PROBABILISTIC METHODS APPLIED
TO SLOPES AND FOOTINGS

by

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ABSTRACT

This thesis compares and contrasts various techniques of probabilistic methods in geotechnical engineering. The focus is on first order methods (FOSM/FORM) and random finite element methods (RFEM) as they pertain to 2-D slopes and shallow footings. The first order methods, first order second moment (FOSM) and first order reliability method (FORM), are shown to be simply extensions of established mathematical techniques. FOSM uses a first order Taylor series expansion and FORM implements optimization strategies. An effective spreadsheet implementation of FORM is also presented. The effects of anisotropy in random field generation for 2-D slopes and shallow footings are also analyzed. Both correlation lengths and principal axes are adjusted in the random field generation. This thesis demonstrates that changing correlation lengths and rotating the principal axes in random field generation have a significant effect (up to an order of magnitude) on the resulting probability of failure. The objective in using these probabilistic methods is to model more accurately field conditions and evaluate a probability of failure for soil structures.
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CHAPTER 1: INTRODUCTION

Probabilistic methods attempt to tackle the problem of modeling soils as spatially random materials. Conventional methods account for variability by using conservative deterministic values for soil properties. This leads to subjective results and potentially costly over designs or even unsafe designs. Two prominent methods are analyzed in this thesis – first order methods (FORM/FOSM) and random finite element methods (RFEM). The purpose of probabilistic methods is to generate an alternative to conventional methods that require a certain arbitrariness in choosing conservative soil properties and subjectiveness in factor of safety requirements.

Probabilistic methods form the basis of new load and resistance factor design (LRFD) and partial factors of safety codes in geotechnical engineering. The merging of probabilistic methods and geotechnical codes has been investigated for example by Becker (1996), Ditlevsen (1982), Ditlevsen (1997), Honjo et al. (2000), Scott et al. (2003), Foye et al. (2006a), and Foye et al. (2006b). The transition to LRFD was initiated to provide a more consistent standard for safety and reliability. In addition, LRFD is able to account for uncertainty in individual properties and loadings. Factor of Safety methods are limited to only taking into account the global uncertainty.

1.1 FORM/FOSM Investigation

Two first order methods are examined in this thesis. One method is the first order second moment (FOSM). The other method is the first order reliability method (FORM). FORM is also referred to as the Hasofer-Lind FORM, invariant FORM, or geometric reliability index. FOSM uses a first order Taylor expansion, and FORM incorporates optimization techniques to calculate a reliability index. FORM and FOSM have been researched by Hasofer and Lind (1974), Ditlevsen (1985), Ditlevsen and Madsen (1996b), Zhao and Ono (1999), and Griffiths et al. (2002) among others.

FOSM and FORM calculate a reliability index, which can be used to calculate a probability of failure. The FOSM method resembles Newton’s Method for root finding and can be performed by hand with the aid of a calculator. The FORM method is an optimization problem that routinely requires the use of computer optimization software.
Microsoft Excel is equipped with a convenient optimization solve that can be used to implement the FORM method.

FORM is superior to FOSM due to its lack of invariance. The result of FOSM depends upon the formulation of a performance function. FORM, however, is independent of this formulation. FORM and FOSM can lead to the same results, but this only occurs in rare cases. This will be expanded upon further in this thesis.

1.2 Anisotropy in RFEM

The main distinction between conventional finite element methods (FEM) and random finite element methods (RFEM) is random field generation. RFEM maps random fields to finite element meshes such that each individual element in the mesh is assigned a random value. This is ideal when modeling inhomogeneous spatially random materials such as soils. This thesis focuses on the 2-D implementation of random fields. Random fields can incorporate correlation lengths to simulate soil layering and sedimentary trends. Correlation lengths are specified using the x and y principal axes, but these axes can be rotated to simulate soil that has been uplifted. This thesis investigates the effects of varying the correlation lengths and rotating the principal axes of the random fields.

The RFEM analysis is performed using the RFEM software developed by Fenton and Griffiths (1993). The programs are available for free download from http://inside.mines.edu/~vgriffit; however, some in-house modifications are made to the programs to accommodate this research. This research uses the Monte Carlo method with 1,000 simulations for each case.

This thesis finds that varying correlation lengths and principal axes orientation can lead to results that vary by up to an order of magnitude. RFEM simulations show that failure mechanisms attempt to seek out the weaker soil regions. This leads to higher probabilities of failure when the correlation lengths are oriented along the preferred failure planes and lower probabilities of failure when the correlation lengths are oriented perpendicular to preferred failure planes.
CHAPTER 2: BASICS REVIEW

Probabilistic methods use conventional methods and techniques as their basis. It is therefore important to have an understanding of the elementary topics in slope stability, bearing capacity, probability and statistics, and safety factor calculations. A short summary of these topics is provided.

2.1 Slope Stability

Two types of 2-D slopes are used in this thesis – finite slopes and infinite slopes. The finite slope (figure 2.1) is probably more associated to real-world engineering applications. The infinite slope (figure 2.2) is more of an idealized theoretical slope that can be used to model long slopes that have failure mechanisms parallel to the slope.

Figure 2.1 Finite slope

Figure 2.2 Infinite slope
The stability of soil slopes is governed both by soil properties, pore pressures, and the slope geometry. For soil strength parameters, $c'$ is the cohesion, $\phi'$ is the friction angle, $\gamma$ is the unit weight, and $u$ is the excess pore water pressure. A prime after the symbols denotes effective parameters. For slope geometry, $\theta$ is the slope angle measured from the horizontal, $H$ is the height of the finite slope, $D$ is a height factor specifying the height of the slope plus the embankment, $H_1$ is the height of the infinite slope measured vertically, and $H_2$ is the height of the infinite slope measured perpendicular to the slope. It is possible for the slope to be subjected to an earthquake load. In this case, $k_h$ multiplied by the acceleration due to gravity yields the maximum horizontal pseudo-acceleration caused by the earthquake. A summary of these properties is provided in Table 2.1.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c'$</td>
<td>Cohesion</td>
</tr>
<tr>
<td>$\phi'$</td>
<td>Friction Angle</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Unit Weight</td>
</tr>
<tr>
<td>$u$</td>
<td>Pore Pressure</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Slope Angle</td>
</tr>
<tr>
<td>$H$</td>
<td>Height of Finite Slope</td>
</tr>
<tr>
<td>$D$</td>
<td>Height Factor</td>
</tr>
<tr>
<td>$H_1$</td>
<td>Vertical Height of Infinite Slope</td>
</tr>
<tr>
<td>$H_2$</td>
<td>Perpendicular Height of Infinite Slope</td>
</tr>
<tr>
<td>$k_h$</td>
<td>Horizontal Acceleration Coefficient</td>
</tr>
</tbody>
</table>

### 2.1.1 Finite Slope

The factor of safety of a finite slope can be found using the method of slices, charts, or finite element modeling. For cases of simple geometry and homogeneous soil properties, an analytical equation can be used for calculating the factor of safety. These methods all yield relatively similar results for the factor of safety calculation.
The method of slices usually assumes a circular failure surface (Figure 2.3) and then back calculates the factor of safety. Numerous failure surfaces must be assumed until a failure surface with the lowest (most conservative) factor of safety is found.

![Image of predicted failure surface](image)

Figure 2.3 Predicted circular failure surface

Charts are a fast and convenient way of finding the factor of safety. Similar to the method of slices, charts assume a circular or log-spiral failure surface. Taylor’s charts and Michalowski’s charts (Michalowski, 2002) are common resources for finding the factor of safety.

The finite element method makes no assumptions about the failure surface. Instead, it attempts to redistribute the gravitational forces throughout the slope. If the forces cannot be redistributed while satisfying the Mohr-Coulomb failure criterion, then it is assumed that the slope failed. The finite element method may find circular failure mechanisms, but the finite element method makes no assumptions about the shape of the failure plane.

### 2.1.2 Infinite Slope

An analytical equation is frequently the only method used for finding the factor of safety of an infinite slope. If the failure surface is assumed to be parallel to the slope (which is the definition of an infinite slope), then force equilibrium leads to two potential equations (Equation 2.1 and Equation 2.2). The difference comes from the choice of how the soil depth is measured. The soil depth can either be measured vertically or perpendicular to the slope. The first equation uses a vertical soil depth measurement. The second equation uses a soil depth measurement that is perpendicular to the slope.
There are two common types of shallow footings used in bearing capacity analysis – strip footings and rectangular footings. Strip footings are often used to approximate a long thin footing where the ratio of the length to width is very large. Only rectangular and strip footings are analyzed in this thesis.

