchpin in understanding the strong interaction. The history of strong interaction physics highlights the importance of symmetries

in nature. Without knowing the dynamics of quarks and gluons (or even what the fundamental degrees of freedom were), physicists in the 60s developed many of the rudiments of chiral nuclear dynamics. These were systematized in the *current algebra* method [18] which takes as premises the commutation relations of partially conserved currents such as $j^{\mu 5a}$. From the algebra, theorems can be proved about the S-matrix for soft (low energy) pion processes. Today we understand the phenomenology of soft pion physics and PCAC encoded in current algebra to be a consequence of the smallness of the masses of the up, down and strange quarks with respect to the QCD scale Λ_{QCD}.

A coherent picture has evolved of the relationship of QCD to the physics of the low energy degrees of freedom. This is the *chiral perturbation theory* [19, 20]. These are effective field theories which take as inputs the small masses of the quarks and other experimentally fixed parameters and predict processes involving the effective or dressed low energy degrees of freedom of QCD. The phenomenological Lagrangians used are nonrenormalizable; but the theory is predictive beyond tree level in the sense that to any given order in the perturbation, only a finite number of parameters need be fixed and effects from higher orders are suppressed by factors proportional to Λ_{QCD}/M_q where M_q is of the order of one of the heavy quark masses.

It is interesting to remark that any field theory can be regarded as effective in this sense. In QED, for instance, there are corrections associated with the muon (the next heavier lepton after the electron) proportional to m_e/m_μ that are suppressed since the muon is much heavier than the electron. The entire Standard Model can be viewed as an effective field theory of the low energy degrees of freedom of some other theory, perhaps strings. The corrections to the Standard Model associated with the true underlying theory would be proportional to, say m_t/M_{Planck}, where the top

quark mass, $m_t \sim 170\text{GeV}$ and the *Planck mass*, $M_{Planck} = (\hbar c/G)^{1/2} \sim 10^{19}\text{GeV}$; obviously these corrections are heavily suppressed! The impressive predictive power of the Standard Model is more a consequence of the size of the masses of the quarks and the leptons in relation to the Planck scale than the fact that the gauge theories that are used are renormalizable.

Chiral perturbation theory has burgeoned into an industry for computing a variety of low energy quantities in QCD. It's results are presumably a rigorous consequence of low energy QCD. Despite its successes it has inherent limitations. It does not address the issue of confinement. All chiral perturbation results are in terms of already-confined degrees of freedom such as nucleons and pions. Confinement is not a consequence solely of chiral symmetry but results from the nonlinear quark-gluon dynamics. Also, as one includes more loops more parameters appear (since the theory is nonrenormalizable). At some point this proliferation of counter-terms becomes unwieldy. Finally, chiral perturbation theory has little or nothing to say about the origin of spontaneously broken chiral symmetry of the strong interactions—an issue we'll shortly take up.

1.2.3 Models of QCD

We now briefly discuss two general classes of models of QCD. A comprehensive review and guide to the literature is given in Ref. [21]. We dwell on these two because they provide examples of, on the one hand, an unabashedly phenomenological and useful model of QCD, and on the other, an unrealistic but exact limit of QCD. Since it is impossible to solve QCD at all energy scales and "watch" the quarks and gluons become confined we have to skirt the issue and do what we can with models and available limiting solutions.

One class of models that incorporate confinement from the outset are the *Bag Models* [22, 21]. Since we see no asymptotic color-charge, the QCD vacuum evidently *excludes* chromoelectric fields. This is analogous to the *Meissner Effect* where magnetic fields are excluded from the interior of superconductors. The analog of the normal conducting state is the perturbative QCD vacuum. The picture is then of weakly interacting (asymptotically free) quarks inside a "bag" of radius R. For $r \leq R$ the vacuum is treated as in perturbative QCD and for $r > R$ we have the nonperturbative QCD vacuum. Just as Cooper pairs form in superconductors, quark-antiquark pairs (mesons) form in the nonperturbative QCD vacuum "shielding" the outside of the bag from chromoelectric fields. The hallmark of this is the formation of a nonzero expectation value, $<0|\overline{q}q|0>$ where $q = u, d, s$. The baryons interact weakly through the mesons outside the bag while inside the quarks and gluons interact weakly. One has enough parameters to do various fits. An example is given in Ref. [23] p.291 where a simultaneous fit to the N (nucleon), Δ (first excited state of the nucleon), and π (pion) states and a chiral constraint are used to determine all the bag parameters. This gives the quark wavefunctions which are then used to compute various matrix elements. Bag models provide a simple picture where many calculations are easily done and the results are in good agreement with many experimental parameters. Bag models, however, slyly avoid the intermediate energy range in between the asymptotically free quarks and the baryons and mesons saying nothing about the mechanism of confinement. Also, it is not clear if there is a single well defined radius that demarks the transition from the quark-gluon to the baryon-meson phase. Lorentz invariance is hard to implement as well with the sharp coordinate-space radius.

Another "model" involves an exact limit of QCD and provides some basis for the assumption that its low energy degrees of freedom are baryons and mesons. This is

the large N_c limit (where N_c = number of colors). We noted in section 1.1 that the $SU(3)$ gauge symmetry of QCD means that quarks come in three varieties of color. 't Hooft [24] originally came up with the idea of using $1/N_c$ as an expansion parameter in QCD. Witten [25] applied the idea to baryons which are bound states of three quarks. Since the real world presumably has just three colors, the first question is what can be understood in terms of the unrealistic limit $N_c \to \infty$? The conclusion is that many of the salient features of low energy QCD become apparent.[8] For example, in this limit QCD becomes a theory of weakly coupled mesons. The baryons have mass $\sim N_c$ (it takes N_c quarks to make a color singlet state) and therefore are infinitely heavy and interact strongly with each other. This is all consistent with nuclear phenomenology and is encouraging as one of the few exact analytic results obtained from QCD (albeit a *limit* of QCD). As such it also provides the basis for understanding the origin of the solitonic models of the baryon (such as the Skyrme model [27]) in terms of QCD.

In this thesis we use a model that incorporates chiral symmetry to compute, using the renormalization group (RG), the "flow equations" that arise when the theory evolves from a high to a low momentum scale. It is a nonperturbative solution of a model that mirrors the chiral symmetry of QCD. Before detailing the RG approach to field theories, we outline some of the trenchant aspects of this model and take up the issue of spontaneous symmetry breaking in field theory.

1.3 σ-models and Spontaneous Chiral Symmetry Breaking

We now consider a specific model field theory, the celebrated σ-model of Gell-Mann and Levy [28]. This is a renormalizable field theory invented to address the incorporation of chiral symmetry into strong interaction physics long before QCD. Lee

[8] For a recent review see Ref.[26].

[29] works out the perturbative renormalization of this model but here we consider the manifestation and aspects of the different symmetries which the theory obeys. As originally conceived the σ-model was a theory of interacting nucleons and mesons. Here we will be interested in the RG solution of this model and what this says about the role of chiral symmetry in medium energy QCD. Thus we will think of the fermions in the model as quarks which interact through mesons.

The Lagrangian for the σ-model is,

$$\mathcal{L} = \overline{\psi}\Big[i\partial\!\!\!/ + g(\sigma + i\gamma^5\vec{\tau}\cdot\vec{\pi})\Big]\psi + \frac{1}{2}\Big[(\partial_\mu\sigma)^2 + (\partial_\mu\vec{\pi})^2\Big] - V(\rho) \tag{1.32}$$

where $\rho^2 = \sigma^2 + \vec{\pi}^2$. The scalar sector is comprised of the scalar-isoscalar σ and the pseudoscalar-isovector $\vec{\pi}$ with components π^i, $i = 1, 2, 3$ (isovectors are indicated with an arrow). The $\{\tau^i\}$ are the $SU(2)$ Pauli matrices. The interaction of the scalars with the fermions is chosen to be chirally symmetric so that $\mathcal{L}$ is chirally symmetric. We list these symmetries:

$\mathcal{L} \to \mathcal{L}$ under,

- $SU_V(2)$:

$$\begin{aligned} \psi &\longrightarrow \Big(1 - i\frac{\vec{\tau}\cdot\vec{\theta}}{2}\Big)\psi \\ \sigma &\longrightarrow \sigma \\ \vec{\pi} &\longrightarrow \vec{\pi} + \vec{\theta}\times\vec{\pi} \end{aligned} \tag{1.33}$$

- $SU_A(2)$:

$$\begin{aligned} \psi &\longrightarrow \Big(1 - i\gamma^5\frac{\vec{\tau}\cdot\vec{\theta}_5}{2}\Big)\psi \\ \sigma &\longrightarrow \sigma - \vec{\theta}_5\cdot\vec{\pi} \end{aligned} \tag{1.34}$$

$$\vec{\pi} \longrightarrow \vec{\pi} + \vec{\theta}_5 \sigma$$

- $U_{V,A}(1)$:

$$\begin{aligned} \psi &\longrightarrow e^{-i\alpha}\psi \\ \psi &\longrightarrow e^{-i\gamma^5\alpha_5}\psi. \end{aligned} \tag{1.35}$$

Besides these continuous *global* symmetries $\mathcal{L}$ has also the discrete symmetries of P (parity), T (time reversal), and C (charge conjugation). The quantum theory generated by Eq.(1.32) is renormalizable as long as $V(\rho)$ is at most quartic in the meson fields [29]. Notice that the $SU_A(2)$ transformation mixes the σ and the $\vec{\pi}$ fields and that for $SU_V(2)$ the pion field is just $O(3)$ rotated. We include the $U_{V,A}(1)$ symmetries just for completeness but will have little to say about them. If we include both the $SU(2)$'s together, as discussed in the previous section, the symmetry is equivalent to $SU_L(2) \times SU_R(2)$; we'll call this simply $SU(2) \times SU(2)$. As a consequence of this $SU(2) \times SU(2)$ symmetry there are two conserved Noether currents which are readily computed as,

$$\begin{aligned} \vec{j}^{\mu} &= \overline{\psi}\gamma^{\mu}\frac{\vec{\tau}}{2}\psi + \vec{\pi} \times \partial_{\mu}\vec{\pi} \\ \vec{j}^{\mu 5} &= \overline{\psi}\gamma^{\mu}\gamma^{5}\frac{\vec{\tau}}{2}\psi - \vec{\pi}\partial_{\mu}\sigma + \sigma\partial_{\mu}\vec{\pi}. \end{aligned} \tag{1.36}$$

The equations of motion are obtained through the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_{\mu}\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi^A)} = 0 \tag{1.37}$$

(where $\phi^A = \{\overline{\psi}, \psi, \sigma, \vec{\pi}\}$) as,

$$
\begin{aligned}
g\overline{\psi}(\sigma + i\gamma^5\vec{\tau}\cdot\vec{\pi}) - (\partial_\mu\overline{\psi})i\gamma^\mu &= 0 && (1.38)\\
g(\sigma + i\gamma^5\vec{\tau}\cdot\vec{\pi})\psi + i\gamma^\mu\partial_\mu\psi &= 0 && (1.39)\\
g\overline{\psi}\psi + \frac{\sigma}{\rho}V'(\rho) - \partial_\mu^2\sigma &= 0 && (1.40)\\
g\overline{\psi}i\gamma^5\vec{\tau}\psi + \frac{\vec{\pi}}{\rho}V'(\rho) - \partial_\mu^2\vec{\pi} &= 0. && (1.41)
\end{aligned}
$$

With these in hand one can easily check that the currents are indeed conserved,

$$\partial_\mu\vec{j}^\mu = \partial_\mu\vec{j}^{\mu 5} = 0; \tag{1.42}$$

thus L has all the symmetries of $\mathcal{L}^{ud}$ with $m_u = m_d = 0$. In addition the mesons correspond to those seen in nature although the σ has a broad resonance. What about the broken $SU_A(2)$?

One way to *explicitly* break $SU_A(2)$ symmetry is to introduce a mass term for the fermions. Equivalently we can add a term $\propto \sigma$,

$$\mathcal{L} \longrightarrow \mathcal{L} - c\sigma. \tag{1.43}$$

Since this changes Eq.(1.40), but none of the others, we have,

$$
\begin{aligned}
\partial_\mu\vec{j}^\mu &= 0\\
\partial_\mu\vec{j}^{\mu 5} &= c\vec{\pi}. && (1.44)
\end{aligned}
$$

When the symmetry is broken by a term added to $\mathcal{L}$ we assume that $\mathcal{L}$ itself is still symmetric. Symmetries can be broken in another way. It's possible that the ground

state that nature chooses does not exhibit the symmetry of the Lagrangian. This is termed "spontaneous (or hidden) symmetry breaking." An example is the Curie magnet which chooses a state of macroscopic magnetization at $T = 0$ even though the Hamiltonian is rotationally symmetric. For the σ-model consider a potential of the form:

$$V(\rho) = B^2(\rho^2 - A)^2, \tag{1.45}$$

with A and B arbitrary parameters. For $A < 0$ the minimum of $V(\rho)$ is at $\rho = 0$ and the corresponding quantum theory will have a chirally symmetric vacuum state. But for $A > 0$, $\rho = 0$ is unstable. The real vacuum will be at either $\rho = \pm\sqrt{A}$. This is the usual "Mexican Sombrero" potential which is still symmetric with respect to $SU(2) \times SU(2)$ but excitations about the minima in the brow of the hat will not be symmetric and these minima define the symmetry of the vacuum upon which we build the quantum theory. The potential for $A > 0$ (shifting to one of the minima, $\sigma' = \sigma - \sqrt{A}$, $\vec{\pi}' = \vec{\pi}$) becomes,

$$V(\rho) = V(\sigma', \vec{\pi}) = B^2\Big[\rho^{4'} - 4\sqrt{A}\sigma'\rho'^2 + 4\sigma'^2 A\Big]. \tag{1.46}$$

Also the fermion bilinear in $\mathcal{L}$ is,

$$g\overline{\psi}(\sigma + i\gamma^5\vec{\tau}\cdot\vec{\pi})\psi + \overline{\psi}g\sqrt{A}\psi, \tag{1.47}$$

from Eqs.(1.46) and (1.47) we can read off the masses of the pion and the fermion,

$$\begin{aligned} m_\pi &= 0 \\ M &= -g\sqrt{A}. \end{aligned} \tag{1.48}$$

Thus in this unsymmetric vacuum the fermions acquire mass in spite of the fact that a mass term violates chiral symmetry. This is because even though $\mathcal{L}$ is chirally symmetric the vacuum is not. The pion, however, is still massless. Indeed this is a specific example of a general result known as *Goldstone's Theorem* [30] which states that the spontaneous symmetry breakdown of a continuous symmetry is always accompanied by the appearance of massless bosons. In the case of the σ-model, the pions are the Goldstone bosons.

If we include the $-c\sigma$ term in V then the minimum is given by,

$$\begin{aligned} 0 &= \frac{\partial V}{\partial \sigma} = 4\sigma B^2(\rho^2 - A) - c \\ 0 &= \frac{\partial V}{\partial \pi^i} = 4\pi^i B^2(\rho^2 - A), \end{aligned} \tag{1.49}$$

which has solutions,

$$\begin{aligned} \sigma_{min}^2 &= \frac{c}{4B^2\sigma_{min}} + A \\ \pi_{min}^i &= 0. \end{aligned} \tag{1.50}$$

As before we substitute $\sigma' = \sigma - \sigma_{min}$ and $\pi^{i'} = \pi^i$. Now the fermion mass is,

$$M = -g\sigma_{min}, \tag{1.51}$$

and the pion mass is given by,

$$V(\sigma, \vec{\pi}) = \cdots + \frac{1}{2}\frac{c}{\sigma_{min}}\vec{\pi}^2 + \cdots \tag{1.52}$$

as,

$$m_\pi^2 = -\frac{c}{\sigma_{min}}. \tag{1.53}$$

So in the case of both *spontaneous* and *explicit* chiral symmetry breaking both the fermions *and* the pions acquire mass.

We can make a closer connection with phenomenology by considering the definition of the *pion decay constant*, f_π defined through the parametrization of the matrix element,

$$<0|j^{\mu 5i}(x)|\pi^j>= ik^\mu f_\pi \delta^{ij} \mathrm{e}^{ik\cdot x}. \tag{1.54}$$

For weak pion decay the measured value is $f_\pi \simeq 92$MeV [10]. The form of the left-hand side is a result of the fact that $j^{\mu 5i}$ cannot change isospin, and the physical vacuum has even parity (so no γ^5 appears). Now differentiating using $k^2 = -m_\pi^2$ we have,

$$\begin{aligned} <0|\partial_\mu j^{\mu 5i}(x)|\pi^j> &= m_\pi^2 f_\pi \delta^{ij} \mathrm{e}^{ik\cdot x} \\ &\underset{x=0}{\longrightarrow} m_\pi^2 f_\pi \delta^{ij}. \end{aligned} \tag{1.55}$$

From Eqs.(1.44,1.53) we have $m_\pi^2 f_\pi = c = -m_\pi^2 \sigma_{min}$ and thus $\sigma_{min} = -f_\pi$ and $M = g f_\pi$ and Eq.(1.44) becomes,

$$\partial_\mu \vec{j}^{\mu 5} = m_\pi^2 f_\pi \vec{\pi}. \tag{1.56}$$

In the σ-model we see that for $m_\pi \to 0$ the conservation of the axial current is restored. For spontaneously broken $SU_A(2)$ the theory has massless pions (which are the Goldstone bosons) and $\vec{j}^{\mu 5}$ is still conserved. But when we explicitly break $SU_A(2)$ through the $c\sigma$ term, then the pions acquire mass and $\vec{j}^{\mu 5}$ is no longer conserved. Since

$|\partial_\mu \vec{j}^{\mu 5}| \propto m_\pi^2$, to the extent that m_π is "small" (relative to, say $M_{nucleon} \sim 1\text{GeV}$) we can consider the axial current to be only "partially" broken, *i.e.* its divergence is nearly zero. Real pions have mass $m_\pi^2 \simeq (139\text{MeV})^2 \ll 1\text{GeV}$ and therefore it appears that this scheme for chiral symmetry breaking in the σ-model might be close to the mark.

This can be understood in terms of QCD by the following argument. Recall Eq.(1.26) which is the Lagrangian for the u-d sector of QCD. In the limit of very light u-d quarks we can simply ignore the heavier quarks which appear to have infinite mass and are therefore frozen out. The pion π^+ has the quantum numbers of a $\overline{d}u$ bound state, so consider the parametrization of the transition matrix element from the vacuum to the one pion state,

$$<\pi^+(k)|\overline{u}\gamma^\mu\gamma^5 d|0>= ik^\mu f_\pi e^{ik\cdot x}, \tag{1.57}$$

(just as in Eq.(1.54) but with no isospin, for simplicity). The divergence is simply,

$$<\pi^+(k)|\partial_\mu(\overline{u}\gamma^\mu\gamma^5 d)|0>= m_{\pi^+}^2 f_\pi e^{ik\cdot x}. \tag{1.58}$$

Also using Eq.(1.26) we have,

$$\partial_\mu(\overline{u}\gamma^\mu\gamma^5 d) = \overline{u}\gamma^5 d\ i(m_u + m_d). \tag{1.59}$$

Using Eqs.(1.58) and (1.59) we can relate the pion mass to the sum of the quark masses:

$$m_{\pi^+}^2 = \frac{1}{f_\pi}<\pi^+|\overline{u}\gamma^5 d|0>e^{ik\cdot x} i(m_u + m_d). \tag{1.60}$$

If we parametrize the pseudoscalar matrix element by,

$$<\pi^+(k)|\overline{u}\gamma^5 d|0>= -ig_\pi e^{ik\cdot x}, \tag{1.61}$$

then,

$$m^2_{\pi^+} = \frac{g_\pi}{f_\pi}(m_u + m_d). \tag{1.62}$$

The scenario is then that the small (relative to Λ_{QCD}) u and d quark masses give rise to the small pion mass and this is the reason why $SU(2) \times SU(2)$ is only an approximate symmetry of QCD. This picture can be extended to $SU(3) \times SU(3)$ symmetries [31]. In that case the symmetry is more strongly broken since the strange quark mass is so large (roughly $m_s \sim \Lambda_{QCD}$—see Table 1.1). This is seen in the masses of the Goldstone bosons of the $SU(3) \times SU(3)$ symmetry breaking, the heaviest of which (the η) is 4 times as heavy as the pion. A very useful review of departures from chiral symmetry in the strong interactions appears here as Ref. [32].

1.4 RG Approach to Field Theories

Perturbation theory has been both a boon and a bane to theoretical physicists for a long time now. In many cases, nature gives us an easily identified small parameter which can be used in a power series expansion with each successive term contributing less and less and so arbitrary accuracy is obtained with a finite amount of labor. The claim is then made that, since the number calculated is consistent with experiment, we understand the physics. Weakly correlated statistical systems and quantum field theories with small couplings are examples of such gifts from nature. As has been discussed above the success and failure of perturbation theory is well exemplified in QED and QCD respectively. Also, statistical systems with strong correlations, *e.g.* a

ferromagnet near the Curie point, defy a perturbative treatment. Another approach is clearly needed.

Based on the evolution of physics over many length scales, the Renormalization Group (RG) is ideally suited for studying problems nonperturbatively. But, as with many aspects of life, the power of the nonperturbative description carries with it responsibilities. The exact RG equations one derives are analytically intractable (at present) and so we must use an approximate solution. Still, these approximate RG equations appear to retain the essence of the nonperturbative behavior (as will be discussed in detail in the next chapter). In this section, after a brief introduction we present a survey of the literature on the RG drawing heavily from Wilson's review articles [33, 34] for the history up until 1971. For further details see Wilson's instructive introduction in the review article [34].

