TWO-DIMENSIONAL INTERNAL GRAVITY WAVES GENERATED BY AN OSCILLATING CIRCLE IN A DENSITY STRATIFIED FLUID

by

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ABSTRACT

Two-dimensional time-harmonic internal gravity waves are generated by an oscillating body in a density stratified, inviscid, and incompressible fluid. The waves are found in columns that form a St. Andrew's Cross. We will solve a hyperbolic partial differential equation for the pressure $p$ in two dimensions using a reciprocal theorem. We obtain an explicit integral representation for the pressure, in terms of single-layer and double-layer potentials. A method is used for calculating the far-field of these potentials of an oscillating circle. Our solution is verified by comparing the results with a known solution.
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CHAPTER 1
INTRODUCTION

Fluids and the atmosphere are stratified due to changes in temperature, composition, and pressure. The density stratification allows oscillations of particles. The restoring force that produces the oscillation is the buoyancy force. The wave phenomena associated with these oscillations are called internal waves [1]. One method for generating internal waves is by oscillating a submerged object with frequency $\omega$ into an incompressible, inviscid, and density stratified fluid. When a fluid particle is displaced from its equilibrium position, gravity and the density gradient provide a restoring force that enables oscillations. The frequency of oscillations is called the Brunt-Väisälä frequency, $N$. These waves are different from more classical acoustic or electromagnetic waves because the particles do not compress, they oscillate. Internal waves are anisotropic and solve a hyperbolic partial differential equation in terms of the pressure $p$. As opposed to solving an elliptic partial differential equation. Two-dimensional internal waves are confined in beams that form a St. Andrew’s Cross see figure 1. We will derive and evaluate an integral representation for the pressure in the far field within the wave beams in two dimensions.

The approaches used previously to solve for the pressure have used Fourier Transforms. The three-dimensional case was discussed extensively by Voisin in [4] and the two-dimensional case by Hurley in [2]. Voisin and Hurley impose different radiation conditions: Sommerfeld’s, Lighthill’s, and Pierce and Hurley’s, depending on the type of problem, the type of object in the fluid. They assume the fluid is incompressible, inviscid, and density stratified. Voisin and Hurley assume $N$ is a constant. They start with a hyperbolic partial differential equation. Their method for determining internal waves is to first look at the elliptic problem, when the frequency of oscillations $\omega$ is greater than the Brunt-Väisälä frequency, i.e. $\omega > N$. Then analytic continuation in the complex $\omega$-plane is used to be able to solve the hyperbolic problem, for $0 < \omega < N$. Then in three dimensions Voisin takes a look at oscillations of different objects, a point mass and sphere. While in two dimensions Hurley takes a look at oscillations due to a wedge or a circular cylinder.

More recently a different method has been developed to solve for three-dimensional internal waves. That method is obtained by Martin and Llewellyn Smith in [3]. Martin and Llewellyn Smith developed a method that uses boundary integral equations. Their method is similar to Voisin’s in that it solves the elliptic problem first and then uses analytic continuation to get to the hyperbolic problem. Instead of using Fourier Transforms, Martin and Llewellyn Smith solve for an integral representation for the pressure $p$. They start by looking at the three-dimensional linearized Boussinesq equations. This allowed them to solve for a hyperbolic partial differential equation given in terms of the pressure $p$. To solve the hyperbolic partial differential equation Martin and Llewellyn Smith created a three-dimensional reciprocal theorem that relates two time harmonic pressure fields, the pressure $p$ and a Green’s function. Using the reciprocal theorem, they solved
for an integral representation of the elliptic problem, when $\omega > N$, for the pressure. Analytic continuation allowed them to get the integral representation to solve the hyperbolic problem for $0 < \omega < N$. The integral representation was separated into two integrals, a single layer and double layer potential. The single layer and double layer potential integrals are then evaluated in the far-field. Their method was applied to specific problems for spheres and point masses. Martin and Llewellyn Smith’s method gave the same results as Voisin’s method.

A methodology similar to Martin and Llewellyn Smith’s will be utilized to solve the two-dimensional problem. We start the thesis with the three-dimensional linearized Boussinesq equations and assume that the pressure $p$ is independent of a horizontal coordinate $y$, in Section 2. By doing this and assuming the effect of the Coriolis frequency is negligible, we will obtain a two-dimensional partial differential equation in terms of the pressure. Time-harmonic pressure fields are related in Section 3 via a reciprocal theorem. A Green’s function is used to set up the integral representation of the pressure using the reciprocal theorem in Section 4. In Section 5 analytic continuation is used to change the elliptic problem into a hyperbolic problem and the integral representation is split into a single layer and a double layer potential. Up to now the curve $C$ has been any curve. We will consider the simplest case when the curve is a circle of radius $a$, $C_a$. The internal waves are generated in columns called wave beams see figure 1. As can be seen there is symmetry about the curve with the wave beams. In Section 6 we parametrize any point on the curve with respect to the angle $\theta$. The far-field is estimated in Section 7 for the single layer and double layer potentials. To verify the results from the previous section we compare our solution of a vertically oscillating circle with the results obtained by Hurley in Section 9.
Figure 1: The St. Andrew’s Cross is created by the oscillating object submerged in a fluid. The two-dimensional internal waves are generated within the columns II, IV, VI, and VIII, called wave beams. Internal waves are not generated outside of the wave beams, regions III, V, VII, IX.
CHAPTER 2
MATHEMATICAL FORMULATION

We will be deriving a partial differential equation for the pressure \( p \). We start by taking the governing equations to be the three-dimensional Boussinesq equations in linearized form [3]:

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{f} \times \mathbf{v} = -\nabla p + b\mathbf{\hat{z}},
\]

\[
\text{div} \mathbf{v} = 0,
\]

and

\[
\frac{\partial b}{\partial t} + N^2 w = 0.
\]

Where \( \mathbf{\hat{z}} \) is the unit vector in the positive \( z \)-direction, with gravity pointing in the negative \( z \)-direction. The velocity vector of the fluid is \( \mathbf{v} = (u, v, w) \). There could be rotation \( \mathbf{f} = (0, 0, f) \) where \( f \) is the Coriolis frequency, a given constant. The buoyancy frequency, \( N(z) \), is positive and \( b \) is the buoyancy. The basic unknowns are \( u, v, w, p \) and \( b \). We are interested in the two-dimensional case, so let us take a look at what happens when the pressure only depends on \( x, z \), and \( t \). As of now we have five unknowns with five equations. We may eliminate \( u, v \) and \( b \). Let us start by substituting (2.3) into \( \frac{\partial}{\partial t} (2.1) \), which gives

\[
\frac{\partial^2 \mathbf{v}}{\partial t^2} + \frac{\partial}{\partial t} (\mathbf{f} \times \mathbf{v}) = -\frac{\partial}{\partial t} \nabla p - N^2 w \mathbf{\hat{z}}.
\]

Rewriting equation (2.4) as its vector components:

\[
\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t} (-fv) = -\frac{\partial^2 p}{\partial t \partial x},
\]

\[
\frac{\partial^2 v}{\partial t^2} + \frac{\partial}{\partial t} (fu) = 0,
\]

\[
\frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2 p}{\partial t \partial z} - N^2 w.
\]

So now if we take \( \frac{\partial}{\partial x} (2.5) \) and \( \frac{\partial}{\partial y} (2.6) \) and add them together we get

\[
\frac{\partial^2}{\partial t^2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + f \left[ -\frac{\partial}{\partial x} \frac{\partial v}{\partial t} + \frac{\partial}{\partial y} \frac{\partial u}{\partial t} \right] = -\frac{\partial^3 p}{\partial t \partial x^2}.
\]

Similarly, (2.1) can be written in its vector components:

\[
\frac{\partial u}{\partial t} - fv = -\frac{\partial p}{\partial x},
\]

\[
\frac{\partial v}{\partial t} + fu = 0,
\]

\[
\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - N^2 w.
\]
So now by using equation (2.2) and equation (2.9) by solving for \( \frac{\partial u}{\partial t} \) and (2.10) by solving for \( \frac{\partial v}{\partial t} \) and substituting them into equation (2.8) gives

\[
\frac{\partial^3 p}{\partial t \partial x^2} = \frac{\partial^2}{\partial t^2} \left[ -\frac{\partial w}{\partial z} \right] + f \left[ -\frac{\partial}{\partial x} (-f u) + \frac{\partial}{\partial y} \left( f v - \frac{\partial p}{\partial x} \right) \right].
\]

Since \( p \) doesn’t depend on \( y \), \( \frac{\partial^2 p}{\partial y \partial x} = 0 \). Simplifying we get

\[
\frac{\partial^3 p}{\partial t \partial x^2} = \frac{\partial^2}{\partial t^2} \left[ -\frac{\partial w}{\partial z} \right] + f \frac{\partial^2}{\partial t^2} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] \quad (2.12)
\]

Taking (2.12) and equation (2.7), we end up with

\[
\left( \frac{\partial^2}{\partial t^2} + f^2 \right) \frac{\partial w}{\partial z} = \frac{\partial^3 p}{\partial t \partial x^2},
\]

and

\[
\left( \frac{\partial^2}{\partial t^2} + N^2 \right) w = -\frac{\partial^2 p}{\partial t \partial z},
\]

now we have two equations with two unknowns, \( p \) and \( w \). We will now eliminate \( w \) from the above equations to obtain a pde in terms of the pressure \( p \). We are looking for a solution that is time-harmonic so assume

\[
p(x, z, t) = \text{Re}\{p(x, z)e^{-i\omega t}\},
\]

\[
u(x, z, t) = \text{Re}\{v(x, z)e^{-i\omega t}\},
\]

\[
v(x, z, t) = \text{Re}\{v(x, z)e^{-i\omega t}\},
\]

and

\[
w(x, z, t) = \text{Re}\{w(x, z)e^{-i\omega t}\}.
\]

Then by substituting (2.15) through (2.18) into equations (2.9), (2.10), and (2.14), we get

\[
-i\omega u - f v = -\frac{\partial p}{\partial x},
\]

\[
-i\omega v + fu = 0,
\]

\[
(-\omega^2 + N^2)w = i\omega \frac{\partial p}{\partial z}.
\]

We want to find equations for the components of the velocity vector \( v \) in terms of the pressure \( p \). If we take equation (2.19) multiplied by \(-f\) and add it with equation (2.20) multiplied by \(-i\omega\), we get

\[
-\omega^2 v - i\omega fu + f i\omega u + f^2 v = f \frac{\partial p}{\partial x}
\]
and therefore we get
\[(\omega^2 - f^2)v = -f \frac{\partial p}{\partial x}. \quad (2.22)\]

Now let us take equation (2.20) multiplied by \(-f\) and add it to equation (2.19) multiplied by \(i\omega\), we get
\[\omega^2 u - i\omega fv + f \prod v - f^2 u = -i\omega \frac{\partial p}{\partial x}
\]
and therefore giving
\[(\omega^2 - f^2)u = -i\omega \frac{\partial p}{\partial x} \quad (2.23)\]

Finally, we multiply (2.21) by \(\frac{\omega^2 - f^2}{N^2 - \omega^2}\) which gives
\[(\omega^2 - f^2)w = i\omega \frac{\omega^2 - f^2}{N^2 - \omega^2} \frac{\partial p}{\partial z}. \quad (2.24)\]

Now we have three equations for the velocity vector \(v\) in terms of the pressure \(p\), by taking equations (2.22), (2.23), and (2.24), they are
\[u = -\frac{i\omega}{\omega^2 - f^2} \frac{\partial p}{\partial x}; \quad (2.25)\]
\[v = -\frac{f}{\omega^2 - f^2} \frac{\partial p}{\partial x}; \quad (2.26)\]
\[w = -\frac{i\omega \gamma}{\omega^2 - f^2} \frac{\partial p}{\partial z}; \quad (2.27)\]

where
\[\gamma(z) = \frac{\omega^2 - f^2}{\omega^2 - N^2(z)}. \quad (2.28)\]

Now that we have equations for the components of the velocity vector, let us solve for the partial differential equation for the pressure. Let us take \(\frac{\partial}{\partial z}(2.27)\), we get
\[
\frac{\partial w}{\partial z} = -\frac{i\omega}{\omega^2 - f^2} \frac{\partial}{\partial z} \left( \frac{\partial p}{\partial z} \right),
\]

now if we substitute (2.15) and (2.18) into (2.13), we obtain
\[(-\omega^2 + f^2) \frac{\partial w}{\partial z} = -i\omega \frac{\partial^2 p}{\partial x^2}. \quad (2.30)\]

Solving for \(\frac{\partial \omega}{\partial z}\) in (2.30) and substituting into (2.29) we get
\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial z} \left( \gamma(z) \frac{\partial p}{\partial z} \right) = 0.
\]

This is the two-variable partial differential equation for \(p\) in three dimensions assuming \(p\) does not depend on \(y\). As mentioned earlier, we want to solve the two-dimensional problem. The boundary conditions will
involve \( \mathbf{v} \). In order to obtain a two-dimensional problem, we set \( f = 0 \). The velocity \( \mathbf{v} = (u, v, w) \) where \( u, v \) and \( w \) are defined by (2.25) through (2.27) becomes

\[
\mathbf{v} = (u, v, w) = \left( \frac{-i \omega}{\omega} \frac{\partial p}{\partial x}, 0, \frac{-i \gamma}{\omega} \frac{\partial p}{\partial z} \right).
\]  

(2.32)

This implies that there is no velocity in the \( y \)-direction and with \( p \) not depending on \( y \), there is no dependence on \( y \) with our equations now, so we no longer need to consider three dimensions, so \( \mathbf{v} = (u, w) \in \mathbb{C}^2 \). Now we can work in two dimensions, i.e. the \( xz \)-plane. Now the pressure \( p \) solves the same partial differential equation (2.31), and (2.28) simplifies to

\[
\gamma = \frac{\omega^2}{\omega^2 - N^2(z)}.
\]  

(2.33)

We are interested in solving (2.31) when the frequency \( \omega \) satisfies \( 0 < \omega^2 < N^2 \). This will give us a hyperbolic partial differential equation and internal waves.
CHAPTER 3
A TIME-HARMONIC RECIPROCAL THEOREM

To obtain a time-harmonic reciprocal theorem, we start with the two-dimensional divergence theorem,
\[ \oint_C \mathbf{u} \cdot \mathbf{n} \, dC = \int_A \text{div} \mathbf{u} \, dA, \]
where \( \mathbf{u} \in \mathbb{R}^2 \) is a continuously differentiable vector field, \( \mathbf{n} \) is the unit normal vector pointing outward from \( C \), and \( A \) is the area inside the closed curve \( C \). Put \( \mathbf{u} = \phi \mathbf{w} \) to obtain
\[ \oint_C \phi \mathbf{w} \cdot \mathbf{n} \, dC = \int_A (\mathbf{w} \cdot \text{grad} \phi + \phi \text{div} \mathbf{w}) \, dA. \tag{3.1} \]
Replacing \( \mathbf{w} \) with \( \mathbf{v}^p \), where \( \mathbf{v}^p \) is the velocity of the pressure field \( p \) and \( \text{div} \mathbf{v}^p = 0 \), gives
\[ \oint_C \phi \mathbf{v}^p \cdot \mathbf{n} \, dC = \int_A (\mathbf{v}^p \cdot \text{grad} \phi) \, dA. \tag{3.2} \]
Using the equations for \( u \) and \( w \) from (2.32),
\[ u = -i \frac{\partial p}{\partial x}, \quad w = -i \gamma \frac{\partial p}{\partial z}, \tag{3.3} \]
(3.2) becomes
\[ \oint_C \phi \mathbf{v}^p \cdot \mathbf{n} \, dC = \int_A (\mathbf{v}^p \cdot \text{grad} \phi) \, dA \]
\[ = \int_A (u, w) \cdot \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial z} \right) \, dA \]
\[ = \int_A \left[ \left( -i \frac{\partial p}{\partial x} \frac{\partial \phi}{\partial x} + \left( -i \gamma \right) \frac{\partial p}{\partial z} \frac{\partial \phi}{\partial z} \right) \right] \, dA \]
\[ = \frac{-i}{\omega} \int_A \left[ \frac{\partial p}{\partial x} \frac{\partial \phi}{\partial x} + \gamma \frac{\partial p}{\partial z} \frac{\partial \phi}{\partial z} \right] \, dA. \]
Therefore we get
\[ \oint_C \phi \mathbf{v}^p \cdot \mathbf{n} \, dC = \frac{1}{\omega} \int_A \left[ \frac{\partial p}{\partial x} \frac{\partial \phi}{\partial x} + \gamma \frac{\partial p}{\partial z} \frac{\partial \phi}{\partial z} \right] \, dA. \tag{3.4} \]
Now if we suppose that \( \phi \) is a valid pressure field, and then interchange \( p \) and \( \phi \) in (3.4), we obtain
\[ \oint_C p \mathbf{v}^\phi \cdot \mathbf{n} \, dC = \frac{1}{\omega} \int_A \left[ \frac{\partial \phi}{\partial x} \frac{\partial p}{\partial x} + \gamma \frac{\partial \phi}{\partial z} \frac{\partial p}{\partial z} \right] \, dA. \tag{3.5} \]
Subtracting equation (3.5) from (3.4), we obtain
\[ \oint_C (\phi \mathbf{v}^p - p \mathbf{v}^\phi) \cdot \mathbf{n} \, dC = 0, \tag{3.6} \]
which is a two-dimensional reciprocal theorem that relates two time-harmonic pressure fields. We will use equation (3.6) in the next section.
CHAPTER 4
AN ELLIPTIC PROBLEM: $\omega > N$

We will now try to use a Green’s function to arrive at an integral representation for the pressure $p$ by using the two-dimensional reciprocal theorem, (3.6). Suppose that $N$ is a positive constant, $\omega > N$. This is a standard simplification, one that also permits analytic progress. That means, from (2.33), that

$$\gamma = \frac{\omega^2}{\omega^2 - N^2} \quad (4.1)$$

is a constant with $\gamma > 1$ and (2.31) is elliptic.