### 2.2 Terzaghi’s Equation

Terzaghi’s equation incorporating various depth and shape correction factors is probably the most common means for finding a shallow footing’s factor of safety. Terzaghi’s equation, Equation 2.3, states that the ultimate bearing capacity of a soil is a function of the cohesion, friction angle, unit weight, overburden pressure, and footing dimensions. To find the factor of safety, divide the ultimate bearing strength calculated with Terzaghi’s equation by the applied load.

\[
FS = \frac{[H_1 \gamma \cos \theta (\cos \theta - k_h \sin \theta) - u] \tan \phi + c}{H_1 \gamma \cos \theta (\sin \theta + k_h \cos \theta)} \tag{2.1}
\]
\[
FS = \frac{[H_2 \gamma (\cos \theta - k_h \sin \theta) - u] \tan \phi + c}{H_2 \gamma (\sin \theta + k_h \cos \theta)} \tag{2.2}
\]

\[
l_{\text{ult}} = c N_c d_c s_c + q N_q d_q s_q + B' \frac{\gamma}{2} N_y d_y s_y \tag{2.3}
\]

In Terzaghi’s equation, the \(N_c, N_q, \text{ and } N_y\) are bearing capacity factors that are functions of the friction angle, \(d_c, d_q, \text{ and } d_y\) are depth correction factors that depend on the footing depth beneath the ground level, and \(s_c, s_q, \text{ and } s_y\) are shape corrections factors that are functions of the footing’s dimensions. There are various formulations for the bearing capacity factors, shape correction factors, and depth correction factors. For this thesis, Meyerhof’s bearing capacity factors are used.
2.2.2 Finite Element Method for Bearing Capacity

The Griffiths and Fenton program is used for the random finite element bearing capacity analysis. The program uses the same stress redistribution algorithm as the slope stability program. The program starts by applying a loading to the foundation. Then, it attempts to redistribute the stresses without violating the Mohr-Coulomb failure criterion. If the soil does not fail, then the program gradually increases the loading until it finds the loading that causes failure. This final loading is assumed to be the ultimate bearing capacity of the soil. To find the factor of safety, simply divide the calculated ultimate bearing capacity by the anticipated loading.

2.3 Probability and Statistics

There are several main distribution characteristics used to describe the random variables in this thesis. They are distribution type, mean, and standard deviation. In random field generation, spatial correlation lengths must also be specified. There can also be a correlation coefficient specified for two correlated random variables. From these basic properties, the variance, covariance, and coefficient of variation can also be calculated. Upper and lower bounds can also be specified depending upon the distribution type.

2.3.1 Distribution Type

The distribution type governs the shape of the probability density function. The most recognizable distribution is probably the ‘bell curve’ for the normal (Gaussian) distribution (Figure 2.4). Other distribution types include lognormal, Gumbel, tanh, exponential, uniform, triangular, Weibull, gamma, beta, PERT, and Bernoulli. The distribution type selected depends upon the data being modeled. The appeal of certain distributions is that they put limits on the possible values for the random variable. A normal distribution allows possible values to extend to positive and negative infinity. The lognormal, Weibull, exponential, and gamma distributions do not allow the random variable to have a negative value. The uniform, triangular, tanh, beta, and PERT distributions put restrictions on the lower and upper bounds of the data. The Bernoulli distribution is used for data that can only take two distinct values, usually zero or one.
Distributions are modeled as continuous or piecewise continuous functions. However, data collected in the field usually consists of discrete data points. It is then necessary to fit a distribution to the data. This is only possible, though, if a sufficient number of data points are collected. There is usually not enough data points for the distribution type to be readily apparent. Thus, a certain engineering judgment must be employed when selecting a distribution type. Collected data points will provide an approximate figure for the mean and standard deviation, but it is up to the engineer to select the distribution type. There is currently no standard for which distribution type should be chosen for soil properties; although, a vast majority of studies in literature use either a normal or lognormal distribution.

It should be noted that the FORM method assumes a normal distribution. Therefore, if a non-normal distribution is used to represent the data, then the non-normal distribution must be transformed to an equivalent normal distribution. The transformation process is fairly straightforward for the lognormal and tanh distributions. The other distributions require evaluating the cumulative distribution or an iterative process.
2.3.2 Mean and Variance

The mean value is also known as the arithmetic average. The mean can be found for both continuous and discrete data sets. The mean is not necessarily the most common value (mode) or middle value (median).

The variance is a measure of dispersion from the mean. A high variance implies that the values are spread out, while a low variance implies that the values are clustered about the mean. The variance (VAR) can be found for both continuous and discrete data sets. The variance is related to the standard deviation by

\[ \text{VAR}(X) = \sigma_X^2 \]  

where \( \sigma_X \) is the standard deviation of \( X \). The mean and standard deviation are related by the coefficient of variation, \( V \). The coefficient of variation is simply the standard deviation divided by the mean.

There is also a property, covariance, which is a measure of how two variables vary with respect to each other. The covariance (Cov) of two random variables, \( X \) and \( Y \), can be written as

\[ \text{Cov}[X, Y] = \sigma_X \sigma_Y \rho_{XY} \]  

where \( \rho_{XY} \) is the correlation coefficient between the variables. The Pearson product-moment correlation coefficient, \( r_{XY} \), can be used to approximate \( \rho_{XY} \). The Pearson correlation coefficient is calculated from discrete data sets.

2.4 Safety Factor Calculation

The safety factor of a structure is found by dividing the capacity by the loading. In other words, the anticipated loading multiplied by the factor of safety equals the ultimate capacity of the structure. Therefore, in theory, a factor of safety of two signifies that the structure is twice as strong as it needs to be. This, however, is often not the case. If this was the case, then building codes would only specify a required a factor of safety of a little over one. The requirements for high factors of safety (e.g. 3 to 5) come from the uncertainty in property measurements and loadings. Thus, a global factor of safety is designed to encompass all possible areas of uncertainty.
Probabilistic methods attempt to eliminate the arbitrariness in safety factor requirements. Rather than make uncertainty an afterthought by an all-encompassing factor of safety, probabilistic methods incorporate uncertainties from the start. Thus, probabilistic methods should ideally prevent structures from being costly overdesigned by more scientifically assessing the uncertainty in material properties and other aspects of design.
CHAPTER 3: REVIEW OF CURRENT PROBABILISTIC METHODS

This section provides a quick overview of FOSM, FORM, and RFEM. The general techniques and implementations of the methods are discussed. All three methods are probabilistic tools, but they are very different in their approach to incorporating probability and statistics into engineering analysis.

3.1 First Order Methods

First order methods, such as FORM, can be applied to any engineering application. They make no assumptions about the problem to be solved. Rather, the application appears in the formulation of a performance function. A performance function is similar to a factor of safety equation. A performance function is generally generated from a safety factor calculation. A performance function must equal zero when the structure is at failure. Therefore, common formulations for the performance function, $g(x_1, x_2, \ldots, x_N)$ are

$$g(x_1, x_2, \ldots, x_N) = \text{capacity}(x_1, x_2, \ldots, x_N) - \text{loading}(x_1, x_2, \ldots, x_N)$$  \hfill (3.1)

or

$$g(x_1, x_2, \ldots, x_N) = \log(\text{capacity}(x_1, x_2, \ldots, x_N) / \text{loading}(x_1, x_2, \ldots, x_N))$$  \hfill (3.2)

or

$$g(x_1, x_2, \ldots, x_N) = \text{FS}(x_1, x_2, \ldots, x_N) - 1$$  \hfill (3.3)

where $x_1, x_2, \ldots, x_N$ are random variables. The random variables are usually either material properties and/or loadings. The formulation of the performance function can have an effect on the results if the first order second moment (FOSM) method. However, the Hasofer-Lind first order reliability method (FORM) is independent of the performance function formulation.

3.1.1 FOSM

FOSM is derived from a first order Taylor expansion of the performance function about the mean values. The formulation for the FOSM reliability index is
\[
\beta = \frac{g(\mu_1, \mu_2, \ldots, \mu_N)}{\sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} \text{Cov}[x_i, x_j]}}
\]

where \( \beta \) is the reliability index, \( g(\mu_1, \mu_2, \ldots, \mu_N) \) is the performance function as described previously evaluated at the means, and \( N \) is the number of random variables in the performance function. \( \text{Cov}[x_i, x_j] \) is the covariance of \( x_i \) and \( x_j \) or, in the case of \( i=j \), the variance of \( x_i \). The derivatives are evaluated at the means. If an analytical expression for the derivative of \( g(x_1, x_2, \ldots, x_N) \) cannot be obtained, then numerical approximations should be used. Smith and Griffiths (2006) suggest a central difference approximation such as in Equation 3.5.

\[
\frac{\partial g}{\partial x} \approx \frac{g(x + \Delta x) - g(x - \Delta x)}{2\Delta x}
\]

It is debatable what value to use for \( \Delta x \) in Equation 3.5. From a pure mathematical sense, the limit of Equation 3.5 should be taken as \( \Delta x \) approaches zero. The limit can be approximated by taking values close to zero. However, some engineers prefer to use a \( \Delta x \) that is equivalent to the standard deviation. In this case, the first derivative is approximated by the slope of a secant line that intersects the performance function at \( \mu \pm \sigma \).

