A COUPLED GEOMECHANICS, THERMAL AND FLUID FLOW MODEL
FOR WELLBORE INTEGRITY ANALYSIS

by
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Wellbore instability often results in nonproductive time and is costly for drilling operations as well as long term integrity of the wellbore. The most important factor that affects the stability of the bore wall is the drilling fluid density. High density drilling fluid can cause a fracture in the wellbore because the hydrostatic pressure in the wellbore causes tensile failure in the bore wall. Using low density fluid often results in the collapse of the wellbore because the hydrostatic pressure is not sufficient to balance the in-situ stresses around wellbore. Hence, determining a proper mud weight is important to safely drill the wellbore minimizing the cost and risk associated with drilling operations.

Currently, analytical models are used to evaluate the stress state around the wellbore. These models provide a quick determination of the stress field, but they have many limitations coming from the assumptions used to obtain the analytical solutions. One of the most important limitations of the analytical models is that they do not often account for the effect of the fluid and rock interaction because only the conservation of momentum equation is considered in these models. Fluid-rock interaction is an important aspect to consider when determining stresses around the wellbore. Hence, these models cannot be used to solve many challenging problems that arise in modern wellbore stability studies.

To address these limitations, a coupled model is formulated from equations of motion, heat transfer, and mass transport. The model accounts for the transport of two-phase water and oil as well as the heat transfer near the wellbore. The coupled model is solved numerically to obtain the stress change around the wellbore and the invasion of the fluid into the formation as well as the change of formation temperature. Using the developed model, case studies are conducted to evaluate the effect of thermal, fluid and rock interaction, and formation transport properties, permeability and wettability, on the time-dependent stability of the wellbore. The numerical results from this study show that the interaction between
fluid and rock is the most considerable factor affecting the long term stability of the wellbore. Hence, changing the wettability of the drilling fluids can be considered as a method to improve the integrity of the wellbore. We observe that the minimum and maximum mud weights decrease with the decrease of temperature. Therefore, controlling the circulation temperature can also be effective to maintain the wellbore integrity. Permeability and wettability also play an important role in the determination of the mud window. Changing the wettability toward oil-wet can improve the stability of the wellbore, especially when the rock mechanical properties are very sensitive to water saturation.

The model developed in this study helps us to study the complexity of the mechanical interaction between drilling fluid and rocks and determine more accurate mud weights for drilling operations. This model can also be used for evaluating the integrity of the cement sheath during hydraulic fracturing operations and other problems related to the wellbore. Hence, this research provides the industry with a new practical tool to solve more challenging problems in wellbore integrity analysis.
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General Nomenclature

material constants of Drucker-Prager model ........................................... $A, B$
Coulomb strength parameters in the Mogi-Coulomb model ............................. $a, b$
coefficient of stiffness tensor ................................................................. $C_{i,j}$
cohesive strength ...................................................................................... $C_o$
total compressibility .................................................................................. $c_t$
oil compressibility ....................................................................................... $c_o$
specific heat of the rock formation .............................................................. $c_r$
water compressibility .................................................................................. $c_w$
rock compressibility .................................................................................... $c_\phi$
Young’s Modulus ......................................................................................... $E$
external force ............................................................................................... $F$
shear modulus ............................................................................................. $G$
gravitational acceleration ............................................................................. $g$
identity matrix ............................................................................................. $I$
the first invariant of the deviatoric stress tensor ............................................ $I_1^d$
the second invariant of the deviatoric stress tensor ........................................ $I_2^d$
wellbore inclination angle ............................................................................ $I_w$
absolute permeability .................................................................................. $k$
thermal conductivity ..................................................................................... $K$
Young’s Modulus change coefficient \( K_E \)

UCS change coefficient \( K_{UCS} \)

friction angle change coefficient \( K_\mu \)

relative oil permeability \( k_{ro} \)

relative water permeability \( k_{rw} \)

empirical constant \( m \)

number of nodes \( N \)

oil exponent for modified Brooks-Corey functions \( n_o \)

water exponent for modified Brooks-Corey functions \( n_w \)

water-oil capillary pressure \( p_{cwo} \)

initial formation pressure \( p_i \)

pressure of oil \( p_o \)

pressure of water \( p_w \)

wellbore pressure \( p_{well} \)

specific volume of oil \( \hat{q}_o \)

total specific flow rate \( \hat{q}_t \)

specific volume of water \( \hat{q}_{gw} \)

radius \( r \)

radius at bore wall \( r_w \)

rotational matrix \( R_2(\alpha_w) \)

oil saturation \( S_o \)

residual oil saturation \( S_{o_\text{r}} \)

water saturation \( S_w \)
residual water saturation \( S_{wr} \)

empirical constant \( s \)

initial formation temperature \( T_i \)

wellbore temperature \( T_w \)

displacement \( u \)

velocity vector \( v \)

velocity of oil \( v_o \)

velocity of water \( v_w \)

**Greek Letters**

Biot’s tensor \( \alpha \)

Biot’s coefficient \( \alpha \)

thermal diffusivity \( \alpha_T \)

wellbore angle \( \alpha_w \)

thermal expansion coefficient \( \beta \)

node size \( \Delta r_i \)

time step \( \Delta t \)

divergence operator \( \nabla \)

elastic strain tensor \( \varepsilon_e \)

thermal strain tensor \( \varepsilon_T \)

total strain tensor \( \varepsilon_t \)

volumetric strain \( \varepsilon_v \)

angle \( \theta \)
Lame’s constant \( \lambda \)

total mobility \( \lambda_t \)
oil mobility \( \lambda_o \)
water mobility \( \lambda_w \)
viscosity \( \mu \)
internal friction coefficient \( \mu_i \)
oil viscosity \( \mu_o \)
water viscosity \( \mu_w \)
Poison’s Ratio \( \nu \)
mud density \( \rho_m \)
density of oil \( \rho_o \)
rock density \( \rho_r \)
density of water \( \rho_w \)
effective stress \( \sigma' \)
maximum principal stress \( \sigma_1 \)
intermediate principal stress \( \sigma_2 \)
minimum principal stress \( \sigma_3 \)
principal stress \( \sigma_e \)
maximum in-situ horizontal stress \( \sigma_H \)
minimum in-situ horizontal stress \( \sigma_h \)
vertical stress \( \sigma_v \)
radial stress \( \sigma_{rr} \)
shear stress \( \sigma_{r\theta}, \sigma_{\theta r} \)
tensile strength of the rock \( \sigma_T \)

hoop stress \( \sigma_{\theta\theta} \)

axial stress \( \sigma_{zz} \)

shear stress \( \tau \)

octahedral shear stress \( \tau_{oct} \)

porosity \( \phi \)

internal friction angle \( \varphi_i \)
LIST OF ABBREVIATIONS

Bottom Hole Assembly ................................................. BHA
Equivalent Circulation Density ................................. ECD
Inflow Performance Relationship ............................. IPR
Mud Weight ................................................................. MW
Mud Weight Window ................................................ MWW
Total Organic Carbon ........................................... TOC
True Vertical Depth .................................................... TVD
Unconfined Compressive Strength ............................ UCS
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In memory of Dr. Azra N. Tutuncu
CHAPTER 1
INTRODUCTION

Maintaining a stable wellbore is one of the main problems during drilling operations. Drilling under highly difficult geological conditions reveals that there is a need for comprehensive understanding of the wellbore stability issues. Drilling operations all around the globe, especially in complex formations, require a great understanding of the wellbore geomechanics in order to avoid losses, time and cost. Wellbore stability modeling is a challenging task to provide an accurate mud weight to safely drill the wellbore and maintain its stability during operations. This requires a comprehensive understanding not only stress around the wellbore, but also fluid and rock interaction as well as the thermal-induced instability. In this chapter, I first discuss the overview of wellbore instability. Then, I discuss the challenge of wellbore stability modeling. Finally, I outline objectives and scope of this thesis along with applications for practical operations.