1.4.1 What is The Renormalization Group?

Wilson characterizes the RG as a tool used to study a particular kind of continuum limit in physics. He makes the distinction between two different kinds of continuum limits, one very familiar and the other one not so familiar. The familiar one is the well known *hydrodynamic limit.* For the purposes of a hydrodynamic description one assumes that fluids (such as a glass of water or the atmosphere) are *continuous.* One then thinks of this continuum as being parametrized in some way by the microscopic details—which need not be well understood.[9] Effectively, one is "ignoring" the physics below a certain length scale by taking some sort of average. The idea is that in the *limit* of large distances (as compared with the size of the constituent particles) the details of the small distance behavior do not effect the physics

[9] *I.e.* the classical trajectories of the particles are too numerous and complex to be accounted for, even though everything is in principle generated by Newton's second law.

at large distances. For much of nature this description works very well; indeed this sort of continuum limit underlies much science and engineering. Historically, progress in the science of ever shortening length scales throughout the 19th and 20th centuries has only been possible to the extent that for most systems of interest in classical physics the large distance behavior is relatively decoupled from the harder to understand (and probe) small distance behavior. Luckily each of the successive layers in nature's onion (thus far) have been easily pulled apart and have had little to do with each other or the underlying core.

The less familiar continuum limit Wilson calls the "statistical continuum limit," and this is the limit that one uses the RG to understand. In the hydrodynamic continuum limit described above one replaces, say, the information about the positions and momenta of all the particles in a box of suitable size with, say, the continuous functions $\rho(\mathbf{x})$, $T(\mathbf{x})$, and $P(\mathbf{x})$, which one calls the "density," "pressure," and "temperature" respectively, of the fluid. One treats $\mathbf{x}$ as a continuous variable and defines things like $\nabla\rho(\mathbf{x})$, the "derivative" of ρ at the point $\mathbf{x}$. The existence of the continuum limit is in this way the basis for the existence of the derivative. Wilson describes the statistical continuum limit as the analogue of the hydrodynamic limit with the continuous variable, $\mathbf{x}$, replaced by a continuously varying field, $\phi(\mathbf{x})$, which is then treated at each point in space as a continuous variable. Now instead of computing the value of the functions $\rho(\mathbf{x})$, $T(\mathbf{x})$, and $P(\mathbf{x})$, one computes correlation functions, $<\phi(\mathbf{x})\phi(0)>$. In quantum field theories these become vacuum expectation values.

Now imagine the critical phenomena of the water-steam system. A glass of water at room temperature is well described by hydrodynamics. All the fluctuations of the water molecules below a certain length scale are happily ignored and supplanted by the continuous functions discussed in the previous paragraph. Now heat the water

until it boils and place it under pressure keeping it at the boiling point. When a pressure and temperature of 218 atm and 374 degrees C is reached [35] the whole boiling phenomena vanishes—this is the so-called critical point where a second order phase transition takes place. At these pressures and temperatures, bubbles of steam and droplets of water exist on all length scales from angstroms to centimeters. This is because the difference in density of the liquid and gas phase has gone to zero and surface tension can no longer be supported. There is simply no way to understand this system quantitatively within the context of classical hydrodynamics since there is no length scale appropriate over which to make averages. Near a second order phase transition fluctuations occur on *all* length scales and thus couple the small and large distance physics;—those fluctuations occurring on the order of microns scatter visible light leading to "critical opalescence."

To understand the water-steam critical point and others like it one uses the RG. One starts at the smallest length scale in the problem and describes the dynamics with a Hamiltonian, $\mathcal{H}(\Lambda)$, which is parametrized by Λ, the inverse length scale and is a functional of, say, Ω_q, the Fourier components of the field variables. Now take a step back from the system and squint, such that the characteristic inverse length scale is $\Lambda/2$ so that all the modes Ω_q, with $|q| > \Lambda/2$ have been "integrated out" (or more loosely, "averaged over"). Imagine doing this a large number of times so that many "equivalent" Hamiltonians are generated,

$$\mathcal{H}(\Lambda), \quad \mathcal{H}(\Lambda/2), \quad \ldots, \mathcal{H}(\Lambda/2^n), \ldots. \tag{1.63}$$

This the kernel of Kadanoff's block spin idea [36]. Each of the Hamiltonians in this series describe an *equivalent* physical system in that they each give the same dynamics for large enough length scales (small enough Λ). If instead of the discrete

transformation, $\Lambda \to \Lambda/2$ at each step, we have an infinitesimal one, $\Lambda \to \Lambda - \Delta\Lambda$, then one obtains an exact RG equation in differential form,

$$\frac{\partial \mathcal{H}}{\partial \Lambda} = F[\mathcal{H}], \tag{1.64}$$

which describes the flow of the Hamiltonian with inverse length or *momentum* scale. Since this is a description of the change of the dynamics with the change in the momentum scale itself, it obviates the need for any length scale in the problem and is thus ideally suited for studying problems with fluctuations on all length scales or—in Fourier space—on all momentum scales.

So one sees that for systems with no well defined length scale, *i.e.* fluctuations on *all* length scales, the RG and the statistical continuum limit supersede the normal derivative and the hydrodynamic continuum limit. The analogue of the usual derivative in hydrodynamics is the RG equation (1.64). In quantum field theories we describe the flow of the dynamics in terms of the *action*, $S^{(\Lambda)}$, which is a functional of the fields and is parametrized by the momentum scale, Λ. The focus of the present work is the derivation and numerical solution of equations like (1.64) for field theories with bosonic and fermionic degrees of freedom that manifest chiral symmetry.

1.4.2 Literature Review—RG

The first intimation of the RG idea occurs with the foundations of renormalization theory in QED during the late forties and early fifties in the work of Dyson [37], Stueckelberg and Petermann [38], and Gell-Mann and Low [39]. Dyson [37] found that he could improve the convergence of the perturbation series by isolating the high frequency terms and summing those, instead of the entire radiation interaction. He motivated the need for renormalization in QED by likening the matter and radi-

ation fields to "fluids" with quantum fluctuations that must be "averaged" over in order to give meaning to calculations and even invoked an analogy with hydrodynamics. Stueckelberg and Petermann [38] found that the relationship between different re-parametrizations of the bare electron charge and mass, e_0 and m_0, and the renormalized quantities, e and m, could be characterized by transformation groups; they called these groups, "groupes de normalization," which translates to "renormalization groups." Gell-Mann and Low [39], in a much deeper study of QED at small distances, derive a differential equation for the electron charge as a function of momentum; this equation is the forerunner of the exact RG equation (1.64) studied in this thesis.

As it stood, however, from the work of Gell-Mann and Low until the late 60s, the RG appeared more as an esoteric technicality than as a fundamental calculational tool. In the early 70s, the work of Wilson [40] and Wilson and Fisher [41] broke the ice in critical phenomena. Wilson, inspired by Kadanoff's block spin hypothesis [36], found an RG based scheme for quantitative calculations and computed critical exponents using an expansion about $\epsilon = 4 - d$ where d is the dimensionality of space [42]. This technique, called the ϵ-expansion, is cogently reviewed by Wilson and Kogut in Ref. [43]. Needless to say the 70s saw an explosion of work in critical phenomena using the RG techniques pioneered by Kadanoff, Wilson, Fisher and others. The developments are reviewed in detail by Wilson in Ref.[33] which is the lecture delivered in Stockholm on the occasion of the presentation of the 1982 Nobel Prize in Physics.

The RG has also seen widespread application in quantum field theory both as an intuitive and more compelling way to understand and prove renormalization without referring to perturbation theory [44] and as a powerful and heuristic calculational tool. The use of RG techniques to calculate the β-function in QCD, for example, was

paramount in the discovery of asymptotic freedom [8].[10] The RG can also be used to improve the speed of lattice QCD calculations. Here, one uses the RG equation to evolve the action before handing off to the full lattice to continue the calculation. This idea was pioneered by Wilson [12] and has since been developed and used by the whole lattice community. The intriguing connection between field theory and statistical mechanics is given a lucid early discussion in section 10 of Ref.[43]. A more modern and comprehensive treatment is given, *e.g.*, in the treatise by Zinn-Justin [46].

Turning now to the more recent developments regarding the exact RG equation, we note that it was first derived independently by Wilson and Kogut [43] and Wegner and Houghton [47][11] in two different but equivalent ways. Wilson and Kogut derive their equation using a so-called "smooth cutoff" between integrated and unintegrated modes. Their rationale was to avoid nonlocalities that naturally arise in position space when one makes sharp divisions in momentum space. The price paid for this approach is that smoothing functions must be introduced to facilitate the integrations. These smoothing functions complicate the equations and make their solution—even after drastic approximations[12] —impossible analytically. Using the so-called "sharp-cutoff" method of Wegner and Houghton [47] is simpler in that there are no smoothing functions to worry about, but nonlocal effects rear their ugly head and have to be faced. Another problem with this approach is that there appear to be ambiguities in

[10]The hallmark of asymptotic freedom is the ***negative*** sign in front of the first term of the β-function, defined as

$$\beta(\lambda) = \lambda \frac{d\alpha(\lambda)}{d\lambda}, \tag{1.65}$$

where α is the coupling and λ is the momentum scale. See, for example, Ramond [45] p.378 or Itzykson and Zuber [3] p. 657.

[11]Also see Ref. [48].

[12]*E.g.* the leading order (LO) in the derivative expansion: see the following paragraph and references cited therein.

the form of the RG equations that depend on a variable change at a particular point in the derivation.[13] At least to leading order (LO) in the derivative expansion (DE) there is no real difficulty with sharp cutoff ambiguities.

One can imagine many different ways to approximate the exact RG equations. A natural approximation scheme involves expanding the action in powers of momentum, or—in real space—in powers of derivatives of the fields. The leading order (LO) approximation is then obtained by setting all of the momenta in the problem equal to zero. (This is sometimes called the "local potential approximation" (LPA).) The next to leading order (NLO) result is obtained by keeping one small momentum and so on. The LO approximation to the full RG equations appears first in the work of Nicoll *et al.* [49]. Hasenfratz and Hasenfratz [50], in an early seminal work, showed that the RG equations to LO in the DE give interesting and nontrivial results. They solve for the RG flows of the effective potential in a pure scalar theory and extract critical exponents to compare with calculations performed by other means.[14] For $d = 3$ they find impressive agreement for the critical exponents ν and ω. Also for $d = 4$ they find no nontrivial fixed point solution which is consistent with the triviality of ϕ^4 field theories.[15] Their work showed that even the apparently amputative LO approximation gives a rich quantitative description of the critical properties of scalar field theories.

There has been a recent resurgence of interest in the RG effective action approach to scalar field theories. Hasenfratz and Nager [56] study the cutoff dependence of the

[13]These issues are discussed in more detail in chapter 2

[14]The "other means" are: high precision Monte Carlo RG (MCRG) analysis [51], field theory calculations [52], and high temperature expansions [54]. For recent MCRG calculations see Ref.[53].

[15]A field theory is "trivial" if, after renormalization, all of the couplings that generate interactions vanish leaving a free field theory in the limit of infinite cutoff. For in depth and rigorous discussions of triviality in ϕ^4 field theories see, *e.g.*, Ref.[55]. For an early discussion see section 13 of Ref.[43].

Higgs mass using RG methods. Wetterich and co-workers [57], [58] essentially extending the work of Hasenfratz and Hasenfratz to NLO use a smooth cutoff procedure to derive their RG equations. This allows them to include the effects of wavefunction renormalization and compute the exponent η which they find to be in rough agreement with calculations done by other means.[16] Morris [59] studys approximations to the exact RG as derived using both smooth and sharp cutoffs and in Ref.[60] computes exponents to NLO noting that the scheme appears to converge. In Ref.[61] the so-called sharp cutoff ambiguities are treated in detail and the DE of the effective action is reviewed in [62]. Alford [63], with an eye toward the Electroweak phase transition, computes exponents with a sharp cutoff procedure and discusses some of the practical difficulties involved with extending the calculations to NLO. Shepard *et al.* [64] discuss a possible bridge between MCRG and continuous RG (C-RG) calculations including lattice artifacts via a "latticized RG" (L-RG). They find impressive agreement to LO between their L-RG and lattice Monte Carlo (MC) results. Some recent related work appears here as Refs. [65], [66], [67].

Also the problem of including fermions in the model has been addressed in the literature recently. Clearly this must be faced if the technique is to have any impact on real life calculations in low energy nuclear physics or on Standard Model issues. The study of the dynamics of fermions with lattice calculations and MCRG has, from its onset, presented well known and difficult problems.[17] So perhaps even more than in the pure scalar case, the benefit of an approximation scheme to exact *continuous* RG equations cannot be overestimated.

Maggiore [68] includes fermions via a generalized Yukawa term and derives LO flows for the scalar and generalized Yukawa potentials. He finds no evidence that the

[16] See note 14.
[17] See Ref.[13].

fixed point structure of the theory is affected by the addition of fermions. In a more comprehensive study, Clark and co-workers [69] derive the exact RG equations for theories of arbitrary field content and derive from this equation the LO flows for the scalar and generalized Yukawa potentials which agree with the results of Ref.[68]. In two related papers Clark *et al.* study the issue of computing mass bounds for scalars and fermions in the standard model [70] and the stability of fine tuned hierarchies [71]. In related work, Ellwanger and Vergara [72] use RG flow equations for generalized NJL models to study the Higgs top quark system to leading order in $1/N_c$. Other work pertaining to the Higgs top system appears as Ref.[73]. Jungnickel and Wetterich [74] compute RG flows for a model action with chirally symmetric fermions coupled to pseudoscalar and scalar bosons via a Yukawa term and extract the pion decay constant, f_π. Hasenfratz and co-workers [75] have presented MCRG calculations of systems with both strongly and weakly coupled Yukawa fermions in the quenched approximation. For strong Yukawa couplings they find a rich phase structure possibly including the existence of a tricritical point indicating the coexistence of ferromagnetic, paramagnetic, and antiferromagnetic phases.

Chapter 2

DERIVATION OF THE EXACT RG EQUATIONS AND APPROXIMATIONS

As stated in section 1.4, the derivation of the exact RG equations for *scalar* only theories was first presented in Refs.[43] and [47]. Shortly after the work of Hasenfratz and Hasenfratz [50], Maggiore [68], beginning with a LO action (presented below as Eq.(2.7)) derived flow equations for a scalar plus fermion system including the fermions via a generalized Yukawa potential. We take the more systematic approach of Clark *et al.* [69] which involves delving into the formalism of supermatrices (this is reviewed in Appendix B). The exact RG equation for an action with *arbitrary* field content is derived. Then with the specification of an action and its degrees of freedom, we begin the series of approximations that deliver us to the RG flow equations for the three models: ϕ^4 scalar field theory, ϕ^4 plus Yukawa coupled fermions and the σ-model.

As mentioned above, we make an approximation in order to solve the RG equation. This approximation is the leading order (LO) approximation in the derivative expansion. Although it appears that the next-to-leading order (NLO) is ill-defined [61], the LO RG equations describe accurately much of the nonperturbative physics [49, 50, 64]. This assertion is based on the results for the LO RG flow equations compared with Monte Carlo and other calculations. Ref. [64] contains a detailed comparison between the results for LO flow equations, Monte Carlo calculations and Schwinger-Dyson equations. The authors conclude that the results for the LO flow equations far surpass those using the Schwinger-Dyson equations in their agreement

with the Monte Carlo potentials. With this in mind we discuss only LO results below.

But to understand the nature of the LO approximation we can introduce it in terms of the derivative expansion. For concreteness consider the example of the pure scalar theory in d Euclidean dimensions. The derivative expansion of the action then takes the form,

$$\begin{aligned} S^{(\Lambda)} &= \int d^d x \left[V^{(\Lambda)}(\phi) + \frac{1}{2} Z^{(\Lambda)}(\phi)(\partial_\mu \phi)^2 \right. \\ &\quad + Y_1^{(\Lambda)}(\phi)(\partial_\mu \phi)^4 + Y_2^{(\Lambda)}(\phi)(\partial_\mu \partial^\mu \phi)^2 \\ &\quad \left. + Y_3^{(\Lambda)}(\phi)(\partial_\mu \partial^\mu)^2 \phi + \cdots \right] \end{aligned} \tag{2.1}$$

where the functions V, Z, Y_1, Y_2, Y_3, ... all depend on the cutoff, Λ, and the scalar field, ϕ which has cutoff dependence itself. (For brevity we will often leave off the superscript and argument indicating this). The form of the terms in Eq.(2.1) is entirely determined by symmetry, in this case Lorentz invariance. If one retains all of the terms in the expansion, the action is considered exact. This would be equivalent to keeping all of the terms in a functional Taylor expansion.[1] The first, and most drastic approximation one could imagine making, would be to set all of the coefficients, Z, Y_1, Y_2, Y_3, ... to zero leaving only the scalar potential, V. This approximation is not very interesting, however, as it cuts out the entire kinetic part of the theory leaving

[1] *I.e.* Eq.(2.1) is an alternative to the standard expansion in terms of connected one-particle irreducible (1PI) n-point functions,

$$S = \sum_n \frac{1}{n!} \int d^d x_1 \cdots d^d x_n \Gamma^{(n)}(x_1 \cdots x_n) \phi(x_1) \cdots \phi(x_n) \tag{2.2}$$

where,

$$\Gamma^{(n)}(x_1 \cdots x_n) = \left. \frac{\delta^n S}{\delta\phi(x_1) \cdots \delta\phi(x_n)} \right|_{\phi=\phi_0}. \tag{2.3}$$

nothing but the potential. The next, more illuminating approximation would involve setting $Y_1, Y_2, Y_3, \ldots = 0$ and $Z = 1$ (*i.e.* keeping Z constant and picking the constant to be unity). This is termed the leading order (LO) or Local Potential approximation (LPA),

$$S_{LO}^{(\Lambda)} = \int d^d x \left[V^{(\Lambda)}(\phi) + \frac{1}{2}(\partial_\mu \phi)^2 \right]. \tag{2.4}$$

Plugging this action into the exact RG equation (derived in the next section) leads to a flow equation for $V^{(\Lambda)}$ with respect to Λ. Similarly the next to leading order (NLO) approximation is obtained by setting Y_1, Y_2, $Y_3, \ldots = 0$ but retaining $Z^{(\Lambda)}$ (*i.e.* allowing Z to *flow* with Λ),

$$S_{NLO}^{(\Lambda)} = \int d^d x \left[V^{(\Lambda)}(\phi) + \frac{1}{2} Z^{(\Lambda)}(\phi)(\partial_\mu \phi)^2 \right]. \tag{2.5}$$

Now one derives *two* coupled equations for the functions $V^{(\Lambda)}$ and $Z^{(\Lambda)}$.

For theories of interacting fermions and bosons the above procedure generalizes. We describe the fermions with Dirac spinors, ψ, suppressing the spinor indices. Picking a generalized Yukawa interaction we have for our NLO action,

$$S_{NLO}^{(\Lambda)} = \int d^d x \left[V^{(\Lambda)}(\phi) + \frac{1}{2} Z_\phi^{(\Lambda)}(\phi)(\partial_\mu \phi)^2 + U^{(\Lambda)}(\phi)\overline{\psi}\psi + Z_\psi^{(\Lambda)}(\phi)\overline{\psi}\partial\!\!\!/\psi \right]. \tag{2.6}$$

Now the LO form is obtained by setting $Z_\phi^{(\Lambda)}(\phi) = Z_\psi^{(\Lambda)}(\phi) = 1$ giving,

$$S_{LO}^{(\Lambda)} = \int d^d x \left[V^{(\Lambda)}(\phi) + \frac{1}{2}(\partial_\mu \phi)^2 + U^{(\Lambda)}(\phi)\overline{\psi}\psi + \overline{\psi}\partial\!\!\!/\psi \right]. \tag{2.7}$$

For the σ-model we'll extend the scalar sector to $O(4)$ with $\vec{\phi} = (\sigma, \pi^i)$ and the fermions will be $SU(2) \times SU(2)$ symmetric spinors. The forms of V and U will allow

for the explicit and spontaneous breaking of chiral symmetry.