We will use the two-dimensional Green’s function

$$G(x, z, x_0, z_0) = \frac{1}{2} \log \{ (x - x_0)^2 + \gamma^{-1} (z - z_0)^2 \}, \quad (4.2)$$

where $(x_0, z_0)$ is a fixed point, $P$ say, that is outside of $C$. It is easy to see that (4.2) solves (2.31) so everything that held for the pressure $p$ holds for the Green’s function.

Let $v^G$ be the velocity field generated by the pressure $G$. Using (3.3) for $u$ and $w$ gives

$$v^G = \frac{-i}{\omega} \frac{(x - x_0, z - z_0)}{(x - x_0)^2 + \gamma^{-1} (z - z_0)^2}. \quad (4.3)$$

Let $C_\epsilon$ be a circle in the $xz$-plane with radius $\epsilon$ centered at $(x_0, z_0)$. Changing to polar coordinates gives $x - x_0 = \epsilon \sin \xi$ and $z - z_0 = \epsilon \cos \xi$. With $n$ pointing out of $C_\epsilon$, we get

$$v^G = \frac{-i}{\omega} \frac{(\epsilon \sin \xi, \epsilon \cos \xi)}{(\epsilon \sin \xi)^2 + \gamma^{-1} (\epsilon \cos \xi)^2}. \quad (4.4)$$

and

$$\int_{C_\epsilon} v^G \cdot n dS = \int_{C_\epsilon} \frac{-i}{\omega} \frac{(\epsilon \sin \xi, \epsilon \cos \xi) \cdot (\sin \xi, \cos \xi)}{\epsilon \sin \xi)^2 + \gamma^{-1} (\epsilon \cos \xi)^2} dC$$

$$= \frac{-i}{\omega} \int_{C_\epsilon} \frac{\epsilon}{\epsilon^2 (\sin^2 \xi + \gamma^{-1} \cos^2 \xi)} d\xi$$

$$= \frac{-i}{\omega} \int_0^{2\pi} \frac{1}{\sin^2 \xi + \gamma^{-1} \cos^2 \xi} d\xi. \quad (4.5)$$

The integrand is $\pi$-periodic so (4.5) becomes

$$= \frac{-2i}{\omega} \int_0^{\pi} \frac{1}{\sin^2 \xi + \gamma^{-1} \cos^2 \xi} d\xi$$

$$= \frac{-2i}{\omega} \int_{-\pi/2}^{\pi/2} \frac{1}{\sin^2 \xi + \gamma^{-1} \cos^2 \xi} d\xi.$$
Also, the integrand is even because of the squared trigonometric functions, so
\[ = -\frac{4i}{\omega} \int_0^\frac{\pi}{2} \frac{1}{\sin^2 \xi + \gamma^{-1} \cos^2 \xi} d\xi. \]

We can factor out a \( \sin^2 \theta \) from the denominator, giving
\[ = -\frac{4i}{\omega} \int_0^\frac{\pi}{2} \frac{\csc^2 \xi}{1 + \gamma^{-1} \cot^2 \xi} d\xi, \]

If we substitute \( \lambda = \gamma^{-\frac{1}{2}} \cot \xi \), we get
\[ = -\frac{4i \gamma^{\frac{1}{2}}}{\omega} \int_0^\infty \frac{1}{1 + \lambda^2} d\lambda \]

which is a known integral, so
\[ \int_{C_\epsilon} v^G \cdot \mathbf{n} dC = -\frac{4i \gamma^{\frac{1}{2}}}{\omega} \tan^{-1} \lambda \bigg|_0^\infty \]
\[ = -\frac{2\pi i}{\omega} \gamma^{\frac{1}{2}}. \quad (4.6) \]

Let \( C \) be a closed contour and let \( P = (x_0, z_0) \) be a point outside \( C \). Surround \( P \) by a small circle \( C_\epsilon \) (as above) and surround \( C \) and \( P \) by a large circle \( C_R \), with radius \( R \). Apply the reciprocal theorem, (3.6), with \( \phi = G \) and the normal vector pointing outward to the region bounded by \( C, C_\epsilon \) and \( C_R \). This gives us
\[ \int_{C_R} (pv^G - Gv^p) \cdot \mathbf{n} dC_R - \int_{C_\epsilon} (pv^G - Gv^p) \cdot \mathbf{n} dC_\epsilon - \int_{C} (pv^G - Gv^p) \cdot \mathbf{n} dC = 0. \]

We expect that the contribution from \( C_R \) goes to zero as \( R \to \infty \) because the velocity and pressure \( p \) decays as \( R \) goes to infinity. Section 7.1 shows how the integrals over a circle in the far field goes to zero if there are no singularities within the integral. Thus
\[ \int_{C_\epsilon} (pv^G - Gv^p) \cdot \mathbf{n} dC_\epsilon + \int_{C} (pv^G - Gv^p) \cdot \mathbf{n} dC = 0. \quad (4.7) \]

We can separate the integral over \( C_\epsilon \) and we can Taylor expand \( p \) about the point \( P \) and taking the first term gives
\[ \int_{C_\epsilon} (pv^G - Gv^p) \cdot \mathbf{n} dC_\epsilon = p(x_0, z_0) \int_{C_\epsilon} v^G \cdot \mathbf{n} dC_\epsilon + \int_{C_\epsilon} Gv^p \cdot \mathbf{n} dC_\epsilon. \]

Changing to polar coordinates, \( x - x_0 = \epsilon \sin \theta \) and \( z - z_0 = \epsilon \cos \theta \), in the second integral and substituting (4.6) for the first integral gives
\[ = p(x_0, z_0) \frac{-2\pi i}{\omega} \gamma^{\frac{1}{2}} + \int_{C_\epsilon} 2 \log(\epsilon) \log(\sin^2 \theta + \gamma^{-1} \cos^2 \theta) v^p \cdot \mathbf{n} d\theta. \quad (4.8) \]
Taking the limit as epsilon tends to zero the second integral tends to zero, therefore we get

\[ \int_{C_\epsilon} (pv^G - G\nabla p) \cdot n \, dC_\epsilon = p(x_0, z_0) \frac{-2\pi i}{\omega} \gamma^{\frac{1}{2}}. \]  

(4.9)

Substituting (4.9) into (4.7) gives

\[ \frac{i}{\omega} 2\pi \gamma^{\frac{1}{2}} p(x_0, z_0) = \int_{C} (pv^G - G\nabla p) \cdot n \, dC, \]

\[ p(x_0, z_0) = \frac{i}{2\pi} (\omega^2 - N^2)^{\frac{1}{2}} \int_{C} (pv^G - G\nabla p) \cdot n \, dC, \]  

(4.10)

where the normal vector points inward and the definition of \( \gamma \), (4.1), has been used. This is the integral formula for the pressure in the fluid in terms of the pressure and the normal velocity on \( C \) where \( P \) is outside of \( C \) when \( \omega > N \).
CHAPTER 5
SINGLE AND DOUBLE LAYER POTENTIALS

Now that the pressure is represented by an integral formula, the formula can be written as a combination of single layer and double layer potentials. They are

\[(S\mu)(P) = i(\omega^2 - N^2)^{1/2} \int_C \mu G \frac{dC}{2\pi}, \quad (5.1)\]
\[(D\mu)(P) = i(\omega^2 - N^2)^{1/2} \int_C \mu \mathbf{v} \cdot \mathbf{n} \frac{dC}{2\pi}. \quad (5.2)\]

Therefore our integral formula (4.10) and the function \(\mu\) are defined by

\[p(P) = D(p) - S(v^p \cdot \mathbf{n}) \quad (5.3)\]
for \(P\) in the fluid.

In the last section we considered \(\omega > N\), but internal waves are generated when \(0 < \omega < N\). To get this we use analytic continuation with respect to \(\omega\), this is called the Pierce-Hurley method [4]. The idea is to impose causality in the time domain, which means there is no motion before a disturbance. We used a time-dependence of \(e^{-i\omega t}\), by causality there are no singularities or branch cuts in the upper half of the complex \(\omega\)-plane.

Let us consider the quantity \((\omega^2 - N^2)^{1/2}\). Take cuts from \(\omega = \pm N\) going vertically downwards. As \((\omega^2 - N^2)^{1/2}\) must be real and positive when \(\omega\) is real and greater than \(N\), we get

\[(\omega^2 - N^2)^{1/2} = \begin{cases} +\sqrt{\omega^2 - N^2}, & \omega > N \\ i\sqrt{N^2 - \omega^2}, & -N < \omega < N \\ -\sqrt{\omega^2 - N^2}, & \omega < -N \end{cases} \quad (5.4)\]

Notice we do not have an even function of \(\omega\).

5.1 Single Layer Potential

First we will simplify the single layer potential, (5.1), by converting the Green’s function, (4.2), into polar coordinates. We will use the following parameterization

\[x_0 - x = R \sin \theta \text{ and } z_0 - z = R \cos \theta \quad (5.5)\]

substituting into our Green’s function, (4.2), we get
\[ G(R, \theta) = \frac{1}{2} \log((x - x_0)^2 + \gamma^{-1}(z - z_0)^2) \]
\[ = \log(R\{\sin^2 \theta + \gamma^{-1} \cos^2 \theta\}^{\frac{1}{2}}). \]

Also by using (4.1), we get
\[ G(R, \theta) = \log \left( R\left\{ \frac{\omega^2 \sin^2 \theta + \omega^2 - N^2}{\omega^2} \cos^2 \theta \right\}^{\frac{1}{2}} \right) \]
\[ = \log \left( \frac{R}{|\omega|}\left\{ \omega^2 - N^2 \cos^2 \theta \right\}^{\frac{1}{2}} \right), \quad (5.6) \]

which gives branch points at \( \omega = \pm N|\cos \theta| \) with cuts extending downward.

We are interested in using (5.6) when \( 0 < \omega < N \), so the absolute value can be removed. Let us take a look at (5.6),
\[ G(R, \theta) = \log \left( \frac{R}{\omega}\left\{ \omega^2 - N^2 \cos^2 \theta \right\}^{\frac{1}{2}} \right) \]
\[ = \log R - \log \omega + \frac{1}{2} \log(\omega - N \cos \theta)(\omega + N \cos \theta) \]
\[ = \log R - \log \omega + \frac{1}{2} \log(\omega - N \cos \theta) + \frac{1}{2} \log(\omega + N \cos \theta). \]

Looking at the third term, we have two cases, when \( 0 < \omega < N \cos \theta \) and \( N \cos \theta < \omega < N \). First taking a look at the easier case, \( N \cos \theta < \omega < N \), our Green’s function, (5.6), stays the same. We can define \( \omega \) by
\[ \omega = N \cos \theta_c \text{ with } 0 < \theta_c < \frac{\pi}{2} \quad (5.7) \]

where \( \theta_c \) is a constant. So substituting (5.6) and (5.7) into the single layer potential, (5.1), and using (5.4), we get
\[ (S\mu)(P) = -(N^2 - \omega^2)^{\frac{1}{2}} \int_C \mu \log \left( \frac{R}{\omega}\left\{ \omega^2 - N^2 \cos^2 \theta \right\}^{\frac{1}{2}} \right) \frac{dC}{2\pi} \]
\[ = -(N^2 - N^2 \cos^2 \theta_c)^{\frac{1}{2}} \int_C \mu \log \left( \frac{R}{N \cos \theta_c}\left\{ N^2 \cos^2 \theta_c - N^2 \cos^2 \theta \right\}^{\frac{1}{2}} \right) \frac{dC}{2\pi} \]
\[ = -N(1 - \cos^2 \theta_c)^{\frac{1}{2}} \int_C \mu \log \left( \frac{R}{\cos \theta_c}\left\{ \cos^2 \theta_c - \cos^2 \theta \right\}^{\frac{1}{2}} \right) \frac{dC}{2\pi} \]
\[ = -N \sin \theta_c \int_C \mu \log \left( \frac{R}{\cos \theta_c}\left\{ \cos^2 \theta_c - \cos^2 \theta \right\}^{\frac{1}{2}} \right) \frac{dC}{2\pi} \]
\[ \quad \text{when } N \cos \theta < \omega < N, \text{ which is satisfied when } |\cos \theta| < \cos \theta_c. \]

Now taking a look at what happens to (5.6) when \( 0 < \omega < N \cos \theta \), our Green’s function will have a log with a negative argument, so we need to discuss what the value of a log is for negative values. Let \( z \) be a
complex number, then the log is given by

\[ \log z = \log r e^{i\theta} = \log r + i\theta. \]

If we define

\[ \omega - N \cos \theta = Te^{i\phi}, \]

then

\[ \log(\omega - N \cos \theta) = \log(T) + i\phi \]

where

\[ T = |\omega - N \cos \theta|, \]

so we get

\[ \log(\omega - N \cos \theta) = \log(|\omega - N \cos \theta|) + i\phi. \]

As before, the branch cut from \( \omega = N \cos \theta \) goes down, so we choose the interval for \( \phi \) to be \( -\frac{\pi}{2} < \phi < \frac{3\pi}{2} \). In particular, when \( \omega \) is real with \( \omega > N \cos \theta \), \( \phi = 0 \), and when \( \omega \) is real with \( \omega < N \cos \theta \), \( \phi = \pi \). Thus

\[ \log(\omega - N \cos \theta) = \log(N \cos \theta - \omega) + i\pi, \quad \omega < N \cos \theta. \]

So (5.6) becomes

\[ G(R, \theta) = \log \left( \frac{R}{\omega} \left( N^2 \cos^2 \theta - \omega^2 \right)^{\frac{1}{2}} \right) + \frac{i\pi}{2} \] (5.9)

when \( 0 < \omega < N \cos \theta \), which is satisfied when \( |\cos \theta| > \cos \theta_c \).

Substituting in the Green’s function, (5.9), and (5.7) into the single layer potential, (5.1), and using (5.4), we get

\[ (S\mu)(P) = -(N^2 - \omega^2)^{\frac{1}{2}} \int_C \mu \left[ \log \left( \frac{R}{\omega} \left( N^2 \cos^2 \theta - \omega^2 \right)^{\frac{1}{2}} \right) + \frac{i\pi}{2} \right] \frac{dC}{2\pi} \]

\[ = -(N^2 - N^2 \cos^2 \theta_c)^{\frac{1}{2}} \int_C \mu \left[ \log \left( \frac{R}{N \cos \theta_c} \left( N^2 \cos^2 \theta - N^2 \cos^2 \theta_c \right)^{\frac{1}{2}} \right) + \frac{i\pi}{2} \right] \frac{dC}{2\pi} \]

\[ = -N(1 - \cos^2 \theta_c)^{\frac{1}{2}} \int_C \mu \left[ \log \left( \frac{R}{\cos \theta_c} \left( \cos^2 \theta - \cos^2 \theta_c \right)^{\frac{1}{2}} \right) + \frac{i\pi}{2} \right] \frac{dC}{2\pi} \]

\[ = -N \sin \theta_c \int_C \mu \left[ \log \left( \frac{R}{\cos \theta_c} \left( \cos^2 \theta - \cos^2 \theta_c \right)^{\frac{1}{2}} \right) + \frac{i\pi}{2} \right] \frac{dC}{2\pi} \] (5.10)

when \( 0 < \omega < N \cos \theta \), which is satisfied when \( |\cos \theta| > \cos \theta_c \).

Therefore the single layer potential is

\[ (S\mu)(x_0, z_0) = \int_C \mu \mathcal{M}(\theta) \frac{dC(x, z)}{2\pi}, \] (5.11)
where \( M(\theta) \) comes from (5.8) and (5.10),

\[
M(\theta) = \begin{cases} 
-N \sin \theta_c \log \left( \frac{R}{\cos \theta_c} \left\{ \cos^2 \theta - \cos^2 \theta_c \right\}^{\frac{1}{2}} \right), & |\cos \theta| < \cos \theta_c \\
-N \sin \theta_c \left[ \log \left( \frac{R}{\cos \theta_c} \left\{ \cos^2 \theta - \cos^2 \theta_c \right\}^{\frac{1}{2}} \right) + \frac{i\pi}{2} \right], & |\cos \theta| > \cos \theta_c.
\end{cases}
\]

(5.12)

### 5.2 Double Layer Potential

Now doing the same steps as for the single layer potential, let us find the double layer potential equation. By using (5.5) and substituting into (4.3), we get

\[
\mathbf{v}^G = \frac{-i}{\omega} \frac{(-R \sin \theta, -R \cos \theta)}{R^2 \sin^2 \theta + \gamma^{-1} R^2 \cos^2 \theta}.
\]

(5.13)

Substituting for \( \gamma \) from equation (4.1) gives

\[
\mathbf{v}^G = \frac{i \omega}{R} \frac{(\sin \theta, \cos \theta)}{\frac{\omega^2}{R} \sin^2 \theta + \frac{\omega^2 - N^2}{\omega^2} \cos^2 \theta} = \frac{i \omega}{R} \frac{(\sin \theta, \cos \theta)}{\omega^2 - N^2 \cos^2 \theta}.
\]

If we let \( \mathbf{n} = (n_1, n_2) \) we get

\[
\mathbf{v}^G \cdot \mathbf{n} = \frac{i \omega}{\omega^2 - N^2 \cos^2 \theta} n_1 \sin \theta + n_2 \cos \theta.
\]

and letting

\[
N(\theta) = n_1 \sin \theta + n_2 \cos \theta,
\]

(5.14)

we get

\[
\mathbf{v}^G \cdot \mathbf{n} = \frac{i \omega}{R} \frac{N(\theta)}{\omega^2 - N^2 \cos^2 \theta}.
\]

(5.15)

So our double layer potential, equation (5.2), and using (5.4), becomes

\[
(D\mu)(P) = -(N^2 - \omega^2)^\frac{1}{2} \int_C \frac{i \omega}{R} \frac{N(\theta)}{\omega^2 - N^2 \cos^2 \theta} \frac{dC}{2\pi}
\]

(5.16)

We are interested in using (5.15) when \( 0 < \omega < N \) and substituting (5.7) into the double layer potential, equation (5.16), becomes

\[
(D\mu)(P) = \int_C \frac{-i N \cos \theta_c (N^2 - N^2 \cos^2 \theta_c)^\frac{1}{2}}{R} \frac{N(\theta)}{N^2 \cos^2 \theta_c - N^2 \cos^2 \theta} \frac{dC}{2\pi}
\]

\[
= \int_C \frac{-i \cos \theta_c (1 - \cos^2 \theta_c)^\frac{1}{2}}{R} \frac{N(\theta)}{\cos^2 \theta_c - \cos^2 \theta} \frac{dC}{2\pi}
\]

\[
= \int_C \frac{-i \cos \theta_c \sin \theta_c}{\cos^2 \theta_c - \cos^2 \theta} N(\theta) \frac{dC}{2\pi R}.
\]

(5.16)
Therefore the double layer potential becomes

\[
(D\mu)(P) = \int_C \mu D(\theta) N(\theta) \frac{dC(x, z)}{2\pi R}
\]  
(5.17)

where

\[
D(\theta) = \frac{i \cos \theta_c \sin \theta_c}{\cos^2 \theta - \cos^2 \theta_c}. 
\]  
(5.18)

Notice that there are singularities for both (5.12) and (5.18) when \(\theta = \theta_c\), \(\theta = \pi - \theta_c\), \(\theta = \pi + \theta_c\), and \(\theta = -\theta_c\).