The primary shortfall of FOSM is its dependence on the formulation of the performance function. Different reliability indices can be found depending upon whether the performance function was formulated as Equation 3.1, 3.2, 3.3 or another allowable formulation. To show FOSM’s dependence on the performance function, let us examine the case where we use a performance function \( h(x_1, x_2, \ldots, x_N) \) such that \( h(x_1, x_2, \ldots, x_N) = g(x_1, x_2, \ldots, x_N)^m \), the performance function is raised to an arbitrary power, \( m \). If we plug \( g(x_1, x_2, \ldots, x_N) \) into Equation 3.4, we obtain a reliability index, \( \beta \). Now, let us plug \( h(x_1, x_2, \ldots, x_N) \) into Equation 3.4.

\[
\beta = \frac{h(\mu_1, \mu_2, \ldots, \mu_N)}{\sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} \text{Cov}[x_i, x_j]}} = \frac{(g(\mu_1, \mu_2, \ldots, \mu_N))^m}{\sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} \text{Cov}[x_i, x_j]}}
\]
Simplifying (3.6a) yields a value of $\beta$ that is a factor of the value obtained when we do not raise the performance function to a value of $m$.

$$\beta = \frac{(g(\mu_1, \mu_2, \ldots, \mu_N))^2}{mg(\mu_1, \mu_2, \ldots, \mu_N) \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial g}{\partial \mu_i} \frac{\partial g}{\partial \mu_j} \text{Cov}[\mu_i, \mu_j]}} = \frac{g(\mu_1, \mu_2, \ldots, \mu_N)}{\sqrt{m \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial g}{\partial \mu_i} \frac{\partial g}{\partial \mu_j} \text{Cov}[\mu_i, \mu_j]}}$$ (3.6b)

This invariance poses difficulties when comparing different reliability index calculations. FOSM can still be used for safety calculations if a consistent performance function is agreed upon. However, it is impossible to apply an absolute reliability index to a structure using FOSM.

3.1.2 Hasofer-Lind FORM
FORM in its current structure was first developed by A.M. Hasofer and N.C. Lind (1974). FORM is an optimization method that finds the shortest distance between the mean values and failures values. FORM maps the random variables to a normalized normal distribution space with a mean of zero and standard deviation of one. The FORM reliability index is found using

$$\beta = \min \sqrt{\left(\frac{x_i - \mu_i^N}{\sigma_i^N}\right)^T \left[R_i\right]^{-1} \left[\frac{x_i - \mu_i^N}{\sigma_i^N}\right]}$$ (3.7)

where $i$ denotes a vector of values, $x_i$ is the value of the design variable at failure and $[R]$ is the correlation matrix. The correlation matrix always contains ones on the diagonal and reduces to the identity matrix, $[I]$, if the variables are uncorrelated. If the variables are correlated, the off-diagonal terms $R_{ij}$ contain the correlation coefficients $r_{ij}$. The optimization problem is bounded by Equation 3.8. The performance function must equal zero at the failure values.

$$g(x_1, x_2, \ldots, x_N) = 0$$ (3.8)

FORM is independent of the performance function formulation because the values that cause failure $(x_1, x_2, \ldots, x_N)$ will be the same for all performance function formulations. To implement FORM, an optimization software package is used with Equation 3.7 as the
objective function and Equation 3.8 as the constraint. A logical search starting point would be the mean values when using a gradient search method.

3.2 RFEM

RFEM combines finite elements with random field generation. It is beyond the scope of this paper to delve into the math and calculations behind finite elements and random fields. For this information, refer to Fenton and Griffiths (2008). Rather, let us look at the logic and capabilities of these two techniques.

3.2.1 FEM

Finite elements is a numerical method for finding approximate solutions to partial differential equations. In engineering analysis of strains and stresses, the partial differential equation governs the deformations and subsequent stresses in the material. Finite elements can be performed in one, two, and three dimensions. Often, 3-dimensional problems are simplified to 2-dimensions by assuming plane-strain, plain-stress, or axisymmetry. This assumes that the body to be analyzed is either extremely thin or extremely thick, or axisymmetric, respectively.

The finite element process starts by discretizing the problem into smaller elements to form a mesh. For 2-D problems, the elements are usually triangles or quadrilaterals. The elements in the mesh do not need to have the same size or orientation. However, it is common to use consistent element types. (All the elements are either triangles or quadrilaterals.) It should be noted that quadrilateral elements can be degenerated to triangular elements by making one of the side lengths equal to zero. The type and number of elements in the mesh influence the final solution. Usually, a greater number of elements leads to a more accurate solution. However, more elements also leads to longer run times.

The finite element program for bearing capacity and slope stability analysis in this thesis uses a stress redistribution algorithm. The finite element program attempts to redistribute the gravitational forces throughout the soil while maintaining the Mohr-Coulomb failure criterion. If the stresses cannot be redistributed, then it is assumed that the soil failed due to excessive deformations.
3.2.2 Random Field Generation

Random field generation is a technique for generating random numbers from a specified distribution and appropriate parameters. The random numbers are then mapped to a one, two, or three dimensional grid. Spatial correlation lengths (also referred to as autocorrelation) can also be specified. These govern how numbers vary over a group of grid cells. Numbers vary more gradually over a space if the correlation length is high (Figure 3.1a); and, numbers vary rapidly if the correlation length is small (Figure 3.1b).

![Figure 3.1 Correlation lengths: (a) High correlation length; (b) Low correlation length](image)

Research into random field generation has been done, for example, by Fenton (1990), Fenton and Vanmarcke (1990), Li and Der Kiureghian (1993), and Robin et al. (1993). There are several common algorithms used for random field generation. These algorithms include, but are not limited to, moving average (MA) method, discrete Fourier transform (DFT) method, covariance matrix decomposition method, fast Fourier transform (FFT) method, turning bands method (TBM), and local average subdivision (LAS) method. The LAS will be used in this thesis and only allows the input of a normal (Gaussian) distribution. If a non-normal distribution is desired, then an appropriate transform must be applied.
CHAPTER 4: FORM/FOSM INVESTIGATION

FORM and FOSM often yield similar results. The methods can yield the same results under appropriate circumstances. Let us examine when and why these two different methods sometimes calculate the same or different reliability indices.

The first order methods only work with normally distributed variables. If a non-normal distribution is desired, then the underlying equivalent normal distribution is needed. There are some inherent problems with a normal distribution. A normal distribution assumes that the variables values can go off unbounded to infinity and negative infinity. However, starting with a non-normal distribution then using the equivalent normal distribution appears to be a reasonable compromise.

4.1 FORM Investigation

Let us examine how one could obtain a solution to FORM graphically. For simplicity, let us start with the case where the performance function consists of only a single random variable. Plotting the normally distributed probability density curve for the case of a single random variable yields Figure 4.1.

![Figure 4.1 Normal distribution for single random variable](image-url)
In Figure 4.1, \( \mu \) is the mean value and \( x_f \) is the value that causes failure. \( x_f \) is found by setting the performance function equal to zero and solving for \( x_f \). FORM transforms this distribution to a normalized normal distribution with a mean of zero and a standard deviation of one. Figure 4.2 shows the normalized distribution.

The reliability index, \( \beta \), measures the distance between the failure value and the mean value in terms of standard deviations. Recall the equation for calculating the FORM reliability index.

\[
\beta = \min \left[ \frac{x_i - \mu_i^N}{\sigma_i^N} \right]^{T} \left[ R^{-1} \right]^{T} \left[ \frac{x_i - \mu_i^N}{\sigma_i^N} \right]
\]  

(4.1)

For the case of a single random variable, Equation 4.1 reduces to

\[
\beta = \min \left[ \frac{x_f - \mu}{\sigma} \right]^2 = \left| \frac{x_f - \mu}{\sigma} \right|
\]  

(4.2)

This is the same result that is found graphically using Figure 4.2. There is, however, an inherent sign problem when using Equation 4.1. The square root prevents negative values of beta. In a logical sense, it would seem appropriate to not be able to calculate negative distances. However, it is important to know whether the value that causes failure, \( x_f \), is
above or below the mean value. A value of $x_t$ above the mean implies a probability of failure greater than 50%, and a value of $x_t$ below the mean implies a probability of failure less than 50%.

FORM can subsequently be expanded to cases with more than one random variable. If the variables are uncorrelated, a $\beta_i$ is found for each individual variable. Then, the $\beta_i$s are combined by taking the square root of the sum of the squares (RSS). Recall Equation 4.1, but let

$$\beta_i = \left( \frac{x_i - \mu_i}{\sigma_i} \right)$$  \hspace{1cm} (4.3)$$

If the variables are uncorrelated, $[R]$ reduces to the identity matrix, $[I]$, and Equation 4.1 can be rewritten as

$$\beta = \min \sqrt{\sum_{i=1}^{N} \beta_i^2}$$  \hspace{1cm} (4.4)$$

This process can be visualized as the addition of orthogonal vectors.

![Diagram showing orthogonal vectors](image)

Figure 4.3 FORM reliability index for two uncorrelated random variables

FORM is similar to partial factor of safety methods (such as LRFD) because of the calculating of individual $\beta$s. FORM finds how much each individual property must vary from the mean in order for failure to occur. This is in sharp contrast to conventional factor of safety methods that only look at the global safety requirements. Partial factor of
safety methods use probabilistic methods as a basis for load and resistance factors (Ditlevsen and Madsen, 1996a).