2.1 General Case

We first present a derivation of an exact RG equation for an arbitrarily complicated quantum field theory described by the action, $S^{(\Lambda)}[\Phi]$, regulated at some large momentum cutoff Λ. The field content of the theory can in principle be anything at all, $\Phi = \{\phi, \psi, A_\mu, F_{\mu\nu}, \ldots\}$ *i.e.* fields described by complex scalars, spinors and/or tensors, though we will only be concerned with theories containing scalars and spinors. We define the effective action at the momentum scale $\Lambda - \Delta\Lambda$, $S^{(\Lambda-\Delta\Lambda)}$ as,[2]

$$e^{-S^{(\Lambda-\Delta\Lambda)}} = \int_{shell} \mathcal{D}\overline{\Phi}\mathcal{D}\Phi \; e^{-S^{(\Lambda)}}, \tag{2.10}$$

where the subscript "*shell*" indicates that only the Fourier components of the fields $\{\Phi_q\}$ with momenta in the shell, $\Lambda - \Delta\Lambda < |q| < \Lambda$ are integrated over. Thus the notation for the measure means,

$$\mathcal{D}\overline{\Phi}\mathcal{D}\Phi = \prod_{q_1} d\overline{\Phi}_{q_1} \prod_{q_2} d\Phi_{q_2}, \tag{2.11}$$

[2]The *generating functional*, W for connected Green's functions is given by,

$$e^{-W^{(\Lambda)}[J]} = \int \mathcal{D}\overline{\Phi}\mathcal{D}\Phi \; e^{-S^{(\Lambda)}(\Phi) - \int d^dx \; J\cdot\phi} \tag{2.8}$$

where $\mathcal{D}\overline{\Phi}\mathcal{D}\Phi = \prod_{q_1 q_2} d\overline{\Phi}_{q_1} d\Phi_{q_2}$ for $0 < |q_1|, |q_2| < \Lambda$. Thus the effective action becomes the generating functional for a field theory (regulated at some cutoff) in the thick shell limit $\Delta\Lambda \to \Lambda$,

$$S^{(\Lambda-\Delta\Lambda)} \longrightarrow W\Big|_{J\to 0}. \tag{2.9}$$

for $\Lambda - \Delta\Lambda < |q_1|, |q_2| < \Lambda$. $\overline{\Phi}$ is the "conjugate" of Φ (*e.g.* complex conjugate for the complex scalar field ϕ, Dirac adjoint for the Dirac spinor ψ, Hermitian conjugate for matrix fields *etc.*). The actions $S^{(\Lambda-\Delta\Lambda)}$ and $S^{(\Lambda)}$ are considered *equivalent* in the sense that for $|q| \ll \Lambda$ they each give the same n-point functions. We now decompose $\Phi(x)$ into uniform and nonuniform pieces,

$$\begin{aligned} \Phi(x) &= \Omega_0 + \Omega(x) \\ \Omega(x) &= \sum_{q\neq 0} \Omega_q e^{iq\cdot x}. \end{aligned} \tag{2.12}$$

Expanding $S^{(\Lambda)}$ in a functional Taylor series about the uniform field components gives,

$$\begin{aligned} S^{(\Lambda)}[\overline{\Phi},\Phi] &= S^{(\Lambda)}[\overline{\Omega}_0,\Omega_0]\Big|_0 + \sum_{q\neq 0}{}' \left[\frac{\delta S}{\delta\Omega_q}\Big|_0 \Omega_q + \overline{\Omega}_q \frac{\delta S}{\delta\overline{\Omega}_q}\Big|_0\right] \\ &\qquad + \sum_{q_1,q_2\neq 0}{}' \overline{\Omega}_{q_1} \frac{\delta^2 S}{\delta\overline{\Omega}_{q_1}\delta\Omega_{q_2}}\Big|_0 \Omega_{q_2} \\ &= S^{(\Lambda)}\Big|_0 + \overline{\mathcal{J}}\Omega + \overline{\Omega}\mathcal{J} + \overline{\Omega}M\Omega. \end{aligned} \tag{2.13}$$

The subscript zero means that all modes, Ω_q, with q in the shell are set to zero; and the primed summation symbol indicates that only momenta in the shell are included in the sum. For the purposes of deriving differential equations with respect to the independent variable Λ we will eventually take the limit $\Delta\Lambda \to 0$, thus we can truncate the series after the quadratic term *without approximation* since all higher order contributions to the RG flows will be at least $\mathcal{O}(\Delta\Lambda^2)$. In the second equality

several definitions have been made:

$$\overline{\mathcal{J}} \equiv \begin{pmatrix} \frac{\delta S}{\delta \Omega_{q_1}}\big|_0 \\ \frac{\delta S}{\delta \Omega_{q_2}}\big|_0 \\ \vdots \end{pmatrix}, \tag{2.14}$$

and its conjugate are the "generalized source" column vectors and the supermatrix[3] of second derivatives is defined as,

$$M \equiv \begin{pmatrix} \Sigma & \mathcal{A} \\ \overline{\mathcal{A}} & \mathcal{F} \end{pmatrix}. \tag{2.15}$$

Also, where momentum subscripts are not present, matrix multiplication over the momentum indices is implied, *e.g.*,

$$\begin{aligned} \overline{\mathcal{J}}\Omega &= \left(\tfrac{\delta S}{\delta \Omega_{q_1}}\big|_0 \quad \tfrac{\delta S}{\delta \Omega_{q_2}}\big|_0 \quad \dots \right) \begin{pmatrix} \Omega_{q_1} \\ \Omega_{q_2} \\ \vdots \end{pmatrix} \\ &= \sum_{q \neq 0} \frac{\delta S}{\delta \Omega_q}\bigg|_0 \Omega_q. \end{aligned} \tag{2.16}$$

Now substituting Eq.(2.13) into Eq.(2.10) gives,

$$\begin{aligned} \mathrm{e}^{-S^{(\Lambda-\Delta\Lambda)}} &= \mathrm{e}^{-S^{(\Lambda)}|_0} \int \mathcal{D}\overline{\Omega}\mathcal{D}\Omega \; \mathrm{e}^{-(\overline{\Omega}M\Omega + \overline{\mathcal{J}}\Omega + \overline{\Omega}\mathcal{J})} \\ &= \mathrm{e}^{-S^{(\Lambda)}|_0} \{ \mathrm{e}^{\overline{\mathcal{J}}M^{-1}\mathcal{J}} \mathrm{sdet}^{-1} M \} \end{aligned} \tag{2.17}$$

where the result of Appendix B (Eq.(C.1)) has been used in the second line and the superdeterminant is defined and reviewed in Appendix B. Irrelevant constant factors

[3]See Appendix B for a review of the supermatrix formalism.

have been dropped wherever they appear. Using the identity,

$$\mathrm{sdet}^{-1}M = \mathrm{e}^{-\mathrm{str}\ \ln M}, \tag{2.18}$$

the exact RG equation for this generalized system is easily obtained as,

$$S^{(\Lambda-\Delta\Lambda)} = S^{(\Lambda)}\Big|_0 + \mathrm{strln}M - \overline{\mathcal{J}}M^{-1}\mathcal{J} + \mathcal{O}(\Delta\Lambda^2). \tag{2.19}$$

This equation relates the action at momentum scale $\Lambda - \Delta\Lambda$ to the action at momentum scale Λ. We will not convert it directly into a functional differential equation (as Clark *et al.* [69] do) until we consider its form for particular actions in the following sections. A terminology has grown up around equations such as (2.19). The "strln" term is generally referred to as the "loop" term and the "$\overline{\mathcal{J}}M^{-1}\mathcal{J}$" term is referred to as the "tree" term. This is because the contribution from the $\overline{\mathcal{J}}M^{-1}\mathcal{J}$ term is present in mean field theory whereas the contribution from the str ln term includes effects from loop integrations. See Ref.[44] for further discussion of why such terms are appropriate.

2.2 Bosons Only

We consider the form of Eq.(2.19) for theories of interacting bosons only. If we describe the bosons with a *complex* scalar field, ϕ, then the source and supermatrix become,

$$\mathcal{J} = \begin{pmatrix} \frac{\delta S}{\delta \phi^*_{q_1}}|_0 \\ \frac{\delta S}{\delta \phi^*_{q_2}}|_0 \\ \vdots \end{pmatrix} \equiv \{S'_q\} \tag{2.20}$$

and,

$$M = \Sigma = \{\Sigma_{q_1 q_2}\} = \begin{pmatrix} \left.\frac{\delta^2 S}{\delta\phi^*_{q_1}\delta\phi_{q_1}}\right|_0 & \left.\frac{\delta^2 S}{\delta\phi^*_{q_1}\delta\phi_{q_2}}\right|_0 & \cdots \\ \left.\frac{\delta^2 S}{\delta\phi^*_{q_2}\delta\phi_{q_1}}\right|_0 & \left.\frac{\delta^2 S}{\delta\phi^*_{q_2}\delta\phi_{q_2}}\right|_0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.21}$$

so that Eq.(2.19) becomes,

$$S^{(\Lambda-\Delta\Lambda)} = S^{(\Lambda)}\Big|_0 + \mathrm{trln}\Sigma - S'_{q_1}\Sigma^{-1}_{q_1 q_2}S'_{q_2}. \tag{2.22}$$

If we use instead a *real* scalar field such that $\phi^*_q = \phi_{-q}$ then factors of 1/2 are introduced[4] giving,

$$S^{(\Lambda-\Delta\Lambda)} = S^{(\Lambda)}\Big|_0 + \frac{1}{2}\mathrm{trln}\Sigma - \frac{1}{2}S'_{q_1}\Sigma^{-1}_{q_1 q_2}S'_{q_2}, \tag{2.23}$$

and this is equivalant to the RG equations derived in Refs.[43, 47, 50] for purely scalar field theories.

2.3 Generalized Yukawa Coupled Fermions

The derivation of the LO flows from Eq.(2.19) for a theory of interacting bosons and Yukawa coupled fermions is now considered. We begin with the LO ansatz for the action discussed at the beginning of the chapter,

$$S^{(\Lambda)} = \int d^d x \left[V^{(\Lambda)}(\phi) + \frac{1}{2}(\partial_\mu\phi)^2 + \overline{\psi}U^{(\Lambda)}(\phi)\psi + \overline{\psi}\partial\!\!\!/\psi\right]. \tag{2.24}$$

[4]See note 1 in Appendix C for details.

where ϕ is a real scalar field and ψ is a Dirac spinor. This choice of action means that we've chosen for our field space $\Phi = \{\phi, \psi\}$ (with $\overline{\Phi} = \{\phi, \overline{\psi}\}$). The source column vector becomes,

$$\overline{\mathcal{J}} = \begin{pmatrix} \overline{J} \\ \overline{\eta} \end{pmatrix} = \begin{pmatrix} J^*_{q_1} \\ J^*_{q_2} \\ \vdots \\ - \\ \overline{\eta}_{q_1} \\ \overline{\eta}_{q_2} \\ \vdots \end{pmatrix} \tag{2.25}$$

where,

$$\begin{aligned} J^*_q &\equiv \left.\frac{\delta S}{\delta \phi_q}\right|_0 \\ \overline{\eta}_q &\equiv \left.\frac{\delta S}{\delta \psi_q}\right|_0 . \end{aligned} \tag{2.26}$$

The matrix of second derivatives is,

$$M = \begin{pmatrix} \Sigma & \mathcal{A} \\ \overline{\mathcal{A}} & \mathcal{F} \end{pmatrix} = \begin{pmatrix} \left.\frac{\delta^2 S}{\delta \phi_{-q} \delta \phi_q}\right|_0 & \left.\frac{\delta^2 S}{\delta \overline{\psi}_{-q} \delta \phi_q}\right|_0 \\ \left.\frac{\delta^2 S}{\delta \phi_{-q} \delta \psi_q}\right|_0 & \left.\frac{\delta^2 S}{\delta \overline{\psi}_{-q} \delta \psi_q}\right|_0 \end{pmatrix} . \tag{2.27}$$

Now rewrite the superdeterminant appearing in Eq.(2.19) using some tricks from Appendix B (compare with Eq.(B.11)),

$$\mathrm{sdet}^{-1} M = (\det{}^{-1/2} N)(\det \mathcal{F}) \tag{2.28}$$

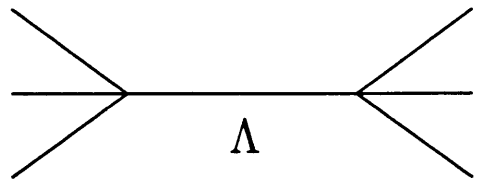

FIG. 2.1.

(where $N \equiv \Sigma - \overline{\mathcal{A}}\mathcal{F}^{-1}\mathcal{A}$) which gives,

$$\mathrm{strln}M = \frac{1}{2}\mathrm{trln}N - \mathrm{trln}\mathcal{F}. \tag{2.29}$$

(The 1/2 coming from the counting in the momentum sums due to the *real* ϕ.)[5] Now Eq.(2.19) becomes,

$$S^{(\Lambda-\Delta\Lambda)} = S^{(\Lambda)}\Big|_0 + \frac{1}{2}\mathrm{trln}N - \mathrm{trln}\mathcal{F} - \overline{\mathcal{J}}M^{-1}\mathcal{J}. \tag{2.30}$$

For the purposes of an LO derivation we can *ignore* the last ("tree") term in this equation. One way to see this diagrammatically is to note that any particular contribution to the tree term in Eq.(2.19) or (2.30) will be comprised of external legs connected by a single *internal* propagator with momentum in the shell (see Fig. 2.1). Since to LO we set the momentum of all external legs to zero, these diagrams cannot conserve momentum at their vertices and therefore they vanish. To make the argument analytically we write a particular term of S'_q down,

$$\frac{\delta S}{\delta\phi_{q'}}\Big|_0 = \cdots + \frac{1}{2!}V''' \sum_{q_1,q_2\neq 0} \phi_{q_1}\phi_{q_2}\delta_{q_1+q_2+q',0} + \cdots. \tag{2.31}$$

[5]See note 1 in Appendix C

where q' is in the shell. When, to LO we set all field components, ϕ_q, with nonzero momentum to zero, it's clear that this term and all higher order terms will vanish. Thus we write Eq.(2.30) as,

$$S^{(\Lambda-\Delta\Lambda)} = S^{(\Lambda)}\Big|_0 - \frac{1}{2}\mathrm{trln}N + \mathrm{trln}\mathcal{F}. \tag{2.32}$$

To facilitate the computation of the derivatives in Eqs.(2.25) and (2.27) we explicitly Fourier transform the action (2.24). Taylor expanding the potential about the uniform mode,

$$V^{(\Lambda)}(\phi) = V^{(\Lambda)}(\phi_0) + V'^{(\Lambda)}(\phi_0)\varphi + \frac{1}{2}V''^{(\Lambda)}(\phi_0)\varphi^2 + \cdots \tag{2.33}$$

(and similarly for U) where, following Eq.(2.12) we've defined,

$$\begin{aligned} \phi(x) &= \phi_0 + \varphi(x); \qquad \varphi(x) = \sum_{q\neq 0}\phi_q \mathrm{e}^{iq\cdot x} \\ \psi(x) &= \psi_0 + \chi(x); \qquad \chi(x) = \sum_{q\neq 0}\psi_q \mathrm{e}^{iq\cdot x}. \end{aligned} \tag{2.34}$$

We use periodic boundary conditions putting the modes in a box of volume, Vol, so that,

$$\int d^d x\ \mathrm{e}^{iq\cdot x} = (\mathrm{Vol})\delta_{q,0}. \tag{2.35}$$

Substituting these expressions into Eq.(2.24) and performing routine but tedious algebra leads to the expression,

$$\begin{aligned} \frac{S^{(\Lambda)}}{\mathrm{Vol}} &= \sum_{q\neq 0}\Big[\frac{1}{2}\phi_{-q}(q^2 + V''^{(\Lambda)} + \overline{\psi}_0\psi_0 U''^{(\Lambda)})\phi_q + \overline{\psi}_{-q}(-i\not{q} + U^{(\Lambda)})\psi_q\Big] \\ &\quad + V^{(\Lambda)} + \overline{\psi}_0\psi_0 U^{(\Lambda)} \end{aligned}$$

$$
\begin{aligned}
&+U'^{(\Lambda)}\Big[\sum_{q_1,q_2\neq 0}\overline{\psi}_{q_1}\psi_{q_2}\phi_{-q_1-q_2}+\sum_{q\neq 0}\Big(\overline{\psi}_{-q}\psi_0\phi_q+\overline{\psi}_0\phi_{-q}\psi_q\Big)\Big]\\
&+\frac{1}{2}U''^{(\Lambda)}\sum_{q_1,q_2\neq 0}\Big(\overline{\psi}_{q_1}\psi_0+\overline{\psi}_0\psi_{q_1}\Big)\phi_{q_2}\phi_{-q_1-q_2}+\cdots\cdot
\end{aligned}
\tag{2.36}
$$

Now N and $\mathcal{F}$ are computed via Eqs.(2.25), (2.27), and (2.36) (adding factors of Vol where appropriate),

$$
\begin{aligned}
\frac{\Sigma}{\mathrm{Vol}} &= q^2+V''+\overline{\psi}_0\psi_0U'' \quad ; \quad \frac{\mathcal{A}}{\mathrm{Vol}}=\psi_0U'\\
\frac{\overline{\mathcal{A}}}{\mathrm{Vol}} &= \overline{\psi}_0U' \quad ; \quad \frac{\mathcal{F}}{\mathrm{Vol}}=\frac{1}{2}(-i\not{q}+U),
\end{aligned}
\tag{2.37}
$$

thus,

$$
\begin{aligned}
\mathrm{N} &= \Sigma-\overline{\mathcal{A}}\mathcal{F}^{-1}\mathcal{A}\\
&= \mathrm{Vol}(q^2+V'')\Big[1+\frac{\overline{\psi}_0\psi_0}{q^2+V''}\Big(U''-\frac{2U'^2}{-i\not{q}+U}\Big)\Big].
\end{aligned}
\tag{2.38}
$$

From Eq.(2.36),

$$
\frac{S^{(\Lambda)}|_0}{\mathrm{Vol}}=V^{(\Lambda)}+\overline{\psi}_0\psi_0U^{(\Lambda)}. \tag{2.39}
$$

Therefore, collecting the results of Eqs. (2.37) and (2.38), Eq. (2.32) becomes:

$$
\begin{aligned}
V^{(\Lambda-\Delta\Lambda)}+\overline{\psi}_0\psi_0U^{(\Lambda-\Delta\Lambda)} &= V^{(\Lambda)}+\overline{\psi}_0\psi_0U^{(\Lambda)}\\
&- \frac{1}{2\mathrm{Vol}}\mathrm{tr}\ \ln\Big\{\mathrm{Vol}(q^2+V'')\Big[1+\frac{\overline{\psi}_0\psi_0}{q^2+V''}\Big(U''-\frac{2U'^2}{-i\not{q}+U}\Big)\Big]\Big\}\\
&+ \frac{1}{\mathrm{Vol}}\mathrm{tr}\ \ln\frac{1}{2}(-i\not{q}+U).
\end{aligned}
\tag{2.40}
$$

We now note that the trace in the last term is over flavor, momentum, and Lorentz indices; thus performing the trace over the flavor and Lorentz indices (in d-dimensions)

and picking up a factor of 1/2 from rationalizing the $-i\not{q}+U$ matrix gives,

$$\text{tr}\ \ln(-i\not{q}+U) = \frac{1}{2}n_f c_d \ln(q^2+U^2) \tag{2.41}$$

where n_f is the number of fermion flavors and $c_d = 2^{d/2}(2^{(d-1)/2})$ for d even (odd).[6] Performing straightforward algebra and ignoring constant factors, one can rewrite Eq.(2.40) as

$$\begin{aligned} V^{(\Lambda-\Delta\Lambda)} + \overline{\psi}_0\psi_0 U^{(\Lambda-\Delta\Lambda)} &= V^{(\Lambda)} + \overline{\psi}_0\psi_0 U^{(\Lambda)} \\ &- \frac{1}{2(\text{Vol})}\Big[\text{tr}\ln(q^2+V'') - n_f c_d \text{tr}\ \ln(q^2+U^2)\Big] \\ &- \frac{1}{2}\overline{\psi}_0\psi_0 \frac{1}{(\text{Vol})}\text{tr}\Big[\frac{1}{q^2+V''}\Big(U'' - \frac{2U'^2}{-i\not{q}+U}\Big)\Big]. \end{aligned} \tag{2.42}$$

Equating coefficients of $\mathcal{O}(1)$ and $\mathcal{O}(\overline{\psi}_0\psi_0)$ on either side of this equation we have the relations:

$$\begin{aligned} V^{(\Lambda-\Delta\Lambda)} &= V^{(\Lambda)} - \frac{1}{2(\text{Vol})}\Big[\text{tr}\ \ln(q^2+V'') - n_f c_d \text{tr}\ \ln(q^2+U^2)\Big] \\ U^{(\Lambda-\Delta\Lambda)} &= U^{(\Lambda)} - \frac{1}{2(\text{Vol})}\text{tr}\Big[\frac{1}{q^2+V''}\Big(U'' - \frac{2U'^2}{-i\not{q}+U}\Big)\Big]. \end{aligned} \tag{2.43}$$

We turn these into differential flow equations by writing,

$$\begin{aligned} \text{tr} f(q) &= \sum_q{}' f(q) = (\text{Vol})\int_{shell}\frac{d^dq}{(2\pi)^d} f(q) = (\text{Vol})A_d\int_{\Lambda-\Delta\Lambda}^{\Lambda} q^{d-1}dq\ f(q) \\ &= (\text{Vol})A_d\Lambda^{d-1}\Delta\Lambda f(\Lambda) \end{aligned} \tag{2.44}$$

[6] See Ref.[77] for a discussion of Dirac γ matrices in d-dimensions.