5.3 Estimating \(M(\theta)\) and \(D(\theta)\)

There are singularities in both the single layer and double layer integrals. We need to take a look at what happens when \(\theta\) is approximately equal to each of the points \(\theta_c\), \(\pi - \theta_c\), \(\pi + \theta_c\), and \(-\theta_c\) to be able to compute the integrals. We will just take a look at when \(\theta \approx \theta_c\), which will make \(0 < \theta < \frac{\pi}{2}\) since \(0 < \theta_c < \frac{\pi}{2}\). The equation for \(M(\theta)\), (5.12), depends on whether you approach \(\theta_c\) from the right or left. So we will have two cases for (5.12). When \(\theta \approx \theta_c\), we can Taylor expand \(\cos^2 \theta\) about \(\theta = \theta_c\). Doing this gives

\[
\cos^2 \theta \approx \cos^2 \theta_c - 2 \cos \theta_c \sin \theta_c (\theta - \theta_c). 
\]  
(5.19)

When \(\cos \theta < \cos \theta_c\) which is the same as \(\theta > \theta_c\) and using (5.19) to estimate (5.12) gives

\[
M(\theta) \approx -N \sin \theta_c \log \left( \frac{R}{\cos \theta_c} \left[ (2 \cos \theta_c \sin \theta_c)^{\frac{1}{2}} (\theta - \theta_c)^{\frac{1}{2}} \right] \right)
\approx -N \sin \theta_c \log \left( R(2 \tan \theta_c)^{\frac{1}{2}} (\theta - \theta_c)^{\frac{1}{2}} \right).
\]

Also when \(\cos \theta > \cos \theta_c\) which is the same as \(\theta < \theta_c\) and using (5.19) to estimate (5.12) gives

\[
M \approx -N \sin \theta_c \left[ \log \left( \frac{R}{\cos \theta_c} \left[ (-2 \cos \theta_c \sin \theta_c)^{\frac{1}{2}} (\theta - \theta_c)^{\frac{1}{2}} \right] \right) + \frac{i\pi}{2} \right]
\approx -N \sin \theta_c \left[ \log \left( \frac{R}{\cos \theta_c} \left[ (2 \cos \theta_c \sin \theta_c)^{\frac{1}{2}} (\theta_c - \theta)^{\frac{1}{2}} \right] + \frac{i\pi}{2} \right]
\approx -N \sin \theta_c \left[ \log \left( R(2 \tan \theta_c)^{\frac{1}{2}} (\theta_c - \theta)^{\frac{1}{2}} \right) + \frac{i\pi}{2} \right].
\]

Therefore \(M(\theta)\) becomes

\[
M(\theta) \approx \begin{cases} 
-N \sin \theta_c \log \left( R(2 \tan \theta_c)^{\frac{1}{2}} (\theta - \theta_c)^{\frac{1}{2}} \right), & \theta > \theta_c \\
-N \sin \theta_c \left[ \log \left( R(2 \tan \theta_c)^{\frac{1}{2}} (\theta_c - \theta)^{\frac{1}{2}} \right) + \frac{i\pi}{2} \right], & \theta < \theta_c.
\end{cases} 
\]  
(5.20)

Also using (5.19) to estimate \(D(\theta)\), (5.18), gives
\[ D(\theta) \approx \frac{-i \cos \theta_c \sin \theta_c}{2 \cos \theta_c \sin \theta_c (\theta - \theta_c)} \]
\[ \approx \frac{-i}{2(\theta - \theta_c)} \quad (5.21) \]

for \( \theta \) near \( \theta_c \). Similar estimates can be made when \( \theta \) is near \( \pi - \theta_c, \pi + \theta_c, \) and \( -\theta_c \). We will not consider those case.
CHAPTER 6
PARAMETERIZING $C_a$

Before we can compute the double layer and single layer potentials, we need a parameterization of any point on the curve $C$. We note that in both integrals (5.11) and (5.17) the curve depends on $x$ and $z$, but inside the integrands, (5.12) and (5.18), depend on $\theta$. We assume that our curve is a circle of radius $a$, $C_a$, the simplest case. Then we will be able to parametrize any point $Q = (x, z)$ on that curve with respect to $\theta$. Also we will parametrize the point $P = (x_0, z_0)$ which is outside of the curve, in terms of $\theta_c$. We will find the bounds for the variable $\theta$ and the Jacobian for the change of variables. In Appendix E, the parameterization is checked.

6.1 Parameterizing $Q$ in terms of $\theta$

Earlier we used the following polar coordinates $x_0 - x = R \sin \theta$ and $z_0 - z = R \cos \theta$ which gave us a relationship between $Q$ and $P$. Using this fact we can solve for $R$ and have an equation that relates $Q$ and $P$ that only depends on $\theta$. That is $x_0 - x = (z_0 - z) \tan \theta$ and let $\tau = \tan \theta$ so

\[ (x - x_0) = (z - z_0) \tau. \]  

(6.1)

Let us introduce local Cartesian coordinates $(X_+, Z_+)$ and $(X_-, Z_-)$ such that

\[ x = X_\pm \cos \theta_c \pm Z_\pm \sin \theta_c \]  

(6.2)

\[ z = \mp X_\pm \sin \theta_c + Z_\pm \cos \theta_c. \]  

(6.3)

This puts $x$ and $z$ in terms of $\theta_c$. Then the following shows that both points $(X_+, Z_+)$ and $(X_-, Z_-)$ are on $C_a$.

\[
a^2 = x^2 + z^2 = (X_\pm \cos \theta_c \pm Z_\pm \sin \theta_c)^2 + (\mp X_\pm \sin \theta_c + Z_\pm \cos \theta_c)^2
\]

\[= X_\pm^2 \cos^2 \theta_c \pm 2X_\pm Z_\pm \cos \theta_c \sin \theta_c + Z_\pm^2 \sin^2 \theta_c \mp 2X_\pm Z_\pm \cos \theta_c \sin \theta_c + Z_\pm^2 \cos^2 \theta_c
\]

\[= X_\pm^2 + Z_\pm^2. \]

Using a similar parametrization for $(x_0, z_0)$ like we did for $(x, z)$, we have

\[ x_0 = X_\pm^0 \cos \theta_c \pm Z_\pm^0 \sin \theta_c \]  

(6.4)

\[ z_0 = \mp X_\pm^0 \sin \theta_c + Z_\pm^0 \cos \theta_c. \]  

(6.5)
So our point $P$ is in terms of $\theta_c$. To make the calculations simpler we will just look at $(X_+, Z_+)$ and $(x_+^0, z_+^0)$. Also to use simpler notation we will set $x_+^0 = \sigma_0$ and $z_+^0 = \zeta_0$. The new point $(\sigma_0, \zeta_0)$ translates the $(x, z)$ coordinate system by the angle $\theta_c$. Where $\sigma_0$ goes across the wave beam and $\zeta_0$ is along the wave beam. This means that $\sigma_0$ is a value between $(-a, a)$. While $\zeta_0$ is the distance from the curve outward in the wave beam. Subtracting (6.4) from (6.2) and subtracting (6.5) from (6.3), we get

$$x - x_0 = (X_+ - \sigma_0) \cos \theta_c + (Z_+ - \zeta_0) \sin \theta_c$$
$$z - z_0 = -(X_+ - \sigma_0) \sin \theta_c + (Z_+ - \zeta_0) \cos \theta_c.$$

Letting $\tilde{X} = X_+ - \sigma_0$ and $\tilde{Z} = Z_+ - \zeta_0$ and using (6.1), we get

$$z - z_0 = \tilde{Z} \cos \theta_c - \tilde{X} \sin \theta_c = \Lambda$$
$$x - x_0 = \tilde{X} \cos \theta_c + \tilde{Z} \sin \theta_c = \Lambda \tau.$$  (6.7)

Taking (6.6) and (6.7), let us solve for $\tilde{X}$ in terms of $\tilde{Z}$,

$$\tilde{X} \cos \theta_c + \tilde{Z} \sin \theta_c = \tilde{Z} \tau \cos \theta_c - \tilde{X} \tau \sin \theta_c$$
$$\tilde{X} (\cos \theta_c + \tau \sin \theta_c) = \tilde{Z} (\tau \cos \theta_c - \sin \theta_c)$$
$$\tilde{X} = \frac{\tilde{Z} \tau \cos \theta_c - \sin \theta_c}{\cos \theta_c + \tau \sin \theta_c}.$$  (6.8)

If we let $c = \cos \theta_c$ and $s = \sin \theta_c$ we get

$$\tilde{X} = \frac{\tilde{Z} \tau c - s}{c + \tau s}.$$  (6.8)

Now we need to find another equation that relates $\tilde{X}$ and $\tilde{Z}$ so we can solve for them in terms of $\theta$ and $\theta_c$. Let us consider

$$(c + \tau s)^2 X_+^2 = (c + \tau s)^2 (\tilde{X} + \sigma_0)^2$$
$$= [(c + \tau s)(\tilde{X} + \sigma_0)]^2$$
$$= [(c + \tau s)\tilde{X} + (c + \tau s)\sigma_0]^2$$
$$= [(\tau c - s)\tilde{Z} + (c + \tau s)\sigma_0]^2.$$  

We also know that $x^2 + z^2 = X_+^2 + Z_+^2 = a^2$ which implies that $X_+^2 = a^2 - Z_+^2 = a^2 - (\tilde{Z} + \zeta_0)^2$. So we get

$$(c + \tau s)^2 X_+^2 = (c + \tau s)^2 [a^2 - (\tilde{Z} + \zeta_0)^2] = [(\tau c - s)\tilde{Z} + (c + \tau s)\sigma_0]^2.$$  

Now we have an equation that only has $\tilde{Z}$ as our unknown. Let us solve for $\tilde{Z}$. 

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\[ 0 = [(c + \tau s)\sigma_0 + (c + \tau s)^2(\sigma_0^2 + \zeta_0^2 - a^2)] \\
= (\tau c - s)^2 \tilde{Z}^2 + 2\tilde{Z}\sigma_0(\tau c - s)(c + \tau s) + (c + \tau s)^2(\tilde{Z}^2 + 2\zeta_0 \tilde{Z} + \zeta_0^2) - a^2(c + \tau s)^2 \\
= [(\tau c - s)^2 + (c + \tau s)^2(\tilde{Z}^2 + 2(c + \tau s)(\sigma_0^2 + \zeta_0^2 - a^2)] \\
\]

This gives us a quadratic in terms of \( \tilde{Z} \). Let \( Q_1 = (c + \tau s)\zeta_0 + (\tau c - s)\sigma_0 \) and \( Q_2 = \sigma_0^2 + \zeta_0^2 - a^2 \). Also the coefficient in front of \( \tilde{Z}^2 \) is \( \sec^2 \theta \). Our quadratic becomes

\[ 0 = \sec^2 \theta \tilde{Z}^2 + 2(c + \tau s)Q_1 \tilde{Z} + (c + \tau s)^2Q_2. \quad (6.9) \]

Using the quadratic formula to solve (6.9), we get

\[ \tilde{Z} = \frac{-2(c + \tau s)Q_1 \pm \sqrt{4(c + \tau s)^2Q_1^2 - 4(c + \tau s)^2Q_2 \sec^2 \theta}}{2 \sec^2 \theta} \\
= \frac{-(c + \tau s)Q_1 \pm (c + \tau s)\sqrt{Q_1^2 - Q_2 \sec^2 \theta}}{\sec^2 \theta} \\
= (c + \tau s) \cos^2 \theta \left[ -Q_1 \pm \sqrt{Q_1^2 - Q_2 \sec^2 \theta} \right] \\
= (c \cos \theta + s \sin \theta) \left[ -Q_1 \cos \theta \pm \sqrt{(Q_1 \cos \theta)^2 - Q_2} \right]. \]

Let

\[ Q_3 = Q_1 \cos \theta \\
= (c \sin \theta - s \cos \theta)\sigma_0 + (c \cos \theta + s \sin \theta)\zeta_0 \\
= \sin(\theta - \theta_c)\sigma_0 + \cos(\theta - \theta_c)\zeta_0, \quad (6.10) \]

and

\[ Q_\pm = -Q_3 \pm \sqrt{\Delta} \quad (6.11) \]

where

\[ \Delta = Q_3^2 - Q_2. \quad (6.12) \]

We get

\[ \tilde{Z} = (c \cos \theta + s \sin \theta)Q_\pm \\
= \cos(\theta - \theta_c)Q_\pm \quad (6.13) \]

and from equation (6.8) we get
\[ \dot{X} = (c \sin \theta - s \cos \theta)Q_{\pm} \]
\[ = \sin(\theta - \theta_c)Q_{\pm}. \quad (6.14) \]

Using (6.14) and (6.13) we have a parameterization for \(X_+\) and \(Z_+\) which are on the circle \(C_a\).

\[ X_+ = \dot{X} + \sigma_0 = \sin(\theta - \theta_c)Q_{\pm} + \sigma_0 \quad (6.15) \]
\[ Z_+ = \dot{Z} + \zeta_0 = \cos(\theta - \theta_c)Q_{\pm} + \zeta_0. \quad (6.16) \]

Now we have equations for the points on \(C_a\) in terms of our variable \(\theta\) and the point \((\sigma_0, \zeta_0)\).

### 6.2 Finding the Integration Limits

Next, we will find the bounds for the variable \(\theta\). If we let \(\nu = \theta - \theta_c\) then (6.10) becomes

\[ Q_3 = \sin(\nu)\sigma_0 + \cos(\nu)\zeta_0. \quad (6.17) \]

Taylor expanding (6.17) to the highest power of 2 gives

\[ Q_3 \approx (\nu - \frac{\nu^3}{3})\sigma_0 + (1 - \frac{\nu^2}{2})\zeta_0 \]
\[ \approx \zeta_0 + \sigma_0\nu - \frac{\zeta_0}{2}\nu^2. \quad (6.18) \]

So then (6.12) becomes

\[ \Delta \approx (\zeta_0 + \sigma_0\nu - \frac{\zeta_0}{2}\nu^2)^2 - \sigma_0^2 - \zeta_0^2 \]
\[ \approx \zeta_0^2 + 2\sigma_0\zeta_0\nu - \zeta_0^2\nu^2 + \sigma_0^2\nu^2 - \sigma_0^2 - \zeta_0^2 \]
\[ \approx a^2 - \sigma_0^2 + 2\sigma_0\zeta_0\nu - (\zeta_0^2 - \sigma_0^2)\nu^2. \quad (6.19) \]

Now let \(Z_0 = \zeta_0^2 - \sigma_0^2\) and we are interested when \(\Delta = 0\), which gives

\[ a^2 - \sigma_0^2 = Z_0\nu^2 - 2\sigma_0\zeta_0\nu \]
\[ \frac{a^2 - \sigma_0^2}{Z_0} = \nu^2 - \frac{2\sigma_0\zeta_0}{Z_0}\nu \quad (6.20) \]

adding the square of \(\nu_0 = \frac{\sigma_0\zeta_0}{Z_0}\) to both sides of (6.20) gives

\[ (\nu - \nu_0)^2 = \frac{a^2 - \sigma_0^2}{Z_0} + \nu_0^2 \]
\[
\nu - \nu_0 = \sqrt{\frac{a^2 - \sigma_0^2}{Z_0} + \nu_0^2} \\
\nu = \pm \sqrt{\frac{a^2 - \sigma_0^2}{Z_0} + \nu_0^2 + \nu_0}.
\]

Now let \( \beta^2 = \frac{a^2 - \sigma_0^2}{Z_0} + \nu_0^2 \)

\[
\theta - \theta_c = \nu = \pm \beta + \nu_0 \\
\theta_{\pm} = \theta + \nu_0 \pm \beta,
\]

(6.21)

this gives us the bounds for \( \theta \), where \( \theta_+ \) is the upper bound and \( \theta_- \) is the lower bound for the integrals.