1.1 Overview of Wellbore Instability

During drilling operations, drilling engineers adjust the drilling fluid density to balance the formation stresses around the wellbore. Sometimes this balance is not proper, and it results in the failure of the wellbore. The wellbore can be failed by too high or too low mud pressure. When the drilling fluid density is not sufficiently high, the in-situ stresses forcing the rock toward the wellbore creating wellbore breakout, compressive failure. This failure causes the rock around the bore wall to fall into the wellbore or reduces the wellbore diameter causing stuck pipe and other problems with the cuttings transport. In some cases, it affects the normal drilling operations. When the mud weight is excessively high, it can cause the fracture of the formation, tensile failure, resulting in loss of drilling fluid and contamination of the formation. Hence, in drilling operations, determining a proper mud weight is highly important.
Huge amounts of money have been spent on wellbore integrity during drilling operations and maintaining the stability during the life cycle of the wellbore. It is estimated that wellbore instability issues result in about 1 billion U.S. dollars losses per year (Zeynali 2012). Wellbore instability is the main factor causing the time loss for over 40% of all drilling related nonproductive time (Zhang et al. 2009). To drill horizontal, inclined, extended reach, multi-laterals, and highly fractured wells, it requires more focus on wellbore stability studies. A variety of problems with the wellbore could happen during pre-production and post-production.

Wellbore integrity is the key to successful and low-cost drilling. Issues related to wellbore instability are the results of mechanical and chemical effects. Due to the change of behavior of rock, time and expenditures for drilling operations could be increased significantly. Also, casing programs as well as choosing right mud density could prevent wellbore from failure due to collapse or fracture. Preventing wellbore instability could be result in other problems related to well design such as formation damage, rate of penetration and hole cleaning ability. Therefore, to optimize drilling, predictive or numerical methods might be used. Numerical simulations could determine mud weight window as well as sand production prevention.

The most important factors affecting wellbore instability are rock mechanical properties, temperature, pore pressure, and salt concentration (Ostadhassan 2013). There are many other factors that impact wellbore instability, however a model that includes every factor will be very complex. Therefore, the main important mechanical aspects that affect wellbore stability are the in-situ stresses existing in different layers of rocks, mechanical properties of rock, variation of pore pressure, mud weight, wellbore azimuth and inclination angles, thermal effects, and formation anisotropy. The focus should be given to the mentioned factors, due to their importance in solving wellbore instability problems (Zeynali 2012).

The second most important controllable factor that often results in time-dependent wellbore instability issue is the physical and chemical interaction between fluid and rock. Near-wellbore stresses or rock strength can be affected by physical and chemical interaction be-
tween fluid and rock. For instance, water can weaken the rock mechanical property, it also can cause swelling in shale formations. Hydration, osmotic pressure, rock softening and dispersion can also cause alteration of stress or rock strength near wellbore. Their effect strongly depends on the interaction of factors such as the nature of formation, the existence of the mud cake and skin factor around a borehole (McLellan 1996).

The tectonic stress is an important uncontrollable factor causing wellbore instability. Drilling in tectonically stressed formations could lead to wellbore instability and produce cavings. Rock is being compressed in the areas where pressure build up occurs by the movement of tectonic plates. To overcome such problems, the hydrostatic pressure should be much higher than the fracture pressure of other open formations (Bowes and Procter 1997). Mountainous regions are good example of such mechanisms.

1.2 Challenges in Wellbore Stability Modeling

To determine the proper drilling fluid to drill a wellbore safely without any issue, we often need to determine the stress variation around the wellbore. To calculate stresses we need to solve the geomechanical governing equation deriving from the conservation of momentum. Often, this equation is quite complex to solve, hence assumptions are made to simplify the problem. Consequently, it creates some limitations when dealing with some complex drilling scenarios and can not account for many important physical phenomena affecting wellbore integrity such as the change of rock mechanical properties due to fluid invasion.

1.2.1 Effect of fluid and rock interaction

Drilling fluid has a strong effect on the chemo-mechanical properties of formations. Yadav et al. (2016) investigated the proper drilling fluid design for unconventional and conventional formations and its impact on such properties. Their experiments showed that water-based drilling fluid imbibition has strong effect on the Young’s Modulus and Poisson’s Ratio of the formation. The Young’s Modulus of the rock is reported to be decreased from 1.416 Mpsi to 1.008 Mpsi, while the Poisson’s Ratio increased from 0.28 to 0.40. Therefore, it is crucial to
account for this effect in wellbore stability analysis.

Shale swelling can also be a significant factor which can cause the change of the mechanical properties of the formation and the alteration of the stresses around wellbore. Bui and Tutuncu (2018a) used a multi-phase flow model to show that imbibition of drilling fluid into formations depends strongly on its wettability as well as relative permeability causing the alteration of rock properties. Therefore, conventional models cannot capture this effect. In their study, geomechanical model is coupled with transport model to determine the imbibition of water into formation caused by pressure gradient, osmosis pressure and capillary pressure. The result by Bui and Tutuncu (2018a) suggests that shale formations with high total organic carbon (TOC) content allows less water to imbibe into the formation than the formation with low TOC content. Also, they recommend to further investigate the interaction between fluid and rock to better evaluate the stability of the wellbore. To investigate the effect of fluid and rock interaction, a model coupling mechanical interaction as well as chemical interaction should be used because the current analytical models cannot account for this phenomenon.

1.2.2 Effect of temperature

Drilling fluid temperature is an important factor that induces the stress change around the wellbore, or thermal-induced stress. Temperature is an important factor affecting not only principal stresses, but also rock strength. During drilling, mud temperature and bottomhole temperature can alter the stresses around wellbore leading to unfavorable wellbore conditions causing wellbore failure. The reduction in drilling fluid temperature decreases the stresses acting on wellbore preventing the rock from collapse (McLellan 1996).