(where $A_d = \int \frac{d\Omega_d}{(2\pi)^d} = (2^{d-1}\pi^{d/2}\Gamma(d/2))^{-1}$), and

$$\lim_{\Delta\Lambda\to 0} \Lambda \frac{X^{(\Lambda-\Delta\Lambda)} - X^{(\Lambda)}}{\Delta\Lambda} = -\Lambda \frac{dX^{(\Lambda)}}{d\Lambda} = A_d \Lambda^d f(\Lambda) \tag{2.45}$$

where X can be either V or U. Neglecting terms of $\mathcal{O}(\hbar)$ since we're using an LO approximation, Eqs. (2.43) can now be written as,

$$\begin{aligned} \Lambda \frac{\partial V}{\partial \Lambda} &= -\frac{A_d}{2}\Lambda^d \Big[\ln(\Lambda^2 + V'') - n_f c_d \ln(\Lambda^2 + U^2)\Big] \\ \Lambda \frac{\partial U}{\partial \Lambda} &= -\frac{A_d}{2}\Lambda^d \Big[\frac{1}{\Lambda^2 + V''}\Big(U'' - \frac{2U'^2 U}{\Lambda^2 + U^2}\Big)\Big]. \end{aligned} \tag{2.46}$$

The *scaled* version of these equations is derived by defining the dimensionless variables,

$$\begin{aligned} t &= \ln(\Lambda_0/\Lambda) \\ x &= \Lambda^{1-d/2}\phi \end{aligned} \tag{2.47}$$

(where Λ_0 is a UV cutoff) and the dimensionless potentials,

$$\begin{aligned} \mathcal{V}(x,t) &= \Lambda^{-d} V(\phi, \Lambda) \\ \mathcal{U}(x,t) &= \Lambda^{-1} U(\phi, \Lambda), \end{aligned} \tag{2.48}$$

and writing Eq.(2.46) in terms of them:

$$\begin{aligned} \dot{\mathcal{V}} &= \frac{A_d}{2}\Big[\ln(1 + \mathcal{V}'') - n_f c_d \ln(1 + \mathcal{U}^2)\Big] + (1 - d/2)x\mathcal{V}' + \mathcal{V} \\ \dot{\mathcal{U}} &= \frac{A_d}{2}\Big[\frac{1}{1 + \mathcal{V}''}\Big(\mathcal{U}'' - \frac{2\mathcal{U}'^2\mathcal{U}}{1 + \mathcal{U}^2}\Big)\Big] + (1 - d/2)x\mathcal{U}' + \mathcal{U}. \end{aligned} \tag{2.49}$$

where dots refer to derivatives with respect to t and primes refer to derivatives with

respect to x. Eqs.(2.49) are precisely the equations obtained by Maggiore [68]. These results have also been reported by the present author in Ref.[76].

To make contact with Clark *et al.* [69] and with Hasenfratz and Hasenfratz [50] we define $F(x,t) = \tilde{V}'(x,t)$ and $G(x,t) = \tilde{U}(x,t)$, then Eqs.(2.49) become,

$$\begin{aligned} \dot{F} &= \frac{A_d}{2}\Big(\frac{F''}{1+F'} - n_f c_d \frac{2GG'}{1+g^2}\Big) + (1-\frac{d}{2})xF + (1+\frac{d}{2})F \\ \dot{G} &= \frac{A_d}{2}\frac{1}{1+F'}\Big(G'' - \frac{2G'^2G}{1+G^2}\Big) + (1-\frac{d}{2})xG' + G. \end{aligned} \tag{2.50}$$

For $d = 4$ and $n_f = 1$ these are identical to Eq. (3.15) in reference [69] *except* for the minus sign in front of the $n_f c_d$ term which Clark *et al.* have (in error) as a plus sign. Setting G equal to zero (*i.e.* ignoring the fermions) in Eqs.(2.50) returns the single equation for F first derived and appearing in slightly different notation in Ref.[50].

2.4 σ-model Flow Equations

We now derive the flow equations for the σ-model. The incorporation of chiral symmetry breaking brings a substantial increase in the complexity of the algebra. Many of the tedious details are relegated to Appendices. The action is (see section 1.3):

$$S^{(\Lambda)} = \int d^dx \left[V^{(\Lambda)}(\rho,\sigma) + \frac{1}{2}(\partial_\mu\phi^a)^2 + \overline{\psi}[\not{\partial} + U^{(\Lambda)}(\rho,\sigma,\Gamma)]\psi\right]. \tag{2.51}$$

This is generalized version of the Lagrangian used in section 1.3. We've gone over to Euclidean space and, to be consistent with the foregoing sections, made a few

definitions to compactify the notation,

$$
\begin{aligned}
\phi^a &= \begin{pmatrix} \sigma \\ \pi^i \end{pmatrix}, \qquad a = 0, i; \qquad i = 1, 2, \ldots, N \\
\Gamma^a &= \begin{pmatrix} 1 \\ i\gamma^5\tau^i \end{pmatrix}, \\
\Gamma &= \Gamma^a\phi^a = \sigma + i\gamma^5\vec{\tau}\cdot\vec{\pi}, \\
\rho^2 &= \Gamma\Gamma^\dagger = (\phi^a)^2 = \sigma^2 + (\vec{\pi})^2.
\end{aligned} \tag{2.52}
$$

In specifying Γ^a we limit ourselves to the $N = 4$ or $O(4)$ case as in section 1.3. We will often use the more general notation but will only be considering $O(4)$ in this thesis. Recall that $SU(2) \times SU(2)$ roations of the fermions induce $O(4)$ rotations in the boson sector. The scalar sector of the σ-model, then is a four component field. To get the action of the previous section simply set all the pion fields to zero and set $U^{(\Lambda)}(\rho, \sigma, \Gamma) = U^{(\Lambda)}(\sigma)$ where σ corresponds to the ϕ of the the previous section. The potentials $V^{(\Lambda)}(\rho, \sigma)$ and $U^{(\Lambda)}(\rho, \sigma, \Gamma)$ are the analogues of the potentials in the previous section but are now allowed to include terms proportional to powers of the zeroth component of the scalar field, σ, which will break chiral symmetry. To keep track of the symmetry breaking as the theory flows from high to low momentum we expand the potentials in the form,

$$
\begin{aligned}
V(\rho, \sigma) &= V_0(\rho) + \sigma V_1(\rho) + \frac{\sigma^2}{2} V_2(\rho) + \cdots \\
U(\rho, \sigma, \Gamma) &= m(\rho, \sigma) + \Gamma g(\rho, \sigma)
\end{aligned} \tag{2.53}
$$

where $m(\rho, \sigma)$ and $g(\rho, \sigma)$ are expanded similarly to $V(\rho, \sigma)$. Each of the functions V_k, m_k and g_k for $k = 0, 1, 2$ is $SU(2) \times SU(2)$ symmetric since they only depend on ρ. Chiral symmetry is broken by V_1, V_2, g_1, g_2 and $m(\rho, \sigma)$. Note that if we retain only

V_0 and g_0 we have precisely (modulo the transformation from Minkowski to Euclidean space) the Lagrangian of section 1.3.

The LO flow of the action Eq.(2.51) is given by Eq.(2.32),

$$S^{(\Lambda-\Delta\Lambda)} = S^{(\Lambda)}\Big|_0 - \mathrm{trln}\mathbf{N} + \mathrm{trln}\mathcal{F}. \tag{2.54}$$

where $\mathbf{N}$ has components N^{ab},

$$\begin{aligned} N^{ab} &\equiv \Sigma^{ab} - \overline{\mathcal{A}}^a \mathcal{F}^{-1} \mathcal{A}^b \\ \begin{pmatrix} \Sigma & \mathcal{A} \\ \overline{\mathcal{A}} & \mathcal{F} \end{pmatrix} &= \begin{pmatrix} \frac{\delta^2 S}{\delta\phi^a_{-q}\delta\phi^b_q}\Big|_0 & \frac{\delta^2 S}{\delta\overline{\psi}_{-q}\delta\phi^b_q}\Big|_0 \\ \frac{\delta^2 S}{\delta\phi^a_{-q}\delta\psi_q}\Big|_0 & \frac{\delta^2 S}{\delta\overline{\psi}_{-q}\delta\psi_q}\Big|_0 \end{pmatrix}. \end{aligned} \tag{2.55}$$

The extra indices now refer to the components of the scalar field. As before we Taylor expand the potentials,

$$V^{(\Lambda)}(\phi^a) = V^{(\Lambda)}(\phi^a_0) + V'^{(\Lambda)a}(\phi^a_0)\varphi^a + \frac{1}{2}V''^{(\Lambda)ab}(\phi^a_0)\varphi^a\varphi^b + \cdots \tag{2.56}$$

(and similarly for U) with normalization Eq.(2.35) to get,

$$\begin{aligned} \frac{S^{(\Lambda)}}{\mathrm{Vol}} &= V^{(\Lambda)}(\phi^a_0) + \overline{\psi}_0 U^{(\Lambda)}(\phi^a_0)\psi_0 \\ &+ \sum_{q\neq 0}\Big[\frac{1}{2}\phi^a_{-q}[\delta^{ab}q^2 + V''^{(\Lambda)ab}(\phi^a_0) + \overline{\psi}_0 U''^{(\Lambda)ab}(\phi^a_0)\psi_0]\phi^b_q \\ &+ \overline{\psi}^\alpha_{-q}[i\not{q} + U^{(\Lambda)}(\phi^a_0)]\psi^\alpha_q \\ &+ U'^{(\Lambda)a}(\phi^a_0)(\overline{\psi}_0\phi^a_{-q}\psi_q + \overline{\psi}_{-q}\phi^a_q\psi_0)\Big] + \cdots \end{aligned} \tag{2.57}$$

From this expression we can compute all the matrices in Eq.(2.56),

$$
\begin{aligned}
\Sigma^{ab} &= \delta^{ab}q^2 + V''^{ab}(\phi_0^a) \\
\Omega^{ab} &\equiv U''^{ab} - \frac{2U'^a U'^b}{i\not{q} + U} \\
\frac{\mathcal{F}}{\mathrm{Vol}} &= \frac{1}{2}(i\not{q} + U) \\
\frac{N^{ab}}{\mathrm{Vol}} &= \Sigma^{ab} + \overline{\psi}_0 \Omega^{ab} \psi_0.
\end{aligned} \tag{2.58}
$$

With these definitions Eq.(2.54) becomes,

$$
\begin{aligned}
\frac{S^{(\Lambda-\Delta\Lambda)}}{\mathrm{Vol}} &= V^{(\Lambda-\Delta\Lambda)}(\phi_0^a) + \overline{\psi}_0 U^{(\Lambda-\Delta\Lambda)}(\phi_0^a)\psi_0 \\
&= V^{(\Lambda)}(\phi_0^a) + \overline{\psi}_0 U^{(\Lambda)}(\phi_0^a)\psi_0 \\
&\quad -\frac{1}{2\mathrm{Vol}}\mathrm{tr}\ \ln\Big[\Sigma^{ab} + \overline{\psi}_0\Big(\Omega^{ab}\Big)\psi_0\Big] \\
&\quad +\frac{1}{\mathrm{Vol}}\mathrm{tr}\ \ln(i\not{q} + U^{(\Lambda)}(\phi_0^a)) + \cdots
\end{aligned} \tag{2.59}
$$

As in the previous section we can write Eq.(2.59) as two coupled equations for the flow of V and U. Before doing so we make a number of modifications. First, we can write,

$$
\begin{aligned}
\frac{1}{i\not{q} + U} &= \frac{i\not{q} + U}{q^2 + UU^\dagger} \simeq \frac{U^\dagger}{D_F}, \\
D_F &\equiv q^2 + UU^\dagger = q^2 + m^2 + 2\sigma m g + \rho^2 g^2,
\end{aligned} \tag{2.60}
$$

where (as in the previous section) we've dropped the term proportional to $\not{q}$ since it will vanish to LO when we take angle averages. Second, we can write the trace over

the inverse fermion propagator term as,

$$\text{tr } \ln(i\not{q} + U) = \frac{1}{2} n_f c_d \sum_q \ln D_F \tag{2.61}$$

with $c_d = 2^{d/2}(2^{(d-1)/2})$ for d even (odd), using the identity $|i\not{q} + U|^2 = q^2 + U^2$ and the $n_f c_d \sum_q$ factor comes from the trace over the flavor and Dirac indices. Also we can write,

$$\text{tr } \ln\left[\Sigma^{ab} + \overline{\psi}_0\left(\Omega^{ab}\right)\psi_0\right] = \ln \det \Sigma^{ab} + \text{tr } [(\Sigma^{ac})^{-1}\overline{\psi}_0\Omega^{cb}\psi_0], \tag{2.62}$$

where the trace in the last term is only over the ab indices. Putting all this together we can write the V and U equations from Eq.(2.59) as,

$$\begin{aligned} V^{(\Lambda-\Delta\Lambda)} &= V^{(\Lambda)} - \frac{1}{2\text{Vol}} \sum_q (\ln \det \mathbf{\Sigma} - n_f c_d \ln D_F) \\ U^{(\Lambda-\Delta\Lambda)} &= U^{(\Lambda)} - \frac{1}{2\text{Vol}} \sum_q \text{tr } \mathbf{\Sigma}^{-1} \cdot \mathbf{\Omega}. \end{aligned} \tag{2.63}$$

And using Eqs.(2.44) and (2.45), taking the limit $\Delta\Lambda \to 0$ we have,

$$\Lambda \frac{\partial V^{(\Lambda)}}{\partial \Lambda} = -\frac{A_d}{2}\Lambda^d \ (\ln \det \mathbf{\Sigma} - n_f c_d \ln D_F) \tag{2.64}$$

$$\Lambda \frac{\partial U^{(\Lambda)}}{\partial \Lambda} = -\frac{A_d}{2}\Lambda^d \text{ tr } (\mathbf{\Sigma}^{-1} \cdot \mathbf{\Omega}), \tag{2.65}$$

where the determinant and trace are only over the ab indices. These equations are the generalizations of Eqs.(2.46) to the case of $O(4)$ scalar fields interacting with chiral $SU(2) \times SU(2)$ fermions. In Eqs.(2.64) and (2.65),

$$\Sigma^{ab} = \delta^{ab}\Lambda^2 + V''^{ab}$$

$$
\begin{aligned}
\Omega^{ab} &= U''^{ab} - \frac{2U'^{a}U^{\dagger}U'^{b}}{D_F} \\
D_F &= \Lambda^2 + m^2 + \rho^2 + 2\sigma m g.
\end{aligned} \tag{2.66}
$$

The matrices V''^{ab}, U''^{ab}, and U'^{a} are worked out in terms of derivatives with respect to ρ in Appendix E. One glance at these expressions is adequate to impress the reader of the proliferation of algebraic complication in the extension of the simple Yukawa coupled fermions to the case of broken chiral symmetry. The matrices can be simplified by a similarity transformation, however, since the determinant and the trace are invariant with respect to such transformations. This is discussed in Appendices D and E. The similarity transformed, $\mathbf{\Sigma}'$ has only six nonzero elements,

$$
\mathbf{\Sigma}' = \begin{pmatrix} \Sigma_{00} & \Sigma_{01} & 0 & 0 \\ \Sigma_{10} & \Sigma_{11} & 0 & 0 \\ 0 & 0 & \Sigma_{22} & 0 \\ 0 & 0 & 0 & \Sigma_{33} \end{pmatrix}, \tag{2.67}
$$

but $\mathbf{\Omega}'$ still has every element nonzero although each of the elements is somewhat simpler. Notice, however, that since only the combination $\mathrm{tr}\mathbf{\Sigma}'^{-1} \cdot \mathbf{\Omega}'$ appears in the flow equations, only six elements of $\mathbf{\Omega}'$ will contribute so that,

$$
\begin{aligned}
\mathrm{tr}\ \mathbf{\Sigma}'^{-1} \cdot \mathbf{\Omega}' &= (\Sigma^{00})^{-1}\Omega^{00} + (\Sigma^{10})^{-1}\Omega^{01} + (\Sigma^{01})^{-1}\Omega^{00} \\
&\quad + (\Sigma^{11})^{-1}\Omega^{11} + (\Sigma^{22})^{-1}\Omega^{22} + (\Sigma^{33})^{-1}\Omega^{33},
\end{aligned} \tag{2.68}
$$

where, *e.g.* $(\Sigma^{00})^{-1}$ means the 00 element of $\mathbf{\Sigma}'^{-1}$.

We can now consider the "chiral limit" where we set all chiral breaking terms

$(m(\rho,\sigma)$, V_1, V_2, g_1 and $g_2)$ to zero. Then,

$$\Sigma^{ab} \longrightarrow (\Lambda^2 + V_0'(\rho)/\rho)\delta^{ab} + (V_0''(\rho) - V_0'/\rho)\frac{\phi^a\phi^b}{\rho^2}, \tag{2.69}$$

which becomes, after a similarity transformation,

$$\Sigma^{ab'} = \begin{pmatrix} D_\phi & 0 & 0 & 0 \\ 0 & D_\pi & 0 & 0 \\ 0 & 0 & D_\pi & 0 \\ 0 & 0 & 0 & D_\pi \end{pmatrix}, \tag{2.70}$$

where $D_\phi = \Lambda^2 + V_0''$ and $D_\pi = \Lambda^2 + V_0'/\rho$. Thus,

$$(\Sigma^{ab'})^{-1} = \begin{pmatrix} D_\phi^{-1} & 0 & 0 & 0 \\ 0 & D_\pi^{-1} & 0 & 0 \\ 0 & 0 & D_\pi^{-1} & 0 \\ 0 & 0 & 0 & D_\pi^{-1} \end{pmatrix}, \tag{2.71}$$

and so in the chiral limit only the diagonal elements of $\mathbf{\Omega}$ will contribute to the trace term. These can be shown to be,

$$\begin{aligned} \Omega^{00} &= \Gamma[g_0'' + \frac{2g'}{\rho} - \frac{2g_0}{D_F}(g_0^2 + 2g_0^2\rho^2 + \rho^2 g_0'^2)] \\ \Omega^{11} &= \Omega^{22} = \Omega^{33} = \Gamma(\frac{g_0'}{\rho} + \frac{2g_0^3}{D_F}). \end{aligned} \tag{2.72}$$

Thus from Eqs. (2.64) and (2.65) the $O(4)$ flow equations in the chiral limit are,

$$\begin{aligned} \Lambda\frac{\partial V_0}{\partial \Lambda} &= -\frac{A_d}{2}\Lambda^d(\ln D_\phi + \ln D_\pi - n_f c_d \ln D_F) \\ \Lambda\frac{\partial g_0}{\partial \Lambda} &= -\frac{A_d}{2}\Lambda^d\Big\{\frac{1}{D_\phi}\Big[g_0'' + \frac{2g'}{\rho} - \frac{2g_0}{D_F}(g_0^2 + 2g_0^2\rho^2 + \rho^2 g_0'^2)\Big] \end{aligned} \tag{2.73}$$

$$+\frac{3}{D_\pi}\left(\frac{g_0'}{\rho}+\frac{2g_0^3}{D_F}\right)\Big\}. \tag{2.74}$$

For $O(N)$ symmetric theories there would be an $N-1$ in place of the 3s in front of the $\ln D_\pi$ and the $(1/D_\pi)(\cdots)$ terms. In Ref.[64] the flow equation for an $O(N)$ scalar-only field theory is derived and is equivalent to Eq.(2.73) with $n_f = 0$ and $N = 4$.

Chapter 3

NUMERICAL RESULTS

In this chapter we discuss the numerical solution of the RG flow equations derived in the last chapter for the Yukawa coupled fermions (Eqs.(2.46)) and the σ-model (Eqs.(2.65)). We first outline the procedure used for numerically solving the flow equations by discussing the Yukawa coupled fermions. In this case the functions $\lambda(t)$ and $g(t)$ ($t = \ln\ \Lambda_0/\Lambda$) are plotted and compared with similar results in Ref.[69] with which they are essentially in aggreement. Also, the flow of the Yukawa coupling is seen to be small (*i.e.* $g(t) \sim$ const). Then the numerical results for the σ-model are discussed. We motivate the general philosophy of the calculation and the choice of the paramters to be fixed and predicted. Many of the parameters are related to the unknown high energy behavior of the theory where nonperturbative QCD effects are important. Results for $\pi\pi$ scattering lengths show an improvement over the results of the perturbative σ-model. These and are other results are systematically compared with experiment and other calculations. We finally mention some extensions to the model.

3.1 Generalized Yukawa Coupled Fermions

To numerically solve Eqs.(2.46) we first expand $V(\phi, \Lambda)$ and $U(\phi, \Lambda)$ in powers of ϕ:

$$V(\phi, \Lambda) \quad = \quad \sum_{i=1}^{M} \frac{1}{2i} v_{2i} \phi^{2i}$$

$$U(\phi,\Lambda) = \sum_{i=1}^{M} \frac{1}{2i-1} u_{2i-1}\phi^{2i-1} \tag{3.1}$$

where we have chosen V even and U odd since this is consistent with the bare action Eq.(2.24)) having the symmetry $\phi \to -\phi$ and $\psi \to e^{i\pi\gamma_5}\psi$. The boundary values are given by the assumptions about the form of the bare potentials (at the UV scale, Λ_0) which we assume to have the form

$$\begin{aligned} V(\phi,\Lambda_0) &= \frac{1}{2}m_0^2\phi^2 + \frac{1}{4}\lambda_0\phi^4 \\ U(\phi,\Lambda_0) &= g_0\phi \end{aligned} \tag{3.2}$$

where m_0, λ_0, and g_0 are the bare boson mass, coupling constant, and Yukawa coupling respectively. In terms of the coefficients in Eqs.(3.1) this implies

$$\begin{aligned} v_2(\Lambda_0) &= m_0^2 \\ v_4(\Lambda_0) &= \lambda_0 \\ u_1(\Lambda_0) &= g_0 \end{aligned} \tag{3.3}$$

with $v_{2i}(\Lambda_0) = u_{2i-1}(\Lambda_0) = 0$ for $2i > 4$ and $2i-1 > 1$. The Λ-integration is divided into N intervals of equal length $\Delta\Lambda = (\Lambda_0 - \Lambda_{IR})/N$ given by

$$\Lambda_n = \Lambda_0 - n\Delta\Lambda. \tag{3.4}$$

The choice of Λ_0 and Λ_{IR} is arbitrary. For comparisons with lattice calculations we pick Λ_0 to yield the same phase space volume as the finite sum over modes [64] (for $d = 4$ this is $\Lambda_0 = (32\pi^2)^{1/4}$) and $\Lambda_{IR} = 2\pi/N_{lat}^d$ where N_{lat}^d is the number of lattice points in the d-dimensional lattice. To compare with Clark *et al.* [69] we fix

$t_{max} = \ln(\Lambda_0/\Lambda_{IR})$ and pick Λ_{IR} and m_0 such that the vacuum expectation value of the scalar field in the broken phase, v, equals that of Ref.[69].