4.2 FOSM Investigation

FOSM uses a first order Taylor series approximation of the performance function to calculate $\beta$. This leads FOSM to bear resemblance to Newton’s Method for root finding. Newton’s Method uses a first order Taylor series approximation to find what $x$ value will lead to a $f(x) = 0$. Equation 4.5 shows the first order expansion for Newton’s Method.

$$f(x + \Delta x) = 0 = f(x) + \frac{df}{dx} \Delta x$$

(4.5)

Recall the equation for calculating the FOSM reliability index.

$$\beta = \frac{g(\mu_1, \mu_2, ..., \mu_N)}{\sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} Cov[x_i, x_j]}}$$

(4.6)

We can rearrange Equation 4.6 to make it appear more like Newton’s Method. For simplicity, let us assume that the variables are uncorrelated. Thus,

$$Cov[x_i, x_j] = \begin{cases} \sigma_{x_i}^2, & i = j \\ 0, & i \neq j \end{cases}$$

(4.7)

This allows us to rewrite Equation 4.6 as

$$0 = g(\mu_1, \mu_2, ..., \mu_N) - \beta \left( \sum_{i=1}^{N} \left( \frac{\partial g}{\partial x_i} \sigma_{x_i} \right)^2 \right)^{\frac{1}{2}}$$

(4.8)

For the case of a single random variable, Equation 4.8 can be written as

$$0 = g(\mu) - \left| \frac{\partial g}{\partial x} \right| \beta \sigma$$

(4.9)
Comparing Equations 4.5 and 4.9 shows that FOSM bears a striking resemblance to Newton’s Method. Comparing the two equations leads us to assume that $\Delta x = \beta \sigma$. Recall that FORM finds a value of $\beta \sigma$ that is the distance between the mean values and the failure values. In FORM,

$$g(\mu - \beta \cdot \sigma) = 0$$

(4.10)

However, FOSM is only able to approximate what $\beta \sigma$ will cause the performance function to equal zero. Similar to Newton’s Method, FOSM will lead to exact answers if the performance function is linear. Figure 4.4 shows how FOSM can miss the exact root if the performance function is nonlinear. The FOSM reliability index greatly depends upon the formulation of the performance function.

In the case of more than one uncorrelated random variables, the $\Delta x$ in FOSM is multiplied by an additional term. Let $\Delta x_i = \beta \sigma B_i$, where

$$B_i = \frac{\frac{\partial g}{\partial x_i} \sigma_i}{\sqrt{\sum_{i=1}^{N} \left( \frac{\partial g}{\partial x_i} \sigma_i \right)^2}}$$

(4.11)
Thus, $g(\mu_1 - \Delta x_1, \mu_2 - \Delta x_2, ..., \mu_n - \Delta x_n) = 0$ if the performance function is a linear combination of the random variables.

Despite FOSM’s dependence on the performance function, it can still potentially be used in engineering applications for comparing alternative designs. The performance function must be formulated the same way for all structures. Then, the reliability index of different structures can be compared. However, this has no real advantage over current factor of safety methods.

4.3 Cases When FORM Equals FOSM

For uncorrelated random variables, FORM will always find a solution such that

$$g(\mu_1 - \beta_1 \sigma_1, \mu_2 - \beta_2 \sigma_2, ..., \mu_n - \beta_n \sigma_n) = 0$$  \hspace{1cm} (4.12)

However, FOSM finds a solution where

$$g(\mu_1 - \beta \sigma_1 B_1, \mu_2 - \beta \sigma_2 B_2, ..., \mu_n - \beta \sigma_n B_n) \approx 0$$  \hspace{1cm} (4.13)

Note how FORM finds individual $\beta$s then combines them to form a global $\beta$ while FOSM only finds a global $\beta$. FOSM will only equal FORM if Equation 4.13 is exactly equal to zero instead of approximately equal to zero. This occurs when the performance function is linear with respect to the random variables.

4.3.1 Proof for Single Random Variable

Let us examine the case of an infinite slope with $\tan \phi = u = k_a = 0$. This allows us to write Equation 2.1 as

$$FS = \frac{c_u}{Hy \sin \theta \cos \theta}$$  \hspace{1cm} (4.14)

Let us assume that $c_u$ is the only random variable. Therefore, let

$$\alpha = Hy \sin \theta \cos \theta$$  \hspace{1cm} (4.15)

and write Equation 4.14 as
\[ FS = \frac{c_u}{\alpha} \]  
This allows us to formulate the performance function as
\[ g(c_u) = FS - 1 = \frac{c_u}{\alpha} - 1 \]  
(4.17)

Recall that for a single random variable, the FORM reliability index equation will reduce to
\[ \beta = \left| \frac{x_f - \mu}{\sigma} \right| \]  
(4.18)

We can use Equation 4.17 to find \( x_f \), the value at failure,
\[ g(c_u) = 0 \Rightarrow x_f = c_u = \alpha \]  
(4.19)

Plug \( x_f = \alpha \) into Equation 4.18 and we find the FORM reliability index to be
\[ \beta = \left| \frac{\alpha - \mu_{c_u}}{\sigma_{c_u}} \right| \]  
(4.20)

Recall that for a single random variable, the FOSM reliability index will simplify to
\[ \beta = \frac{g(\mu)}{\frac{\partial g}{\partial x} \sigma} \]  
(4.21)

The derivative of the performance function, Equation 4.17, is
\[ \frac{\partial}{\partial c_u} g(c_u) = \frac{1}{\alpha} \]  
(4.22)

Plugging Equations 4.17 and 4.22 into Equation 4.21 yields
\[ \beta = \frac{\mu_{c_u} - 1}{\frac{1}{\alpha} \sigma_{c_u}} \]  
(4.23)

Multiplying Equation 4.23 by \( \alpha / \alpha \) yields
\[ \beta = \frac{\mu_{c_u} - \alpha}{\sigma_{c_u}} \]  
(4.24)
Comparing Equations 4.20 and 4.24 reveals that FORM and FOSM will calculate the same reliability index if the performance function is formulated as

\[ g(x) = ax + c \]  
(4.25)

where \( a \) and \( c \) are constants. The signs will also be the same if the failure value is below the mean value. The general solution for a performance function formulated as Equation 4.25 is

\[ \beta = \frac{\mu + c/a}{\sigma} \]  
(4.26)

Letting \( a = 1/\alpha \) and \( c = -1 \), we obtain the same reliability index as before.

\[ \beta = \frac{\mu - \alpha}{\sigma} \]  
(4.27)

### 4.3.2 Proof for Two Random Variables

Let us examine the case of two uncorrelated random variables, and the performance function is formulated as

\[ g(x_1, x_2) = a x_1 + b x_2 + c \]  
(4.28)

The FOSM reliability index for two uncorrelated random variables can be written as

\[ \beta = \frac{g(\mu_1, \mu_2)}{\sqrt{\left(\frac{\partial g}{\partial x_1} \sigma_1\right)^2 + \left(\frac{\partial g}{\partial x_2} \sigma_2\right)^2}} \]  
(4.29)

Plugging the performance function and its derivatives into Equation 4.28 yields

\[ \beta = \frac{a \mu_1 + b \mu_2 + c}{\sqrt{(a \sigma_1)^2 + (b \sigma_2)^2}} \]  
(4.31)

The FORM reliability index for two uncorrelated variables is formulated as

\[ \beta = \sqrt{\left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2} \]  
(4.32)
In FORM, the performance function must equal zero. Therefore,

\[ x_2 = \frac{-c - ax_1}{b} \]  \hspace{1cm} (4.33)

Plugging Equation 4.33 into Equation 4.32 yields an equation for \( \beta \) that is only a function of one variable, \( x_1 \).

\[ \beta = \left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2 + \left( \frac{-c - ax_1 - \mu_2}{b} \right)^2 \]  \hspace{1cm} (4.34)

Using minimization techniques, it can be shown that Equation 4.34 is minimized when

\[ x_1 = \frac{-ac \sigma_1^2 - ab \mu_1 \sigma_1^2 + b^2 \sigma_2^2 \mu_1}{a^2 \sigma_1^2 + b^2 \sigma_2^2} \]  \hspace{1cm} (4.35)

If the FORM and FOSM reliability indices (Equations 4.31 and 4.34) are set equal to each other, it is found that \( x_1 \) must satisfy Equation 4.35. It can also be shown in FOSM that \( \mu_1 - \beta \sigma B_1 \) equals Equation 4.35. Thus, FORM equals FOSM if the performance function is formulated as a linear combination of uncorrelated random variables. Through induction, it can be concluded that FORM will equal FOSM if the performance function is formulated as a linear combination of any number of uncorrelated random variables.

4.3.3 Numerical Comparison of FORM and FOSM

Recall that FORM will equal FOSM for the infinite slope equation if the performance function is formulated as a linear function of a single random variable. Let us examine the case when the performance function is not linear and there are two uncorrelated random variables. Using Equation 2.1 with the properties given in Table 4.1, the FORM and FOSM reliability indices are compared.
Table 4.1 Soil properties and slope dimensions for FORM/FOSM comparison

<table>
<thead>
<tr>
<th>Property</th>
<th>Description</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Distribution Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_u$</td>
<td>Cohesion</td>
<td>20.0</td>
<td>4.0</td>
<td>Normal</td>
</tr>
<tr>
<td>$\phi'$</td>
<td>Friction Angle</td>
<td>0</td>
<td>0</td>
<td>Deterministic</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Unit Weight</td>
<td>18.0</td>
<td>2.0</td>
<td>Normal</td>
</tr>
<tr>
<td>$H_1$</td>
<td>Vertical Depth</td>
<td>2.0</td>
<td>0</td>
<td>Deterministic</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Slope Angle</td>
<td>30°</td>
<td>0</td>
<td>Deterministic</td>
</tr>
<tr>
<td>$u$</td>
<td>Pore Pressure</td>
<td>0</td>
<td>0</td>
<td>Deterministic</td>
</tr>
<tr>
<td>$k_h$</td>
<td>Horizontal Acceleration</td>
<td>0</td>
<td>0</td>
<td>Deterministic</td>
</tr>
</tbody>
</table>

FORM calculates a reliability index of 1.0121. The FOSM reliability index depends upon the formulation of the performance function. Using Equations 3.1, 3.2, and 3.3 for the performance function, FOSM calculates a reliability index of 1.0121, 1.0892, and 0.9641, respectively. A summary of the results is provided in Table 4.2. For comparison, the safety factor is 1.2830.