### 6.3 Approximating the Variables as the Point \( P \) goes to Infinity

We will be interested later in what happens as the point, \( P \), tends to infinity. So we get

\[
Z_0 = \zeta_0^2 - \sigma_0^2 \approx \zeta_0^2, \\
\nu_0 = \frac{\sigma_0 \zeta_0}{Z} \approx \frac{\sigma_0}{\zeta_0}, \\
\beta = \frac{a^2 - \sigma_0^2}{Z} + \nu_0 \approx \frac{a}{\zeta_0}, \\
Q_3 = \zeta_0 + \sigma_0 \nu - \frac{\zeta_0^2}{Z} \nu^2 \approx \zeta_0.
\]

Also letting \( \zeta_0 \sim r_0 \). Our bounds for our \( \theta \), (6.21), as \( P \) goes to infinity become

\[
\theta_{\pm} = \theta_c \pm \frac{a + \sigma_0}{r_0}.
\]

(6.22)

Also we get

\[
Q_3 \approx r_0.
\]

(6.23)

We also need to approximate (6.19),

\[
\Delta \approx a^2 - \sigma_0^2 + 2\sigma_0 \zeta_0 \theta - 2\sigma_0 \zeta_0 \theta_c - (\zeta_0^2 - \sigma_0^2)(\theta^2 - 2\theta \theta_c + \theta_c^2) \\
= (a - \sigma_0)(a + \sigma_0) + 2\sigma_0 \zeta_0 \theta - 2\sigma_0 \zeta_0 \theta_c - (\zeta_0^2 - \sigma_0^2)\theta^2 + (\zeta_0^2 - \sigma_0^2)2\theta \theta_c - (\zeta_0^2 - \sigma_0^2)\theta_c^2 \\
= (a - \sigma_0)(a + \sigma_0) + 2\sigma_0 \zeta_0 \theta + \theta_0 a - \theta_0 a - 2\sigma_0 \zeta_0 \theta_c + \theta_c \zeta_0 a - \theta_c \zeta_0 a - (\zeta_0^2 - \sigma_0^2)\theta_c^2 \\
+ (\zeta_0^2 - \sigma_0^2)2\theta \theta_c - (\zeta_0^2 - \sigma_0^2)\theta_c^2,
\]

now substituting in \( \zeta_0 \approx r_0 \) and \( \zeta_0^2 - \sigma_0^2 \approx r_0^2 \).
\[ \Delta \approx (a - \sigma_0)(a + \sigma_0) + 2\sigma_0r_0\theta + \theta r_0a - \theta r_0a - 2\sigma_0r_0\theta c + \theta cr_0a - \theta cr_0a - r_0^2\theta^2 + r_0^22\theta\theta c - r_0^2\theta^2_c \\
= -r_0^2\theta^2 + 2\theta\theta_c r_0^2 + \sigma_0r_0\theta + \theta r_0a - \sigma_0r_0\theta c + \theta cr_0a - \sigma_0r_0\theta c - \theta cr_0a - r_0^2\theta^2_c \\
+ (a - \sigma_0)(a + \sigma_0) \\
= -r_0^2\theta^2 + 2\theta\theta_c r_0^2 + \theta r_0(a + \sigma_0) - \theta r_0(a - \sigma_0) - r_0^2\theta^2_c - \theta cr_0(a + \sigma_0) + \theta cr_0(a - \sigma_0) \\
+ (a - \sigma_0)(a + \sigma_0) \\
\]

\[
= -r_0^2 \left[ \frac{\theta^2 - \theta \theta_c - \theta \frac{a + \sigma_0}{r_0} - \theta \theta_c + \theta \frac{a - \sigma_0}{r_0}}{} \right] - r_0^2 \left[ \frac{\theta^2 + \theta \theta_c \frac{a + \sigma_0}{r_0} - \theta \theta \frac{a - \sigma_0}{r_0} - (a + \sigma_0)(a - \sigma_0)}{} \right] \\
= -r_0^2 \left[ \frac{\theta^2 - \theta \theta_c - \theta \frac{a + \sigma_0}{r_0} - \theta \theta_c + \theta \frac{a - \sigma_0}{r_0}}{} \right] - r_0^2 \left[ \frac{\theta \theta_c - \frac{a - \sigma_0}{r_0}}{} \right] \left[ \frac{\theta \theta_c + \frac{a + \sigma_0}{r_0}}{} \right] \\
= -r_0^2 \left[ \frac{\theta^2 - \theta \left( \frac{\theta \theta_c + \frac{a + \sigma_0}{r_0}}{} \right)}{} \right] - \theta \left( \frac{\theta \theta_c - \frac{a - \sigma_0}{r_0}}{} \right) + \left( \frac{\theta \theta_c - \frac{a - \sigma_0}{r_0}}{} \right) \left( \frac{\theta \theta_c + \frac{a + \sigma_0}{r_0}}{} \right) \\
= -r_0^2 \left[ \theta - \left( \frac{\theta \theta_c - \frac{a - \sigma_0}{r_0}}{} \right) \right] \left[ \theta - \left( \frac{\theta \theta_c + \frac{a + \sigma_0}{r_0}}{} \right) \right] \\
\]

and by using (6.22) we get,

\[ \Delta \approx -r_0^2(\theta - \theta_\pm)(\theta - \theta_\pm) \]

\[ \approx r_0^2(\theta - \theta_\pm)(\theta_\pm - \theta). \quad (6.24) \]

From here we can use (6.23) and (6.24) to approximate (6.11),

\[ Q_\pm \approx -r_0. \quad (6.25) \]

### 6.4 The Jacobian for the Change of Variables

We found a parameterization for any point \( Q = (x, z) \) on the curve \( C_a \) in terms of the variable \( \theta \). Finally, we need to find the Jacobian for the change of variables to be able to solve the single layer and double layer integrals, which is \( |r'(\theta)| \). So our vector parameterization using (6.15) and (6.16) becomes

\[ r(\theta) = (X_+, Z_+) \]

\[ = (\sin(\theta - \theta_c)Q_\pm + \sigma_0, \cos(\theta - \theta_c)Q_\pm + \zeta_0). \]

Taking \( r'(\theta) = \left( \frac{\partial X_+}{\partial \theta}, \frac{\partial Z_+}{\partial \theta} \right) \) we have
\[
\begin{align*}
\frac{\partial X_+}{\partial \theta} &= \cos(\theta - \theta_c) Q_+ + \sin(\theta - \theta_c) \frac{\partial Q_+}{\partial \theta} \\
\frac{\partial Z_+}{\partial \theta} &= -\sin(\theta - \theta_c) Q_+ + \cos(\theta - \theta_c) \frac{\partial Q_+}{\partial \theta}.
\end{align*}
\]  
(6.26)

Now we need to find \(|\mathbf{r}'(\theta)| = \sqrt{\left(\frac{\partial X_+}{\partial \theta}\right)^2 + \left(\frac{\partial Z_+}{\partial \theta}\right)^2}\). Let us take a look at \(|\mathbf{r}'(\theta)|^2\).

\[
|\mathbf{r}'(\theta)|^2 = (\cos(\theta - \theta_c) Q_+ + \sin(\theta - \theta_c) Q_+)^2 + (-\sin(\theta - \theta_c) Q_+ + \cos(\theta - \theta_c) Q_+)^2
\]

\[
= \cos^2(\theta - \theta_c) Q_+^2 + 2Q_+ \frac{\partial Q_+}{\partial \theta} \cos(\theta - \theta_c) \sin(\theta - \theta_c) + \left(\frac{\partial Q_+}{\partial \theta}\right)^2 \sin^2(\theta - \theta_c)
\]

\[
+ \sin^2(\theta - \theta_c) Q_+^2 - 2Q_+ \frac{\partial Q_+}{\partial \theta} \cos(\theta - \theta_c) \sin(\theta - \theta_c) + \left(\frac{\partial Q_+}{\partial \theta}\right)^2 \cos^2(\theta - \theta_c)
\]

\[
= Q_+^2 + \left(\frac{\partial Q_+}{\partial \theta}\right)^2.
\]  
(6.28)

Now we need to simplify \(|\mathbf{r}'(\theta)|^2\). So we need to find \(\frac{\partial}{\partial \theta} (6.12)\), which is

\[
\frac{\partial \Delta}{\partial \theta} = 2Q_3 \frac{\partial Q_3}{\partial \theta}.
\]  
(6.29)

We can find \(\frac{\partial}{\partial \theta} (6.11)\).

\[
\frac{\partial Q_+}{\partial \theta} = -\frac{\partial Q_3}{\partial \theta} \pm \frac{1}{2} \sqrt{\frac{\Delta}{\partial \theta}} \frac{\partial \Delta}{\partial \theta}
\]

\[
= -\frac{\partial Q_3}{\partial \theta} \pm \frac{1}{2} \sqrt{\Delta} \frac{2Q_3}{\partial Q_3} \frac{\partial Q_3}{\partial \theta}
\]

\[
= -\frac{\partial Q_3}{\partial \theta} \pm \frac{Q_3}{\sqrt{\Delta}} \frac{\partial Q_3}{\partial \theta}
\]

\[
= \frac{1}{\sqrt{\Delta}} \frac{\partial Q_3}{\partial \theta} (-\sqrt{\Delta} \pm Q_3)
\]

\[
= \mp Q_+ \frac{\partial Q_3}{\partial \theta}.
\]  
(6.30)

So (6.28) becomes

\[
|\mathbf{r}'(\theta)|^2 = Q_+^2 \left[1 \mp \frac{1}{\Delta} \left(\frac{\partial Q_3}{\partial \theta}\right)^2\right]
\]

\[
= Q_+^2 \left[\Delta \mp \left(\frac{\partial Q_3}{\partial \theta}\right)^2\right].
\]  
(6.31)

Taking \(\frac{\partial}{\partial \theta} (6.10)\), we get

\[
\frac{\partial Q_3}{\partial \theta} = \cos(\theta - \theta_c) \sigma_0 - \sin(\theta - \theta_c) \zeta_0
\]  
(6.32)
Now substituting in (6.10) and \(Q_2\) into (6.12) gives

\[
\Delta = (\sin(\theta - \theta_c)\sigma_0 + \cos(\theta - \theta_c)\zeta_0)^2 + a^2 - \sigma_0^2 - \zeta_0^2
\]

\[
= \sin^2(\theta - \theta_c)\sigma_0^2 + 2\sigma_0\zeta_0\cos(\theta - \theta_c)\sin(\theta - \theta_c) + \cos^2(\theta - \theta_c)\zeta_0^2 + a^2 - \sigma_0^2 - \zeta_0^2.
\]

(6.33)

So now adding (6.33) to the square of (6.32), we get

\[
\Delta + \left(\frac{\partial Q_3}{\partial \theta}\right)^2 = \sin^2(\theta - \theta_c)\sigma_0^2 + 2\sigma_0\zeta_0\cos(\theta - \theta_c)\sin(\theta - \theta_c) + \cos^2(\theta - \theta_c)\zeta_0^2 + a^2 - \sigma_0^2 - \zeta_0^2
\]

\[
+ \cos^2(\theta - \theta_c)\sigma_0^2 - 2\sigma_0\zeta_0\cos(\theta - \theta_c)\sin(\theta - \theta_c) + \sin^2(\theta - \theta_c)\zeta_0^2
\]

\[
= -\sigma_0^2 - \zeta_0^2 + a^2 + \sigma_0^2 + \zeta_0^2
\]

\[
= a^2.
\]

So therefore (6.31) becomes

\[
|r'(\theta)|^2 = \frac{Q_2^2}{\Delta}a^2.
\]

So we get

\[
|r'(\theta)| = \frac{a|Q_\pm|}{\sqrt{\Delta}},
\]

(6.34)

which is the Jacobian for a change of variables.
CHAPTER 7
THE FAR-FIELD

This section will consider what happens to the waves as the viewpoint, $P$, goes to infinity. We will consider the single layer and the double layer potentials that were found earlier. Looking at the figure 1, there is symmetry amongst the wave beams. Therefore we are just taking a look at the case when we are in region II. By symmetry similar calculations can be made for the other regions. When our circle is oscillating at a frequency $\omega = N \cos \theta_c$ the internal waves are generated within the wave beams region II, IV, VI, VII. Outside the wave beams the waves are negligible. The following sections will talk about what happens to the internal waves inside the wave beams and outside the wave beams in the far field.

7.1 The Far-Field Outside the Wave Beams

When $P = (x_0, z_0)$ is in region III, $-\theta_c < \theta < \theta_c$, we have that $\cos \theta > \cos \theta_c$, so we can take a look at (5.10) the single layer potential and (5.17). There are no singularities in this region so in the far field $R \approx r_0$, so for the single layer

$$ (S\mu)(P) \approx -\frac{N \sin \theta_c \log r_0}{2\pi} \int_{C_a} \mu dC. \quad (7.1) $$

If we want $S\mu$ to go to zero as $r_0 \to \infty$, we must have

$$ \int_{C_a} \mu dC = 0. \quad (7.2) $$

For the single layer potential, $\mu = v^p \cdot n$, from (5.3). When $C_a$ oscillates as a rigid body, we have $v^p \cdot n = v_0^p \cdot n$, where $v_0^p$ is a constant vector. Hence, $\int_{C_a} \mu dC = 0$, by the divergence theorem.

Outside of the wave beams, the double layer potential becomes

$$ (D\mu)(P) \approx iN(\theta_0) \frac{\cos \theta_c \sin \theta_c}{(\cos^2 \theta_0 - \cos^2 \theta_c)r_0} \int_{C_a} \mu dC, \quad (7.3) $$

where $\theta_0$ is the constant angle from the origin to the point $P$ from the $z$ axis. Therefore the double layer potential decays at least like $r_0^{-1}$, it also depends on what the integral over the curve of $\mu = p$, the pressure.

Similar arguments can be made for both the single and double layer potentials in the other non-beam regions.
7.2 Single Layer Potential Inside the Wave Beams

With the parameterization of $C_a$ and the change of variables found in the previous section, we can substitute into our single layer potential. Equation (5.11) becomes

$$ (S_{\mu})_{\pm} \approx \int_{\theta_-}^{\theta_+} \mu_{\pm}(\theta) M(\theta) \frac{|Q_{\pm}|}{\sqrt{\Delta}} \frac{d\theta}{2\pi} $$  \hspace{1cm} (7.4)

where the $\pm$ refers to the two sides of $C_a$, and the two have to be summed to obtain $S_{\mu}$.

In the integrand we have two cases for $M(\theta)$ from (5.20). The integrand will be approximated for the two cases. The first case is when $\theta > \theta_c$. Then, we get

$$ M(\theta) \frac{|Q_{\pm}|}{\sqrt{\Delta}} = -aN \sin \theta_c \log \left[ R(2 \tan \theta_c)^{1/2} (\theta - \theta_c)^{1/2} \right] \frac{|Q_{\pm}|}{\sqrt{\Delta}} $$

$$ \approx -aN \sin \theta_c \log \left[ r_0(2 \tan \theta_c)^{1/2} (\theta - \theta_c)^{1/2} \right] \sqrt{(\theta - \theta_-)(\theta_+ - \theta)} $$ \hspace{1cm} (7.5)

using the approximations (6.24) and (6.25). Now for the second case, $\theta < \theta_c$, we get

$$ M(\theta) \frac{|Q_{\pm}|}{\sqrt{\Delta}} = -aN \sin \theta_c \left[ \log \left( R(2 \tan \theta_c)^{1/2} (\theta_+ - \theta) \right) + \frac{i\pi}{2} \right] \frac{|Q_{\pm}|}{\sqrt{\Delta}} $$

$$ \approx -aN \sin \theta_c \log \left[ r_0(2 \tan \theta_c)^{1/2} (\theta_+ - \theta) \right] + \frac{i\pi}{2} \sqrt{(\theta - \theta_-)(\theta_+ - \theta)} $$ \hspace{1cm} (7.6)

using the approximations (6.24) and (6.25). By substituting in equations (7.5) and (7.6) into (7.4), the single layer potential gets split into two integrals, giving

$$ (S_{\mu})_{\pm} \approx \frac{-aN \sin \theta_c}{2\pi} \left\{ \int_{\theta_+}^{\theta^+} \mu_{\pm}(\theta) \frac{\log \left[ r_0(2 \tan \theta_c)^{1/2} (\theta - \theta_c)^{1/2} \right] \sqrt{(\theta - \theta_-)(\theta_+ - \theta)}}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta + \int_{\theta_-}^{\theta_c} \mu_{\pm}(\theta) \frac{\log \left[ r_0(2 \tan \theta_c)^{1/2} (\theta - \theta_c)^{1/2} \right] \sqrt{(\theta - \theta_-)(\theta_+ - \theta)}}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta \right\} $$

$$ = \frac{-aN \sin \theta_c}{2\pi} \left\{ \int_{\theta_+}^{\theta^+} \mu_{\pm}(\theta) \frac{\log \left[ r_0(2 \tan \theta_c)^{1/2} (\theta - \theta_c)^{1/2} \right] \sqrt{(\theta - \theta_-)(\theta_+ - \theta)}}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta + \int_{\theta_-}^{\theta_c} \mu_{\pm}(\theta) \frac{\log \left[ r_0(2 \tan \theta_c)^{1/2} (\theta - \theta_c)^{1/2} \right] \sqrt{(\theta - \theta_-)(\theta_+ - \theta)}}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta \right\} $$

$$ + \int_{\theta_+}^{\theta^+} \mu_{\pm}(\theta) \frac{\log \left[ r_0(2 \tan \theta_c)^{1/2} (\theta - \theta_c)^{1/2} \right] \sqrt{(\theta - \theta_-)(\theta_+ - \theta)}}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta $$

$$ + \int_{\theta_-}^{\theta_c} \mu_{\pm}(\theta) \frac{i\pi}{2} \sqrt{(\theta - \theta_-)(\theta_+ - \theta)} d\theta $$

29
\[(S\mu)_\pm = -aN \sin \theta_c \left\{ \int_{\theta_-}^{\theta_+} \mu_\pm(\theta) \log \left[ r_0 (2 \tan \theta_c) \frac{\frac{1}{2} \theta - \theta_c}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} \right] d\theta \right\} + \frac{i\pi}{2} \int_{\theta_-}^{\theta_+} \frac{\mu_\pm(\theta)}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta \]

Adding the two sides from the curve \(C_a\), our single layer potential is

\[(S\mu) = (S\mu)_+ + (S\mu)_- \]

\[
\approx -aN \sin \theta_c \left\{ \int_{\theta_-}^{\theta_+} \left( \mu_+(\theta) + \mu_-(\theta) \right) \log \left[ r_0 (2 \tan \theta_c) \frac{\frac{1}{2} \theta - \theta_c}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} \right] d\theta \right\} + \frac{1}{2} \int_{\theta_-}^{\theta_+} \left( \mu_+(\theta) + \mu_-(\theta) \right) \log |\theta - \theta_c| d\theta + \frac{i\pi}{2} \int_{\theta_-}^{\theta_+} \frac{\mu_+(\theta) + \mu_-(\theta)}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta \].