Similarly the ϕ-integration is decomposed via

$$\phi_0 < \cdots < \phi_j < \phi_{j+1} < \cdots < \phi_J \tag{3.5}$$

and at each step in the integration a least squares fit is made of the polynomials on the l.h.s. of Eq.(3.1) to V and U to determine v_{2i} and u_{2i-1} and from them the RG effective potentials, $V(\phi_j, \Lambda_{n+1})$ and $U(\phi_j, \Lambda_{n+1})$ computed from the finite step versions of Eqs.(2.46):

$$\begin{aligned} V(\phi_j, \Lambda_{n+1}) &= V(\phi_j, \Lambda_n) + \frac{A_d}{2}\Lambda_n^{d-1}\Delta\Lambda \\ &\times \left[\ln(\Lambda_n^2 + V''(\phi_j, \Lambda_n) - n_f c_d \ \ln(\Lambda_n^2 + U^2(\phi_j, \Lambda_n))\right] \\ U(\phi_j, \Lambda_{n+1}) &= U(\phi_j, \Lambda_n) + \frac{A_d}{2}\Lambda_n^{d-1}\Delta\Lambda \\ &\times \frac{1}{\Lambda_n^2 + V''(\phi_j, \Lambda_n)}\left[U''(\phi_j, \Lambda_n) - \frac{2U'^2(\phi_j, \Lambda_n)U(\phi_j, \Lambda_n)}{\Lambda_n^2 + U^2(\phi_j, \Lambda_n)}\right]. \end{aligned} \tag{3.6}$$

The functions, $V(\phi_j, \Lambda_{n+1})$ and $U(\phi_j, \Lambda_{n+1})$, obtained by this calculation we refer to as the boson and fermion effective potentials respectively. For our calculations we typically use

$$M \simeq 10, \quad N \simeq 4000, \quad \text{and} \quad J \simeq 50. \tag{3.7}$$

Figure 3.1 displays the effective potentials $V(\phi)$ (solid line) and $U(\phi)$ (dotted line) calculated in the symmetric phase with $\Lambda_0 = (32\pi^2)^{1/4}$ and $\Lambda_{IR} = 2\pi/N_{lat}^d$ with $N_{lat}^d = 8$. We also use $m_0^2 = 0.1^2$, $\lambda = 10$, $n_f = 1$, and $g_0 = 2$. We note that the renormalized fermion effective potential, $U(\phi)$, is essentially of the bare Yukawa form with $g_0 \approx 2$. We find that the effect of the RG flow on $U(\phi)$ is minimal *except*

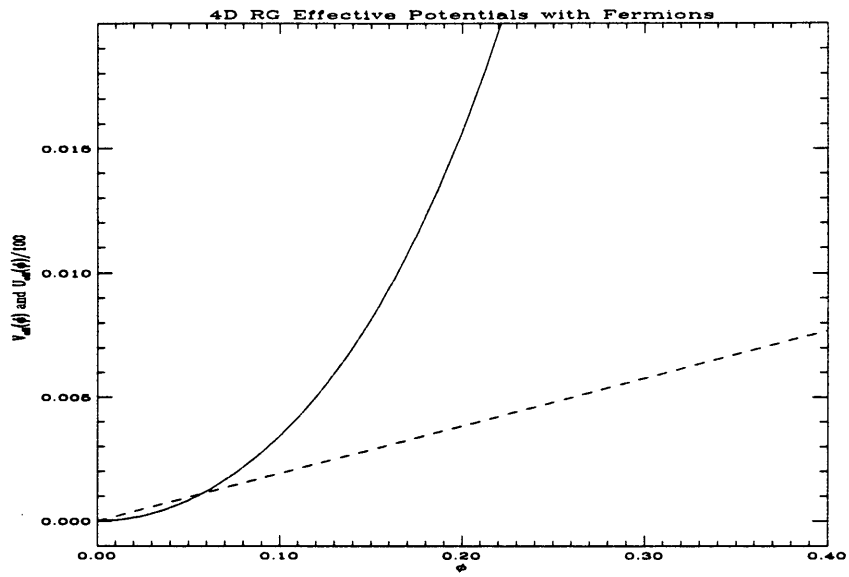

FIG. 3.1. RG effective potentials V (solid line) and U (dotted line) in $d = 4$ for $\Lambda_0 = (32\pi^2)^{1/4}$ and $\Lambda_{IR} = 2\pi/N_{lat}^d$ with $N_{lat}^d = 8$. The bare parameters are $m_0^2 = 0.1^2$, $\lambda_0 = 10$, and $g_0 = 2$.

when evolving far into the broken phase. Figure 3.2 displays similar results but for $\Lambda_0 = 0.75$ and $\Lambda_{IR} = 0.25$ as well as $m_0^2 = -0.1^2$, $\lambda = 5$ and $g_0 = 1$. Figures 3.3 through 3.5 display the flow of the renormalized boson and fermion couplings in a manner which allows comparison with the numerical results of Ref.[69]. Figure 3.3 displays our calculation of the running boson coupling, $\lambda(t)$, for comparison with the curve labeled $g(0) = 1$, $\Lambda = 38.6v$ in Fig.1 of Ref.[69]. This curve is obtained by evolving V into the broken phase until v, the vacuum expectation value of ϕ, equals 0.08 as specified in Ref.[69]. We choose Λ_{IR}=0.25 in order to obtain this value of v given $t_{max} = 2.6$ as also specified in Ref.[69]. Figure 3.4 displays a similar plot of the running boson coupling for comparison with the curve labeled $g(0) = 1$, $\Lambda = 5.4v$ in Fig.1 of Ref.[69]. For this plot we choose $\Lambda_{IR} = 0.5$ so as to give $v = 0.14$ with $t_{max} = 1.1$. Figure 3.5 shows the running Yukawa coupling, $g(t)$, for comparison with

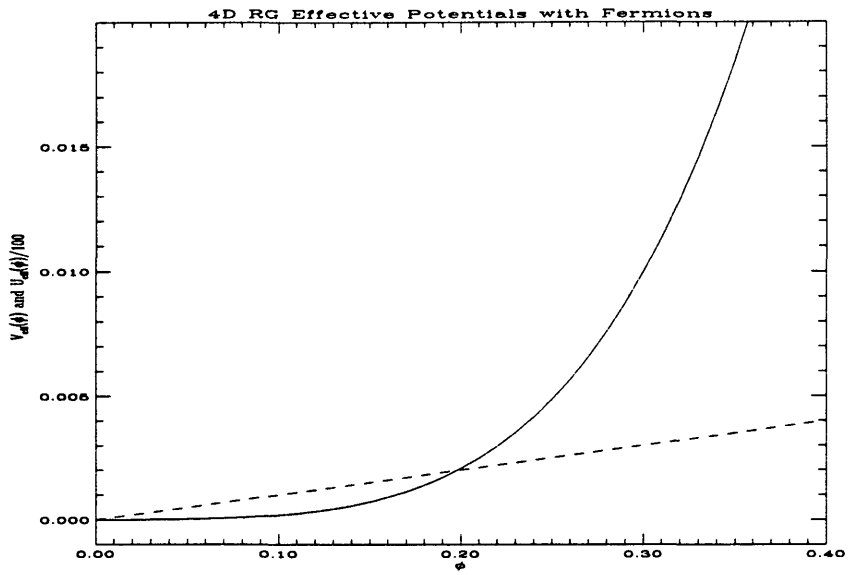

FIG. 3.2. RG effective potentials V (solid line) and U (dotted line) for $d = 4$ with $\Lambda_0 = 0.75$ and $\Lambda_{IR} = 0.25$. The bare parameters are $m_0^2 = -0.1^2$, $\lambda_0 = 5$, and $g_0 = 1$.

the curve labeled $\lambda(0) = 5$, $\Lambda = 5.4v$ in Fig.1 of Ref.[69]. Here we set $\Lambda_{IR} = 0.5$ to yield $v = 0.28$ for $t_{max} = 1.1$. It is apparent that the change in the Yukawa coupling is minimal in this case; in contrast, the change in $V_{eff}(\phi)$ is dramatic. We note that these numerical results have been reported previously in Ref.[76].

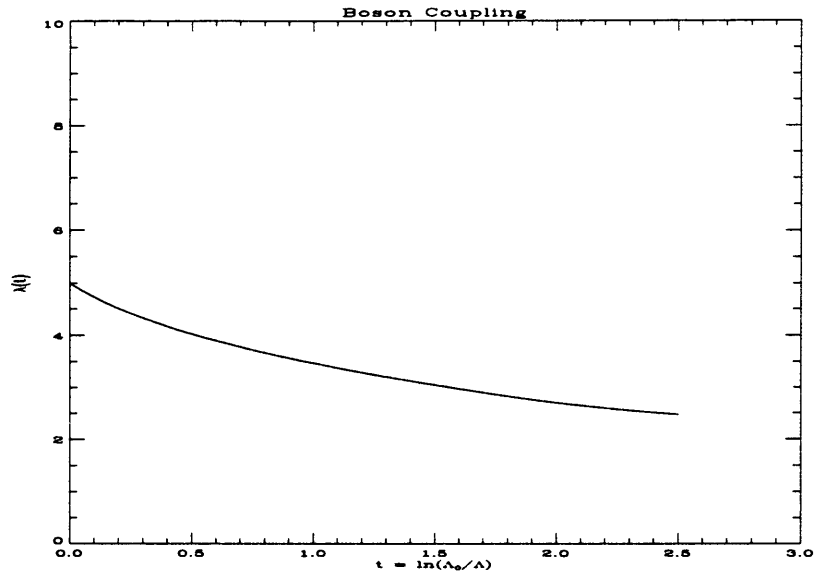

FIG. 3.3. The flow of the the boson coupling for comparison with the curve labeled $g(0) = 1$, $\Lambda = 38.6v$ in Fig.1 of Ref.[44]. The bare parameters are $m_0^2 = -0.747^2$, $\lambda_0 = 5$, and $g_0 = 1$. We use $\Lambda_{IR} = 0.25$ which implies $v = 0.08$.

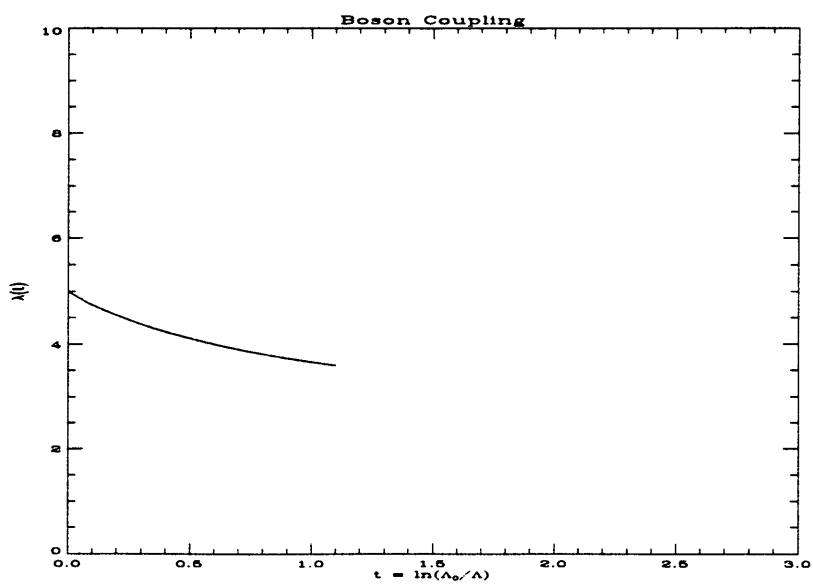

FIG. 3.4. The flow of the the boson coupling for comparison with the curve labeled $g(0) = 1$, $\Lambda = 5.4v$ in Fig.1 of Ref.[44]. The bare parameters are $m_0^2 = -0.325^2$, $\lambda_0 = 5$, and $g_0 = 1$. We used $\Lambda_{IR} = 0.25$ which implies $v = 0.14$.

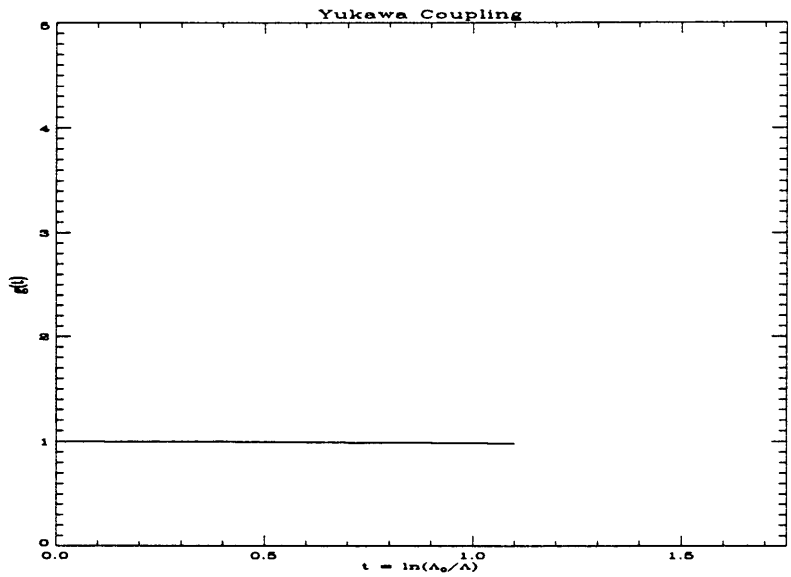

FIG. 3.5. The flow of the the fermion coupling for comparison with the curve labeled $\lambda(0) = 5$, $\Lambda = 5.4v$ in Fig.2 of Ref.[44]. The bare parameters are $m_0^2 = -0.660^2$, $\lambda_0 = 5$, and $g_0 = 1$. We use $\Lambda_{IR} = 0.5$ which implies $v = 0.28$.

3.2 σ-model

We now discuss the numerical solution of the LO σ-model RG flow equations, (2.65). As mentioned in Appendix E the specific form of these equations is quite complicated. Since we expand (see section 2.4),

$$\begin{aligned} V(\rho,\sigma) &= V_0(\rho) + \sigma V_1(\rho) + \frac{\sigma^2}{2} V_2(\rho) + \cdots \\ U(\rho,\sigma,\Gamma) &= m(\rho,\sigma) + \Gamma g(\rho,\sigma) \end{aligned} \tag{3.8}$$

(with m and g expanded similarly to V) to $\mathcal{O}(\sigma)$—first order in the chiral symmetry breaking parameter—there are six coupled flow equations, which for

$$\det \Sigma' = F_0(\rho) + \sigma F_1(\rho)$$

$$\text{tr } \mathbf{\Sigma}'^{-1} \cdot \mathbf{\Omega} = \Delta m_0(\rho) + \Gamma \Delta g_0(\rho) + \sigma \Big[\Delta m_1(\rho) + \Gamma \Delta g_1(\rho) \Big] \tag{3.9}$$

take the form:

$$\begin{aligned}
\Lambda \frac{\partial V_0}{\partial \Lambda} &= -\frac{A_d}{2} \Lambda^d \Big(\ln F_0(\rho) - n_f c_d \ln D_F \Big) \\
\Lambda \frac{\partial V_1}{\partial \Lambda} &= -\frac{A_d}{2} \Lambda^d \frac{F_1(\rho)}{F_0(\rho)} \\
\Lambda \frac{\partial m_0}{\partial \Lambda} &= -\frac{A_d}{2} \Lambda^d \Delta m_0(\rho) \\
\Lambda \frac{\partial m_1}{\partial \Lambda} &= -\frac{A_d}{2} \Lambda^d \Delta m_1(\rho) \\
\Lambda \frac{\partial g_0}{\partial \Lambda} &= -\frac{A_d}{2} \Lambda^d \Delta g_0(\rho) \\
\Lambda \frac{\partial g_1}{\partial \Lambda} &= -\frac{A_d}{2} \Lambda^d \Delta g_1(\rho).
\end{aligned} \tag{3.10}$$

Mathematica greatly facilitates the determination of the functions F_0, F_1, Δm_0 etc. ($\Delta g_1(\rho)$, for instance, requires several pages of output!—see Appendix G). Once these functions are determined, however, the numerical solution of Eqs.(3.11) procedes exactly as in the previous section except with more potentials (six now instead of two). There are a number of subtleties in the solution of Eq.(3.11) that are either not present or are more complicated to handle than for the simpler case presented in section 3.1. At present we have two different codes to solve these equations which give essentially the same results.

Just as in the previous section we set our "initial condition" by specifying the value of the parameters in the UV action,

$$\begin{aligned}
V^{(\Lambda_0)}(\rho, \sigma) &= \frac{1}{2} \mu_0^2 \rho^2 + \frac{1}{4} \lambda_0 \rho^4 \\
U^{(\Lambda_0)}(\rho, \sigma, \Gamma) &= m_q^0 + g_0 \Gamma.
\end{aligned} \tag{3.11}$$

Then we expand in powers of ρ,

$$
\begin{aligned}
V_k^{(\Lambda)}(\rho) &= \sum_{i=1}^{M} \frac{1}{2i} v_k^{(2i)}(\Lambda)\rho^{2i} \\
m_k^{(\Lambda)}(\rho) &= \xi_k^{(0)}(\Lambda) + \sum_{i=1}^{M} \frac{1}{2i} \xi_k^{(2i)}(\Lambda)\rho^{2i} \\
g_k^{(\Lambda)}(\rho) &= y_k^{(0)}(\Lambda) + \sum_{i=1}^{M} \frac{1}{2i} y_k^{(2i)}(\Lambda)\rho^{2i},
\end{aligned} \tag{3.12}
$$

for $k = 0, 1, 2$ (see Eq.(2.53)). Thus at the UV scale we have

$$
\begin{aligned}
v_0^{(2)}(\Lambda_0) &= \mu_0^2 \\
v_0^{(4)}(\Lambda_0) &= \lambda_0 \\
\xi_0^{(0)}(\Lambda_0) &= m_q^0 && (3.13) \\
y_0^{(0)}(\Lambda_0) &= g_0, && (3.14)
\end{aligned}
$$

with all higher order coefficients for $k = 1, 2$ set to zero. Just as in the last section we perform a fit of the functions V_k, m_k, and g_k to power series' in ρ at each Λ step. Also, since we're interested in spontaneous symmetry breaking we will have the parameter $f_\pi = <\sigma>_{vac}$, which sets the scale of the symmetry breaking. Thus we consider $\{m_\pi$, f_π, μ_0^2, λ_0, m_q^0, g_0, Λ_0, $\Lambda_{IR}\}$ as the input parameters to the model.

The basic philosophy of the calculation is as follows. Since there are as yet no phenomenological constraints at the UV scale we look to the IR scale to determine the free parameters of the model. The two parameters in the model most easily fixed by low energy experiments are m_π and f_π. So we tune other parameters to get m_π and f_π to be their experimental values. This still leaves too many parameters; we must decide which out of $\{\mu_0^2$, λ_0, m_q^0, g_0, Λ_0, $\Lambda_{IR}\}$ to fix and which to tune to m_π and f_π. Results for scalar only calculations as well as for Yukawa coupled fermions

indicate that the two parameters λ_0 and μ_0^2 are related, *i.e.* we can tune to particular values of IR parameters with any number of values for λ_0 and μ_0^2 so long as $\lambda_0 > 0$. So we fix $\lambda_0 = 10$ and use μ_0 to tune the IR parameters. Next there is the bare current quark mass, m_q^0. From the Table 1.1 we see that there is about $\pm$5MeV spread of the values for m_u and m_d. We will tune m_q^0 to get the average of the means of these values at the IR scale, *i.e.* $m_0^{(\Lambda_{IR})} \equiv m_q = \frac{1}{2}(\overline{m}_u + \overline{m}_d) = 7.5$MeV. Also we have the two scales Λ_0 and Λ_{IR}, one of whixh we need to fix, the other we need to have free to tune. We pick $\Lambda_{IR} = m_{\pi^\pm} \simeq 140$MeV and tune Λ_0. (It does not matter what the lower cutoff is, so we pick the mass of the pion—the lightest particle in nuclear physics.) This still leaves g_0. We know that the quark-meson coupling is strong at these energy scales and the Goldberger-Treiman relation for the constituent quark model gives $g \simeq M_{nucleon}/3f_\pi \simeq 3.366$. But this is not satisfactory, since we have no nucleons in our model. We therefore perform three fits one for $g >, \simeq, < 3.366$ and compare the results. So for each of these values of g_0 we tune $\{\mu_0, m_q^0, \Lambda_0\}$ to $\{m_\pi, f_\pi, m_q\}$, then all the parameters calculated at the IR scale are predictions of the model; among these are $\{m_\sigma$ (sigma mass), λ_{4-pt}, λ_{3-pt} (the sigma four- and three-point couplings), g, a_0^0, and $a_0^2\}$. The last two are $\pi\pi$ scattering lengths discussed in more detail later (see Appendix G for background).