Table 4.2 FORM/FOSM reliability index comparison

<table>
<thead>
<tr>
<th>Method</th>
<th>Performance Function</th>
<th>Reliability Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>FORM</td>
<td>Any</td>
<td>1.0121</td>
</tr>
<tr>
<td>FOSM</td>
<td>Equation 3.1</td>
<td>1.0121</td>
</tr>
<tr>
<td>FOSM</td>
<td>Equation 3.2</td>
<td>1.0892</td>
</tr>
<tr>
<td>FOSM</td>
<td>Equation 3.3</td>
<td>0.964</td>
</tr>
</tbody>
</table>

Note that FORM equals FOSM if the performance function is formulated as Equation 3.1. This is because Equation 3.1 simplifies to

$$g(c_u, \gamma) = c_u - H_1 \sin(\theta)\cos(\theta)\gamma$$

which is a linear combination of the random variables. From the results in Table 4.2, it is unclear whether FOSM will lead to higher or lower reliability indices than FORM.
However, it is apparent that FOSM and FORM will generally lead to roughly the same reliability index. Figure 4.5 demonstrates how the curvature of the performance function can effect the reliability index calculation.

Figure 4.5 FOSM reliability index depends upon curvature of performance function

FOSM approximates the distance between the mean value, $\mu$, and the failure value, $x_f$, using a gradient search method. It can be seen from Figure 4.5 that it is possible for FOSM to overestimate or underestimate the distance. The FOSM reliability index will find the same reliability index as FORM if the performance function is linear. However, if the performance function is not linear, then the difference in the reliability index from FORM and FOSM depends upon the curvature of the performance function.

4.4 Probability of Failure Calculation

The probability of failure is calculated using the reliability index. The probability of failure is found by integrating the normalized Gaussian (normal) probability density function from negative infinity to negative $\beta$. The Gaussian distribution is commonly used because the reliability index is found in a Gaussian space. However, it is possible to integrate other distributions. Microsoft Excel is equipped with an intrinsic function,
NORMSDIST(), which performs this integration. Recall that a positive reliability index indicates failure values below the mean. Therefore, to calculate the probability of failure for a reliability index of 4, call NORMSDIST(-4).

4.5 FORM Spreadsheet Implementation

FORM requires the solving of an optimization problem in order to find the reliability index. There has been extensive research into efficient search algorithms to solve optimizations problems. The search algorithm used for this thesis is the generalized reduced gradient (GRG) algorithm in the Excel solver (Figure 4.6). The Excel solver is readily available to most computer users and is an ideal platform for the implementation of FORM.

Low and Tang (1997, 2007), Low and Phoon (2002), and Massih et al. (2003) developed methods for implementing FORM into an Excel spreadsheet. The Excel solver allows the user to input an objective function and constraints. The FORM reliability index and constraint \((g(x_1, x_2, \ldots, x_N)=0)\) can be programmed as functions into cells. These two cells can then be designated as the objective function and constraint in the Excel solver. The solver also requires the user to select which cells to change during the optimization process. The straight forward choice would be to select the \('x_1, x_2, \ldots, x_N'\)
cells that the performance function depends upon. Thus, the performance is satisfied as
the solver attempts to minimize the FORM reliability index.

Low and Tang suggest a substitution of variables for the Excel solver. They proposed a substitution similar to

$$\beta_i = \frac{\mu_i - x_i}{\sigma_i}, \quad i = 1 \ldots N$$  \hspace{1cm} (4.37)

The solver then adjusts the $\beta_i$s until the FORM reliability index is minimized. The performance function is satisfied by back calculating $x$ using $\beta_i$. From Equation 4.24, $x_i$ can be calculated as

$$x_i = \mu_i - \beta_i \sigma_i, \quad i = 1 \ldots N$$  \hspace{1cm} (4.38)

This is somewhat more intuitive than adjusting the ‘$x_1, x_2, \ldots, x_N$’ values. With this method, we see the individual $\beta_i$s that lead to the global $\beta$.

One potential problem with FORM is the sign of reliability index, $\beta$. FORM only allows the calculation of a positive reliability index. Thus, the FORM program developed in this thesis checks to see whether the failure values are above or below the mean values. This is actually done in the program by checking whether the mean values lead to a positive or negative performance function value. It should be noted that the equivalent normal distribution mean values must be used when performing this check. The mean value of the underlying normal distribution may be higher or lower than the mean of the original non-normal distribution.

The FORM programs accompanying this thesis currently only support distribution types that can be described solely by their means and standard deviations. The user is required to input the mean, standard deviation, and distribution type. It is possible for cells to depend upon each other. For example, the standard deviation can be specified as a fraction of the mean. This can be helpful when performing a sensitivity analysis (described later). A screenshot of the program is provided in Figure 4.7.

Macros can be incorporated into Excel to simplify the FORM program operation. One problem with optimization algorithms is search starting points. Excel’s solver will start at the initial values in the ‘cells to be changed’. Programming a macro allows the
Figure 4.7 Screenshot of FORM program for slope stability
programmer control over the starting point of the solver. Macros are programmed using Microsoft Visual Basic for Applications. Macros are somewhat equivalent to specifying what keystrokes the computer should automatically perform. Macros can perform any function that can normally be done by the user. The FORM macro sets the $\beta$s to zero. Then, it invokes the Excel solver with the appropriate objective function and constraint specified. Macros can be attached to pictures in the spreadsheet. This enables the user to click on the picture to activate the macro.

Macros also allow the user to enter ‘batch file’ type commands to perform numerous operations. A sensitivity analysis macro is programmed into the FORM spreadsheet. The macro analyzes the sensitivity of the reliability index to changes in the mean values. This provides insight into which properties should be measured more precisely by the engineer in the field. The sensitivity macro adjusts the mean values individually by factors of 0, 0.25, 0.5, 0.75, 1.25, 1.5, 1.75, and 2.0. It then invokes the solver and calculates the reliability index. The results are then plotted to show how the reliability index depends on the mean. A subsequent plot is generated comparing the probability of failure to the means. Sample plots of the sensitivity analysis are shown in Figures 4.8a and 4.8b.

The sensitivity analysis reveals the extent to which the reliability index depends on the different soil properties. If the reliability index is highly sensitive to certain soil properties, those properties should be measured more precisely. Also, those soil properties can be looked at as areas for improvement if a higher reliability index is desired.

4.5 Concluding Remarks

FORM and FOSM both find a reliability index and subsequent probability of failure. FOSM is closely related to Newton’s method and will not find the ‘correct’ reliability index except for rare cases. FORM will find a reliability index that causes the performance function to exactly equal zero. However, FORM requires the solving of an optimization problem. The optimization problem can easily be solved using the solver built into Excel. The ease of FORM implementation and its invariance of the performance function makes it a preferable choice to FOSM as a probabilistic method.
Figure 4.8a FORM sensitivity analysis

Figure 4.8b FORM sensitivity analysis
The Griffiths and Fenton RFEM bearing capacity program is used for the anisotropy analysis. The program calculates a mean bearing capacity of the soil using the Monte Carlo method. This study analyzes the effects of varying the horizontal and vertical correlation lengths in the random field generation. Varying the correlation lengths leads to a significant effect on the mean bearing capacity calculation.

5.1 Soil Properties and Study Parameters

This study looks at a 2 feet wide strip footing on a frictional, cohesive soil. The RFEM program assumes a weightless soil. If the soil properties were modeled with deterministic values, Terzaghi’s equation, Equation 2.3, could be used to calculate the ultimate bearing capacity of the soil. In this case, Terzaghi’s equation would reduce to

\[ q_{\text{ult}} = cN_c \]  
(5.1)

because the soil is weightless and the strip footing rests on the surface of the soil. The soil properties used in this study are summarized in Table 5.1.

<table>
<thead>
<tr>
<th>Property</th>
<th>Distribution Type</th>
<th>Mean</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cohesion</td>
<td>Lognormal</td>
<td>50</td>
<td>15</td>
</tr>
<tr>
<td>Friction Angle</td>
<td>Lognormal</td>
<td>25</td>
<td>5</td>
</tr>
<tr>
<td>Dilation Angle</td>
<td>Deterministic</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Elastic Modulus</td>
<td>Deterministic</td>
<td>257143</td>
<td>0</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>Deterministic</td>
<td>0.2857</td>
<td>0</td>
</tr>
</tbody>
</table>

The soil properties entered into the data file are unitless. Therefore, a consistent set of unit must be established. For this study, all forces are in kilonewtons (kN) and all lengths are in meters (m).

The other parameters in the data file are chosen to allow the failure mechanism to freely form. The size of the mesh is such that the failure mechanism will not be bounded
by the edges of the mesh. The tolerances and iterations are such that the program should accurately find the failure mechanism. Figure 5.1 shows a sample data file.