Let us call \(\mu_S(\theta) = \mu_+(\theta) + \mu_-(\theta)\). That gives us the single layer potential

\[(S\mu) \approx -aN \sin \theta_c \left\{ \log \left[ r_0 (2 \tan \theta_c) \frac{\frac{1}{2} \theta - \theta_c}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} \right] \int_{\theta_-}^{\theta_+} \frac{\mu_S(\theta)}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta \right\} + \frac{1}{2} \int_{\theta_-}^{\theta_+} \frac{\mu_S(\theta) \log |\theta - \theta_c|}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta + \frac{i\pi}{2} \int_{\theta_-}^{\theta_+} \frac{\mu_S(\theta)}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta \] (7.7)

Let us expand \(\mu_S(\theta)\) as a Taylor series about \(\theta = \theta_c\), if the Taylor coefficient of \((\theta - \theta_c)^n\) is \(c_n r_0^n\), we suppose that \(m_n^S = \lim_{r_0 \to \infty} c_n r_0^n\) exists and we get

\[
\mu_S(\theta) = \sum_{n=0}^{\infty} m_n^S r_0^n (\theta - \theta_c)^n. \quad (7.8)
\]

Substituting (7.8) into (7.7)

\[
(S\mu) \approx -aN \sin \theta_c \left\{ \log \left[ r_0 (2 \tan \theta_c) \frac{\frac{1}{2} \theta - \theta_c}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} \right] \sum_{n=0}^{\infty} m_n^S r_0^n \int_{\theta_-}^{\theta_+} \frac{(\theta - \theta_c)^n}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta \right\} + \frac{1}{2} \sum_{n=0}^{\infty} m_n^S r_0^n \int_{\theta_-}^{\theta_+} \frac{(\theta - \theta_c)^n \log |\theta - \theta_c|}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta + \frac{i\pi}{2} \sum_{n=0}^{\infty} m_n^S r_0^n \int_{\theta_-}^{\theta_+} \frac{(\theta - \theta_c)^n}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta \right\}. \quad (7.9)
\]

Let us label the three integrals from (7.9), \(I_1\), \(I_2\), and \(I_3\), respectively. We will take a look at each integral
separately. First we will look at $I_1$.

$$ I_1 = \sum_{n=0}^{\infty} m_n^S a_n^r 0 \int_{\theta_-}^{\theta_+} \frac{(\theta - \theta_c)^n}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta $$

(7.10)

Performing a change of variables with $\theta = \alpha x + \beta$ with $\beta = \frac{\theta_+ + \theta_-}{2}$ and $\alpha = \frac{\theta_+ - \theta_-}{2}$.

$$ \theta - \theta_- = \alpha x + \beta - \theta_- = \alpha x + \frac{\theta_+ + \theta_-}{2} - \theta_- = \alpha x + \frac{\theta_+ - \theta_-}{2} = \alpha x + \alpha $$

(7.11)

and

$$ \theta_+ - \theta = \theta_+ - \alpha x - \beta = \theta_+ - \alpha x - \frac{\theta_+ + \theta_-}{2} = \frac{\theta_+ - \theta_-}{2} - \alpha = \alpha - \alpha x $$

(7.12)

with

$$ \theta - \theta_c = \alpha x + \beta - \theta_c = \alpha \left( x + \frac{\beta - \theta_c}{\alpha} \right) = \alpha (x - x_c) $$

(7.13)

where the variable $x_c = \frac{\theta_+ - \beta}{\alpha}$, $-1 < x_c < 1$, now using (7.11), (7.12), and (7.13), the integral (7.10) becomes

$$ I_1 = \sum_{n=0}^{\infty} m_n^S a_n^r 0 \int_{-1}^{1} \frac{(\alpha(x - x_c))^n}{\alpha(x + 1)\alpha(1-x)} \alpha dx 
= \sum_{n=0}^{\infty} m_n^S a_n^r \alpha \int_{-1}^{1} \frac{x - x_c)^n}{\sqrt{1-x^2}} dx. $$

(7.14)

In the far field we have

$$ \alpha \approx \frac{a}{r_0} $$

(7.15)

and

$$ \beta \approx \theta_c + \frac{\sigma_0}{r_0} $$

(7.16)

Doing another change of variables with $x = \sin \phi$ and letting there be an angle $\phi_c$ such that $x_c = \sin \phi_c$, $-\frac{\pi}{2} < \phi_c < \frac{\pi}{2}$, (7.14) becomes

$$ I_1 = \sum_{n=0}^{\infty} m_n^S a_n^r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\sin \phi - \sin \phi_c)^n}{\sqrt{1 - \sin^2 \phi}} \cos \phi d\phi 
= \sum_{n=0}^{\infty} m_n^S a_n^r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \phi - \sin \phi_c)^n d\phi, $$

(7.17)

We will take out the term $n = 0$ from (7.17), giving

$$ I_1 = m_0^S \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi + \sum_{n=1}^{\infty} m_n^S a_n^r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \phi - \sin \phi_c)^n d\phi $$

$$ = \pi m_0^S + \sum_{n=1}^{\infty} m_n^S a_n^r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \phi - \sin \phi_c)^n d\phi. $$

(7.18)
Using the Binomial Theorem the integral in (7.18) becomes

$$I_1 = \pi m_0^S + \sum_{n=1}^{\infty} m_n a_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!(-1)^n}{(2k)!(n-2k)!} \sin^{n-2k} \phi_c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^k \phi d\phi.$$  \hspace{1cm} (7.19)

Now we have an integral that is the sine function raised to an integer power. When the power $k$ is an odd integer the integral is equal to zero so we need $k$ when it is an even number.

$$I_1 = \pi m_0^S + \sum_{n=1}^{\infty} m_n a_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!(-1)^n}{(2k)!(n-2k)!} \sin^{n-2k} \phi_c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^k \phi d\phi,$$  \hspace{1cm} (7.20)

where $\lfloor x \rfloor$ is defined by rounding the value $x$ down to the nearest integer. The integral in (7.19) can be solved using the recursion relationship (A.3) so our equation for $I_1$ becomes

$$I_1 = \pi m_0^S + \sum_{n=1}^{\infty} m_n a_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!(-1)^n}{(2k)!(n-2k)!} \sin^{n-2k} \phi_c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^k \phi d\phi,$$  \hspace{1cm} (7.21)

Second we will look at $I_2$

$$I_2 = \sum_{n=0}^{\infty} m_n a_n \int_{\theta_+}^{\theta_-} \frac{(\theta - \theta_c)^n \log |\theta - \theta_c|}{\sqrt{(\theta - \theta_+)(\theta_+ - \theta)}} d\theta,$$  \hspace{1cm} (7.22)

Using the same change of variables as for $I_1$, $\theta = \alpha x + \beta$ and equations (7.11), (7.12), and (7.13), the integral (7.21) becomes

$$I_2 = \sum_{n=0}^{\infty} m_n a_n \int_{-1}^{1} \frac{(x - x_c)^n \log |x - x_c|}{\sqrt{\alpha(x + 1)\alpha(1 - x)}} dx$$

$$= \sum_{n=0}^{\infty} m_n a_n \int_{-1}^{1} \frac{(x - x_c)^n \log |x - x_c|}{\sqrt{1 - x^2}} dx$$

$$= \log(\alpha) \sum_{n=0}^{\infty} m_n a_n \int_{-1}^{1} \frac{(x - x_c)^n \log |x - x_c|}{\sqrt{1 - x^2}} dx + \sum_{n=0}^{\infty} m_n a_n \int_{-1}^{1} \frac{(x - x_c)^n \log |x - x_c|}{\sqrt{1 - x^2}} dx,$$  \hspace{1cm} (7.23)

Notice that the first integral in (7.22) is the exact same as $I_1$, (7.20), multiplied by a constant, so we get

$$I_2 = \log(\alpha) I_1 + \sum_{n=0}^{\infty} m_n a_n \int_{-1}^{1} \frac{(x - x_c)^n \log |x - x_c|}{\sqrt{1 - x^2}} dx.$$  \hspace{1cm} (7.24)

Now we will do the change of variables, $x = \sin \phi$, so (7.23) becomes
\[ I_2 = \log(\alpha)I_1 + \sum_{n=0}^{\infty} m_n^S a^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \phi - \sin \phi_c)^n \log |\sin \phi - \sin \phi_c| \cos \phi d\phi \]
\[ = \log(\alpha)I_1 + \sum_{n=0}^{\infty} m_n^S a^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \phi - \sin \phi_c)^n \log |\sin \phi - \sin \phi_c| d\phi. \quad (7.24) \]

We will take out the term \( n = 0 \) from (7.24), giving
\[ I_2 = \log(\alpha)I_1 + m_0^S \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |\sin \phi - \sin \phi_c| d\phi \]
\[ + \sum_{n=1}^{\infty} m_n^S a^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \phi - \sin \phi_c)^n \log |\sin \phi - \sin \phi_c| d\phi. \quad (7.25) \]

Let us label the two integrals from (7.25), \( I_2^0 \) and \( I_2^n \) respectively. So \( I_2^0 \) is
\[ I_2^0 = m_0^S \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |\sin \phi - \sin \phi_c| d\phi. \quad (7.26) \]

Using equation (B.5), (7.26) becomes
\[ I_2^0 = -m_0^S \pi \log 2. \quad (7.27) \]

Also \( I_2^n \) is
\[ I_2^n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \phi - \sin \phi_c)^n \log |\sin \phi - \sin \phi_c| d\phi. \quad (7.28) \]

In Appendix D, equation (7.28) becomes (D.15). Using (7.27) and (D.15) equation (7.25) becomes
\[ I_2 = \log(\alpha)I_1 - m_0^S \pi \log 2 \]
\[ + \sum_{n=1}^{\infty} m_n^S a^n \left\{ \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{n!\psi(1+n) - \psi(1+n-k)}{(n-k)!} \right) \sin^k \phi_c \int_{-\frac{\pi}{2}}^{\phi_c} \sin^{n-k} \phi d\phi \right. \]
\[ + \sum_{k=0}^{n} \frac{(-1)^{n+k}}{k!} \left( \frac{n!\psi(1+n) - \psi(1+n-k)}{(n-k)!} \right) \sin^{n-k} \phi_c \int_{-\frac{\phi_c}{
\[ + \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{n!\psi(1+n) - \psi(1+n-k)}{(n-k)!} \right) \sin^k \phi_c \int_{\phi_c}^{\frac{\pi}{2}} \sin^{n-k} \phi d\phi \right\} \quad (7.29) \]

Then we will look at \( I_3 \).
\[ I_3 = \sum_{n=0}^{\infty} m_n^S r_0^n \int_{\theta_-}^{\theta_+} \frac{(\theta - \theta_c)^n}{\sqrt{(\theta - \theta_-)(\theta_+ - \theta)}} d\theta. \quad (7.30) \]

The integral (7.30) is almost exactly the same as (7.10), the only difference is the upper bound of the integral.

By using the same change of variables as for \( I_1 \), \( \theta = \alpha x + \beta \) and equations (7.11), (7.12), and (7.13), the integral (7.30) becomes
Using the Binomial Expansion on (7.33) we get

\[ I_3 = \sum_{n=0}^{\infty} m_n^S a^n \int_{-1}^{x_c} \frac{(\alpha(x - x_c))^n}{\sqrt{\alpha(1 - x)}} \, dx \]
\[ = \sum_{n=0}^{\infty} m_n^S a^n \int_{-1}^{x_c} \frac{(x - x_c)^n}{\sqrt{1 - x^2}} \, dx. \]  

(7.31)

Doing the change of variables \( x = \sin \phi \), (7.31) becomes

\[ I_3 = \sum_{n=0}^{\infty} m_n^S a^n \int_{-\pi/2}^{\phi_c} \frac{(\sin \phi - \sin \phi_c)^n}{\sqrt{1 - \sin^2 \phi}} \cos \phi \, d\phi \]
\[ = \sum_{n=0}^{\infty} m_n^S a^n \int_{-\pi/2}^{\phi_c} (\sin \phi - \sin \phi_c)^n \phi \, d\phi. \]  

(7.32)

We will take out the term \( n = 0 \) from (7.32).

\[ I_3 = m_0^S \left( \sin^{-1}(x_c) + \frac{\pi}{2} \right) + \sum_{n=1}^{\infty} m_n^S a^n \int_{-\pi/2}^{\phi_c} (\sin \phi - \sin \phi_c)^n \phi \, d\phi \]
\[ = m_0^S \left( \phi_c + \frac{\pi}{2} \right) + \sum_{n=1}^{\infty} m_n^S a^n \int_{-\pi/2}^{\phi_c} (\sin \phi - \sin \phi_c)^n \phi \, d\phi \]
\[ = m_0^S \left( \sin^{-1}(x_c) + \frac{\pi}{2} \right) + \sum_{n=1}^{\infty} m_n^S a^n \int_{-\pi/2}^{\phi_c} (\sin \phi - \sin \phi_c)^n \phi \, d\phi. \]  

(7.33)

Using the Binomial Expansion on (7.33) we get

\[ I_3 = m_0^S \left( \sin^{-1}(x_c) + \frac{\pi}{2} \right) + \sum_{n=1}^{\infty} m_n^S a^n \sum_{k=0}^{n} \frac{(-1)^n n!}{k!(n-k)!} \sin^{n-k} \phi_c \int_{-\pi/2}^{\phi_c} \sin^k \phi \, d\phi. \]  

(7.34)

Finally let us substitute (7.20), (7.29), and (7.34) into (7.9) to get our single layer potential.

\[ S \mu \sim \frac{-aN \sin \theta_c}{2\pi} \left\{ m_0^S \left[ \pi \log \left[ r_0(2 \tan \theta_c)^{\frac{1}{2}} \right] + \frac{\pi}{2} \log(\alpha) - \frac{\pi}{2} \log(2) + i\frac{\pi}{2} \left( \sin^{-1}(x_c) + \frac{\pi}{2} \right) \right] \right. \]
\[ + \left\{ \pi \log \left[ r_0(2 \tan \theta_c)^{\frac{1}{2}} \right] + \frac{\pi}{2} \log(\alpha) \right\} \sum_{n=1}^{\infty} m_n^S a^n \left\{ \sum_{k=0}^{n} \frac{(-1)^n n! \psi(1+n) - \psi(1+n-k)}{(n-k)!} \right\} \sin^{n-k} \phi_c \int_{-\pi/2}^{\phi_c} \sin^k \phi \, d\phi \]
\[ + \frac{1}{2} \sum_{k=0}^{n} \frac{(-1)^{n+k} n! \psi(1+n) - \psi(1+n-k)}{(n-k)!} \sin^{n-k} \phi_c \int_{-\pi/2}^{\phi_c} \sin^k \phi \, d\phi \]
\[ + \frac{1}{2} \sum_{k=0}^{n} \frac{(-1)^k n! \psi(1+n) - \psi(1+n-k)}{(n-k)!} \sin^k \phi_c \int_{-\pi/2}^{\phi_c} \sin^{n-k} \phi \, d\phi \]
\[ + \frac{i\pi}{2} \sum_{n=1}^{\infty} m_n^S a^n \sum_{k=0}^{n} \frac{(-1)^n n! \sin^{n-k} \phi_c \int_{-\pi/2}^{\phi_c} \sin^k \phi \, d\phi}{k!(n-k)!} \right\} \]  

(7.35)
7.3 Double Layer Potential Inside the Wave Beams

As we did for the single layer potential, substituting the parametrization of \( C_a \) into our double layer potential, equation (5.17) becomes

\[
(D\mu)_\pm \approx \int_{\theta_-}^{\theta_+} \mu_{\pm}(\theta) D(\theta) N(\theta) \frac{a Q_\pm}{\sqrt{\Delta}} \frac{d\theta}{2\pi R} \quad (7.36)
\]

where the \( \pm \) refers to the two sides of \( C_a \), and the two have to be summed to obtain \( D\mu \). So our normal vector \( \mathbf{n} = (n_1, n_2) \) becomes

\[
a \mathbf{n} = (x, z) = (X_+ \cos \theta_c + Z_+ \sin \theta_c, -X_+ \sin \theta_c + Z_+ \cos \theta_c).
\]