The results of the three fits for $g >, \simeq, < 3.366$ are displayed in Table 3.1. The middle column is th result for fitting g_0 to $g \simeq 3.36$; the right and left columns are the results for arbitrarily choosing $g_0 = 2.500$ and 3.100 respectively. The last three rows (in the top section) contain the actual fits to m_π, f_π, and m_q. The point was to get $(m_\pi, f_\pi, m_q) \simeq$ (140MeV, 92.4MeV, 7.5MeV) [10]. The * and † indicate that m_q^0 and (μ_0,Λ_0) were used to fix m_q and (m_π,f_π). The values of m_q^0 and m_q all fall within the uncertainty quoted in the particle data book [10]. Rows 9 through 15 represent

Table 3.1. Results for numerical solution of σ-model RG flow equations with 6 quark flavors for three different values of the UV quark-meson coupling g_0.

Parameter	Fit 1	Fit 2	Fit 3
g	2.967	3.358	3.893
g_0	2.500	2.765	3.100
μ_0(MeV)†	666.0	739.4	818.0
m_q^0(MeV)*	6.54	6.42	6.36
Λ_0(MeV)†	950.0	937.9	927.0
m_π(MeV)†	140	140	140
f_π(MeV)†	92.60	92.49	92.44
m_q(MeV)*	7.48	7.51	7.54
m_σ(MeV)	507.3	536.1	550.4
λ_{4-pt}	17.3	20.8	30.1
λ_{3-pt}	43.0	50.8	64.2
$<\overline{\psi}\psi>^{1/3}$(MeV)	-190	-195	-201
$a_0^0(m_\pi^{-1})$	0.232	0.225	0.220
$a_0^2(m_\pi^{-1})$	-0.042	-0.043	-0.043
$(2a_0^0 - 5a_0^2)(m_\pi^{-1})$	0.677	0.664	0.656

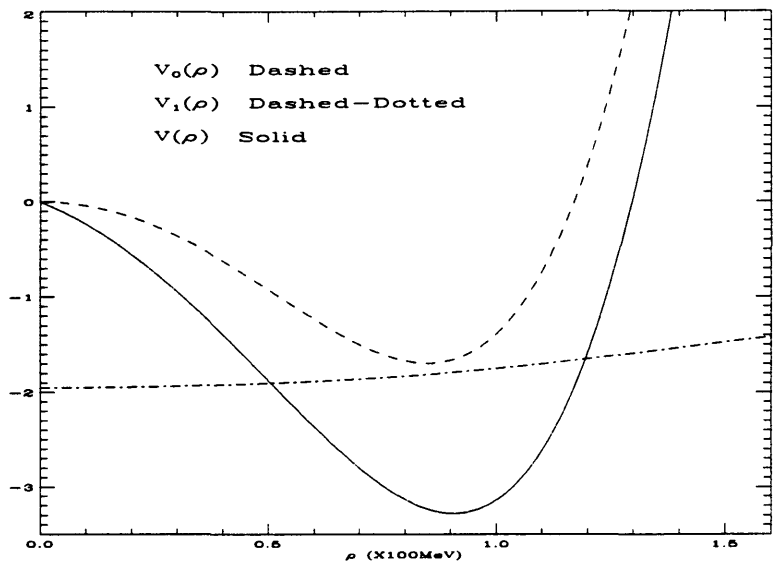

FIG. 3.6. RG boson potentials for the σ-model. The parameters of the calculation are displayed in Table 3.1

some of the predictions of the calculation. Since the sigma is less a "particle" and more a broad resonance, the values for m_σ, λ_{4-pt}, and λ_{3-pt} are hard to compare quantitatively with experiment. The values for the scalar density, $<\overline{\psi}\psi>$ are close to the typical calculated values of $<\overline{\psi}\psi>\sim -(240 \pm 25)$MeV [78].

Fig. 3.6 shows the boson potentials as a function of ρ. As expected for weakly broken chiral symmetry, the first order term, V_1 is just a small correction. One can see clearly that the minimum is at $\rho = <\sigma>_{vac} = f_\pi \simeq 92$ MeV. Fig. 3.7 displays the other potentials computed in the model. At $\rho \simeq 93$ MeV the values of m and g are just the IR values quoted in Table 3.1.

Another set of predictions from the RG solution of the σ-model come in the form of the parametrization of low energy $\pi\pi$ scattering. The expansion of the real part of

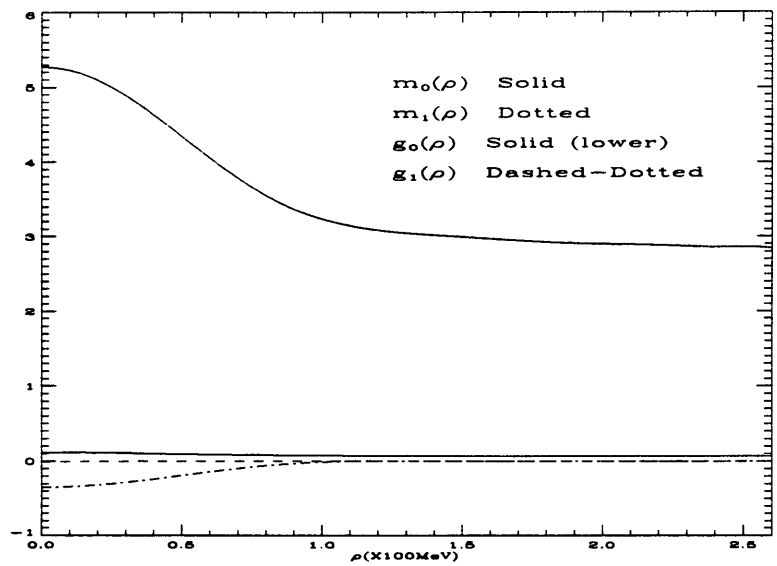

FIG. 3.7. RG potentials for the σ-model. The parameters of the calculation are displayed in Table 3.1

the partial-wave amplitude can be written as

$$\text{Re } A_l^I(s) = 32\pi \left(\frac{q^2}{m_\pi^2}\right)^l \left(a_l^I + b_l^I \frac{q^2}{m_\pi^2} + \cdots\right), \tag{3.15}$$

where $I = 0, 1, 2$ denotes the isospin channel and l is the partial wave index.[1] So for low energy ($q^2 \ll m_\pi^2$) scattering a_l^I and b_l^I will be the most relevent quantities to study. Table 3.2[2] displays a comparison of our three fits and a number of different calculations and experiment for a_0^0 and a_0^2 in dimensions of inverse pion mass. Also, the quantity $2a_0^0 - 5a_0^2$ is included since for s-wave scattering it provides a constraint

[1]See, *e.g.*, Refs.[23] section VI-4 and [79]. Appendix G contains a review of the perturbative calculation and the connection to our model.

[2]Experimental results in the first row are from Ref.[80]. The calculations in the second through the third row are quoted from Refs.[81, 82, 84] respectively and were performed using the perturbative σ-model, chiral perturbation theory (χPT), and lattice QCD respectively. In the last three rows are the results from our model for the three fits used in Table 3.1. Comprehensive reviews of $\pi\pi$ scattering are given in Ref.[23] section VI-4 and Ref.[85].

Table 3.2. Comparison of s-wave $\pi\pi$ scattering lengths obtained by measurement and various calculations in dimensions of inverse pion mass. See footnote for references.

	$a_0^0(m_\pi^{-1})$	$a_0^2(m_\pi^{-1})$	$2a_0^0 - 5a_0^2$
Experiment	0.26 ± 0.05	-0.028 ± 0.012	0.66 ± 0.12
Pert. σ-model	0.16	-0.045	0.56
χPT	0.20	-0.042	0.65
Lattice QCD	0.22	-0.042	0.65
Fit 1	0.232	-0.42	0.677
Fit 2	0.225	-0.043	0.664
Fit 3	0.220	-0.043	0.656

[81]. Each of the three fits give results that are consistent with experiment and with χPT and lattice QCD calculations. We discuss how the $\pi\pi$ scattering lengths are computed in our model in Appendix G.

There are a number of possible extensions to the present calculation. Perhaps the easiest is the extension to $\mathcal{O}(\sigma^2)$. Indeed much of the analytical work has been done. The expected result is that, for small current quark masses, the second order potentials will be small corrections to the first order potentials. We've performed $\mathcal{O}(\sigma^2)$ calculations with only the bosonic potentials flowing (*i.e.* fixing $m(\rho) = m_q^0$ and $g(\rho) = g_0$ for all Λ) and have seen that V_2 is small compared to V_1. The results for the other calculated parameters in the model shouldn't change substantially at

$\mathcal{O}(\sigma^2)$ since m_q^0 is small.

Another possible extension would be to include finite density. All of the calculations we've done so far have been for the vacuum and we've not been able to discuss bound states. Roughly, in this case there would be a momentum scale, k_F, below which the fermion parts of the flow equations would cease to contribute due to Pauli blocking, while the boson parts would still contribute to the flow. But this is a kluge. Handling finite density with the RG for relativistic field theories requires some care. For one thing, for $k_F > 0$ the theory is not Lorentz invariant. Preliminary calculations with the crude Pauli-blocking have been carried out and show interesting results such as the restoration of chiral symmetry at finite density. Thus far the treatment of finite density has been rather primitive and needs to be systematically elaborated upon.

Yet another extension is to incorporate strangeness into the scheme, to go to $SU(3) \times SU(3)$ symmetry breaking. As mentioned in section 1.2.2, since $m_s \sim \Lambda_{QCD}$, chiral $SU(3) \times SU(3)$ is *strongly* broken. Whereas in the $SU(2) \times SU(2)$ case outlined above the spontaneous overwhelmed the explicit symmetry breaking, for $SU(3) \times SU(3)$ just the opposite is the case. So the higher orders in σ might be ill-behaved or may grow to a maximum at some order and decrease from there. It may be that only the first or second order flow equations make sense.

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Appendix A

CONVENTIONS

In this appendix we present a listing of the conventions used in this thesis for Dirac matrices and $SU(2)$ Pauli matrices.

Most results obtained depend only on representation independent properties of the γ-matrices which are defined to satisfy,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{A.1}$$

where we use a metric with signature, $\mathrm{diag}(g^{\mu\nu}) = (+---)$. Also, one defines,

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5^\dagger, \tag{A.2}$$

and thus,

$$\begin{aligned} \gamma_5^2 &= \mathbf{1} \\ \{\gamma^\mu, \gamma^5\} &= 0. \end{aligned} \tag{A.3}$$

The representation used in this thesis is the "chiral representation" where (in 2×2 block form),

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & -\mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix}, \qquad i = 1, 2, 3 \\ \gamma^5 &= \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \end{aligned} \tag{A.4}$$

where the $\{\tau^i\}$ are the Pauli matrices,

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.5}$$

which have the property $\tau^i\tau^j = \delta^{ij} + i\epsilon^{ijk}\tau^k$ and thus,

$$\{\tau^i, \tau^j\} = 2\delta^{ij}. \tag{A.6}$$

The "Dirac adjoint," $\overline{\psi}$ is defined as,

$$\overline{\psi} = \psi^\dagger\gamma^0, \tag{A.7}$$

so that the currents such as,

$$j^\mu = \overline{\psi}\gamma^\mu\psi, \tag{A.8}$$

tranform as a Lorentz 4-vectors and the scalar density, $\rho \equiv \psi^\dagger\psi = j^0 > 0$. In relativistic quantum mechanics the Lorentz invariant combination $\gamma^\mu a_\mu$ frequently appears for which we employ the "Feynman" slash shorthand,

$$\not{a} \equiv \gamma^\mu a_\mu. \tag{A.9}$$

Appendix B

REVIEW OF SUPERMATRICES

A "supermatrix" is a square matrix of the form,

$$M = \begin{pmatrix} M_{BB} & M_{BF} \\ M_{FB} & M_{FF} \end{pmatrix}, \tag{B.1}$$

where the square submatrices M_{BB} and M_{FF} are both *even* elements of the Grassmann algebra[1] while M_{BF} and M_{FB} are *odd* elements of the Grassmann algebra.

The supertrace is defined as,

$$\mathrm{str}M = \mathrm{tr}M_{BB} - \mathrm{tr}M_{FF} \tag{B.4}$$

so that the familiar commutative property of the normal trace still holds for the supertrace. The superdeterminant is defined as,

$$\mathrm{sdet}M = \mathrm{e}^{\mathrm{str}\ln M}, \tag{B.5}$$

which preserves the familiar property of determinants, $\det MN = \det M \det N$.

[1] Grassmann or anticommuting variables allow the incorporation of Fermi statistics into the path integral formalism of quantum field theory. An arbitrary Grassmann number can be written as,

$$f(\theta) = a + \beta\theta, \tag{B.2}$$

where a is a regular number and β and θ obey the Grassmann algebra,

$$\{\theta, \theta\} = \theta\theta + \theta\theta = 0 \Rightarrow \theta^2 = 0 \tag{B.3}$$

this is why the above series terminates. For more details see any good field theory book, *e.g.*, Ref.[45] p.214-219.

Now consider the decomposition of M:

$$M = \begin{pmatrix} M_{BB} & 0 \\ M_{FB} & 1 \end{pmatrix} \begin{pmatrix} 1 & M_{BB}^{-1} M_{BF} \\ 0 & N_{FF} \end{pmatrix} \tag{B.6}$$

where $N_{FF} = M_{FF} - M_{FB} M_{BB}^{-1} M_{BF}$. Now it's easy to show

$$\mathrm{sdet} \begin{pmatrix} M_{BB} & 0 \\ M_{FB} & 1 \end{pmatrix} = \det M_{BB} \tag{B.7}$$

and,

$$\mathrm{sdet} \begin{pmatrix} 1 & M_{BB}^{-1} M_{BF} \\ 0 & N_{FF} \end{pmatrix} = \det N_{FF} \tag{B.8}$$

so that,

$$\mathrm{sdet} M = \det M_{BB} \det{}^{-1} N_{FF}. \tag{B.9}$$

Similarly we could have chosen the decomposition,

$$M = \begin{pmatrix} N_{BB} & M_{BF} M_{FF}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ M_{FB} & M_{FF} \end{pmatrix} \tag{B.10}$$

where $N_{BB} = M_{BB} - M_{BF} M_{FF}^{-1} M_{FB}$. Then we would be led to,

$$\mathrm{sdet} M = \det N_{BB} \det{}^{-1} M_{FF}. \tag{B.11}$$

Appendix C

PROOF OF THE GAUSSIAN INTEGRAL FORMULA

In this Appendix a proof for the formula,

$$I = \int \mathcal{D}\overline{\Omega}\mathcal{D}\Omega \; \mathrm{e}^{-\overline{\Omega}M\Omega - \overline{\mathcal{J}}\Omega - \overline{\Omega}\mathcal{J}} \propto \mathrm{e}^{\overline{\mathcal{J}}M^{-1}\mathcal{J}} \mathrm{sdet}^{-1} M \tag{C.1}$$

(which is used in section 2.1) will be given. Begin by writing out,

$$\begin{aligned} \overline{\Omega}M\Omega &= (\overline{\phi} \;\; \overline{\psi}) \begin{pmatrix} \Sigma & \mathcal{A} \\ \overline{\mathcal{A}} & \mathcal{F} \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \\ &= \overline{\phi}\Sigma\phi + \overline{\psi\mathcal{A}}\phi + \overline{\phi}\mathcal{A}\psi + \overline{\psi}\mathcal{F}\psi \end{aligned} \tag{C.2}$$

and similarly for the $\overline{\mathcal{J}}\Omega + \overline{\Omega}\mathcal{J}$ term where, *e.g.*

$$\overline{\phi} = \begin{pmatrix} \phi^*_{q_1} \\ \phi^*_{q_2} \\ \vdots \end{pmatrix}. \tag{C.3}$$

The measure is given by,

$$\begin{aligned} \mathcal{D}\overline{\Omega}\mathcal{D}\Omega &= \mathcal{D}\phi^*\mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi \\ &= \prod_{q_1} d\phi^*_{q_1} \prod_{q_2} d\phi_{q_2} \prod_{q_3} d\overline{\psi}_{q_3} \prod_{q_4} d\psi_{q_4}. \end{aligned} \tag{C.4}$$

Now we have,

$$I = \int \mathcal{D}\phi^*\mathcal{D}\phi \; \mathrm{e}^{-S_B} \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \; \mathrm{e}^{-S_F - S_{mix}} \tag{C.5}$$

with, *e.g.*

$$S_B = \phi^*_{q_1} \Sigma_{q_1 q_2} \phi_{q_2} + J^*_{q_1} \phi_{q_1} + \phi^*_{q_1} J_{q_1} \tag{C.6}$$

(summing over repeated momentum indices). The integral over the ψ variables is performed using the standard formula (see, *e.g.*, Ref.[3] p. 442) and after rearranging terms we have,

$$I = (\det\mathcal{F}) e^{\overline{\eta}\mathcal{F}^{-1}\eta} \int \mathcal{D}\phi^* \mathcal{D}\phi \; e^{-\phi^*(\Sigma - \mathcal{A}\mathcal{F}\overline{\mathcal{A}})\phi - j^*\phi - \phi^* j} \tag{C.7}$$

with $j \equiv J^* - \overline{\eta}\mathcal{F}^{-1}\overline{\mathcal{A}}$. The integral over the ϕ variables is performed using another standard formula[1] (Ref.[3] p. 439) so that, after more rearranging we arrive at

$$\begin{aligned} I &= (2\pi i)(\det{}^{-1} N)(\det\mathcal{F}) e^{\overline{\eta}\mathcal{F}^{-1}\eta + j^* N^{-1} j} \\ &= (2\pi i)(\mathrm{sdet}^{-1} M) e^{\overline{\eta}\mathcal{F}^{-1}\eta + j^* N^{-1} j} \end{aligned} \tag{C.9}$$

using Eq.(B.11) where $N = \Sigma - \mathcal{A}\mathcal{F}^{-1}\overline{\mathcal{A}}$. It remains to show that,

$$\overline{\eta}\mathcal{F}^{-1}\eta + j^* N^{-1} j = \overline{\mathcal{J}} M^{-1} \mathcal{J}. \tag{C.10}$$

Using the l.h.s. of this expression we obtain,

$$M^{-1} = \begin{pmatrix} N^{-1} & -N^{-1}\mathcal{A}\mathcal{F}^{-1} \\ -\mathcal{F}^{-1}\overline{\mathcal{A}}N^{-1} & \mathcal{F}^{-1} + \mathcal{F}^{-1}\overline{\mathcal{A}}N^{-1}\mathcal{A}\mathcal{F}^{-1} \end{pmatrix}. \tag{C.11}$$

[1]The factors of 1/2 introduced when ϕ is *real* appear here via the variant of the standard formula (Ref.[3] p. 586),

$$\int \mathcal{D}\phi \; e^{-\phi K \phi - \phi \cdot j} = (\det{}^{-1/2} K) e^{j K^{-1} j/2} \tag{C.8}$$

Now using the decomposition (Eq.(B.10)) for M one readily finds that $M^{-1}M = \mathbf{1}$ proving our assertion.

Appendix D

SIMILARITY TRANSFORMATIONS

We define the similarity tranformation matrix as the rotation matrix that brings $\vec{\phi} = (\sigma, \pi^i)$ coincident with the σ-axis, *i.e.*,

$$\mathbf{S} \cdot \vec{\phi} = \rho \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \equiv \rho\hat{\sigma}. \tag{D.1}$$

In the $O(3)$ case $\mathbf{S} = \mathbf{R}_2 \cdot \mathbf{R}_1$ where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the ususal rotation matrices in 3-space. We choose $\mathbf{R}_1$ to rotate $\vec{\phi}$ through angle α_1 into the $\sigma - \pi_1$ plane and $\mathbf{R}_2$ to rotate through angle α_2 to the σ-axis; thus,

$$\mathbf{S}^{O(3)} = \mathbf{R}_2 \cdot \mathbf{R}_1 = \begin{pmatrix} \cos\alpha_2 & \sin\alpha_2 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix} \tag{D.2}$$

$$= \frac{1}{\rho} \begin{pmatrix} \sigma & \pi_1 & \pi_2 \\ -\pi_1' & \frac{\sigma\pi_1}{\pi_1'} & \frac{\sigma\pi_2}{\pi_1'} \\ 0 & -\frac{\pi_2\rho}{\pi_1'} & \frac{\pi_1\rho}{\pi_1'} \end{pmatrix}, \tag{D.3}$$

where we've substituted,

$$\begin{aligned} \sigma &= \rho\cos\alpha_2 \\ \pi_1 &= \rho\sin\alpha_2\cos\alpha_1 \\ \pi_2 &= \rho\sin\alpha_2\sin\alpha_1 \\ \rho &= \sqrt{\sigma^2 + \pi_1^2 + \pi_2^2} \end{aligned} \tag{D.4}$$

$$\pi_1^{'} = \sqrt{\pi_1^2 + \pi_2^2},$$

in the last step. The fact that $\mathbf{S}^{O(3)} \cdot \vec{\phi} = \rho\hat{\sigma}$ is easily checked.