![Sample data file for bearing capacity RFEM](image)

Figure 5.1 Sample data file for bearing capacity RFEM

The correlation lengths and property correlation coefficient are the only two properties that change between the various cases in this study. Table 5.2 summarizes the different cases analyzed in this study. The majority of cases involve analyzing the case when the horizontal correlation length is larger than the vertical correlation length. This is because it is commonly believed that this more accurately represents field conditions. However, it is unknown to what degree the horizontal correlation length is larger than the vertical correlation length.
Table 5.2 Bearing capacity cases

<table>
<thead>
<tr>
<th>Case</th>
<th>Horizontal Correlation Length (Lh)</th>
<th>Vertical Correlation Length (Lv)</th>
<th>Correlation Coefficient (r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>1,2,5,10</td>
<td>-0.5</td>
</tr>
<tr>
<td>2</td>
<td>1,5,10,20,30</td>
<td>1</td>
<td>-1.0</td>
</tr>
<tr>
<td>3</td>
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<td>-0.75</td>
</tr>
<tr>
<td>4</td>
<td>1,5,10,20,30</td>
<td>1</td>
<td>-0.5</td>
</tr>
<tr>
<td>5</td>
<td>1,5,10,20,30</td>
<td>1</td>
<td>-0.25</td>
</tr>
<tr>
<td>6</td>
<td>1,5,10,20,30</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1,5,10,20,30</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>8</td>
<td>1,5,10,20,30</td>
<td>1</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Case 1 examines the effects of varying the vertical correlation length while holding the horizontal correlation length and correlation coefficient constant. Cases 2 through 8 examine the effects of varying the horizontal correlation length while holding the vertical correlation length and correlation coefficient constant.

5.2 Results

The results are plotted in Figures 5.2 and 5.3. Figure 5.3 shows the results from case 1. Figure 5.3 shows the results from cases 2 through 8. There is an anomalous results point in Figure 5.3. This particular point is correct. The data file associated with it was run multiple times with similar results achieved each time.

The approximate value for the ultimate bearing capacity can be found by applying Equation 5.1. Using the equivalent normal distribution means for the soil properties, the value for the ultimate bearing capacity from Equation 5.1 is approximately 1,034 kN.
Figure 5.2 Mean bearing capacity vs. vertical correlation length
Figure 5.3 Mean bearing capacity vs. horizontal correlation length
The results show that RFEM will lead to lower values of ultimate bearing capacity than Terzaghi’s equation. This is due to the program’s ability to seek out the weaker soil regions. Figure 5.4 shows a non-symmetric failure mechanism. This is due to the program’s ability to seek out the weaker soil regions. It appears that correlation lengths can have a little over 5% effect on the ultimate bearing capacity calculation. Figure 5.2 suggests that an increase in vertical correlation length will lead to higher bearing capacity strength. Figure 5.3 is less clear on the effect of the horizontal correlation length.

Figure 5.4 Non-symmetric failure mechanism

The horizontal correlation length appears to both increase and decrease the bearing capacity of the soil according to Figure 5.3. There appears to be an initial decrease in bearing capacity. Then, the bearing capacity increases with horizontal correlation length after the initial drop.

The correlation coefficient has a predictable effect on the bearing capacity of the soil. A negative correlation coefficient implies that a decrease in either cohesion or friction will lead to an increase in the other property. Thus, a sort of balance of strength is maintained. This leads to no comparatively weak regions of soil. A positive correlation coefficient, however, exaggerates weak and strong regions of soil. The cohesion and friction either both increase or decrease when the correlation coefficient is positive.
5.3 Concluding Remarks

Correlation lengths and correlation coefficients both have an effect on the ultimate bearing capacity strength calculation. It is important to take accurate field measurements of the soil properties so that the footing can be modeled properly in the computer. If enough field data is not available, it is important to be aware of the worst case scenarios. Surprisingly, the worst case scenario does not occur when the correlation lengths are maximized or minimized. Thus, parametric studies are required to find the worst case scenario.
CHAPTER 6: ANISOTROPY IN PROBABILISTIC SLOPE STABILITY

The RLEM program used for slope stability analysis was developed by Griffiths and Fenton (Griffiths and Fenton 1993, Fenton and Griffiths 1993). The program uses 8-noded quadrilateral elements to model 2-dimensional finite slopes. The user controls the size and angle of the slope as well as the number of elements in the mesh. The soil properties are input by specifying the distribution type and appropriate parameters. The random field is generated using the Local Average Subdivision (LAS) algorithm. The program uses a stress redistribution algorithm to determine whether the slope failed under its own gravitational loading for each Monte Carlo simulation. The mean and standard deviation of the probability of failure is found by fitting a Bernoulli (binomial) distribution to the results.

The RFEM slope stability program produces PostScript files as part of the output. This allows the user to see a visual representation of the slope with the mesh and random field mapping if desired. The user can also view the final deformed mesh with exaggerated displacements. These PostScript files yield insight into the failure mechanism. It is found that the failed slopes often generate circular failure mechanisms that go through weaker soil regions.

6.1 Soil Layering and Principal Axes Rotation

The original RFEM slope stability program only allowed users to input vertical and horizontal correlation lengths. The program’s Fortran 77 source code was modified to allow soil layering and rotation of the principal axes. The modifications are included in Appendix B. Soil layering refers to setting the horizontal correlation length to infinity by making all elements in a row of the 2-dimensional random field equivalent. The principal axes rotation adjusts the reference angle for horizontal and vertical correlation length measurements. An angle of rotation of 90° would make the horizontal reference a vertical line.

The soil layering could potentially lead to problems with satisfying the mean and standard deviation requirements of the distribution. When soil layering is activated in the program, the program still generates a completely random 2-dimensional field. It then
overwrites all entries in a row with the first entry of that row. Therefore, it is possible for any given trial run to have a mean and standard deviation different than the ones specified. Random fields without layering can also suffer from this problem of plucking values from a larger random field. However, the law of large numbers states that the numbers will average out after a sufficient number of Monte Carlo simulations (von Mises, 1939). 1,000 Monte Carlo simulations are used for each case in this thesis. Plots showing convergence by 1,000 simulations are provided at the end of this chapter.

The principal axes rotation programming takes advantage of symmetry. If the principal axes are rotated 180°, then they will end up in the same position that they started. Thus, the program allows the user to enter any value on the real domain for the angle of rotation, but it will be converted to a value between -90° and +90°. This thesis looks at angles of rotation from 0° to 180°, which can easily be copied to form results for a full circle from 0° to 360° by taking advantage of symmetry.

6.2 Correlation Length Investigation

The first set of runs for the slope stability program examines the effects of varying the correlation lengths for a 1:1 slope comprised of undrained clay. There is no soil layering or principal axes rotation in these runs. The horizontal and vertical correlation lengths are varied from 10% of the slope height (size of one element) to 3x the slope height. 1,000 Monte Carlo simulations are used for each case.

The soil properties used in this analysis are given in Table 6.1. The lognormal distribution is chosen for unit weight and cohesion to prevent the properties from taking negative values. The mean values are chosen to be fairly representative of an undrained clay. The standard deviation is a somewhat arbitrary value chosen for this research. All the variables are uncorrelated. It should be noted that all values entered into the RFEM program are unitless. Therefore, a consistent set of units must be chosen. For this study, all forces are in kilonewtons (kN) and all lengths are in meters (m).
Table 6.1 Soil properties used in slope stability RFEM analyses

<table>
<thead>
<tr>
<th>Property</th>
<th>Distribution Type</th>
<th>Mean</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cohesion</td>
<td>Lognormal</td>
<td>20</td>
<td>4</td>
</tr>
<tr>
<td>Friction Angle</td>
<td>Deterministic</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Dilation Angle</td>
<td>Deterministic</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Unit Weight</td>
<td>Lognormal</td>
<td>20</td>
<td>4</td>
</tr>
<tr>
<td>Elastic Modulus</td>
<td>Deterministic</td>
<td>250000</td>
<td>0</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>Deterministic</td>
<td>0.3</td>
<td>0</td>
</tr>
</tbody>
</table>

A picture of the slope is provided in Figure 6.1. The figure shows the cohesion random field mapped to it. The lighter regions represent weaker soil regions and the darker regions represent stronger regions. The random field mapping in Figure 6.1 is only for one of the Monte Carlo simulations.

![Figure 6.1 1:1 Slope with random field mapping](image)

An example data file is shown in Figure 6.2. Only the correlation length values change between runs. The correlation lengths entered into the data file are absolute lengths. Correlation lengths in the results plots are normalized by the height of the slope.
6.2.1 Results

A total of 65 correlation length combinations are tested. The results of the runs are shown in Figure 6.3 and Figure 6.4. Figure 6.3 shows how the probability of failure varies with respect to the horizontal correlation length if the vertical correlation length is held constant. Figure 6.4 plots the cases when the horizontal correlation length equals the vertical correlation length (isotropic cases). It should be noted that the correlation lengths specified in the plots have been normalized by the height of the slope. Therefore, a correlation length of one implies that the correlation length is equivalent to the height of the slope, H (Figure 2.1).

There are several abbreviations used in the analysis results. lx and ly are abbreviations for the correlation lengths in the x and y directions, respectively. Pf stands for the mean probability of failure found from the Bernoulli distribution of the results.
Figure 6.3 Probability of failure vs. horizontal correlation length
Figure 6.4 Probability of failure vs. correlation length for isotropic case
The results indicate that an increase in correlation lengths leads to higher probabilities of failure. This is true for both horizontal and vertical correlation lengths. The probabilities of failure begin to level off when the correlation lengths are equal to the height of the slope. These results can be explained by examining the failure mechanism and recalling basic slope stability theory. Theory suggests that the preferred failure surface will be circular in nature. The circular surface that causes failure is the one with the lowest factor of safety. When correlation lengths are high, it is possible for there to be a failure mechanism that seeks out primarily weaker soil regions. When correlation lengths are low, the failure mechanism will have to pass through both weak and strong soil regions because these soil regions are interspersed at low correlation lengths.