Then (5.14) becomes

\[
a N = X_+ \sin \theta \cos \theta_c + Z_+ \sin \theta_c \sin \theta - X_+ \sin \theta_c \cos \theta + Z_+ \cos \theta \cos \theta_c = X_+ \sin(\theta - \theta_c) + Z_+ \cos(\theta - \theta_c).
\]

We now substitute in (6.15) and (6.16) to get

\[
a N = (\sin(\theta - \theta_c)Q_\pm + \sigma_0) \sin(\theta - \theta_c) + (\cos(\theta - \theta_c) + \zeta_0) \cos(\theta - \theta_c)
\]

\[
= \sin^2(\theta - \theta_c)Q_\pm + \sigma \sin(\theta - \theta_c) + \cos^2(\theta - \theta_c)Q_\pm + \zeta_0 \cos(\theta - \theta_c)
\]

\[
= Q_\pm + \sigma \sin(\theta - \theta_c) + \zeta_0 \cos(\theta - \theta_c)
\]

\[
= Q_\pm + Q_3
\]

\[
= -Q_3 \pm \sqrt{\Delta} + Q_3
\]

\[
= \pm \sqrt{\Delta}.
\]

(7.37)

Now substituting in (5.21) and (7.37) into (7.36) and simplifying, we get

\[
(D\mu)_\pm \approx \frac{a}{2\pi} \int_{\theta_-}^{\theta_+} \mu_{\pm}(\theta) \frac{-i}{2(\theta - \theta_c)} \frac{\pm \sqrt{\Delta} Q_\pm}{a} \frac{d\theta}{\sqrt{\Delta} R}
\]

\[
= \mp \frac{i}{4\pi} \int_{\theta_-}^{\theta_+} \mu_{\pm}(\theta) \frac{Q_\pm}{(\theta - \theta_c)} \frac{d\theta}{R}.
\]

If we use the approximations (6.24) and (6.25), the two sides of the double layer become

\[
(D\mu)_\pm \approx \pm \frac{i}{4\pi} \int_{\theta_-}^{\theta_+} \mu_{\pm}(\theta) \frac{d\theta}{(\theta - \theta_c)}.
\]

(7.38)
Adding the two sides from the curve $C_a$, our double layer potential is

$$
(D\mu) = (D\mu)_+ + (D\mu)_-
$$

$$
\approx \frac{i}{4\pi} \int_{\theta_-}^{\theta_+} (\mu_+(\theta) - \mu_-(\theta)) \frac{d\theta}{(\theta - \theta_c)}.
$$

Let us call $\mu_D(\theta) = \mu_+(\theta) - \mu_-(\theta)$. That gives us the double layer potential

$$
(D\mu) \approx \frac{i}{4\pi} \int_{\theta_-}^{\theta_+} \mu_D(\theta) \frac{d\theta}{(\theta - \theta_c)}.
$$

(7.39)

Next, we expand $\mu_D(\theta)$ as a Taylor series about $\theta = \theta_c$, as we did with $\mu_S$

$$
\mu_D(\theta) = \sum_{n=0}^{\infty} m_n^D r^n_0 (\theta - \theta_c)^n.
$$

(7.40)

Substituting (7.40) into (7.39)

$$
D\mu \approx \frac{i}{4\pi} \sum_{n=0}^{\infty} m_n^D r^n_0 \int_{\theta_-}^{\theta_+} (\theta - \theta_c)^{n-1} d\theta.
$$

(7.41)

We will have two cases for the integral (7.41). We will first take a look at when $n = 0$ then $n > 0$, the first case gives

$$
D\mu_0 \approx \frac{i}{4\pi} m_0^D \int_{\theta_-}^{\theta_+} (\theta - \theta_c)^{-1} d\theta.
$$

(7.42)

Using Cauchy’s Principal Value with $\epsilon > 0$, we can integrate

$$
D\mu_0 \approx \frac{i}{4\pi} m_0^D \lim_{\epsilon \to 0^+} \left[ \int_{\theta_-}^{\theta_- - \epsilon} (\theta - \theta_c)^{-1} d\theta + \int_{\theta_+ + \epsilon}^{\theta_+} (\theta - \theta_c)^{-1} d\theta \right]
$$

$$
= \frac{i}{4\pi} m_0^D \lim_{\epsilon \to 0^+} \left[ \log(\theta - \theta_c) |_{\theta_-}^{\theta_- - \epsilon} + \log(\theta - \theta_c) |_{\theta_+ + \epsilon}^{\theta_+} \right]
$$

$$
= \frac{i}{4\pi} m_0^D \lim_{\epsilon \to 0^+} \left[ \log(\theta_c - \epsilon - \theta_c) - \log(\theta_- - \theta_c) + \log(\theta_+ - \theta_c) - \log(\theta_c + \epsilon - \theta_c) \right]
$$

$$
= \frac{i}{4\pi} m_0^D \lim_{\epsilon \to 0^+} \left[ \log(\theta_+ - \theta_c) - \log(\theta_- - \theta_c) + \log(-\epsilon) - \log(\epsilon) \right].
$$

(7.43)

Let us consider when $r > 0$ then $-r = re^{i\pi}$ and $\log(-r) = \log(r) + i\pi$. Using this fact, equation (7.43) becomes

$$
D\mu_0 \approx \frac{i}{4\pi} m_0^D \lim_{\epsilon \to 0^+} \left[ \log(\theta_+ - \theta_c) - \log(\theta_- - \theta_c) - i\pi + \log(\epsilon) + i\pi - \log(\epsilon) \right]
$$

$$
= \frac{i}{4\pi} m_0^D \lim_{\epsilon \to 0^+} \left[ \log(\theta_+ - \theta_c) - \log(\theta_- - \theta_c) \right]
$$

$$
= \frac{i}{4\pi} m_0^D \left[ \log(\theta_+ - \theta_c) - \log(\theta_- - \theta_c) \right].
$$
Now simplifying some more.

\[
D\mu_0 = \frac{i}{4\pi} m_0^D \log(\theta_+ - \theta_c) - \log(\theta_c - \theta_-)
\]

\[
= \frac{i}{4\pi} m_0^D \left[ \log \left( \frac{a + \sigma_0}{r} \right) - \log \left( \frac{\sigma_0 - a}{r} \right) \right]
\]

\[
= \frac{i}{4\pi} m_0^D \log \left( \frac{a + \sigma_0}{\sigma_0 - a} \right).
\]  

(7.44)

When \( n > 0 \), (7.41) gives us an integral of the form

\[
\int_{\theta_-}^{\theta_+} (\theta - \theta_c)^{\nu-1} d\theta = \frac{1}{\nu} \left[ (\theta_+ - \theta_c)^{\nu} - e^{i\nu\pi} (\theta_c - \theta_-)^{\nu} \right]
\]  

(7.45)

substituting our bounds (6.22) into (7.45), we get

\[
\int_{\theta_-}^{\theta_+} (\theta - \theta_c)^{\nu-1} d\theta = \frac{1}{\nu} \left[ \left( \frac{a + \sigma_0}{r_0} \right)^{\nu} - e^{i\nu\pi} \left( \frac{\sigma_0 - a}{r_0} \right)^{\nu} \right]
\]

\[
= \frac{1}{\nu r_0^{\nu}} [(a + \sigma_0)^{\nu} - (-1)^{\nu} (\sigma_0 - a)^{\nu}].
\]  

(7.46)

Using (7.46) with \( \nu = n \), (7.41) becomes

\[
D\mu_n = \frac{i}{4\pi} \sum_{n=1}^{\infty} m_n^D r_0^n \frac{1}{m_0^n} [(a + \sigma_0)^n - (-1)^{n} (\sigma_0 - a)^n]
\]

\[
= \frac{i}{4\pi} \sum_{n=1}^{\infty} m_n^D \frac{1}{n} [(a + \sigma_0)^n + (-1)^{n+1} (\sigma_0 - a)^n].
\]  

(7.47)

Our double layer potential is (7.44) added with (7.47), namely

\[
D\mu \sim \frac{i}{4\pi} m_0^D \log \left( \frac{a + \sigma_0}{\sigma_0 - a} \right) + \frac{i}{4\pi} \sum_{n=1}^{\infty} \frac{m_n^D}{n} [(a + \sigma_0)^n + (-1)^{n+1} (\sigma_0 - a)^n].
\]  

(7.48)
COMBINING THE SINGLE AND DOUBLE LAYER POTENTIALS

In order to obtain the general solution, we combine the single and double layer potentials. Using equations (7.35) and (7.48), our equation (5.3) for the pressure $p$ becomes

$$
p(P) = D(p) - S(v^p \cdot n)
$$

$$
\sim \frac{i}{4\pi} m_0^D \log \left( \frac{a + \sigma_0}{\sigma_0 - a} \right) + \frac{i}{4\pi} \sum_{n=1}^{\infty} \frac{m_0^D}{n} [(a + \sigma_0)^n + (-1)^{n+1}(\sigma_0 - a)^n]
$$

$$
+ \frac{a N \sin \theta_c}{2\pi} \left\{ m_0^S \left[ \pi \log \left( r_0 (2 \tan \theta_c)^{\frac{1}{2}} \right) + \frac{\pi}{2} \log(\alpha) - \frac{\pi}{2} \log(2) + \frac{i\pi}{2} \left( \sin^{-1}(x_c) + \frac{\pi}{2} \right) \right] \right. 
$$

$$
+ \left\{ \pi \log \left[ r_0 (2 \tan \theta_c)^{\frac{1}{2}} \right] + \frac{\pi}{2} \log(\alpha) \right\} \sum_{n=1}^{\infty} m_n^S a^n \sum_{k=0}^{\infty} \frac{n!(-1)^n}{2^{2k}k!(n-2k)!} \sin^{n-2k} \phi_c
$$

$$
+ \sum_{n=1}^{\infty} m_n^S a^n \left\{ \frac{1}{2} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{n!\left[\psi(1+n) - \psi(1+n-k)\right]}{(n-k)!} \right) \sin^k \phi_c \int_{-\phi_c}^{\phi_c} \sin^{n-k} \phi d\phi \right. 
$$

$$
+ \frac{1}{2} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{k!} \left( \frac{n!\left[\psi(1+n) - \psi(1+n-k)\right]}{(n-k)!} \right) \sin^{n-k} \phi_c \int_{-\phi_c}^{\phi_c} \sin^{n-k} \phi d\phi \right. 
$$

$$
+ \frac{1}{2} \sum_{k=0}^{n} \frac{(-1)^{k}}{k!} \left( \frac{n!\left[\psi(1+n) - \psi(1+n-k)\right]}{(n-k)!} \right) \sin^k \phi_c \int_{-\phi_c}^{\phi_c} \sin^{n-k} \phi d\phi \right. 
$$

$$
+ \left. \frac{i\pi}{2} \sum_{n=1}^{\infty} m_n^S a^n \sum_{k=0}^{n} \frac{(-1)^{n-k}n!}{k!(n-k)!} \sin^{n-k} \phi_c \int_{-\phi_c}^{\phi_c} \sin^k \phi d\phi \right\}, \quad (8.1)
$$

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CHAPTER 9
VERTICALLY OSCILLATING RIGID CIRCLE

This section will look at when the circle $C_a$ is oscillating vertically, in the $z$-direction. Hurley’s results are compared with the results obtained in the previous section. To achieve vertical oscillations $\mu_\pm = \mathbf{v} \cdot \mathbf{n} = \frac{W_z}{a}$, for the single layer potential, where $W$ is a constant. We need to parameterize $z$ with respect to the angle $\theta$. We already did this in Section 6. Using the equations (6.15), and (6.16), the equation for $z$, (6.3), in terms of $\theta$ and $\theta_c$ becomes

$$
z = -(\sin(\theta - \theta_c)Q_\pm + \sigma_0) \sin \theta_c + (\cos(\theta - \theta_c)Q_\pm + \zeta_0) \cos \theta_c
= -Q_\pm \sin(\theta - \theta_c) \sin \theta_c + \sigma_0 \sin \theta_c + Q_\pm \cos(\theta - \theta_c) \cos \theta_c + \zeta_0 \cos \theta_c
= Q_\pm \cos \theta - \sigma_0 \sin \theta_c + \zeta_0 \cos \theta_c
$$

(9.1)

To estimate the far field, we need to calculate $\mu_S$ using the approximations $Q_\pm \sim -r_0$ and $\zeta_0 \sim r_0$. Our $\mu_\pm$ is equal to (9.1),

$$
\mu_\pm \sim -r_0 \cos \theta - \sigma_0 \sin \theta_c + r_0 \cos \theta_c
\sim -r_0 (\cos \theta - \cos \theta_c).
$$

(9.2)

Recall that

$$
\mu_S = \mu_+ + \mu_-
\sim -2r_0 (\cos \theta - \cos \theta_c).
$$

(9.3)

We are now going to prove equation (7.2) with $\mu$ being defined by (9.3) and using the Jacobian (6.34) with the approximations (6.24) and (6.25), we get

$$
\int_{C_a} \mu_S dC \approx \int_{\theta_-}^{\theta_+} -2r_0 (\cos \theta - \cos \theta_c) \frac{|Q_\pm| d\theta}{\sqrt{\Delta}}
= -2r_0 a \int_{\theta_-}^{\theta_+} \frac{r_0 (\cos \theta - \cos \theta_c)}{\sqrt{r_0^2 (\theta_+ - \theta)(\theta - \theta_-)}} d\theta
= -2r_0 a \int_{\theta_-}^{\theta_+} \frac{(\cos \theta - \cos \theta_c)}{\sqrt{(\theta_+ - \theta)(\theta - \theta_-)}} d\theta.
$$

Performing the same change of variables as in Section 7, $\theta = \alpha x + \beta$, we get

$$
\int_{C_a} \mu_S dC \approx -2r_0 a \int_{-1}^{1} \frac{(\cos(\alpha x + \beta) - \cos \theta_c)}{\sqrt{\alpha (1 - x) \alpha (1 + x)}} \alpha dx.
$$
\[
\int_{C_a} \mu_S dC \approx -2r_0a \int_{-1}^{1} \frac{(\cos(\alpha x + \beta) - \cos \theta_c)}{\sqrt{1-x^2}} dx
\]
\[
= -2r_0a \int_{-1}^{1} \frac{(\cos(\alpha x) \cos \beta - \sin(\alpha x) \sin(\beta) - \cos \theta_c)}{\sqrt{1-x^2}} dx
\]
\[
= -2r_0a \left[ \int_{-1}^{1} \frac{\cos(\alpha x) \cos \beta - \cos \theta_c}{\sqrt{1-x^2}} dx - \int_{-1}^{1} \frac{\sin(\alpha x) \sin(\beta)}{\sqrt{1-x^2}} dx \right]
\]
(9.4)

Taking a look at when \( r_0 \to \infty \), equations (7.15) and (7.16) become respectively \( \alpha \approx 0 \) and \( \beta \approx \theta_c \). Therefore (9.4) becomes

\[
\int_{C_a} \mu_S dC = -2r_0a \left[ \int_{-1}^{1} \frac{\cos(0) \cos \theta_c - \cos \theta_c}{\sqrt{1-x^2}} dx - \int_{-1}^{1} \frac{\sin(0) \sin(\theta_c)}{\sqrt{1-x^2}} dx \right]
\]
\[
= -2r_0a [0 - 0]
\]
\[
= 0.
\]
(9.5)

Therefore (7.2) holds for \( \mu_S = z \).