For $O(4)$ we choose the same order for the rotation: $\mathbf{R}_1$ rotates $\vec{\phi}$ through angle α_1 into the $\sigma - \pi_1 - \pi_2$ space and $\mathbf{R}_2$ rotates through angle α_2 to the $\sigma - \pi_1$ plane and $\mathbf{R}_3$ rotates through angle α_3 to the σ axis:

$$\mathbf{S}^{O(4)} = \mathbf{R}_3 \cdot \mathbf{R}_2 \cdot \mathbf{R}_1 = \begin{pmatrix} \cos\alpha_3 & \sin\alpha_3 & 0 & 0 \\ -\sin\alpha_3 & \cos\alpha_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{D.5}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_2 & \sin\alpha_2 & 0 \\ 0 & -\sin\alpha_2 & \cos\alpha_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & 0 & -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix} \tag{D.6}$$

$$= \frac{1}{\rho} \begin{pmatrix} \sigma & \pi_1 & \pi_2 & \pi_3 \\ -\pi_1' & \frac{\sigma\pi_1}{\pi_1'} & \frac{\sigma\pi_2}{\pi_1'} & \frac{\sigma\pi_3}{\pi_1'} \\ 0 & -\frac{\rho\pi_2'}{\pi_1'} & \rho\frac{\pi_2\pi_1}{\pi_2'\pi_1'} & \rho\frac{\pi_3\pi_1}{\pi_2'\pi_1'} \\ 0 & 0 & -\rho\frac{\pi_3}{\pi_2'} & \rho\frac{\pi_2}{\pi_2'} \end{pmatrix}, \tag{D.7}$$

where we've substituted,

$$\begin{aligned} \sin\alpha_1 &= \frac{\pi_3}{\pi_2^{'}}; \qquad \sin\alpha_2 = \frac{\pi_3}{\pi_2^{'}}; \qquad \sin\alpha_3 = \frac{\pi_1^{'}}{\rho} \\ \cos\alpha_1 &= \frac{\pi_2}{\pi_2^{'}}; \qquad \cos\alpha_2 = \frac{\pi_1}{\pi_1^{'}}; \qquad \cos\alpha_3 = \frac{\sigma}{\rho}, \end{aligned} \tag{D.8}$$

and,

$$
\begin{aligned}
\rho &= \sqrt{sigma^2 + \pi_1^2 + \pi_2^2 + \pi_3^2} \\
\pi_1^{'} &= \sqrt{\pi_1^2 + \pi_2^2 + \pi_3^2} \\
\pi_2^{'} &= \sqrt{\pi_2^2 + \pi_3^2}, \qquad \text{(D.9)}
\end{aligned}
$$

in the last equality. Once again it's easy to check that $\mathbf{S}^{O(4)} \cdot \vec{\phi} = \rho\hat{\sigma}$. Also $\mathbf{S} \cdot \mathbf{S}^T = \mathbf{1}$ for both $O(3)$ and $O(4)$. In the next appendix we use $\mathbf{S}$ for $O(4)$ to perform similarity transformations of $\mathbf{\Sigma}$ and $\mathbf{\Omega}$ which greaty simplify them.

Appendix E

DERIVATIVES OF THE CHIRAL FUNCTIONS

In the derivation of the flow equations for the σ-model, derivatives with respect to ϕ^a of functions of ρ, σ, and $\Gamma = \Gamma^a\phi^a$ are taken. In this appendix we work out what these derivatives are. Consider the potential,

$$U(\rho,\sigma,\Gamma) = m(\rho,\sigma) + \Gamma g(\rho,\sigma) \tag{E.1}$$

where,

$$m(\rho,\sigma) = m_0(\rho) + \sigma m_1(\rho) + \frac{\sigma^2}{2} m_2(\rho) + \cdots \tag{E.2}$$

and similarly for g. Now,

$$\begin{aligned} \frac{\partial U}{\partial \phi^a} &\equiv U'^a = m'^a + \Gamma g'^a + \Gamma^a g \\ \frac{\partial^2 U}{\partial \phi^a \partial \phi^b} &\equiv U''^{ab} = m''^{ab} + \Gamma g''^{ab} + \Gamma^a g'^b + g'^a \Gamma^b. \end{aligned} \tag{E.3}$$

We need to work out the derivatives of functions of ρ and σ. Consider $f(\rho,\sigma)$, $f = m$ or g:

$$\begin{aligned} \frac{\partial f}{\partial \phi^a} \equiv f'^a &= f^{(0)'a} + \sigma f^{(1)'a} + \frac{\sigma^2}{2} f^{(2)'a} + \delta^{a0}(f^{(1)} + \sigma f^{(2)}) \\ &= \frac{\phi^a}{\rho} f' + \delta^{a0}(f^{(1)} + \sigma f^{(2)}) \\ \frac{\partial^2 f}{\partial \phi^a \partial \phi^b} \equiv f''^{ab} &= f^{(0)''ab} + \sigma f^{(1)''ab} + \frac{\sigma^2}{2} f^{(2)''ab} + \delta^{a0}(f^{(1)'b} + \sigma f^{(2)'b}) \\ &\quad + (f^{(1)'a} + \sigma f^{(2)'a})\delta^{0b} + \delta^{a0}\delta^{0b} f^{(2)} \end{aligned}$$

$$
\begin{aligned}
&= \delta^{ab}\frac{f'}{\rho} + \frac{\phi^a\phi^b}{\rho^2}\left(f'' - \frac{f'}{\rho}\right) \\
&\quad + \frac{\delta^{a0}\phi^b + \phi^a\delta^{0b}}{\rho}\left(f^{(1)'} + \sigma f^{(2)'}\right) + \delta^{a0}\delta^{0b} f^{(2)}. \qquad \text{(E.4)}
\end{aligned}
$$

In the last of each of the above equalities we've used,

$$
\begin{aligned}
h'^a &= \frac{\phi^a}{\rho} h'(\rho) \\
h''^{ab} &= \delta^{ab}\frac{h'}{\rho} + \frac{\phi^a\phi^b}{\rho^2}\left(h'' - \frac{h'}{\rho}\right), \qquad \text{(E.5)}
\end{aligned}
$$

where $h = h(\rho)$, $h = m^{(i)}$ or $g^{(i)}$ for $i = 0, 1, 2$. Also,

$$
\begin{aligned}
f'(\rho) &= f^{(0)'} + \sigma f^{(1)'} + \frac{\sigma^2}{2} f^{(2)'} \\
f''(\rho) &= f^{(0)''} + \sigma f^{(1)''} + \frac{\sigma^2}{2} f^{(2)''}. \qquad \text{(E.6)}
\end{aligned}
$$

With these derivatives, we can compute the matrices in section 2.4:

$$
\begin{aligned}
\Sigma^{ab} &\equiv \Lambda^2\delta^{ab} + V''^{ab}(\rho,\sigma) \\
&= \left(\Lambda^2 + \frac{V'}{\rho}\right)\delta^{ab} + \left(V'' - \frac{V'}{\rho}\right)\frac{\phi^a\phi^b}{\rho^2} \\
&\quad + \tilde{V}' \frac{\delta^{a0}\phi^b + \phi^a\delta^{0b}}{\rho} + V^{(2)}\delta^{a0}\delta^{0b} \\
&= \Sigma_\delta\delta^{ab} + \Sigma_{\phi\phi}\frac{\phi^a\phi^b}{\rho^2} + \Sigma_{\delta\phi}\frac{\delta^{a0}\phi^b + \phi^a\delta^{0b}}{\rho} + V^{(2)}\delta^{a0}\delta^{0b} \qquad \text{(E.7)}
\end{aligned}
$$

and,

$$
\begin{aligned}
\Omega^{ab} &\equiv U''^{ab}(\rho,\sigma,\Gamma) - \frac{2}{D_F}U'^a(\rho,\sigma,\Gamma)U^\dagger(\rho,\sigma,\Gamma)U'^b(\rho,\sigma,\Gamma) \\
&= \frac{U'}{\rho}\delta^{ab} + \left(U'' - \frac{U'}{\rho} - \frac{2U^\dagger}{D_F}U'^2\right)\frac{\phi^a\phi^b}{\rho^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\tilde{U}' - \frac{2U^\dagger}{D_F}U'\tilde{U}\right)\frac{\delta^{a0}\phi^b + \phi^a\delta^{0b}}{\rho} \\
&+\frac{2U^\dagger}{D_F}\Gamma g'\tilde{m}\frac{\phi^a\delta^{0b}}{\rho} + \frac{2U^\dagger}{D_F}\Gamma m'\tilde{g}\frac{\delta^{a0}\phi^b}{\rho} \\
&+\left(U^{(2)} - \frac{2U^\dagger}{D_F}\tilde{U}^2\right)\delta^{a0}\delta 0b + \left(g' - \frac{2U^\dagger}{D_F}gU'\right)\frac{\Gamma^a\phi^b + \phi^a\Gamma^b}{\rho} \\
&+\left(\tilde{g} - \frac{2U^\dagger}{D_F}\tilde{U}\right)\left(\Gamma^a\delta^{0b} + \delta^{a0}\Gamma^b\right) - \frac{2g^2}{D_F}\Gamma^a U^\dagger\Gamma^b \\
= \;& \Omega_\delta\delta^{ab} + \Omega_{\phi\phi}\frac{\phi^a\phi^b}{\rho^2} + \Omega^S_{\delta\phi}\frac{\delta^{a0}\phi^b + \phi^a\delta^{0b}}{\rho} + \Omega_{\phi\delta}\frac{\phi^a\delta^{0b}}{\rho} + \Omega_{\delta\phi}\frac{\delta^{a0}\phi^b}{\rho} \\
&+\Omega_{\delta\delta}\delta^{a0}\delta^{0b} + \Omega_{\Gamma\phi}\frac{\Gamma^a\phi^b + \phi^a\Gamma^b}{\rho} + \Omega_{\Gamma\delta}(\Gamma^a\delta^{0b} + \delta^{a0}\Gamma^b) - \Gamma^a\Omega_{\Gamma\Gamma}\Gamma^b, \quad \text{(E.8)}
\end{aligned}
$$

where,

$$
\begin{aligned}
\tilde{X} &= X^{(1)} + \sigma X^{(2)}, \qquad X = V, U, m, g \\
U' &= m' + \Gamma g'. \qquad \text{(E.9)}
\end{aligned}
$$

Note that the Σs are all functions of ρ and σ and the Ωs are all functions of ρ, σ and Γ. The ordering in all the terms containing Γs is nontrivial since the Pauli matrices dont commute with each other.

Now using the similarity transformation matrix from Appendix D we can compute,

$$
\begin{aligned}
\mathbf{\Sigma}' &= \mathbf{S}\cdot\mathbf{\Sigma}\cdot\mathbf{S}^T \\
\mathbf{\Omega}' &= \mathbf{S}\cdot\mathbf{\Omega}\cdot\mathbf{S}^T, \qquad \text{(E.10)}
\end{aligned}
$$

this amounts to performing similarity transformations on each of the tensors

$$
\delta^{ab}, \qquad \frac{\phi^a\phi^b}{\rho^2}, \qquad \frac{\phi^a\delta^{0b}}{\rho}, \qquad \frac{\delta^{a0}\phi^b}{\rho}, \qquad \delta^{a0}\delta^{0b},
$$

$$\frac{\Gamma^a\phi^b + \phi^a\Gamma^b}{\rho}, \qquad (\Gamma^a\delta^{0b} + \delta^{a0}\Gamma^b), \qquad \Gamma^a\Gamma^b, \tag{E.11}$$

as can be read off from Eqs.(2.70) and (E.8). After performing the similarity transformation $\mathbf{\Sigma}'$ has the form,

$$\mathbf{\Sigma}' = \begin{pmatrix} \Sigma_{00} & \Sigma_{01} & 0 & 0 \\ \Sigma_{10} & \Sigma_{11} & 0 & 0 \\ 0 & 0 & \Sigma_{22} & 0 \\ 0 & 0 & 0 & \Sigma_{33} \end{pmatrix}, \tag{E.12}$$

since $\Sigma^{01} = \Sigma^{10}$ we can write,

$$\begin{aligned} \det \mathbf{\Sigma}' &= (\Sigma^{00}\Sigma^{11} - (\Sigma^{01})^2)\Sigma^{22}\Sigma^{33} \\ \mathrm{tr}\, \mathbf{\Sigma}'^{-1} \cdot \mathbf{\Omega}' &= (\Sigma^{00})^{-1}\Omega^{00} + (\Sigma^{10})^{-1}\Omega^{01} + (\Sigma^{01})^{-1}\Omega^{00} \\ &\quad +(\Sigma^{11})^{-1}\Omega^{11} + (\Sigma^{22})^{-1}\Omega^{22} + (\Sigma^{33})^{-1}\Omega^{33}. \end{aligned} \tag{E.13}$$

Thus only 12 elements—6 from $\mathbf{\Sigma}'$ and 6 from $\mathbf{\Omega}'$—need be computed to determine the flow equations for the σ-model. As is probably clear this substantially reduces the complexity of the expressions but they they are still quite complicted. The derivation is facilitated by using Mathematica to compute the Σ and Ω functions in Eqs.(E.7) and (E.8). Then Eqs.(E.13) can be written in terms of these expressions.

Appendix F

SPECIFIC FORMS OF THE FLOW EQUATIONS

In this appendix we list the expressions used in the right hand side of Eqs.(3.11). The output is in FORTRAN form so that it could simply be cut and pasted from the code (instead of typed in!). A translation table for the notation in the body of this thesis is included first.

$$
\begin{aligned}
Dphi &= \Lambda^2 + V_0'' + \sigma V_1'' + \frac{\sigma}{2} V'' \\
Dpi &= \Lambda^2 + (V_0' + \sigma V_1' + \frac{\sigma}{2} V')/\rho \\
V_k' &= ukp; \qquad k = 0,1,2 \\
V_k'' &= ukpp; \qquad k = 0,1,2 \\
m_k' &= mkp; \qquad k = 0,1,2 \\
m_k'' &= mkpp; \qquad k = 0,1,2 \\
g_k' &= gkp; \qquad k = 0,1,2 \\
g_k' &= gkpp; \qquad k = 0,1,2 \\
D_F &= D_f = \Lambda^2 + m^2 + 2\sigma mg + \rho^2 g^2 \\
cdnf &= c_d n_f
\end{aligned}
\tag{F.1}
$$

Now the expressions in terms of this notation are:

```
denom=Dpi**3*Dphi - Dpi**2*u1p**2
```

```
denom0=Dpi0**3*Dphi0

F0(ρ)=denom/denom0

F1(ρ)=-cdnf*(2*g0*m0+2*m0*m1+2*g0*g1*x**2)/Df
- + (3*Dphi*Dpi**2*u1poverx+2*Dpi**3*u1poverx
- - 2*Dpi*u1p**2*u1poverx+Dpi**3*u1pp)

Δm0(ρ)=(m0pp*Dpi**3-
- 4*m0p*g0*x*(g0+g0p*x)*Dpi**3/Df -
- 2*m0*(m0p**2 - (g0 + g0p*x)**2)*Dpi**3/Df +
- 3*m0poverx*Dpi**2*Dphi -
- 4*g0*(m1 + g0)*g1*(x*Dpi)**2*Dphi/Df +
- 2*g1*Dpi**2*Dphi + 4*m0*g0**2*Dpi**2*Dphi/Df -
- 2*m0*(m1**2 + 2*m1*g0 + g0**2 - g1**2*x**2)*
- Dpi**2*Dphi/Df-2*m1p*Dpi**2*U1p-2*g0p*Dpi**2*U1p -
- 4*m0*(-(m0p*m1)-m0p*g0+g0*g1*x+g0p*g1*x**2)*Dpi**2*U1p/Df-
- 4*g0*x*(-(m0p*g1*x)-(m1+g0)*(g0+g0p*x))*
- Dpi**2*U1p/Df-2*m0p*x*Dpi*U1poverx**2-
- 4*m0*g0**2*Dpi*U1p**2/Df)/denom

Δm1(ρ)=-((2*Dphi*Dpi**2*g0**2*m0/Df+2*Dpi**3*g0**2*m0/Df+
- 3*Dphi*Dpi**2*m0p/x+
- 4*Dpi**3*g0*(g0p*m0-g0*m0p)*x/Df-
- 2*Dphi*Dpi**2*(m0*m1**2-g1**2*m0*x**2+2*g0*g1*m1*x**2)/
```

```
- Df+2*Dphi*Dpi**2*(g1-2*g0*(m0*m1+g0*g1*x**2)/Df)+
- Dpi**3*(m0pp-2*
- (m0*m0p**2+g0p*(-(g0p*m0)+2*g0*m0p)*x**2)/Df)+
- 4*Dpi**2*g0**3*x*u1p/Df-
- 4*Dpi**2*g0*(g1*m0-g0*m1)*x*u1p/Df-
- 2*Dpi**2*(g0p-2*g0*(m0*m0p+g0*g0p*x**2)/Df)*u1p-
- 2*Dpi**2*(m1p-2*
- (g1*(-(g0p*m0)+g0*m0p)*x**2+
- m1*(m0*m0p+g0*g0p*x**2))/Df)*u1p-
- 4*Dpi*g0**2*m0*u1p**2/Df-2*Dpi*m0p*u1p**2/x)*
- (3*Dphi*Dpi**2*u1p+2*Dpi**3*u1p-2*Dpi*u1p**3+
- Dpi**3*x*u1pp)/(Dphi*Dpi**3-Dpi**2*u1p**2)**2)+
- (-4*Dphi*Dpi**2*g0**3*x/Df-4*Dpi**3*g0**3*x/Df-
- 4*Dphi*Dpi**2*g0*(g1*m0-g0*m1)*x/Df+
- 4*Dpi**3*g0*(g1*m0-g0*m1)*x/Df+
- 2*Dpi**3*(m1p-2*(g1*(-(g0p*m0)+g0*m0p)*x**2+
- m1*(m0*m0p+g0*g0p*x**2))/Df)+
- Dpi**3*(m1pp*x-2*
- (-((m0*m0p**2+g0p*(-(g0p*m0)+2*g0*m0p)*x**2)*
- (2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2)+
- (2*m0*m0p*m1p*x+m0p**2*(2*g0*x+m1*x)+
- x**2*(g1p*(-(g0p*m0)+2*g0*m0p)*x+
- g0p*(-(g1p*m0*x)-g0p*m1*x+
- 2*(g1*m0p*x+g0*m1p*x))))/Df))+
- 8*Dpi**2*g0**2*m0*u1p/Df+6*Dpi**2*m0p*u1p/x+
```

```
- 4*Dpi**2*g0*(g0p*m0-g0*m0p)*x*u1p/Df+
- 2*Dpi**2*(g1-2*g0*(m0*m1+g0*g1*x**2)/Df)*u1p+
- 3*Dpi**2*(m0pp-2*(m0*m0p**2+g0p*(-(g0p*m0)+2*g0*m0p)*x**2)/
- Df)*u1p+8*Dpi*g0**3*x*u1p**2/Df+
- 2*(Dpi**3*(-(g0**2*m0*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/
- Df**2)+(2*g0*g1*m0*x+g0**2*(2*g0*x+m1*x))/Df)
- +3*Dpi**2*g0**2*m0*u1p/Df)+
- 4*x*(Dpi**3*(g0*(g1p*m0*x-g1*m0p*x+g0p*m1*x-
- g0*m1p*x)/Df+
- (g0p*m0-g0*m0p)*
- (g1*x/Df-g0*
- (2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2))+
- 3*Dpi**2*g0*(g0p*m0-g0*m0p)*u1p/Df)+
- Dphi*(-2*Dpi**2*(-((m0*m1**2-g1**2*m0*x**2+
- 2*g0*g1*m1*x**2)*
- (2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2)+
- (g1**2*m1*x**3+m1**2*(2*g0*x+m1*x))/Df)-
- 4*Dpi*(m0*m1**2-g1**2*m0*x**2+2*g0*g1*m1*x**2)*u1p/Df)
- +4*x*(Dpi**2*(3*g0**2*g1*x/Df-
- g0**3*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2)*u1p
- +2*Dpi*g0**3*u1p**2/Df)-
- 4*x*(Dpi**2*(g1*(g1*m0-g0*m1)*x/Df-
- g0*(g1*m0-g0*m1)*
- (2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2)*u1p+
- 2*Dpi*g0*(g1*m0-g0*m1)*u1p**2/Df)-
```

```
- 2*(Dpi**2*(g1p*x-
- 2*(g0*(2*g0*m0p*x+m0p*m1*x+m0*m1p*x+
- x**2*(g0p*g1*x+g0*g1p*x))/Df+
- (m0*m0p+g0*g0p*x**2)*
- (g1*x/Df-
- g0*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2))
- )*u1p+2*Dpi*(g0p-2*g0*(m0*m0p+g0*g0p*x**2)/Df)*u1p**2
- )-2*(-2*Dpi**2*(-((2*g0*m0*x+2*m0*m1*x+
- 2*g0*g1*x**3)*
- (g1*(-(g0p*m0)+g0*m0p)*x**2+
- m1*(m0*m0p+g0*g0p*x**2))/Df**2)+
- (g1*x**2*(-(g1p*m0*x)+g1*m0p*x-g0p*m1*x+
- g0*m1p*x)+
- m1*(m0*m1p*x+x**2*(g0p*g1*x+g0*g1p*x)+
- m0p*(2*g0*x+m1*x)))/Df)*u1p+
- 2*Dpi*(m1p-2*(g1*(-(g0p*m0)+g0*m0p)*x**2+
- m1*(m0*m0p+g0*g0p*x**2))/Df)*u1p**2)-
- 4*(Dpi*(-(g0**2*m0*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/
- Df**2)+(2*g0*g1*m0*x+g0**2*(2*g0*x+m1*x))/Df)*
- u1p**2+g0**2*m0*u1p**3/Df)-
- 2*(Dpi*m1p*x*u1p**2+m0p*u1p**3)/x-
- 2*Dpi**2*x*(m0*m1**2-g1**2*m0*x**2+2*g0*g1*m1*x**2)*u1pp/
- Df+2*(Dphi*(Dpi**2*
- (-(g0**2*m0*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/
- Df**2)+
```

```
- (2*g0*g1*m0*x+g0**2*(2*g0*x+m1*x))/Df)+
- 2*Dpi*g0**2*m0*u1p/Df)+Dpi**2*g0**2*m0*x*u1pp/Df)+
- 3*(Dphi*(Dpi**2*m1p*x+2*Dpi*m0p*u1p)+Dpi**2*m0p*x*u1pp)/
- x+2*(Dphi*(-2*Dpi**2*
- (g0*(g1**2*x**3+m1*(2*g0*x+m1*x))/Df+
- (m0*m1+g0*g1*x**2)*
- (g1*x/Df-
- g0*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2))
- +2*Dpi*(g1-2*g0*(m0*m1+g0*g1*x**2)/Df)*u1p)+
- Dpi**2*x*(g1-2*g0*(m0*m1+g0*g1*x**2)/Df)*u1pp))/
- (Dphi*Dpi**3-Dpi**2*u1p**2)
```