The downhole temperature affects the viscosity and density of the drilling fluid, resulting in the different measurements with what are obtained on the surface (Zhang et al. 2009). In high geothermal gradient areas, the thermal expansion of drilling fluids can result in drilling under-balance, which may lead to a kick (Zhang et al. 2009). During drilling, the wellbore temperature is affected by the penetration rate, flow rate, shut-in intervals, pump
and rotary speeds, fluid and formation properties (Eppelbaum et al. 2014). One main factor that governs the relationship between the temperature effect and pore pressure is the fluid circulation rate. The higher the circulation rates are, the higher the bottomhole temperature changes, thus the higher induced tensile stresses around the wellbore. In addition, adjusting the circulation rate can be considered as a method to manage bottomhole temperature and stress conditions (Wu et al. 2017).

Rocks can be classified as ductile and brittle. Ductility is a measure of the ability of rocks to deform plastically before fracture occurs. Ductile rocks are the materials which undergo large amounts of plastic deformation before they break. These rocks are often fractured at higher strains. Temperature also has an impact on rock ductility. Ductile rocks can become brittle material and fail faster at lower temperatures. Brittle rocks can be fractured at very low strains with little to no plastic deformation. Griggs et al. (1960) described the behavior of the granite under different temperatures a confining stress of 500 MPa. Rock is brittle at room temperature 25 °C, but it becomes ductile at 800 °C. They observed that brittle-ductile transition occurs during both confining stress and temperature increase. However, if there is no confining stress some rock can be brittle due to melting temperatures Murrell and Chakravarty (1973).

1.2.3 Effect of natural fractures

Natural fractures in the formation can be the main contributor to wellbore instability events such as stuck pipe and tight hole during drilling operations. Chen et al. (2020) used geomechanical model to address wellbore instability problems. Their model includes image data, well logs and laboratory core experiment. Their results show that the main problem for loss circulation and stuck pipe events is the natural fractures in the reservoir. Naturally fractured formations are discovered in a wide range of scale, near faults, micro-fissures, and multi kilometer-long attributes. Faults could cause near rock to break into tiny or large parts. In the case of loose compaction between rock grains, formation fragments fall into the borehole and stick the string in the wellbore (Nguyen et al. 2009). Drill string vibrations
could also cause these parts to fall into wellbore, even though the compaction between rock grains is strong. Unfavorable well inclination to weak bedding planes might be the cause of wellbore collapse. The existence of faults or fractures could lead to drilling fluid invasion to such formations resulting in formation softening or hole collapsing. Also, minimization of drill string vibrations is a key to stable wellbore in these formations (Bowes and Procter 1997).

The naturally over-pressured shale formations also cause wellbore instability. These formations are naturally over-pressured because of the geological phenomena such as compaction, uplift, naturally removed overburden, and hydrocarbon generation. Usually in over-pressured shale formations, pore pressure is greater than normal hydrostatic pressure. Drilling with insufficient mud density will cause the wellbore to collapse or become unstable in these formations (Bowes and Procter 1997; Tan et al. 1999). A shorter exposure time and an adequate drilling fluid density can help to stabilize these formations. The other related issues that also cause wellbore stability issue is the induced over-pressured shale collapse. Pore pressure excess decreases confining stresses around wellbore which lead to borehole instability conditions. Water-based drilling fluids, long exposure time, and the reduction of mud weight can be an example of this mechanism. However, the conventional model cannot properly model the effect of this factor on the long term stability of the wellbore. Therefore, numerical model is recommended to properly capture this complexity.

1.2.4 Effect of formation bedding

Drilling into weak bedding layers can be problematic for maintaining wellbore stability. The trajectory of the wellbore such as the azimuth and inclination angles is also a determining factor causing wellbore instability issue. Well azimuth and inclination angles with reference to the principal in-situ stresses could be a crucial consideration affecting the possibility of fracture breakdown or the collapse of wellbore. These angles play a major role, especially in the estimation of the breakdown pressure in tectonically stressed formations where there is a high anisotropy of the in-situ stresses (McLellan 1996).
Wellbore can enlarge during drilling into weak bedding planes. There has been a case study in the Monterey field, where a vertical well is deviated into a weak formation. The bedding-plane slippage was the main cause to wellbore instability (Moinfar and Tajer 2013). Moreover, weak planes reduce the cohesive strength and internal friction angle which may result in the failure of wellbore. Moinfar and Tajer (2013) developed a model for proper well trajectory design to avoid wellbore instability problems. The model is consist of in-situ stress measurements, mechanical properties, pore pressure and mud weight. After the calculations have been made, a failure criterion was used to identify possible bedding slippage. Their results also show that the wellbore enlargement is significant when inclination angle is between 20° and 30° (Moinfar and Tajer 2013).

1.2.5 Modeling time dependent wellbore stability

It is also observed that wellbore stability is a time dependent. Exposure time, the time that drilling fluid contacts with formation, is an important factor, which is not accounted by the Kirsch model, a popular model used in wellbore stability modeling. Exposure time also plays big role in a rate of change of mechanical properties of the formation as the result of rock and drilling fluid interaction. Mohamed et al. (2019) showed that high density mud can impact the Young’s Modulus and unconfined compressive strength (UCS) of sandstone and limestone. Uniaxial compression and ultra-sonic pulse velocity experiments were used to calculate the mechanical properties of core samples. After exposing to the fluids in two days, the Young’s Modulus and UCS of their sandstone samples increased from 9.8 GPa to 19 GPa and from 31 MPa to 43 MPa, respectively. While for limestone samples, they observe a decrease from 29.81 GPa to 19.8 GPa and to 27 MPa to 18 MPa. However, the conventional wellbore stability models do not take into account exposure time during drilling operations. Hence, it is critical to develop a model to study the time dependent wellbore stability, and that is the one of the objectives of this thesis.
1.2.6 Effect of nonlinear elasticity

Most of the conventional models used in wellbore stability modeling assume that the rock is linear elastic. Linear elasticity is the behavior of rock that stress is linearly dependent on strain during the deformation. Also, the elasticity of rock is its ability to return to its original shape after acting force is being removed. However, rocks naturally behave nonlinearly due to the presence of pore space and complex composition of rock grain as well as natural fractures. To overcome wellbore instability issues, linear elastic stress analysis is frequently used to compute the stress state around a wellbore, due to its simplicity. But in reality, it underestimates fracture pressures (Zeynali 2012). There are several published researches studying this effect on the stress change around wellbore. Lozovyi et al. (2020) compared linear and nonlinear elasticity impacts on stress path during depletion. They used numerical modeling, which is based on finite-element method, to replicate the real data for nonlinear model taken from their real experimental data. While the formation data were taken from confined undrained triaxial tests. It is observed that the vertical strain during the depletion are considerably different between linear and nonlinear models. Their results suggest that linear elastic assumption could lead to errors in the estimation of stress path and strain. Therefore, it is important to account for this factor in wellbore stability modeling. However, the models accounted for nonlinear elasticity are often difficult to solve analytically. Hence, the development of numerical model can help to overcome this challenge. The model developed in this thesis can be modified to account for this nonlinearity.