The RFEM program does not assume a circular failure surface. However, the stress redistribution algorithm will often lead to circular failure surfaces. Figure 6.5 shows an example of a failed slope with a circular failure surface.

![Figure 6.5 Slope failure showing circular failure mechanism](image)

Examining Figure 6.5 reveals that the failure mechanism attempts to cut through weaker soil regions when possible. The failure path waited until it reached a weaker (lighter colored) soil region to cut out through the toe. Figure 6.6a identifies this weaker soil region of Figure 6.5. It would be feasible to suggest that it is a coincidence that the weaker soil region lies along the failure path. Perhaps the failure mechanism always cuts through 3 elements from the bottom of the mesh. Figure 6.6b shows another case for the same data file (but a different Monte Carlo Simulation) where the failure mechanism sought out the weaker soil region.
Not taking into account correlation lengths can lead to lower probabilities of failure. It is important to take into account correlation lengths because worst case scenarios often govern engineering design. It is also feasible to suggest that soil properties in the field may have high correlation lengths due to sedimentary deposits. The first law of geography (Tobler, 1970) also suggests that soil properties will vary gradually over space.
6.3 Soil Layering and Principal Axes Rotation Investigation

2-dimensional finite slopes comprised of undrained clay are used to analyze the effects of soil layering and principal axes rotation. The purpose of this study is to demonstrate the possibility and effects of modeling layered soil and uplifted soil. Monte Carlo simulations are run for both 1:1 and 2:1 slopes. If the previous anisotropy study is an indication, it is to be expected that higher probabilities of failure will be calculated when weak soil regions align with the predicted failure surface.

The random properties for these runs are cohesion and unit weight. The other properties are assumed to be deterministic. The soil properties used in these runs are summarized in Table 6.2. As before, the unit system used assumes all forces to be in kilonewtons (kN) and all lengths in meters (m).

Table 6.2 Soil properties for slope stability RFEM analyses

<table>
<thead>
<tr>
<th>Property</th>
<th>Distribution Type</th>
<th>Mean</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cohesion</td>
<td>Lognormal or Deterministic</td>
<td>20</td>
<td>4</td>
</tr>
<tr>
<td>Friction Angle</td>
<td>Deterministic</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Dilation Angle</td>
<td>Deterministic</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Unit Weight</td>
<td>Lognormal or Deterministic</td>
<td>20</td>
<td>4</td>
</tr>
<tr>
<td>Elastic Modulus</td>
<td>Deterministic</td>
<td>250000</td>
<td>0</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>Deterministic</td>
<td>0.3</td>
<td>0</td>
</tr>
</tbody>
</table>

The principal axes are rotated at 15° increments in the random field generation. The results for an angle of rotation of 180° were simply copied from the results for an angle of rotation of 0° because of symmetry arguments. The purpose of doing this is to generate one period of results that can be copied to form results for 0° to 360° using symmetry arguments.

The vertical correlation length is also adjusted in addition to soil layering and principal axes rotation. The vertical correlation length is applied before the rotation. Thus, the vertical correlation length is always applied perpendicular to the soil layering.
Cases are run for various combinations of random cohesion and random unit weight. Table 6.3 shows all the possible combinations that are run for this study. In total 960 different data files are used in this analysis.

Table 6.3 Slope stability cases

<table>
<thead>
<tr>
<th>Case</th>
<th>Slope Angle</th>
<th>Random Property</th>
<th>Correlation Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1:1</td>
<td>Cohesion</td>
<td>N/A</td>
</tr>
<tr>
<td>2</td>
<td>2:1</td>
<td>Cohesion</td>
<td>N/A</td>
</tr>
<tr>
<td>3</td>
<td>1:1</td>
<td>Unit Weight</td>
<td>N/A</td>
</tr>
<tr>
<td>4</td>
<td>2:1</td>
<td>Unit Weight</td>
<td>N/A</td>
</tr>
<tr>
<td>5</td>
<td>1:1</td>
<td>Cohesion &amp; Unit Weight</td>
<td>-1.0</td>
</tr>
<tr>
<td>6</td>
<td>2:1</td>
<td>Cohesion &amp; Unit Weight</td>
<td>-1.0</td>
</tr>
<tr>
<td>7</td>
<td>1:1</td>
<td>Cohesion &amp; Unit Weight</td>
<td>0.0</td>
</tr>
<tr>
<td>8</td>
<td>2:1</td>
<td>Cohesion &amp; Unit Weight</td>
<td>0.0</td>
</tr>
<tr>
<td>9</td>
<td>1:1</td>
<td>Cohesion &amp; Unit Weight</td>
<td>+1.0</td>
</tr>
<tr>
<td>10</td>
<td>2:1</td>
<td>Cohesion &amp; Unit Weight</td>
<td>+1.0</td>
</tr>
</tbody>
</table>

Twelve different angles of rotation are analyzed for each case, and eight different vertical correlation lengths are analyzed for each angle of rotation. Thus, 96 different runs are used for each case. 96 runs multiplied by 10 cases yields the 960 different data files used in this thesis.

6.3.1 Results

The results for this analysis are shown in Figures 6.7a – 6.16b. The results are plotted as probability of failure versus principal axes rotation angle and probability of failure versus vertical correlation length. The second method of plotting is simply to confirm the results from the previous section that higher correlation lengths lead to higher probabilities of failure. In the plots, Pf is the probability of failure, ly is the correlation length perpendicular to the layering, and rfrot is the angle of random field rotation.
Figure 6.7a Probability of failure vs. angle of random field rotation for 1:1 slope with random cohesion

Figure 6.7b Probability of failure vs. correlation length perpendicular to soil layering for 1:1 slope with random cohesion
Figure 6.8a Probability of failure vs. angle of random field rotation for 2:1 slope with random cohesion

Figure 6.8b Probability of failure vs. correlation length perpendicular to soil layering for 2:1 slope with random cohesion
Figure 6.9a Probability of failure vs. angle of random field rotation for 1:1 slope with random unit weight

Figure 6.9b Probability of failure vs. correlation length perpendicular to soil layering for 1:1 slope with random unit weight
Figure 6.10a Probability of failure vs. angle of random field rotation for 2:1 slope with random unit weight

Figure 6.10b Probability of failure vs. correlation length perpendicular to soil layering for 2:1 slope with random unit weight
Figure 6.11a Probability of failure vs. angle of random field rotation for 1:1 slope with random cohesion, random unit weight, and correlation coefficient of -1.0.

Figure 6.11b Probability of failure vs. correlation length perpendicular to soil layering for 1:1 slope with random cohesion, random unit weight, and correlation coefficient of -1.0.
Figure 6.12a Probability of failure vs. angle of random field rotation for 2:1 slope with random cohesion, random unit weight, and correlation coefficient of -1.0

Figure 6.12b Probability of failure vs. correlation length perpendicular to soil layering for 2:1 slope with random cohesion, random unit weight, and correlation coefficient of -1.0
Figure 6.13a Probability of failure vs. angle of random field rotation for 1:1 slope with random cohesion, random unit weight, and correlation coefficient of 0.0

Figure 6.13b Probability of failure vs. correlation length perpendicular to soil layering for 1:1 slope with random cohesion, random unit weight, and correlation coefficient of 0.0
Figure 6.14a Probability of failure vs. angle of random field rotation for 2:1 slope with random cohesion, random unit weight, and correlation coefficient of 0.0

Figure 6.14b Probability of failure vs. correlation length perpendicular to soil layering for 2:1 slope with random cohesion, random unit weight, and correlation coefficient of 0.0
Figure 6.15a Probability of failure vs. angle of random field rotation for 1:1 slope with random cohesion, random unit weight, and correlation coefficient of +1.0

Figure 6.15b Probability of failure vs. correlation length perpendicular to soil layering for 1:1 slope with random cohesion, random unit weight, and correlation coefficient of +1.0
Figure 6.16a Probability of failure vs. angle of random field rotation for 2:1 slope with random cohesion, random unit weight, and correlation coefficient of +1.0

Figure 6.16b Probability of failure vs. correlation length perpendicular to soil layering for 2:1 slope with random cohesion, random unit weight, and correlation coefficient of +1.0
It should be noted that low correlation lengths can lead to some misleading results. The random field generation algorithm used in this study, LAS, may significantly reduce the standard deviation of the random field if the correlation lengths are low. Thus, the cases involving low correlation lengths are essentially modeling a slope with lower variability in soil properties.

The results agree with those of the previous section. Higher correlation lengths generally lead to higher probabilities of failure. This occurs even with soil layering and principal axes rotation.

The results indicate that higher probabilities of failure occur when the soil layering is aligned with the slope, and lower probabilities of failure occur when the soil layering is perpendicular to the slope. The exception to this is when the correlation coefficient is equal to positive one. The PostScript files demonstrate potential reasons for the higher and lower probabilities of failure.

The PostScript files produce two potential failure mechanism types – parallel and series. The parallel mechanism occurs when the layering is parallel to the slope and the program seeks out the weak line of soil. The series mechanism occurs when the layering is perpendicular to the slope and the failure mechanism must pass through both strong and weak layers of soil.