$\Delta g_0(\rho)$=(2*g0poverx*Dpi**3+g0pp*Dpi**3 -

```
- 2*(2*m0*m0poverx*(g0 + g0p*x) +
- g0*(-m0p**2 + (g0 + g0p*x)**2))*Dpi**3/Df +
- 3*g0poverx*Dpi**2*Dphi +
- 4*g0**3*Dpi**2*Dphi/Df - 2*(2*m0*(m1 + g0)*g1 +
- g0*(-m1**2-2*m1*g0-g0**2+g1**2*x**2))*
- Dpi**2*Dphi/Df -
- 2*g1*Dpi**2*u1poverx-2*g1p*Dpi**2*U1p -
- 4*(g0*x*(m0p*m1+m0p*g0-g0*g1*x -g0p*g1*x**2)+
- m0*(-(m0p*g1*x) - (m1 + g0)*(g0+g0p*x)))*
- Dpi**2*u1poverx/Df -
- 2*g0p*(x*Dpi)*U1poverx**2 -
- 4*g0**3*Dpi*U1p**2/Df)
```

$\Delta g_1(\rho)$=(-((3*Dphi*Dpi**2*g0p+6*Dphi*Dpi**2*g0**3*x/Df -

- 2*Dpi**3*g0**3*x/Df-

- 4*Dphi*Dpi**2*g0*(g1*m0-g0*m1)*x/Df-

- 2*Dphi*Dpi**2*x*(2*g1*m0*m1-g0*m1**2+g0*g1**2*x**2)/

- Df+2*Dpi**3*(g0p-2*g0*(m0*m0p+g0*g0p*x**2)/Df)+

- Dpi**3*x*(g0pp-

- 2*(2*g0p*m0*m0p-g0*m0p**2+g0*g0p**2*x**2)/Df)+

- 4*Dpi**2*g0**2*m0*u1p/Df+

- 4*Dpi**2*g0*(g0p*m0-g0*m0p)*x*u1p/Df-

- 2*Dpi**2*(g1-2*g0*(m0*m1+g0*g1*x**2)/Df)*u1p-

- 2*Dpi**2*x*(g1p-

- 2*((g0p*m0-g0*m0p)*m1+g1*(m0*m0p+g0*g0p*x**2))/Df)*

- u1p-2*Dpi*g0p*u1p**2-4*Dpi*g0**3*x*u1p**2/Df)*

- (3*Dphi*Dpi**2*u1p+2*Dpi**3*u1p-2*Dpi*u1p**3+

- Dpi**3*x*u1pp)/(Dphi*Dpi**3-Dpi**2*u1p**2)**2)+

- (4*Dphi*Dpi**2*g0**2*m0/Df-4*Dpi**3*g0**2*m0/Df-

- 8*Dpi**3*g0*(g0p*m0-g0*m0p)*x/Df-

- 2*Dphi*Dpi**2*(g1-2*g0*(m0*m1+g0*g1*x**2)/Df)+

- 2*Dpi**3*(g1-2*g0*(m0*m1+g0*g1*x**2)/Df)+

- 2*Dpi**3*x*(g1p-

- 2*((g0p*m0-g0*m0p)*m1+g1*(m0*m0p+g0*g0p*x**2))/Df)+

- 6*Dpi**2*g0p*u1p+4*Dpi**2*g0*(g1*m0-g0*m1)*x*u1p/Df+

- 2*Dpi**2*(g0p-2*g0*(m0*m0p+g0*g0p*x**2)/Df)*u1p-

- 2*x*(Dpi**3*(3*g0**2*g1*x/Df-

```
- g0**3*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2)
- +3*Dpi**2*g0**3*u1p/Df)+
- 2*(Dpi**3*(g1p*x-
- 2*(g0*(2*g0*m0p*x+m0p*m1*x+m0*m1p*x+
- x**2*(g0p*g1*x+g0*g1p*x))/Df+
- (m0*m0p+g0*g0p*x**2)*
- (g1*x/Df-
- g0*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2
- )))+3*Dpi**2*
- (g0p-2*g0*(m0*m0p+g0*g0p*x**2)/Df)*u1p)+
- x*(Dpi**3*(g1pp*x-
- 2*(-((2*g0p*m0*m0p-g0*m0p**2+g0*g0p**2*x**2)*
- (2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2)
- +(-(g1*m0p**2*x)-2*g0*m0p*m1p*x+
- 2*g0*g0p*g1p*x**3+
- g0p**2*(2*m0*x+g1*x**3)+
- 2*(g0p*m0*m1p*x+m0p*(g1p*m0*x+g0p*m1*x)))/
- Df))+3*Dpi**2*
- (g0pp-2*(2*g0p*m0*m0p-g0*m0p**2+g0*g0p**2*x**2)/Df)*
- u1p)+4*(Dpi**2*
- (-(g0**2*m0*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/
- Df**2)+(2*g0*g1*m0*x+g0**2*(2*g0*x+m1*x))/Df
- )*u1p+2*Dpi*g0**2*m0*u1p**2/Df)+
- 4*x*(Dpi**2*(g0*(g1p*m0*x-g1*m0p*x+g0p*m1*x-
- g0*m1p*x)/Df+
```

```
- (g0p*m0-g0*m0p)*
- (g1*x/Df-
- g0*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2))*
- u1p+2*Dpi*g0*(g0p*m0-g0*m0p)*u1p**2/Df)-
- 2*(-2*Dpi**2*(g0*(g1**2*x**3+m1*(2*g0*x+m1*x))/Df+
- (m0*m1+g0*g1*x**2)*
- (g1*x/Df-
- g0*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2))*
- u1p+2*Dpi*(g1-2*g0*(m0*m1+g0*g1*x**2)/Df)*u1p**2)-
- 2*x*(-2*Dpi**2*(-((2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)*
- ((g0p*m0-g0*m0p)*m1+g1*(m0*m0p+g0*g0p*x**2))/
- Df**2)+(m1*
- (g1p*m0*x-g1*m0p*x+g0p*m1*x-g0*m1p*x)+
- g1*(m0p*m1*x+x**2*(g0p*g1*x+g0*g1p*x)+
- m0*(2*g0p*x+m1p*x)))/Df)*u1p+
- 2*Dpi*(g1p-2*((g0p*m0-g0*m0p)*m1+
- g1*(m0*m0p+g0*g0p*x**2))/Df)*u1p**2)-
- 4*x*(Dpi*(3*g0**2*g1*x/Df-
- g0**3*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2)*
- u1p**2+g0**3*u1p**3/Df)-
- 2*(Dpi*g1p*x*u1p**2+g0p*u1p**3)+
- 6*x*(Dphi*(Dpi**2*
- (3*g0**2*g1*x/Df-
- g0**3*(2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2
- )+2*Dpi*g0**3*u1p/Df)+Dpi**2*g0**3*x*u1pp/Df)+
```

```
- 3*(Dphi*(Dpi**2*g1p*x+2*Dpi*g0p*u1p)+
- Dpi**2*g0p*x*u1pp)-
- 4*x*(Dphi*(Dpi**2*
- (g1*(g1*m0-g0*m1)*x/Df-
- g0*(g1*m0-g0*m1)*
- (2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2)+
- 2*Dpi*g0*(g1*m0-g0*m1)*u1p/Df)+
- Dpi**2*g0*(g1*m0-g0*m1)*x*u1pp/Df)+
- x*(Dphi*(-2*Dpi**2*
- (-((2*g1*m0*m1-g0*m1**2+g0*g1**2*x**2)*
- (2*g0*m0*x+2*m0*m1*x+2*g0*g1*x**3)/Df**2)
- +(g1*m1**2*x+g1**2*(2*m0*x+g1*x**3))/Df)-
- 4*Dpi*(2*g1*m0*m1-g0*m1**2+g0*g1**2*x**2)*u1p/Df)-
- 2*Dpi**2*x*(2*g1*m0*m1-g0*m1**2+g0*g1**2*x**2)*u1pp/Df
- ))/(Dphi*Dpi**3-Dpi**2*u1p**2))/x
```

Appendix G

$\pi\pi$ SCATTERING LENGTHS

In this appendix we will shetch the calculation of $\pi\pi$ scattering lengths. Before addressing the calculation in our model we first discuss the calculation using the perturbative σ-model. This was first done by Weinberg [81]; useful reviews appear here as Refs. [85, 23].

Differential cross sections in field theory are computed by squaring the "invariant amplitude", A which is usually computed to a given order in perturbation theory using diagrammatic techniques[1] ,

$$\frac{d\sigma}{d\Omega} \propto |A|^2. \tag{G.1}$$

Consider the amplitude for processes involving two pions in and two pions out (see Fig. G.1). As indicated, it will depend on the isospin channel $I = 0, 1, 2$, the Mandelstam momentum variables[2] $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_1 + p_4)^2$ and the isospin

[1]For details on conventions see *e.g.* Ref.[3] Appendix A-3

[2]In the center-of-mass frame these become $s = 4(m_\pi^2 + \vec{k}^2)$, $t = -2(1 - \cos\theta)\vec{k}^2$, and $u = -2(1 + \cos\theta)\vec{k}^2$, where $\vec{k}$ is the 3-momentum of the incident pion and θ is the angle between the incident pion and the out going pion.

$$A^I_{ijkl}(s,t,u) =$$

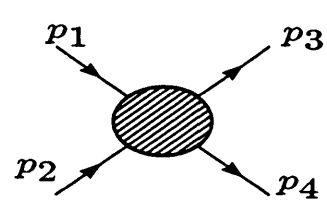


FIG. G.1.

indices of each of the pions, i, j, k, l. We can expand the amplitude in partial waves,

$$A^I(s,t,u) = \sum_{l=0}^{\infty}(2l+1)A_l^I(s,t,u)P_l(\cos\theta), \tag{G.2}$$

where $P_l(\cos\theta)$ are the Legendre ploynomials. Amplitudes for each of the isopin channels are not independent, however; since all the particles are bosons $A_{ijkl}^I(s,t,u)$ is totally symmetric in all indices. This can be exploited to show that there is really only one independent amplitude $A_l(s,t,u)$. Each of the amplitudes A_l^I can be written in terms of $A_l(s,t,u)$:

$$\begin{aligned} A_l^0(s,t,u) &= 3A_l(s,t,u) + A_l(t,s,u) + A_l(u,t,s) \\ A_l^1(s,t,u) &= A_l(t,s,u) - A_l(u,t,s) \\ A_l^2(s,t,u) &= A_l(t,s,u) + A_l(u,t,s). \end{aligned} \tag{G.3}$$

So we need only compute $A_l(s,t,u)$.

At tree level, or lowest order in the coupling constant λ, the calculation is quite simple. The Feynman rules can be read off of the potential in Eq(1.32),

$$V(\rho) = \frac{1}{2}\mu^2\rho^2 + \frac{1}{4}\lambda\rho^4 = \cdots - \lambda\sigma_{min}\sigma\vec{\pi}^2 - \frac{\lambda}{4}\vec{\pi}^4 + \cdots. \tag{G.4}$$

We consider the process $\pi^+\pi^- \longrightarrow \pi^0\pi^0$ with $\pi_0 = \pi^3$, $\pi_\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm \pi^2)$, then we have

$$V(\rho) = \cdots - \lambda\sigma_{min}\sigma(\pi_0^2 + 2\pi_+\pi_-) - \frac{\lambda}{4}(\pi_0^4 + 4\pi_+^2\pi_-^2 + 4\pi_0^2\pi_+\pi_-) + \cdots. \tag{G.5}$$

Diagramatically the amplitude can be written as in Fig. G.2 or as

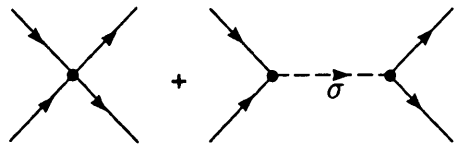
$A_l(s,t,u) =$ 

FIG. G.2.

$$
\begin{aligned}
A_l(s,t,u) &= -2i\lambda + 4(-i\lambda\sigma_{min})\frac{i}{s-m_\sigma^2} \qquad \text{(G.6)}\\
&= -2i\lambda\Big(1+\frac{m_\sigma^2-m_\pi^2}{s-m_\sigma^2}\Big)\\
&= \Big(\frac{s-m_\pi^2}{f_\pi^2}\Big)\Big(\frac{m_\sigma^2-m_\pi^2}{m_\sigma^2-s}\Big) \simeq \frac{s-m_\pi^2}{f_\pi^2}
\end{aligned}
$$

where we used $m_\pi^2 \ll m_\sigma^2$ and $s = 4m_\pi^2$ (at threshold) in the last approximation. Also we used the relations

$$
\begin{aligned}
\sigma_{min} &= -f_\pi \qquad \text{(G.7)}\\
\lambda &= \frac{m_\sigma^2-m_\pi^2}{2f_\pi^2} \qquad \text{(G.8)}\\
\mu^2 &= \frac{1}{2}(m_\sigma^2-3m_\pi^2) \qquad \text{(G.9)}
\end{aligned}
$$

to replace $(\sigma_{min}, \lambda, \mu^2)$ with the observables (m_σ, m_π, f_π). From Eq.(G.4) we can compute the amplitudes for the isospin channels,

$$
\begin{aligned}
A_0^0(s,t,u) &= 3A(s,t,u) + A(t,s,u) + A(u,t,s) \qquad \text{(G.10)}\\
A_1^1(s,t,u) &= A(t,s,u) - A(u,t,s)\\
A_0^2(s,t,u) &= A(t,s,u) + A(u,t,s).
\end{aligned}
$$

using Eq.(G.7), $s+t+u=4m_\pi^2$, $s=4(m_\pi^2+\vec{k}^2)$

$$
\begin{aligned}
A_0^0 &= \frac{2s-m_\pi^2}{f_\pi^2} = 7\frac{m_\pi^2}{f_\pi^2} + 8\frac{m_\pi^2}{f_\pi^2}\frac{\vec{k}^2}{m_\pi^2} \\
A_1^1 &= \frac{t-u}{f_\pi^2} = \frac{4}{3\pi}\frac{m_\pi^2}{f_\pi^2}\frac{\vec{k}^2}{m_\pi^2} \\
A_0^2 &= \frac{t+u-2m_\pi^2}{f_\pi^2} = -2\frac{m_\pi^2}{f_\pi^2} - \frac{4}{\pi}\frac{m_\pi^2}{f_\pi^2}\frac{\vec{k}^2}{m_\pi^2}.
\end{aligned}
\tag{G.11}
$$

The "scattering length" a_l^I and "slope parameter" b_l^I are defined by

$$
\mathrm{Re}\ A_l^I(s) = 32\pi\left(\frac{q^2}{m_\pi^2}\right)^l\left[a_l^I + b_l^I\frac{q^2}{m_\pi^2} + \mathcal{O}(\frac{\vec{k}^4}{m_\pi^4}) + \cdots\right],
\tag{G.12}
$$

which gives the tree level values [81, 23]

$$
\begin{aligned}
a_0^0 &= \frac{7}{32\pi}\frac{m_\pi^2}{f\pi^2} \simeq 0.16 \quad ; \quad b_0^0 = \frac{8}{32\pi}\frac{m_\pi^2}{f\pi^2} \simeq 0.18 \\
a_1^1 &= \frac{1}{24\pi}\frac{m_\pi^2}{f\pi^2} \simeq 0.0 \quad ; \quad b_1^1 = 0 \\
a_0^2 &= -\frac{1}{16\pi}\frac{m_\pi^2}{f\pi^2} \simeq -0.045 \quad ; \quad b_0^2 = -\frac{1}{16\pi}\frac{m_\pi^2}{f\pi^2} \simeq -0.09
\end{aligned}
\tag{G.13}
$$

The calculation in our model proceeds similarly. Since we compute the potential $V(\rho)$ we must relate the tree diagrams in the invariant amplitude to this potential. In the leading order (LO) approximation, all the momenta on the external legs of the diagrams are zero and therefore we can only compute the s-wave ($l=0$) scattering lengths. We equate the vertex between two pions and one sigma with the third derivative of the potential (see Fig G.3) and the 4-pion vertex with the fourth derivative (see Fig G.4). Computation of these is staightforward using Eq.(E.4) as a starting

$$\frac{\partial^3 V}{\partial\sigma\partial\pi^i\partial\pi^j}\Big|_{\substack{\vec{\pi}=0\\ \rho,\sigma=f_\pi}} =$$

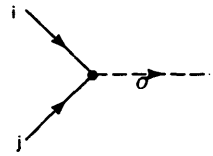


FIG. G.3.

$$\frac{\partial^4 V}{\partial\pi^i\partial\pi^j\partial\pi^k\partial\pi^l}\Big|_{\substack{\vec{\pi}=0\\ \rho,\sigma=f_\pi}} =$$

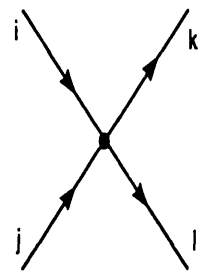


FIG. G.4.

point,

$$\begin{aligned}
\frac{\partial^3 V}{\partial\sigma\partial\pi^i\partial\pi^j}\Big|_{\substack{\vec{\pi}=0\\ \rho,\sigma=f_\pi}} &= \delta^{ij}\Big(\frac{V''}{f_\pi}-\frac{V'}{f_\pi^2}\Big) \qquad \text{(G.14)}\\
\frac{\partial^4 V}{\partial\pi^i\partial\pi^j\partial\pi^k\partial\pi^l}\Big|_{\substack{\vec{\pi}=0\\ \rho,\sigma=f_\pi}} &= (\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk})\Big(\frac{V''}{f_\pi^2}-\frac{V'}{f_\pi^3}\Big)
\end{aligned}$$

where

$$V' = V_0'(\rho)+\sigma V_1'(\rho)+\frac{\sigma}{2}V_2'(\rho) \qquad \text{(G.15)}$$

and similarly for V''. Defining $F(\rho)=\frac{V''}{f_\pi}-\frac{V'}{f_\pi^2}$ and $G(p)=F(\rho)/f_\pi+F^2(\rho)/(p-m_\sigma^2)$ we can construct the invariant amplitude (see Fig G.5),

$$A(s,t,u) = \delta^{ij}\delta^{kl}G(s)+\delta^{ik}\delta^{jl}G(t)+\delta^{il}\delta^{jk}G(u). \qquad \text{(G.16)}$$

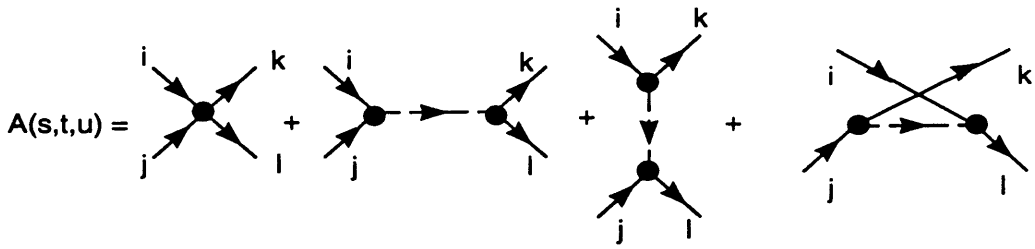


FIG. G.5.

At threshold $s = 4m_\pi^2$, $t = u = 0$, so

$$A(s) = \delta^{ij}\delta^{kl}\Big(\frac{F(\rho)}{f_\pi} + \frac{F^2(\rho)}{4m_\pi^2 - m_\sigma^2}\Big). \tag{G.17}$$

Thus $A(t,s,u) = A(u,t,s) = A(0)$ and our crossing relations (Eq.(G.11)) give,

$$\begin{aligned} A_0^0 &= 3A(s) + 2A(0) \\ A_0^2 &= 2A(s) \end{aligned} \tag{G.18}$$

from which we have,

$$\begin{aligned} a_0^0 &= \frac{1}{32\pi}A_0^0 = \frac{1}{32\pi}(3A(s) + 2A(0)) \\ a_0^2 &= \frac{1}{32\pi}A_2^0 = \frac{1}{16\pi}A(s). \end{aligned} \tag{G.19}$$