1.2.7 Other challenges in wellbore stability modeling

There are other problems related to wellbore instability that the conventional models cannot handle such as the instability of the cement sheath during hydraulic fracture operations. If cement fails it impacts not only the drilling operations, but also affecting all stages during the lifecycle of the wellbore (Bui and Tutuncu 2013). The most prevalent failure in cement sheath is tensile failure. The reasons for cement failure are excessively high
pressure during hydraulic fracturing stimulation, contamination caused by drilling fluids and thermal-induced stress.

In addition wellbore is not in circular shape, and analytical models could not account for this geometry. During drilling operations wellbore hole can change its shape due to wellbore instability events, the friction caused by drilling string, and the erosion of rock by drilling fluid. Stresses around wellbore, erosion and washouts could change borehole’s size and shape as suggested by Kiran et al. (2019). They investigated the impact of wellbore geometry on several aspects such as drilling fluid loss, cuttings transport and cementing. In their study, wellbore geometry was evaluated by using caliper logs, and fluid dynamics was simulated by using commercial software. Also, a geomechanical model was used to determine the hoop stresses near the wellbore. Their results suggest that the hoop stress profile shows different patterns between elliptical and circular wellbores. Also, they observed that in an elliptical borehole, the bottomhole pressure to maintain wellbore stability is higher. However, this effect cannot be captured by conventional model utilizing the Kirsch solution because it assumes the circular shape of wellbore. Therefore, it is important to develop a numerical model to better estimate the stresses around wellbore for different well geometries.

In summary, the listed factors above are significantly important in calculating stress distribution and determining the proper mud weight. However, current conventional models are very limited and do not take into account the discussed factors which could result in the inaccurate determination of stress distribution around wellbore as well as the mud weight window. Moreover, such errors in calculation lead to the significant increase of economic expenses as well as nonproductive time during drilling. There are multiple method to address the wellbore instability, but the main focus is on adjusting the drilling fluid density. In this thesis, we developed a coupled model that composed of geomechanical, fluid-flow and heat transfer models to overcome the limitation of the conventional model and provide a better estimation of the mud weight.
1.3 Objectives and Scope

The objective of this thesis is to develop a coupled geomechanics and fluid flow model to address wellbore instability issues that accounts for flow-induced stress, thermal-induced stress as well as fluid and rock interaction. Specifically, a coupled fluid flow, heat transfer and geomechanical model is proposed to analyze wellbore stability during drilling operations. The scope of this thesis is to investigate the effect of transport parameters namely formation permeability and wettability as well as fluid and rock interaction on the time-dependent wellbore stability.

The main objectives of this thesis are:

• To formulate a coupled model for wellbore stability analysis

• To investigate the effect of temperature as well as fluid and rock interaction on the safe mud weight window

• To determine the effect of permeability and wettability on the safe mud weight window

1.4 Practical Applications

Developing the coupled model for wellbore stability analysis is important task to solve the wellbore instability issues when drilling more challenging environment such as high temperature and high pressure. The coupled model can be used to evaluate the effect of formation transport properties such as porosity and permeability on the stability of the wellbore that the conventional models cannot. The model can also be used to evaluate the integrity of the cement sheath to determine the maximum injection pressure during hydraulic fracturing operations to prevent the failure of the cement sheath.
CHAPTER 2
LITERATURE REVIEW

In this chapter, I provide a review of the conventional method to determine the mud weight as well as the recent development in coupled modeling in the literature. First, the Kirsch solution and procedure to determine mud weight are presented. Then, different failure criteria are summarized. Eventually, a brief literature review of the models that have been developed to model wellbore stability is presented.

2.1 Conventional Mud Weight Determination

Determining the Mud Weight Window (MWW) will provide suitable drilling fluid density to effectively drill the formations needed to reach the target zone. Theoretically, when drilling in conventional reservoirs without mechanical instabilities, the MWW can be determined by using the Kirsch solution. This determination needs to be accompanied by a safety factor that gives a straightforward and reliable calculation. To calculate the MWW, different mechanical factors should be taken into account. The most important ones are the behavior of rock under stress, deformation characteristics, tectonic activity, mechanical strength, failure characteristics, inclination and azimuth angles.

The method of determining the MWW has changed as the challenges in the industry get more complex. Many years ago, reservoirs were typically at shallow depths. Formations drilled were considered as a continuous rock, where the thermal effects were ignored, and linear elasticity was used to describe the behavior of most reservoirs (Liz-Losada and Alejano 2000). Basically, the MWW is the density range between pore and fracture pressures, which is not correlated with the inclination and the direction of the well. This means that the mud static density will always have to be above the pore pressure to prevent a kick and the ECD (Equivalent Circulation Density) below the fracture pressure in order to prevent mud losses (Charlez 1999). When the mud pressure is maintained between these two pressures,
the wellbore will maintain stable. The real task in determining the MWW is to properly calculate the stress around wellbore. Assuming a homogenous formation and an anisotropic stress field, where the horizontal stresses are denoted as $\sigma_h$ and $\sigma_H$ ($\sigma_H > \sigma_h$) and the vertical stress is denoted as $\sigma_v$. The induced fracture is typically parallel to $\sigma_H$ and the fracture pressure should not be greater than $\sigma_h$. At this stress, the rock will fail.

The most important controllable factor is the bottomhole pressure, or mud density. The supporting pressure offered by the drilling fluid pressure during either drilling, stimulation, and workover or production of a well, will determine the stress concentration near the wellbore. Because rock failure is dependent on the effective stress, the consequence for stability is highly dependent on whether and how rapidly fluid pressure penetrate the wellbore wall. That is not to say however, that high mud density or bottomhole pressure is always optimal for avoiding instability in a given well. In the absence of an efficient filter cake, such as in fractured formations, a rise in a bottomhole pressure may be detrimental to stability and can compromise other criteria, e.g., formation damage, differential sticking risk, mud properties, or hydraulic parameters (Tan and Willoughby 1993; McLellan 1996; Mohiuddin et al. 2001).

Maintaining wellbore stability over time is also important in wellbore stability modeling. Numerical simulation by Lin et al. (2019) showed that proper mud weight can increase wellbore stability time in fractured formations. At low densities 1.05 g/cm$^3$ failure could occur immediately after drilling. Densities from 1.05 g/cm$^3$ to 1.35 g/cm$^3$ could keep wellbore stable. However, due to drilling fluid interaction with the formation, the pore pressure and the strength reduction near wellbore result in wellbore failure. Higher density mud weigh 1.40 g/cm$^3$ could help to stabilize wellbore for 20 days.