9.1 Hurley’s Results for the Pressure

From Hurley’s paper to have a vertically oscillating circle the boundary condition on the curve \( C_a \) is

\[
F(\vartheta) = \cos \vartheta,
\]
where \( \vartheta \) is angle for the usual polar coordinates. His variables are converted to the variables we use throughout this section. We need to find his solution using the boundary condition. From his paper he uses a stream function, \( \Psi \), and than has a relationship to get to the pressure \( p \). Using the boundary condition the stream function becomes

\[
F(\vartheta) = \Psi(\vartheta)
\]
\[
= S_0 \log \left\{ \frac{ie^{i(\vartheta-\mu)}}{ie^{-i(\vartheta+\mu)}} \right\} + \sum_{n=1}^{\infty} c_n (ie^{i(\vartheta-\mu)})^{-n} + \sum_{n=1}^{\infty} d_n (ie^{-i(\vartheta+\mu)})^{-n}
\]
(9.6)

Substituting in \( \mu = \frac{\pi}{2} - \theta_c \), we get

\[
\cos \vartheta = S_0 2i \vartheta + \sum_{n=1}^{\infty} c_n (e^{-i(\vartheta-\theta_c)})^n + \sum_{n=1}^{\infty} d_n (e^{i(\vartheta+\theta_c)})^n
\]
\[
= S_0 2i \vartheta + \sum_{n=1}^{\infty} c_n e^{-in(\vartheta-\theta_c)} + \sum_{n=1}^{\infty} d_n e^{in(\vartheta-\theta_c)}.
\]
To be able to get \( \cos \vartheta \), \( S_0 = 0 \) because \( \vartheta \) is an odd function while \( \cos \vartheta \) is an even function. Also to just have \( \vartheta \) in the argument the constants \( c_n \) and \( d_n \) for \( n > 1 \) should be equal to 0. Therefore we get

\[
\cos \vartheta = c_1 e^{-i(\vartheta - \theta_c)} + d_1 e^{i(\vartheta - \theta_c)} = c_1 e^{i\theta_c} e^{-i\vartheta} + d_1 e^{-i\theta_c} e^{i\vartheta}. \tag{9.7}
\]

To have the boundary condition satisfied we need \( c_1 = \frac{1}{2} e^{-i\theta_c} \) and \( d_1 = \frac{1}{2} e^{i\theta_c} \). Therefore the stream function from Hurley becomes

\[
\Psi = \frac{1}{2} e^{-i\theta_c} \left\{ \frac{\sigma_0}{a} + \left[ \frac{\sigma_0^2}{a^2} - 1 \right]^{\frac{1}{2}} \right\}^{-1} + \frac{1}{2} e^{i\theta_c} \left\{ \frac{\sigma_0}{a} + \left[ \frac{\sigma_0^2}{a^2} - 1 \right]^{\frac{1}{2}} \right\}^{-1}. \tag{9.8}
\]

Using the relationship from Hurley’s paper the pressure \( p \) is

\[
p = \frac{i \rho_0 \omega \eta}{2} \left\{ e^{-i\theta_c} \left\{ \frac{\sigma_0}{a} + \left[ \frac{\sigma_0^2}{a^2} - 1 \right]^{\frac{1}{2}} \right\}^{-1} + e^{i\theta_c} \left\{ \frac{\sigma_0}{a} + \left[ \frac{\sigma_0^2}{a^2} - 1 \right]^{\frac{1}{2}} \right\}^{-1} \right\}. \tag{9.9}
\]

Substituting in \( \omega = N \cos \theta_c \) and \( \eta = \tan \theta_c \) into the pressure equation (9.9) with \( \sigma_+ = \sigma_0 \) and \( \sigma_- \approx \zeta_0 \) gives

\[
p = \frac{i \rho_0 N \sin \theta_c}{2} \left\{ e^{-i\theta_c} \left\{ \frac{\sigma_0}{a} + \left[ \frac{\sigma_0^2}{a^2} - 1 \right]^{\frac{1}{2}} \right\}^{-1} + e^{i\theta_c} \left\{ \frac{\zeta_0}{a} + \left[ \frac{\zeta_0^2}{a^2} - 1 \right]^{\frac{1}{2}} \right\}^{-1} \right\}. \tag{9.10}
\]

We are interested in the far field, so using the approximation from Section 7, \( \zeta \sim r_0 \). Let us take a look at the second term in (9.10),

\[
\left\{ \frac{r_0}{a} + \left[ \frac{r_0^2}{a^2} - 1 \right]^{\frac{1}{2}} \right\}^{-1} \approx \left\{ \frac{r_0}{a} + \frac{r_0}{a} \right\}^{-1} = \frac{a}{2r_0} \approx 0. \tag{9.11}
\]

Therefore Hurley’s pressure (9.10) in the far field is approximately

\[
p \sim \frac{i \rho_0 N \sin \theta_c}{2} e^{-i\theta_c} \left\{ \frac{\sigma_0}{a} + \left[ \frac{\sigma_0^2}{a^2} - 1 \right]^{\frac{1}{2}} \right\}^{-1}. \tag{9.12}
\]

Doing some more algebra on (9.12)

\[
p \sim \frac{i \rho_0 N \sin \theta_c}{2a} \left\{ \sigma_0 \cos \theta_c - i \sigma_0 \sin \theta_c - i (a^2 - \sigma_0^2)^{\frac{1}{2}} \cos \theta_c - (a^2 - \sigma_0^2)^{\frac{1}{2}} \frac{1}{2} \sin \theta_c \right\}. \tag{9.13}
\]

\section{9.2 Our Results}

For vertical oscillations we defined \( \mu = v^\theta \cdot n = \frac{\omega \gamma}{a} \). The double layer potential doesn’t depend on this \( \mu \) but the single layer does. Let us examine the single layer potential. We are looking in the far field so \( \theta \approx \theta_c \)
so we can Taylor expand \( \cos \theta \) about \( \theta = \theta_c \) from (9.2), which gives

\[
\mu_S \sim -2\sigma_0 \sin \theta_c + 2r_0 \sin \theta_c (\theta - \theta_c). \tag{9.14}
\]

Looking at the Taylor expansion we used earlier for \( \mu_S \), (7.8), the constants \( m_n^S \) can be calculated. Using (9.14), \( m_0^S = -2\sigma_0 \sin \theta_c \), \( m_1^S = 2 \sin \theta_c \), and \( m_n^S = 0 \) for \( n > 1 \). Equation (7.10), \( I_1 \) is the integral over the curve \( C_a \) of \( \mu_S \). Therefore the integral \( I_1 = 0 \). So we don’t need to solve for those terms of the single layer potential. Substituting in \( \sin \phi_c = \frac{\sigma_0}{a} \) and \( x_c = \frac{\sigma_0}{a} \). We can simplify the rest of the single layer potential (7.35) using our boundary condition on the curve \( C_a \) by taking the integrals now.

The constant term with \( m_0^S \) becomes

\[
-2\sigma_0 \sin \theta_c \left[ -\frac{\pi}{2} \log(2) + \frac{i\pi}{2} \left( \sin^{-1} \left( \frac{\sigma_0}{a} \right) + \frac{\pi}{2} \right) \right]
\]

\[
= \sigma_0 \pi \log(2) \sin \theta_c - \sigma_0 i\pi \sin \theta_c \left( \sin^{-1} \left( \frac{\sigma_0}{a} \right) + \frac{\pi}{2} \right) \tag{9.15}
\]

The different terms multiplied by \( m_1^S \) are as follows

\[
\sum_{n=1}^{\infty} m_n^S a_n \frac{1}{2} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{n! \left[ \psi(1+n) - \psi(1+n-k) \right]}{(n-k)!} \right) \sin^k \phi_c \int_{-\frac{\pi}{2}}^{-\phi_c} \sin^{n-k} \phi d\phi = m_1^S a_1 \frac{1}{2} \sum_{k=0}^{1} \frac{(-1)^k}{k!} \left( \frac{\psi(2) - \psi(2-k)}{1-k} \right) \sin^k \phi_c \int_{-\frac{\pi}{2}}^{-\phi_c} \sin^{1-k} \phi d\phi
\]

\[
= m_1^S a_1 \frac{1}{2} \left[ 0 - (\psi(2) - \psi(1)) \sin \phi_c \int_{-\frac{\pi}{2}}^{-\phi_c} d\phi \right] = -m_1^S a_1 \frac{1}{2} (\psi(2) - \psi(1)) \sin \phi_c \left( \frac{\pi}{2} - \phi_c \right)
\]

\[
= -2 \sin \theta_c a \frac{\sigma_0}{a} \left( \frac{\pi}{2} - \sin^{-1} \left( \frac{\sigma_0}{a} \right) \right) = -2\sigma_0 \sin \theta_c \frac{\pi}{2} - \sin^{-1} \left( \frac{\sigma_0}{a} \right) \tag{9.16}
\]

and

\[
\sum_{n=1}^{\infty} m_n^S a_n \frac{1}{2} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{k!} \left( \frac{n! \left[ \psi(1+n) - \psi(1+n-k) \right]}{(n-k)!} \right) \sin^{n-k} \phi_c \int_{-\phi_c}^{\phi_c} \sin^k \phi d\phi
\]

\[
= m_1^S a_1 \frac{1}{2} \sum_{k=0}^{1} \frac{(-1)^{1+k}}{k!} \left( \frac{\psi(2) - \psi(2-k)}{1-k} \right) \sin^{1-k} \phi_c \int_{-\phi_c}^{\phi_c} \sin^k \phi d\phi = 0. \tag{9.17}
\]

The next integral gives the exact same answer as equation (9.16),
\[ \sum_{n=1}^{\infty} m_n^a a^n \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{n!\psi(1+n) - \psi(1+n-k)}{(n-k)!} \right) \sin^k \phi_c \int_{\frac{\pi}{2}}^{\phi_c} \sin^{n-k} \phi d\phi \]

\[= -2\sigma_0 \sin \theta_c \left( \frac{\pi}{2} - \sin^{-1} \left( \frac{\sigma_0}{a} \right) \right) \]  

(9.18)

and the last integral

\[ \frac{i\pi}{2} \sum_{n=1}^{\infty} m_n^b a^n \sum_{k=0}^{n} \frac{(-1)^{n-k}n!}{k!(n-k)!} \sin^{n-k} \phi_c \int_{-\frac{\pi}{2}}^{\phi_c} \sin^k \phi d\phi \]

\[= \frac{i\pi}{2} \sum_{n=1}^{\infty} m_n^b a^n \sum_{k=0}^{n} \frac{(-1)^{n-k}n!}{k!(1-k)!} \sin^{n-k} \phi_c \int_{-\frac{\pi}{2}}^{\phi_c} \sin^k \phi d\phi \]

\[= \frac{i\pi}{2} m_1^b a \left\{ -\sin \phi_c \int_{-\frac{\pi}{2}}^{\phi_c} d\phi + \int_{-\frac{\pi}{2}}^{\phi_c} \sin \phi d\phi \right\} \]

\[= \frac{i\pi}{2} m_1^b a \left\{ -\sin \phi_c \left[ \phi_c + \frac{\pi}{2} - \cos \phi_c \right] \right\} \]

\[= \left\{ -i\pi \sigma_0 \sin \theta_c \left[ \sin^{-1} \left( \frac{\sigma_0}{a} \right) + \frac{\pi}{2} \right] - i\pi \sin \theta_c \sqrt{a^2 - \sigma_0^2} \right\} \]  

(9.19)

Summing up equations (9.15), (9.16), (9.17), (9.18), and (9.19) and multiplying by the constant \( \frac{W}{a} \) gives

\[ \frac{W}{a} \left\{ -2i\pi \sigma_0 \sin \theta_c \left[ \sin^{-1} \left( \frac{\sigma_0}{a} \right) + \frac{\pi}{2} \right] + \sigma_0 \pi \sin \theta_c \log 2 \right\} \]

\[-2\sigma_0 \sin \theta_c \left[ \frac{\pi}{2} - \sin^{-1} \left( \frac{\sigma_0}{a} \right) \right] - i\pi \sin \theta_c \sqrt{a^2 - \sigma_0^2} \right\}. \]

Therefore the pressure (8.1) in the far field for a vertically oscillating circle becomes

\[ p(P) \sim \frac{i}{4\pi} m_0^b \log \left( \frac{a + \sigma_0}{\sigma_0 - a} \right) + \frac{i}{4\pi} \sum_{n=1}^{\infty} m_n^b \right\}

\[\left[ (a + \sigma_0)^n + (-1)^{n+1}(\sigma_0 - a)^n \right] \]

\[+ \frac{WN \sin \theta_c}{2\pi} \left\{ -2i\pi \sigma_0 \sin \theta_c \left[ \sin^{-1} \left( \frac{\sigma_0}{a} \right) + \frac{\pi}{2} \right] + \sigma_0 \pi \sin \theta_c \log 2 \right\} \]

\[-2\sigma_0 \sin \theta_c \left[ \frac{\pi}{2} - \sin^{-1} \left( \frac{\sigma_0}{a} \right) \right] - i\pi \sin \theta_c \sqrt{a^2 - \sigma_0^2} \right\}. \]  

(9.20)

Noticing if we set \( W = \frac{\rho a}{\pi} \), we will have the same constant value in front of the values as Hurley does.
CHAPTER 10
CONCLUSION

This thesis presents a method for solving two-dimensional time-harmonic internal gravity waves generated by an oscillating object submerged in a density stratified fluid using an integral representation. We first solved for the hyperbolic partial differential equation in terms of the pressure $p$ from the governing linearized Boussinesq equations. We developed a reciprocal theorem to relate two time-harmonic pressure fields. The reciprocal theorem was used to find the solution of the elliptic problem, $\omega > N$, in terms of an integral representation. Analytic continuation in the complex $\omega$-plane provided us with the solution to the hyperbolic problem. Up to this point we solved for the pressure over any curve $C$, the most general case. We took the simplest case when the curve $C$ is a circle of radius $a$, $C_a$. This allowed easy parameterization for the integrals. We calculated the solution for any oscillating circle in the far field. Finally, we compared our solution in the far field with Hurley’s solution from [2] for a vertically oscillating circle.

Our work has shown some similarities and differences compared to the results obtained by Hurley. Both works conclude that internal gravity waves are constant within the wave beams and don’t depend on the distance from the object in the far field. One difference between the solutions is that the current representation contains $\sin^{-1}(a_0)$ terms while Hurley’s doesn’t possess an inverse sine function. This could be due to the fact that we only used the first two terms of the Taylor expansion for the function $\mu$ in the single layer potential. We speculate that the inverse sine functions cancel within higher order terms that we neglected.

Additionally, there is still work to be done to determine how close the solutions are. Different types of objects other than a circular geometry, and differing oscillations could also be the focus of future study.
REFERENCES CITED


This solves for the recursion relationship for the integral in (7.19). The integral is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k} \phi d\phi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k-1} \phi \sin \phi d\phi. \quad (A.1)$$

Doing an integration by parts we get

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k-1} \phi \sin \phi d\phi = -\sin^{2k-1} \phi \cos \phi |_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + (2k - 1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k-2} \phi \cos^2 \phi d\phi$$

$$= (2k - 1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k-2} \phi d\phi - (2k - 1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k} \phi d\phi.$$

The recursion relationship is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k} \phi d\phi = \frac{(2k - 1)}{2k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k-2} \phi d\phi. \quad (A.2)$$

Now let us find a summation for the integral (A.2). When we have different values of $k$ the integral (A.2) becomes

\begin{align*}
k = 0 & \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi = \pi \\
k = 1 & \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \phi d\phi = \frac{1}{2} \pi \\
k = 2 & \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 \phi d\phi = \frac{3}{4} \pi \\
k = 3 & \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^6 \phi d\phi = \frac{5}{6} \pi \\
k = 4 & \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^8 \phi d\phi = \frac{7}{8} \pi \quad \vdots \\
k = k & \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k} \phi d\phi = \frac{(2k)!}{2^{2k} k!} \pi.
\end{align*}

Therefore (A.2) becomes

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k} \phi d\phi = \frac{(2k)!}{2^{2k} k!} \pi. \quad (A.3)$$
APPENDIX B

SOLVING THE INTEGRAL $I_2^0$

Looking at equation (7.26), we have an absolute value so we should split up the integral into two parts.

$$I_2^0 = m_0^S \int_{-\frac{\pi}{2}}^{\phi_c} \log |\sin \phi - \sin \phi_c| d\phi$$

$$= m_0^S \int_{-\frac{\pi}{2}}^{\phi_c} \log(\sin \phi_c - \sin \phi) d\phi + m_0^S \int_{-\frac{\pi}{2}}^{\phi_c} \log(\sin \phi - \sin \phi_c) d\phi.$$  \hspace{1cm} (B.1)

Using the trig identity $\sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$, (B.1) becomes

$$I_2^0 = m_0^S \int_{-\frac{\pi}{2}}^{\phi_c} \log\left[2 \cos\left(\frac{\phi + \phi_c}{2}\right) \sin\left(\frac{\phi_c - \phi}{2}\right)\right] d\phi$$

$$+ m_0^S \int_{-\frac{\pi}{2}}^{\phi_c} \log\left[2 \cos\left(\frac{\phi + \phi_c}{2}\right) \sin\left(\frac{\phi - \phi_c}{2}\right)\right] d\phi.$$  \hspace{1cm} (B.2)

Let us call the two integrals from (B.2), $I_a$ and $I_b$, respectively. Taking a look at $I_a$ we get

$$I_a = m_0^S \int_{-\frac{\pi}{2}}^{\phi_c} \log\left[2 \cos\left(\frac{\phi + \phi_c}{2}\right) \sin\left(\frac{\phi_c - \phi}{2}\right)\right] d\phi$$

$$= m_0^S \int_{-\frac{\pi}{2}}^{\phi_c} \log 2 d\phi + m_0^S \int_{-\frac{\pi}{2}}^{\phi_c} \log \left[\cos\left(\frac{\phi + \phi_c}{2}\right)\right] d\phi + m_0^S \int_{-\frac{\pi}{2}}^{\phi_c} \log \left[\sin\left(\frac{\phi_c - \phi}{2}\right)\right] d\phi$$

$$= m_0^S \log 2 \left(\phi_c + \frac{\pi}{2}\right) + m_0^S \int_{-\frac{\pi}{2}}^{\phi_c} \log \left[\cos\left(\frac{\phi + \phi_c}{2}\right)\right] d\phi + m_0^S \int_{-\frac{\pi}{2}}^{\phi_c} \log \left[\sin\left(\frac{\phi_c - \phi}{2}\right)\right] d\phi$$

Using the identities (C.1) and (C.2)

$$I_a = m_0^S \log 2 \left(\phi_c + \frac{\pi}{2}\right) - m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \int_{-\frac{\pi}{2}}^{\phi_c} \cos(k(\phi + \phi_c)) d\phi - m_0^S \log 2 \int_{-\frac{\pi}{2}}^{\phi_c} d\phi$$

$$- m_0^S \sum_{k=1}^{\infty} \frac{1}{k} \int_{-\frac{\pi}{2}}^{\phi_c} \cos(k(\phi_c - \phi)) d\phi - m_0^S \log 2 \int_{-\frac{\pi}{2}}^{\phi_c} d\phi$$

$$= -m_0^S \log 2 \left(\phi_c + \frac{\pi}{2}\right) - m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \int_{-\frac{\pi}{2}}^{\phi_c} \cos(k(\phi + \phi_c)) d\phi$$

$$- m_0^S \sum_{k=1}^{\infty} \frac{1}{k} \int_{-\frac{\pi}{2}}^{\phi_c} \cos(k(\phi_c - \phi)) d\phi$$

$$= -m_0^S \log 2 \left(\phi_c + \frac{\pi}{2}\right) - m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin(k(\phi + \phi_c)) \bigg|_{-\frac{\pi}{2}}^{\phi_c}$$

$$+ m_0^S \sum_{k=1}^{\infty} \frac{1}{k^2} \sin(k(\phi_c - \phi)) \bigg|_{-\frac{\pi}{2}}^{\phi_c}.$$
\[ I_a = -m_0^S \log 2 \left( \phi_c + \frac{\pi}{2} \right) - m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin(2k\phi_c) + m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin \left( k \left( \phi_c - \frac{\pi}{2} \right) \right) \]
\[ - m_0^S \sum_{k=1}^{\infty} \frac{1}{k^2} \sin \left( k \left( \phi_c + \frac{\pi}{2} \right) \right) \] (B.3)