The parallel failure mechanism is shown in Figures 6.17a, 6.17b. They show failed 1:1 slopes. In both figures, the failure mechanism sought out the weaker regions of soil. The failure mechanism followed two distinctly different paths. The failure mechanism was not confined to a particular geometry; rather, it sought out the weaker regions. Recall that the lighter regions represent weaker soil regions, and darker regions represent stronger soil regions.
The series failure mechanism is shown in Figures 6.18a and 6.18b. The failure mechanism attempts to pass through both weak and strong soil regions. This decreases the probability of failure.
The exact angle that leads to the highest and lowest probabilities of failure is a little unclear. Without further investigation, it can only be stated that the highest probability of failure occurs when the weak soil regions are align between the angle of the slope and $45^\circ$ (except when the correlation coefficient is positive one).

The correlation coefficient has an interesting effect on the results. A correlation coefficient of negative one implies that areas with low cohesion will also have high unit weight. This expedites the failure mechanism in weak soil regions. A correlation coefficient of zero denotes that cohesion and unit weight are independent of each other. Thus, the results should be similar to when only cohesion or only unit weight is random. A correlation coefficient of positive one implies that the cohesion and unit weight values move in line with each other. This yields anomalous results in this analysis.

The cause of the anomalous results when the correlation coefficient equals positive one is unclear. A correlation coefficient of one denotes that the unit weight and cohesion will both increase or decrease. Thus, areas of high unit weight will have high cohesion, and areas of low unit weight will have low cohesion. Usually, weak soil regions govern the failure mechanisms. However, in this case, it might be possible for the strong soil regions to govern the failure mechanism. An area of high unit weight and high cohesion might prevent the failure mechanism from forming. Thus, there is an overall lower probability of failure for this case. Also, the probability of failure is minimized when the strong soil regions are aligned perpendicular to the preferred failure surface.
Similarly, the highest probabilities of failure occur when the strong failure regions are aligned parallel to the preferred failure surface.

6.4 Rate of Convergence for Monte Carlo Simulations

There is always a lingering question of how many Monte Carlo simulations is enough to generate reasonable results. 1,000 Monte Carlo simulations are used for the slope stability analysis in this study. Thus, the results only have a precision of 0.001. Figure 6.19 shows reasonable convergence by 1,000 simulations.

![Figure 6.19 Probability of failure vs. number of simulations](image)

6.5 Concluding Remarks

Soil layering and principal axes rotation provide two additions to the established methods of random finite elements. Anisotropy in slope stability analysis can have up to an order of magnitude effect on the results. Thus, it is vital that soil properties are modeled properly in finite elements. It is not enough to simply use RFEM to model soil slopes. Correlation lengths and orientations also have a significant effect on the probability of failure calculation.
CHAPTER 7: FUTURE RESEARCH

This thesis presents a method for implementing FORM into an Excel spreadsheet by using the Excel solver to solve the FORM optimization problem. However, the FORM programs accompanying this thesis only include applications for infinite slopes, strip footings, and rectangular footings. The FORM spreadsheet implementation could be expanded to include numerous other engineering applications. In addition, the programs with this thesis only include a sensitivity analysis for the mean values of the random variables. Macros could be written for sensitivity analyses of standard deviations and distribution types. A macro for distribution type sensitivity analysis would yield valuable insight into the effects of using different distribution types to model soil properties.

The RFEM anisotropy analysis in this thesis expands the conventional RFEM program for 2-dimensional finite slopes by including the ability to layer the soil and rotate the principal axes of the random field. Further investigations can look into the effects of soil layering and principal axes rotation in bearing capacity analysis and other geotechnical applications.
CHAPTER 8: CONCLUSIONS

Probabilistic methods provide a means for incorporating statistical distributions of soil properties and loadings into engineering analysis. There is an uncertainty in soil properties and loadings that simply cannot be accounted for accurately using a global factor of safety. Probabilistic methods establish the foundation for a more consistent standard for safety and reliability by incorporating property and loading uncertainties directly into the calculations. The dominant probabilistic methods in geotechnical analyses are first order methods (FORM and FOSM) and the random finite element method (RFEM).

The first order methods, FORM and FOSM, incorporate property and loading uncertainty into probability of failure calculations. Both methods require the formulation of a performance function that must equal zero when the structure is at failure. The result obtained by FOSM is dependent upon this performance function formulation because it uses a first order Taylor series expansion of the performance function about the means to approximate the failure values. Thus, the curvature of the performance function at the mean values influences the result of FOSM. FORM, however, is independent of the performance function formulation but requires the solving of an optimization problem. FORM is therefore superior to FOSM for first order analysis because of its invariance of the performance function formulation. It is possible for FORM and FOSM to yield the same results if the performance function is a linear combination of the random variables. If the performance function is not linear, FOSM may lead to higher or lower reliability index calculations depending upon the curvature of the performance function. First order methods, though, do not account for the spatial variability of the random variables.

RFEM combines conventional FEM with random field generation. RFEM, unlike first order methods, is able to account for the spatial variability of soil properties through the use of correlation lengths. Adjusting correlation lengths and rotating principal axes in random field generation can lead to significant differences in RFEM results. In 2-D bearing capacity analysis, the lowest bearing capacity is found when the correlation length is between 2 to 5 times the length of the footing. In 2-D slope stability analysis, higher correlation lengths generally lead to higher probabilities of failure. Also, aligning
soil layers parallel to the slope generally leads to higher probabilities of failure because the failure mechanism can seek out a weak soil layer. The RFEM PostScript files yield insight into the failure mechanisms that lead to these differences in results. Both the bearing capacity and slope stability programs attempt to seek out weaker soil regions. The correlation lengths and principal axes orientation can both positively and negatively influence the program's ability to seek out the weaker soil regions. In slope stability analysis, two significant failure mechanisms are found to be the parallel mechanism and series mechanism. The parallel mechanism usually leads to higher probabilities of failure because the failure mechanism primarily passes through weaker soil regions while the series mechanism leads to lower probabilities of failure because the failure mechanism must pass through both weak and strong soil regions. In all cases, correlation lengths and principal axes orientation have an effect on the results of RFEM.
REFERENCES CITED


APPENDIX A: FORM PROGRAMS

FORM programs for slope stability and bearing capacity are included on the CD accompanying this thesis. There are three programs in total. There is a program for analyzing an infinite slope, a strip footing, and a rectangular footing. The programs are appropriately named depending upon its application. There are directions in the programs for running the FORM solver and sensitivity analyses.

The FORM programs are written in Microsoft Excel 2003. However, they should work in any version of Excel with Visual Basic for Applications capabilities. To ensure that the macros run properly, the security settings in Excel must be set to allow macros to run. The process for adjusting the security settings varies depending upon personal settings and Excel’s version number. If the macros produce error messages, the references may need to be specified. The references are specified by entering the visual basic editor (alt+F11) and going to the appropriate menu item (depending upon the version of Excel). The references dialog box will specify which references are missing and need to be specified.
APPENDIX B: RFEM MODIFICATIONS

The RFEM source code modifications for the 2-dimensional finite slope are included on the CD accompanying this thesis. Additionally, an executable version of the program is provided with a sample data file. The original RFEM programs by Fenton and Griffiths are available from http://www.mines.edu/~vgriffit. The modifications are made to the source code in the sim\rslope2d folder. The original source code was written in Fortran 77 and is modified to accommodate soil layering and principal axes rotation.

The modified subroutines on the CD accompanying this thesis contain some annotations on the changes made to them. A quick summary of the changes is provided here. To engage soil layering, the first column of the random field is mapped horizontally across the finite element mesh. To rotate the principal axes, an origin is established at the bottom left corner of the random field. Then, x and y coordinates are determined for each random field element. These coordinates are then rotated by the specified angle and mapped onto the finite element mesh.

The rslope2d program can be run by either using the executable or compiling the source code. The executable is run by copying the file to a computer hard drive then double clicking on it or invoking it from a blank command prompt. A data file must then be entered with the extension ‘.dat’. A sample data file is provided on the CD accompanying this thesis. The program then outputs a results file (.stt) and several PostScript files. The PostScript files require software such as GhostView to view them. The Postscript files contain visual representations of the problems solved.

To generate the rslope2d executable using the source code, download the original software from the internet (www.mines.edu/~vgriffit). Then, copy the Fortran (.f) files from the accompanying CD into the sim\rslope2d folder and overwrite existing files as necessary. Then, copy the library folders (VGlib and gaf77) from the CD accompanying this thesis to c:\rfem\. (Another location may be chosen, but the runrslope.bat file will need to be modified). Then, compile the libraries by running the build_lib.bat and buildlib.bat batch files in the library folders. A Fortran compiler is required to compile these libraries. A free compiler is provided from g95.org. After the libraries are compiled,
invoke the runrslope.bat batch file in the rslope2d folder. The batch file will generate an executable, rslope2d.exe, for the RFEM slope stability program.

The RFEM slope stability program used in this study was run on a Windows PC. The modified program may not run on other machines. If the modified program does not run properly, ensure that the original Griffiths and Fenton programs run properly. In addition, compiler errors may occur if the Fortran compiler is not set up properly. The compiler may require the setting of environment variables. Please consult the compiler manual or other literature for instructions on how to set environment variables.