2.1.1 In-situ stresses in wellbore coordinates

The determination of the in-situ stresses are important during wellbore stability analysis. The in-situ stress data consist of the in-situ horizontal and vertical stresses. Since the directions of the principal stresses are not always the same as of the directions of the coordinate system associated with the wellbore, or wellbore coordinates, we need to rotate
our coordinate system to obtain the stress tensor in the wellbore coordinates (Figure 2.1). To do that, first we need to rotate the about the \( z \)-axis by an angle \( \alpha_w \) from the vertical direction to the direction of the wellbore with the following rotation matrix:

\[
R_z(\alpha_w) = \begin{bmatrix}
\cos \alpha_w & \sin \alpha_w & 0 \\
-\sin \alpha_w & \cos \alpha_w & 0 \\
0 & 0 & 1
\end{bmatrix}
\] (2.1)

where \( R_z(\alpha_w) \) is the rotational matrix around \( z \)-axis an angle \( \alpha_w \).

Then, we need to rotate the new coordinate system around \( y \)-axis an angle \( I_w \) to the direction of the wellbore using following rotation matrix:

\[
R_y(I_w) = \begin{bmatrix}
\cos I_w & 0 & -\sin I_w \\
0 & 1 & 0 \\
\sin I_w & 0 & \cos I_w
\end{bmatrix}
\] (2.2)

The stress tensor in the wellbore coordinates can be obtain as
\[ \hat{\sigma} = R_y \cdot (R_z \cdot \sigma \cdot R_z^T) \cdot R_y^T = (R_y \cdot R_z) \cdot \sigma \cdot (R_z^T \cdot R_y^T) = (R_y \cdot R_z) \cdot \sigma \cdot (R_y \cdot R_z)^T = T \cdot \sigma \cdot T^T \]  
(2.3)

where \( T = R_y \cdot R_z \) is the transformation matrix defined as

\[
T = \begin{bmatrix}
\cos I_w & 0 & -\sin I_w \\
0 & 1 & 0 \\
\sin I_w & 0 & \cos I_w
\end{bmatrix}
\begin{bmatrix}
\cos \alpha_w & \sin \alpha_w & 0 \\
-\sin \alpha_w & \cos \alpha_w & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[ = \begin{bmatrix}
\cos I_w \cos \alpha_w & \cos I_w \sin \alpha_w & -\sin I_w \\
-\sin \alpha_w & \cos \alpha_w & 0 \\
\sin I_w \cos \alpha_w & \sin I_w \sin \alpha_w & \cos I_w
\end{bmatrix} \]  
(2.4)

and

\[
T^T = \begin{bmatrix}
\cos I_w \cos \alpha_w & -\sin \alpha_w & \sin I_w \cos \alpha_w \\
\cos I_w \sin \alpha_w & \cos \alpha_w & \sin I_w \sin \alpha_w \\
-\sin I_w & 0 & \cos I_w
\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{bmatrix}
\]

\[ = \begin{bmatrix}
\sin I_w \cos \alpha_w & \sin I_w \sin \alpha_w & -\sigma_v \sin I_w \\
-\sigma_H \sin \alpha_w & \sigma_h \cos \alpha_w & 0 \\
\sigma_H \sin I_w \cos \alpha_w & \sigma_h \sin I_w \sin \alpha_w & \sigma_v \cos I_w
\end{bmatrix} \cdot \begin{bmatrix}
\sigma_H & 0 & 0 \\
0 & \sigma_h & 0 \\
0 & 0 & \sigma_v
\end{bmatrix}
\]

Expanding 2.5 yields
\[
\begin{align*}
\sigma_{xx} &= \sigma_H \cos^2 I_w \cos^2 \alpha_w + \sigma_h \sin^2 \alpha_w \cos^2 I_w + \sigma_v \sin^2 I_w \\
\sigma_{yy} &= \sigma_H \sin^2 \alpha_w + \sigma_h \cos^2 \alpha_w \\
\sigma_{zz} &= \sigma_H \cos^2 \alpha_w \sin^2 I_w + \sigma_h \sin^2 I_w \sin^2 \alpha_w + \sigma_v \cos^2 I_w \\
\tau_{xy} &= -\sigma_H \sin \alpha_w \cos \alpha_w \cos I_w + \sigma_h \sin \alpha_w \cos \alpha_w \cos I_w \\
\tau_{yz} &= -\sigma_H \sin \alpha_w \cos \alpha_w \sin I_w + \sigma_h \sin \alpha_w \cos \alpha_w \sin I_w \\
\tau_{xz} &= \sigma_H \cos^2 \alpha_w \sin I_w \cos I_w + \sigma_h \sin^2 \alpha_w \sin I_w \cos I_w - \sigma_v \sin I_w \cos I_w 
\end{align*}
\]

(2.6)

or in vector form:

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{xz}
\end{bmatrix}
= \begin{bmatrix}
\cos^2 I_w \cos^2 \alpha_w & \sin^2 \alpha_w \cos^2 I_w & \sin^2 I_w \\
\sin^2 \alpha_w & \cos^2 \alpha_w & 0 \\
\cos^2 \alpha_w \sin^2 I_w & \sin^2 I_w \sin^2 \alpha_w & \cos^2 I_w \\
-\sin \alpha_w \cos \alpha_w \cos I_w & \sin \alpha_w \cos \alpha_w \cos I_w & 0 \\
-\sin \alpha_w \cos \alpha_w \sin I_w & \sin \alpha_w \cos \alpha_w \sin I_w & 0 \\
\cos^2 \alpha_w \sin I_w \cos I_w & \sin^2 \alpha_w \sin I_w \cos I_w & -\sin I_w \cos I_w
\end{bmatrix}
\begin{bmatrix}
\sigma_H \\
\sigma_h \\
\sigma_v
\end{bmatrix} 
\]

(2.7)

2.1.2 Governing equation

For wellbore integrity analysis, the deformation is a slow process comparing to the change of stress, hence, the acceleration term is ignored. Also, no body force is acting on the rock. The governing equations for stress around the wellbore can be written in cylindrical coordinates as

In \( r \)-direction

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0
\]

(2.8)

In \( \theta \)-direction

\[
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = 0
\]

(2.9)

In \( z \)-direction

\[
\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0
\]

(2.10)
It should be noted that in wellbore integrity, $\sigma_{rr}$ is often called radial stress, $\sigma_{\theta\theta}$ is called hoop stress, $\sigma_{zz}$ is called axial stress, and other stresses ($\sigma_{r\theta}$, $\sigma_{\theta z}$, $\sigma_{zr}$) are shear stresses. The most common solution of this governing equations, with assumptions of circular wellbore, linear elastic, homogeneous formation, no fluid and rock interaction, no thermal stress, and constant pore pressure, is the solution by Gustav Kirsch in 1898.