Now taking a look at \( I_b \) we get

\[ I_b = m_0^S \int_{\phi_c}^{\frac{\pi}{2}} \log \left[ 2 \cos \left( \frac{\phi + \phi_c}{2} \right) \sin \left( \frac{\phi - \phi_c}{2} \right) \right] d\phi \]
\[ = m_0^S \int_{\phi_c}^{\frac{\pi}{2}} \log \left[ \cos \left( \frac{\phi + \phi_c}{2} \right) \right] d\phi + m_0^S \int_{\phi_c}^{\frac{\pi}{2}} \log \left[ \sin \left( \frac{\phi - \phi_c}{2} \right) \right] d\phi \]
\[ = m_0^S \log 2 \left( \frac{\pi}{2} - \phi_c \right) + m_0^S \log 2 \left( \phi_c + \frac{\pi}{2} \right) \]
\[ - m_0^S \sum_{k=1}^{\infty} \frac{1}{k^2} \int_{\phi_c}^{\frac{\pi}{2}} \cos(k(\phi + \phi_c)) d\phi - m_0^S \log 2 \int_{\phi_c}^{\frac{\pi}{2}} d\phi \]
\[ = -m_0^S \log 2 \left( \frac{\pi}{2} - \phi_c \right) - m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin(k(\phi + \phi_c)) \bigg|_{\phi_c}^{\frac{\pi}{2}} \]
\[ - m_0^S \sum_{k=1}^{\infty} \frac{1}{k^2} \sin(k(\phi - \phi_c)) \bigg|_{\phi_c}^{\frac{\pi}{2}} \]
\[ = -m_0^S \log 2 \left( \frac{\pi}{2} - \phi_c \right) - m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin \left( k \left( \phi_c + \frac{\pi}{2} \right) \right) + m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin(2k\phi_c) \]
\[ - m_0^S \sum_{k=1}^{\infty} \frac{1}{k^2} \sin \left( k \left( \frac{\pi}{2} - \phi_c \right) \right) \] (B.4)

Now we need to add (B.3) with (B.4).

\[ I_b^\prime = -m_0^S \log 2 \left( \phi_c + \frac{\pi}{2} \right) - m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin(2k\phi_c) + m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin \left( k \left( \phi_c - \frac{\pi}{2} \right) \right) \]
\[ - m_0^S \sum_{k=1}^{\infty} \frac{1}{k^2} \sin \left( k \left( \phi_c + \frac{\pi}{2} \right) \right) - m_0^S \log 2 \left( \frac{\pi}{2} - \phi_c \right) - m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin \left( k \left( \phi_c + \frac{\pi}{2} \right) \right) \]
\[ + m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin(2k\phi_c) - m_0^S \sum_{k=1}^{\infty} \frac{1}{k^2} \sin \left( k \left( \frac{\pi}{2} - \phi_c \right) \right) \]
\[ = -m_0^S \pi \log 2 + m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \left[ \sin \left( k \left( \phi_c - \frac{\pi}{2} \right) \right) - \sin \left( k \left( \phi_c + \frac{\pi}{2} \right) \right) \right] \]
\[ + m_0^S \sum_{k=1}^{\infty} \frac{1}{k^2} \left[ \sin \left( k \left( \phi_c - \frac{\pi}{2} \right) \right) - \sin \left( k \left( \phi_c + \frac{\pi}{2} \right) \right) \right] \]
\begin{align*}
I_2^0 &= -m_0^S \pi \log 2 + m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos \left( \frac{k\phi_c - k\frac{\pi}{2} + k\phi_c + k\frac{\pi}{2}}{2} \right) \sin \left( \frac{k\phi_c - k\frac{\pi}{2} - k\phi_c - k\frac{\pi}{2}}{2} \right) \\
&\quad + m_0^S \sum_{k=1}^{\infty} \frac{1}{k^2} \cos \left( \frac{k\phi_c - k\frac{\pi}{2} + k\phi_c + k\frac{\pi}{2}}{2} \right) \sin \left( \frac{k\phi_c - k\frac{\pi}{2} - k\phi_c - k\frac{\pi}{2}}{2} \right) \\
&= -m_0^S \pi \log 2 + m_0^S \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(k\phi_c) \sin \left( -\frac{k\pi}{2} \right) + m_0^S \sum_{k=1}^{\infty} \frac{1}{k^2} \cos(k\phi_c) \sin \left( -\frac{k\pi}{2} \right) \\
&= -m_0^S \pi \log 2 + m_0^S \sum_{k=1}^{\infty} \frac{2}{k^2} \cos(k\phi_c) \left[ (-1)^k \sin \left( -\frac{k\pi}{2} \right) + \sin \left( -\frac{k\pi}{2} \right) \right] \\
&= -m_0^S \pi \log 2.
\end{align*}

Therefore we get

\begin{equation}
I_2^0 = -m_0^S \pi \log 2. \tag{B.5}
\end{equation}
This section proves the following summations.

\[- \log(\sin x) = \sum_{k=1}^{\infty} \frac{\cos(2kx)}{k} + \log(2)\]  \hspace{1cm} (C.1)

\[- \log(\cos x) = \sum_{k=1}^{\infty} \left(\frac{-1}{k}\right) \frac{\cos(2kx)}{k} + \log(2).\]  \hspace{1cm} (C.2)

We will use the known Taylor expansion for \(\log(1-x)\) for \(|x| < 1\), which is

\[\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^n}{n}.\]  \hspace{1cm} (C.3)

To start with we will prove (C.1). Let us begin by considering the following

\[\sum_{k=1}^{\infty} \frac{\cos(2kx)}{k} = \sum_{k=1}^{\infty} \frac{e^{2ikx} + e^{-2ikx}}{2k} = \sum_{k=1}^{\infty} \frac{e^{2ikx}}{2k} + \sum_{k=1}^{\infty} \frac{e^{-2ikx}}{2k}.\]

We know that \(|e^{ix}| \leq 1\) for all \(x\) values. So using (C.3) we get,

\[\sum_{k=1}^{\infty} \frac{\cos(2kx)}{k} = -\frac{1}{2} \log(1 - e^{2ix}) - \frac{1}{2} \log(1 - e^{-2ix}) = -\frac{1}{2} \log(1 - e^{-2ix} - e^{2ix} + e^{2ix}) = -\frac{1}{2} \log(4 \sin^2 x) = -\frac{1}{2} \log(4) - \frac{1}{2} \log(\sin^2 x).\]

Therefore we get,

\[\sum_{k=1}^{\infty} \frac{\cos(2kx)}{k} = - \log(2) - \log(\sin x).\]

This yields equation (C.1).
Now we will prove equation (C.2). Let us start by considering the following
\[
\sum_{k=1}^{\infty} \frac{(-1)^k \cos(2kx)}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k (e^{i2kx} + e^{-i2kx})}{2k}
\]
\[
= \sum_{k=1}^{\infty} \frac{(-1)^k e^{i2kx}}{2k} + \sum_{k=1}^{\infty} \frac{(-1)^k e^{-i2kx}}{2k}
\]
\[
= \sum_{k=1}^{\infty} \frac{(-e^{i2x})^k}{2k} + \sum_{k=1}^{\infty} \frac{(-e^{-i2x})^k}{2k}.
\]

We know that \(|-e^{ix}| \leq 1\) for all \(x\) values. So using (C.3) we get,
\[
\sum_{k=1}^{\infty} \frac{(-1)^k \cos(2kx)}{k} = -\frac{1}{2} \log(1 + e^{i2x}) - \frac{1}{2} \log(1 + e^{-i2x})
\]
\[
= -\frac{1}{2} \log(2 + e^{i2x} + e^{-i2x})
\]
\[
= -\frac{1}{2} \log(2 + 2 \cos(2x))
\]
\[
= -\frac{1}{2} \log(4 \cos^2(x))
\]
\[
= -\frac{1}{2} \log(4) - \frac{1}{2} \log(\cos^2 x).
\]

Therefore we get,
\[
\sum_{k=1}^{\infty} \frac{(-1)^k \cos(2kx)}{k} = -\log(2) - \log(\cos x).
\]

This yields equation (C.2).
We want to solve the integral (7.28). Notice that
\[
\frac{d}{d\nu} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \phi - \sin \phi_c)^\nu d\phi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \phi - \sin \phi_c)^\nu \log |\sin \phi - \sin \phi_c| d\phi. \tag{D.1}
\]
We will find a general formula for the derivative with respect to $\nu$ and set $\nu = n$. We will use the known Maclaurin Series
\[
(1 - x)^\nu = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(1 + \nu)}{k! \Gamma(1 + \nu - k)} x^k \tag{D.2}
\]
for $|x| < 1$ and $\nu \in \mathbb{R}$. This will be used to find the derivatives of some integrals. Let us consider the function
\[
\frac{d}{d\nu} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^n \phi - \sin^n \phi_c) (1 - \sin \phi \sin \phi_c) \nu d\phi. \tag{D.3}
\]
First let us take a look at the integral in (D.3). For the bounds that we have for the integral, we have $|\sin \phi \sin \phi_c| < 1$. We can use (D.2) to get
\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^n \phi - \sin^n \phi_c) (1 - \sin \phi \sin \phi_c) \nu d\phi = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(1 + \nu)}{k! \Gamma(1 + \nu - k)} \sin^k \phi_c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{n-k} \phi d\phi. \tag{D.4}
\]
But we want $\frac{d}{d\nu}$ (D.4), which gives us
\[
\frac{d}{d\nu} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^n \phi - \sin^n \phi_c) (1 - \sin \phi \sin \phi_c) \nu d\phi = \frac{d}{d\nu} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(1 + \nu)}{k! \Gamma(1 + \nu - k)} \sin^k \phi_c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{n-k} \phi d\phi

= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{d}{d\nu} \left( \frac{\Gamma(1 + \nu)}{\Gamma(1 + \nu - k)} \right) \sin^k \phi_c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{n-k} \phi d\phi
\]
which gives us
\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{\Gamma(1 + \nu)[\psi(1 + \nu) - \psi(1 + \nu - k)]}{\Gamma(1 + \nu - k)} \right) \sin^k \phi_c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{n-k} \phi d\phi. \tag{D.5}
\]
Where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. We want (D.5) when $\nu = n$. Which gives
\[
\sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{\Gamma(1 + n)[\psi(1 + n) - \psi(1 + n - k)]}{\Gamma(1 + n - k)} \right) \sin^k \phi_c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{n-k} \phi d\phi. \tag{D.6}
\]
Also let us consider the function
\[
\frac{d}{d\nu} \int_{-\phi_c}^{\phi_c} (-1)^n (\sin^n \phi_c) \left( 1 - \frac{\sin \phi}{\sin \phi_c} \right) \nu d\phi. \tag{D.7}
\]
First let us take a look at the integral in (D.7). For the bounds on the integral, we have \( \left| \frac{\sin \phi_c}{\sin \phi} \right| < 1 \). We can use (D.2) to get

\[
\int_{-\phi_c}^{\phi_c} (\sin^n \phi_c \left( 1 - \frac{\sin \phi}{\sin \phi_c} \right) \nu d\phi = \sum_{k=0}^{\infty} \frac{(-1)^{n+k} \Gamma(1 + \nu)}{k! \Gamma(1 + \nu - k)} \sin^{n-k} \phi_c \int_{-\phi_c}^{\phi_c} \sin^k \phi d\phi. \tag{D.8}
\]

But we want \( \frac{d}{d\nu} \) (D.8), which gives us

\[
\frac{d}{d\nu} \int_{-\phi_c}^{\phi_c} (\sin^n \phi_c \left( 1 - \frac{\sin \phi}{\sin \phi_c} \right) \nu d\phi = \frac{d}{d\nu} \sum_{k=0}^{\infty} \frac{(-1)^{n+k} \Gamma(1 + \nu)}{k! \Gamma(1 + \nu - k)} \sin^{n-k} \phi_c \int_{-\phi_c}^{\phi_c} \sin^k \phi d\phi 
\]

which gives us

\[
= \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{k!} \left( \frac{\Gamma(1 + \nu) \psi(1 + \nu) - \psi(1 + \nu - k)}{\Gamma(1 + \nu - k)} \right) \sin^{n-k} \phi_c \int_{-\phi_c}^{\phi_c} \sin^k \phi d\phi. \tag{D.9}
\]

We want (D.9) when \( \nu = n \). Which gives

\[
= \sum_{k=0}^{n} \frac{(-1)^{n+k}}{k!} \left( \frac{\Gamma(1 + n) \psi(1 + n) - \psi(1 + n - k)}{\Gamma(1 + n - k)} \right) \sin^{n-k} \phi_c \int_{-\phi_c}^{\phi_c} \sin^k \phi d\phi. \tag{D.10}
\]

We will also consider the function

\[
\frac{d}{d\nu} \int_{\phi_c}^{\pi/2} \left( 1 - \frac{\sin \phi_c}{\sin \phi} \right) \nu d\phi. \tag{D.11}
\]

First let us take a look at the integral in (D.11). For the bounds on the integral, we have \( \left| \frac{\sin \phi_c}{\sin \phi} \right| < 1 \). We can use (D.2) to get

\[
\int_{\phi_c}^{\pi/2} (\sin^n \phi_c \left( 1 - \frac{\sin \phi}{\sin \phi_c} \right) \nu d\phi = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(1 + \nu)}{k! \Gamma(1 + \nu - k)} \sin^{n-k} \phi_c \int_{\phi_c}^{\pi/2} \sin^k \phi d\phi. \tag{D.12}
\]

But we want \( \frac{d}{d\nu} \) (D.12), which gives us

\[
\frac{d}{d\nu} \int_{\phi_c}^{\pi/2} (\sin^n \phi_c \left( 1 - \frac{\sin \phi_c}{\sin \phi} \right) \nu d\phi = \frac{d}{d\nu} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(1 + \nu)}{k! \Gamma(1 + \nu - k)} \sin^{n-k} \phi_c \int_{\phi_c}^{\pi/2} \sin^k \phi d\phi 
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{d}{d\nu} \left( \frac{\Gamma(1 + \nu)}{\Gamma(1 + \nu - k)} \right) \sin^{n-k} \phi_c \int_{\phi_c}^{\pi/2} \sin^k \phi d\phi.
\]
which gives us

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{\Gamma(1+\nu)[\psi(1+\nu) - \psi(1+\nu-k)]}{\Gamma(1+\nu-k)} \right) \sin^k \phi_c \int_{-\phi_c}^{\frac{\pi}{2}} \sin^{n-k} \phi d\phi. \tag{D.13}
\]

We want (D.13) when \( \nu = n \) which gives

\[
= \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{\Gamma(1+n)[\psi(1+n) - \psi(1+n-k)]}{\Gamma(1+n-k)} \right) \sin^k \phi_c \int_{-\phi_c}^{\frac{\pi}{2}} \sin^{n-k} \phi d\phi. \tag{D.14}
\]

So the integral (D.1) evaluated for \( \nu = n \) is equal to the sum of equations (D.6), (D.10), and (D.14), giving

\[
\int_{-\phi_c}^{\frac{\pi}{2}} (\sin \phi - \sin \phi_c)^n \log |\sin \phi - \sin \phi_c| d\phi
\]

\[
= \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{n![\psi(1+n) - \psi(1+n-k)]}{(n-k)!} \right) \sin^k \phi_c \int_{-\phi_c}^{\frac{\pi}{2}} \sin^{n-k} \phi d\phi
\]

\[
+ \sum_{k=0}^{n} \frac{(-1)^{n+k}}{k!} \left( \frac{n![\psi(1+n) - \psi(1+n-k)]}{(n-k)!} \right) \sin^{n-k} \phi_c \int_{-\phi_c}^{\phi_c} \sin^k \phi d\phi
\]

\[
+ \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{n![\psi(1+n) - \psi(1+n-k)]}{(n-k)!} \right) \sin^k \phi_c \int_{-\phi_c}^{\frac{\pi}{2}} \sin^{n-k} \phi d\phi \tag{D.15}
\]
This is to show that the integral over the curve with our parameterization is

\[ \int_{C_a} dC = 2a\pi \]

Using the parameterization from section 6 and looking at the far field gives

\[
\int_{C_a} dC = \int_{\theta_-}^{\theta_+} \frac{a|Q_\pm|d\theta}{\sqrt{\Delta}}
\]

\[
= a \int_{\theta_-}^{\theta_+} \frac{(|Q_+| + |Q_-|)d\theta}{\sqrt{\Delta}}
\]

\[
= a \int_{\theta_-}^{\theta_+} \frac{2r_0}{\sqrt{r_0^2(\theta_+ - \theta)(\theta - \theta_-)}} d\theta
\]

\[
= 2a \int_{\theta_-}^{\theta_+} \frac{1}{\sqrt{(\theta_+ - \theta)(\theta - \theta_-)}} d\theta.
\]

Using the same change of variables as section 7.2, we get

\[
\int_{C_a} dC = 2a \int_{-1}^{1} \frac{1}{\sqrt{\alpha(1-x)\alpha(1+x)}} dx
\]

\[
= 2a \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx
\]

\[
= 2a (\sin^{-1}x)|_{-1}^{1}
\]

\[
= 2a\pi.
\]

Therefore our parameterization works.