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Exploring Fractional Derivatives and Trig Functions

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Introduction

Continuous fractional derivatives have been around for several centuries, but there is little known in the discrete case. An example of a fractional derivative is shown below:

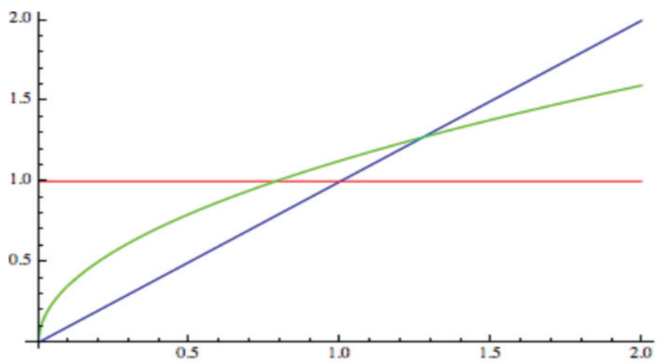

***Figure 1:** An example of a fractional derivative, where blue: f(x) = x, red: f'(x) = 1, green: 1/2-order fractional derivative of f(x)*

Here, the blue line corresponds to f (x) = x, the red line corresponds to f'(x) = 1, and the green curve corresponds to the ½ - order fractional derivative of f (x). We define our discrete difference operator, the **nabla** (∇) **operator**, as follows:

$$(\nabla f)(t) = f(t) - f(t-1) \qquad t \in \mathbb{N}_{a+1}$$

This is known as a backwards difference operator.

Objectives

The objective of this research is to become familiar with **fractional discrete calculus** to the extent that fractional derivatives of **discrete trigonometric functions** can be taken and understood. Before specifically taking fractional derivatives of trig functions, it is important to:

- Have a geometric interpretation of discrete trig functions
- Have a list of trig identities that that we can use when deriving and integrating trig functions
- Have a better understanding of the discrete trig identities to be able to compare the nabla case with other potential discrete operators

Existing Functions

Using discrete fractional nabla calculus, we can define a **nabla exponential function.** Let *p* be in the set of regressive functions, then:

$$E_p(t,s) := \begin{cases} \prod_{\tau=s+1}^{t} \frac{1}{1-p(\tau)}, & t \in \mathbb{N}s, \\ \prod_{\tau=t+1}^{s} [1-p(\tau)], & t \in \mathbb{N}_a^{s-1}. \end{cases}$$

Nabla sine and cosine are defined as a real case of the above exponential. Let p be any value excluding i, $-i$, then:

$$Sin_p(\theta,0) = \frac{E_{ip}(\theta,0) - E_{-ip}(\theta,0)}{2i}$$

$$Cos_p(\theta,0) = \frac{E_{ip}(\theta,0) + E_{-ip}(\theta,0)}{2}$$

One trig identity has been defined and is as follows:

$$Sin_p^2(\theta,0) + Cos_p^2(\theta,0) = E_{-p^2}(\theta,0)$$

Our Proven Trig Identities

In the real case, the half-angle formulas are defined as:

$$\sin^2\theta = \frac{1}{2} - \frac{\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1}{2} + \frac{\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

We have come up with similar identities in the nabla case, using exponential definition of the nabla sine and cosine. Again, we let $p = 1$ for simplicity. We have:

$$\left[Sin_1(\theta,0)\right]^2 = \frac{E_{-1}(\theta,0)}{2} - \frac{Cos_1(2\theta,0)}{2}$$

$$\left[Cos_1(\theta,0)\right]^2 = \frac{E_{-1}(\theta,0)}{2} + \frac{Cos_1(2\theta,0)}{2}$$

Since nabla tan is defined as nabla sine over nabla cosine, similarly to real tan, we have:

$$\left[Tan_1(\theta,0)\right]^2 = \frac{E_{-1}(\theta,0) - Cos_1(2\theta,0)}{E_{-1}(\theta,0) + Cos_1(2\theta,0)}$$

If we use $p = 1$ for the Pythagorean identity already found, then we get:

$$Sin_1^2(\theta,0) + Cos_1^2(\theta,0) = E_{-1}(\theta,0)$$

In all these identities that we have found an exponential term. This term shows up consistently instead of a constant, or, more specifically, 1. If we use the definition of the nabla exponential, we know that:

$$E_{-1}(\theta,0) = (2)^{-\theta}$$

This decaying exponential is consistent with the graphs that we have found nabla sine and cosine.

Relations to Continuous Case

We can define our nabla trig functions in terms of real sine and cosine, using the definition in terms of exponentials. In order to do this, we let $p = 1$. The relationship is defined as below:

$$Sin_1(\theta,0) = \frac{\sin(\frac{\pi}{4}\theta)}{2^{\pi\theta/4}} \qquad Cos_1(\theta,0) = \frac{\cos(\frac{\pi}{4}\theta)}{2^{\pi\theta/4}}$$

Since there is a clear relationship between continuous trig functions and nabla trig functions, we can graph our nabla trig functions in terms of real since and cosine. ***Figure 2*** is a graph of nabla sine and cosine. Notice that both functions decay rather fast due to the exponential.

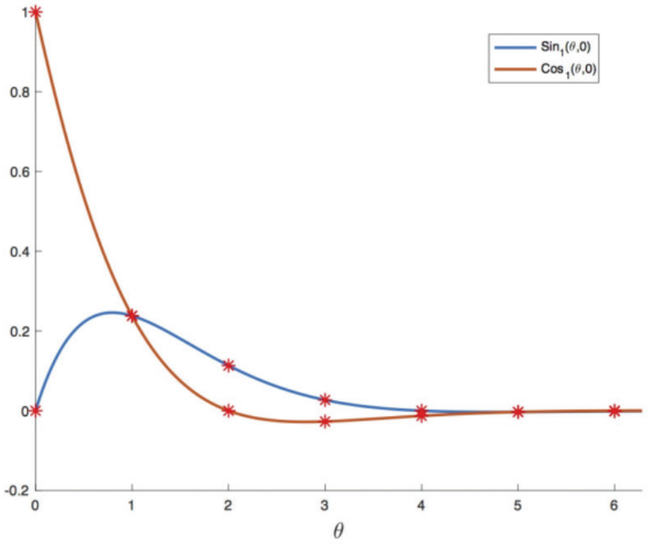

Figure 2: *Graph of nabla sine and cosine*

Future Research

We plan on continuing to look at the exponential that is common to the identities for potential implications. Additionally, we plan on looking further into other discrete operators to see if they make more sense geometrically. As for a larger next step, we hope to be able to look more into taking a fractional derivative of the nabla trig functions.

References

C. Goodrich and A. C. Peterson. *Discrete Fractional Calculus.* Springer, 1st edition, 2015.