2.1.3 Kirsch Solution

The solution of the above equations for a circular wellbore with internal pressure ($p_w$) located in the in-situ stress field with vertical stress ($\sigma_v$), and horizontal stresses ($\sigma_H$ and $\sigma_h$) was provided by Kirsch (1898) as

$$\begin{align*}
\sigma_{rr} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} \left( 1 - \frac{r_w^2}{r^2} \right) + \frac{\sigma_{xx} - \sigma_{yy}}{2} \left( 1 + 3 \frac{r_w^4}{r^4} - 4 \frac{r_w^2}{r^2} \right) \cos 2\theta \\
&\quad + \tau_{xy} \left( 1 + 3 \frac{r_w^4}{r^4} - 4 \frac{r_w^2}{r^2} \right) \sin 2\theta + p_w \frac{r_w^2}{r^2} \\
\sigma_{\theta\theta} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} \left( 1 + \frac{r_w^2}{r^2} \right) - \frac{\sigma_{xx} - \sigma_{yy}}{2} \left( 1 + 3 \frac{r_w^4}{r^4} \right) \cos 2\theta - \tau_{xy} \left( 1 + 3 \frac{r_w^4}{r^4} \right) \sin 2\theta - p_w \frac{r_w^2}{r^2} \\
\tilde{\sigma}_{zz} &= \sigma_{zz} - \nu \left[ 2 \left( \sigma_{xx} - \sigma_{yy} \right) \frac{r_w^2}{r^2} \sin 2\theta + 4 \tau_{xy} \frac{r_w^2}{r^2} \cos 2\theta \right] \\
\tau_{r\theta} &= \frac{\sigma_{xx} - \sigma_{yy}}{2} \left( 1 - 3 \frac{r_w^4}{r^4} + 2 \frac{r_w^2}{r^2} \right) \sin 2\theta + \tau_{xy} \left( 1 - 3 \frac{r_w^4}{r^4} + 2 \frac{r_w^2}{r^2} \right) \cos 2\theta \\
\tau_{\theta z} &= \left( -\tau_{xz} \sin \theta + \tau_{yz} \cos \theta \right) \left( 1 + \frac{r_w^2}{r^2} \right) \\
\tau_{rz} &= \left( -\tau_{xz} \cos \theta + \tau_{yz} \sin \theta \right) \left( 1 - \frac{r_w^2}{r^2} \right)
\end{align*}$$

(2.11)

where $p_w$ is wellbore pressure, $r_w$ is wellbore radius, $r$ is radial distance, $\theta$ is angle with the respect to the $x$-axis of the wellbore coordinate, and $\nu$ is Poisson’s Ratio.

$$p_w = \rho_m g \text{TVD}$$

(2.12)

where $\rho_m$ is mud weight, TVD is true vertical depth, and $g$ is gravitational acceleration.

At the bore wall: $r = r_w$
\[
\begin{align*}
\sigma_{rr} &= p_w \\
\sigma_{\theta \theta} &= (\sigma_{xx} + \sigma_{yy}) - 2 (\sigma_{xx} - \sigma_{yy}) \cos 2\theta - 2\tau_{xy} \sin 2\theta - p_w \\
\sigma_{zz} &= \sigma_{zz} - \nu [2 (\sigma_{xx} - \sigma_{yy}) \sin 2\theta + 4\tau_{xy} \cos 2\theta] \\
\tau_{r\theta} &= 0 \\
\tau_{\theta z} &= 2 (\tau_{yz} \cos \theta - \tau_{xz} \sin \theta) \\
\tau_{rz} &= 0
\end{align*}
\]

(2.13)

Hence, the stress tensor at any point on the bore wall in the cylindrical coordinates has the following form:

\[
\boldsymbol{\sigma}_{r\theta z} = \begin{bmatrix}
\sigma_{rr} & 0 & 0 \\
0 & \sigma_{\theta \theta} & \sigma_{\theta z} \\
0 & \sigma_{z\theta} & \sigma_{zz}
\end{bmatrix}
\]

(2.14)

To determine if a point on the bore wall fails or not, principal stresses should be obtained. Although, most failure criteria are formulated for cartesian coordinates, we do not need to convert stress tensor from cylindrical coordinates to cartesian coordinates, because the principal stresses do not change with the change of the coordinates. The principal stress \(\sigma_e\) can be obtained by solving the following characteristic equation

\[
\det (\boldsymbol{\sigma}_{r\theta z} - \sigma_e \mathbf{I}) = \begin{vmatrix}
\sigma_{rr} - \sigma_e & 0 & 0 \\
0 & \sigma_{\theta \theta} - \sigma_e & \sigma_{\theta z} \\
0 & \sigma_{z\theta} & \sigma_{zz} - \sigma_e
\end{vmatrix} = 0
\]

(2.15)

or

\[
(\sigma_{rr} - \sigma_e) \left[ \sigma_e^2 - (\sigma_{zz} + \sigma_{\theta \theta}) \sigma_e - \sigma_{z\theta}^2 + \sigma_{\theta \theta} \sigma_{zz} \right] = 0
\]

(2.16)

The characteristic equation has 3 solutions:

\[
\begin{align*}
\sigma_1 &= \sigma_{rr} \\
\sigma_2 &= \frac{\sigma_{\theta \theta} + \sigma_{zz}}{2} \sqrt{\frac{\sigma_{\theta \theta} - \sigma_{zz}}{2} + \tau_{\theta z}^2} \\
\sigma_3 &= \frac{\sigma_{\theta \theta} + \sigma_{zz}}{2} \sqrt{\frac{\sigma_{\theta \theta} - \sigma_{zz}}{2} + \tau_{\theta z}^2}
\end{align*}
\]

(2.17)
Knowing the order of principal stresses is often required when checking rock failure. Hence, the maximum and minimum principal stresses are often calculated.

\[
\begin{align*}
\sigma_{\text{max}} &= \max (\sigma_1, \sigma_2, \sigma_3) \\
\sigma_{\text{min}} &= \min (\sigma_1, \sigma_2, \sigma_3)
\end{align*}
\]  

(2.18)

From the principal stresses, we can use the failure criteria to determine if the rock element fails or not.

2.1.4 Failure criteria

Determining when the rock fails is the main focus of wellbore stability studies. After knowing the stresses, the selection of a proper failure criterion, to determine when the rock fails, is needed. Also, failure differs for ductile and brittle rock formations. For example, during uniaxial tests ductile rock fails when plastic deformation starts. In other words, when normal stress reaches the yield strength of rock, the rock fails. For brittle rocks, the failure happens at fracture initiation. However, for other complex cases estimating point of failure can be difficult. Therefore, it is necessary to choose an appropriate failure criterion.

2.1.4.1 Tensile failure

The failure of rock due to tension is tensile failure. Tensile failure typically occurs when the tension or tensile stress exceeds the tensile strength of the rock. It should be noted that the hydraulic fracture will form in the direction perpendicular to the minimum horizontal in-situ stress ($\sigma_h$). Rock is under tension when at least one of the principal stresses is negative. If all principal stresses are positive, rock is only under compression, and tensile failure will not occur. When at least one of the principal stresses is negative, the rock element is under highest tension at the direction of the minimum principal stress ($\sigma_{\text{min}}$). Hence, the failure of rock occurs when

\[ [\sigma_T] + \sigma_{\text{min}} \leq 0 \]  

(2.19)

where $[\sigma_T]$ is the tensile strength of rock.
It should be noted that in calculation, tensile strength is a positive number \( [\sigma_T] > 0 \). In case no experimental data is available, we often use \( [\sigma_T] = 0 \).

2.1.4.2 Compressive failure

Compressive failure, or shear failure, occurs when compressive stress exceeds the compressive strength of the rock. The most common method to determine to what extend any rock can be deformed is triaxial experiment. Triaxial test helps to replicate rock under reservoir conditions. In this test, two different compressive stresses applied. One of them is axial stress and the second stress type is confining stress. Typically, axial stress is increasing, while confining stresses remain the constant, because increasing confining stresses make the rock harder to break. Under the range of stresses applied the rock could behave differently depending on mineralogy, microstructure and temperature Paterson (1979). There are several models to determine the failure of the rock under compression. In this section, various failure models are discussed.

*Mohr-Coulomb criterion*

The Mohr-Coulomb criterion takes into account only the maximum and minimum principal stresses and assumes that the intermediate stress has no effect on the rock failure. The Mohr-Coulomb criterion is the simplest and most used failure criterion in the petroleum industry. The main assumption made by the Mohr-Coulomb model is that the failure occurs along a plane due to shear stress. Shear stress equals to the sum of cohesive strength and the product of internal friction coefficient and normal stress. The cohesive strength and internal friction coefficient are usually determined experimentally. The rock fails in shear in case of circle touching failure envelop. It is often appropriate to determine the failure of the rock using the Mohr-Coulomb failure criterion when \( \sigma_2 \) is slightly greater than or equal to \( \sigma_3 \). This makes it highly applicable in bi-axial experiments. The only drawback of the Mohr-Coulomb criterion is that it assumes the intermediate principal stress has zero influence on the rock failure. Using the maximum and minimum effective stresses, the normal and shear stress acting on a cross section can be obtained as
\[
\begin{align*}
\sigma_n &= \frac{\sigma_1' + \sigma_3'}{2} + \frac{\sigma_1' - \sigma_3'}{2} \cos 2\theta \\
\tau &= -\frac{\sigma_1' - \sigma_3'}{2} \sin 2\theta
\end{align*}
\] (2.20)

where \( \theta \) is angle with respect to the direction of \( \sigma_3 \).

Hence, the Mohr’s circle has the following form:

\[
\left( \sigma_n - \frac{\sigma_1' + \sigma_3'}{2} \right)^2 + \tau^2 = \left( \frac{\sigma_1' - \sigma_3'}{2} \right)^2
\] (2.21)

If Mohr’s circle touches the failure envelope the rock fails. The equation of the straight line is:

\[
\tau = C_o + \sigma_n \mu_i = C_o + \sigma_n \tan \varphi_i
\] (2.22)

where \( \mu_i \) is the internal friction coefficient or friction coefficient, \( \varphi_i \) is the internal friction angle, and \( C_o \) is the cohesive strength, which is the intercept of the failure envelope with the vertical axis. These parameters are properties of a rock, hence, we can estimate these parameters based on lithology when laboratory or log data is not available.

It can be verified that the rock will fail if

\[
\sigma_1 > UCS + \frac{1 + \sin \varphi_i}{1 - \sin \varphi_i} (\sigma_3 - \alpha p_p) + \alpha p_p
\] (2.23)

where \( p_p \) is pore pressure and \( \alpha \) is Biot’s coefficient.

**Hoek-Brown criterion**

The Hoek-Brown criterion is recently used in engineering applications as its ability to show non-linear yielding for different rocks. However, some failure criterions could give misleading results. For example, using the Hoek-Brown criterion for greywacke in New Zealand can give erroneous results. Read et al. (1999) investigated issues related to determining the rock strength. Laboratory tests showed higher results of the rock material constant for greywacke sandstones than by the Hoek-Brown criterion. Also, using directly the Hoek-Brown criterion when estimating the rock strength of intact greywacke sandstones can give
unrealistic results (Read et al. 1999).

The Hoek-Brown:

\[ \sigma'_1 > \sigma'_3 + C_o \sqrt{m \frac{\sigma'_3}{C_o} + s} \]  \hspace{1cm} (2.24)

where \( m \) and \( s \) are dimensionless empirical constants, and \( m \) depends on rock type, and \( s \) depends on the characteristics of the rock. For intact rocks, \( s = 1 \), and for granulated rocks, \( s = 0 \) (Tutuncu 2019).

Table 2.1: The Hoek-Brown criterion parameters (Tutuncu 2019)

<table>
<thead>
<tr>
<th>Typical range of ( m )</th>
<th>Types of rocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5 &lt; m &lt; 8 )</td>
<td>Carbonate rocks with well developed crystal cleavage</td>
</tr>
<tr>
<td></td>
<td>(dolomite, limestone, marble)</td>
</tr>
<tr>
<td>( 4 &lt; m &lt; 10 )</td>
<td>Lithified argillaceous rocks</td>
</tr>
<tr>
<td></td>
<td>(mudstone, siltstone, shale, slate)</td>
</tr>
<tr>
<td>( 15 &lt; m &lt; 24 )</td>
<td>Arenaceous rocks with strong crystals and poorly developed crystal cleavage (sandstone, quartzite)</td>
</tr>
<tr>
<td>( 16 &lt; m &lt; 19 )</td>
<td>Fine-grained polyminallic igneous crystalline rocks</td>
</tr>
<tr>
<td></td>
<td>(andesite, dolerite, diabase, rhyolite)</td>
</tr>
<tr>
<td>( 22 &lt; m &lt; 33 )</td>
<td>Coarse-grained polyminallic igneous and metamorphic rocks</td>
</tr>
<tr>
<td></td>
<td>(amphibolite, gabbro, gneiss, granite, norite, quartz-diorite)</td>
</tr>
</tbody>
</table>

**Mogi-Coulomb criterion**

The Mohr’s assumption is that only the maximum and minimum stresses cause failure of a rock. However, the evidence accumulating over couple of years showed that the intermediate stress has effect on rock’s strength. The Mogi’s assumption takes into account this intermediate stress. Al-Ajmi and Zimmerman (2007) compared the Mogi-Coulomb and the Mohr-Coulomb to analyze wellbore stability for vertical wells. Also, they calculated
overbalance mud weight for both the Mohr-Coulomb and the Mogi-Coulomb criteria. Their results suggest that the minimum overbalance pressure is different for both criteria. Thus, authors showed that taking into account the intermediate stress can lead to different results in specific case. The Mogi-Coulomb criterion can be written as

\[ \tau_{oct} = a + b \frac{\sigma'_1 + \sigma'_3}{2} \]  
(2.25)

where \( \tau_{oct} \) is the octahedral shear stress, and \( a \) and \( b \) are the Coulomb strength parameters defined as

\[ \tau_{oct} = \frac{1}{3} \sqrt{\left( \sigma'_1 - \sigma'_2 \right)^2 + \left( \sigma'_2 - \sigma'_3 \right)^2 + \left( \sigma'_3 - \sigma'_1 \right)^2} \]  
(2.26)

\[ a = \frac{2\sqrt{2}}{3} C_o \cos \phi_i \]  
(2.27)

\[ b = \frac{2\sqrt{2}}{3} \sin \phi_i \]  
(2.28)

**Drucker-Prager criterion**

The Drucker-Prager criterion is also one of the commonly used rock failure criteria in wellbore stability analyses. The Drucker-Prager criterion incorporates the intermediate principal stress as well as the major and minor principal stresses. If two criteria are both fit to the same set of triaxial compression strength data, then the Drucker-Prager criterion will predict higher rock strength than the Mohr-Coulomb criterion for all stress states other than the triaxial compression. The Drucker-Prager model is also called the extended von Mises yield criterion and was initially developed for soil mechanics

\[ \sqrt{I_2^d} > A + BI_1 \]  
(2.29)

where \( A \) and \( B \) are material constants determined from experiments; \( I_1^d \) is the first invariant of the deviatoric stress tensor; and \( I_2^d \) is the second invariant of the deviatoric stress tensor. It should be emphasized that the effective stresses are used.

\[ I_1^d = \sigma'_1 + \sigma'_2 + \sigma'_3 \]  
(2.30)
\[ I_2^d = \frac{(\sigma'_1 - \sigma'_2)^2 + (\sigma'_2 - \sigma'_3)^2 + (\sigma'_3 - \sigma'_1)^2}{6} \quad (2.31) \]

At the failure under uniaxial tension, \( I_1^d = [\sigma_T] \) and \( I_2^d = \frac{[\sigma_T]}{3} \), applying the Drucker-Prager failure criterion yields

\[ \frac{1}{\sqrt{3}} [\sigma_T] = A - B[\sigma_T] \quad (2.32) \]

Similarly, at the failure under uniaxial compression, \( I_1^d = UCS \) and \( I_2^d = \frac{UCS}{3} \), applying the Drucker-Prager failure criterion yields

\[ \frac{1}{\sqrt{3}} UCS = A + B \times UCS \quad (2.33) \]

From Equation 2.32 and 2.33, the model parameters \( A \) and \( B \) can be estimated from tensile stress and \( UCS \) as

\[ \begin{cases} 
A = \frac{2}{\sqrt{3}} \frac{UCS[\sigma_T]}{UCS + [\sigma_T]} \\
B = \frac{1}{\sqrt{3}} \frac{UCS - [\sigma_T]}{UCS + [\sigma_T]} 
\end{cases} \quad (2.34) \]

The model parameters \( A \) and \( B \) can also be determined from the friction angle and cohesive strength for three cases, the circumscribed, middle circumscribed, and inscribed the Drucker-Prager models. The material constants of this criterion are determined as

The circumscribed Drucker-Prager:

\[ \begin{cases} 
A = \frac{6C_o \cos \varphi_i}{\sqrt{3} (3 - \sin \varphi_i)} \\
B = \frac{2 \sin \varphi_i}{\sqrt{3} (3 - \sin \varphi_i)} 
\end{cases} \quad (2.35) \]

The middle circumscribed Drucker-Prager:

\[ \begin{cases} 
A = \frac{6C_o \cos \varphi_i}{\sqrt{3} (3 + \sin \varphi_i)} \\
B = \frac{2 \sin \varphi_i}{\sqrt{3} (3 + \sin \varphi_i)} 
\end{cases} \quad (2.36) \]

The inscribed Drucker-Prager:
\[
\begin{array}{l}
A = \frac{3C_0 \cos \varphi_i}{\sqrt{9 + 3 \sin^2 \varphi_i}} \\
B = \frac{3 \sin \varphi_i}{\sqrt{9 + 3 \sin^2 \varphi_i}}
\end{array}
\] (2.37)

2.2 Coupled Modeling

Reservoir modeling is one of the most predominate fields to use coupled modeling. Coupled modeling is a method that combines two or more different processes to make an overall more accurate predictions or calculations in petroleum engineering. In other words, coupling refers to communication and interchange of information between models. There are three methods which used to couple with reservoir simulation such as “one-way”, “two-way” and “full” coupling methods. In the first methods, fluid flow model solved separately. Then, after estimation of pressure, pressure is sent to geomechanical model to calculate deformation. However, the geomechanical model does not send any feedback to fluid flow model and iterations are not necessary between two processes. In “two-way” coupling, the deformation obtained by geomechanical model is sent to fluid flow model in order to estimate change in porosity, and iterations is used until achieving convergence. “Fully” coupling method is a method when two models or processes solved simultaneously. Some authors suggest that “full” coupling methods theoretically obtain the same solution as in “two-way” coupling iterations made until convergence. Thus, two methods satisfied simultaneously at every time step. “Two-way” method is used by iterations between systems, while “full” coupling method solved all equations simultaneously. In case of limiting random access memory, “two-way” coupling is more preferable because the construction and storage of the full matrix in fully coupled modeling require a lot more memory. Gu et al. (2011) examined effect of the connection between reservoir flow and reservoir deformation using different coupling method approaches. Their results show that fluid flow equations should be solved by “two-way” and “fully” coupling methods to take into account geomechanical aspects. In a field, due to complexity of formations these coupling methods should be used. If deformation is not
determined, “two-way” or “full” coupling methods should be used to avoid significant errors to response of fluid flow pattern.

Another way coupling is used in literature is for production and determine methane-hydrates production. The numerical simulation combines the results produced by a methane-hydrate production simulation and a 3D finite-element geomechanics simulator (Qiu et al. 2015). When these two simulations were combined to create a model to determine the potential risk related to well integrity with respect to hydrate precipitant. This process could possibly be used to determine sand transport from the reservoir to the wellbore. The coupled modeling would be important to predict and mitigate issues during production, since sand production could cause damage to the wellbore, tubing, flow lines, and separator.

Another way that coupled modeling is used, found in literature, is to calculate modified Inflow Performance Relationship, IPR, curves. The original IPR curves do not take into account near well bore reservoir effects, but with the use of the coupled modeling, original IPR models can be used with reservoir simulation studies (Cao et al. 2015). This would make the IPR a better representation of real industry data after each time step. Hence, reducing errors on each time step.

Coupled modeling has been also used in sand production modeling. Wang et al. (2004) investigated the effect of sand production on wellbore stability using fully coupled reservoir-geomechanical model. Governing equations are solved by Galerkin’s method with Newton-Raphson iterative approach. For failure criterion, the Mohr-Coulomb criterion was considered to determine the failure of the rock. Their results show that sand production is strongly related to geomechanics. Sand activity treated as erosion in their study. Thus, as erosion activity occurs, porosity increases, and this leads to decreasing of rock strength. The reduction in strength gradually increases plastic deformation, which will finally result in well failure.

One method to use coupled modeling for well integrity is by combining two models of production through the well and how it affects on the aging materials in the well such as tubing, casing, cement, and other well components (Guen et al. 2012). Coupled modeling for fluid
flow, heat, and geomechanics showed that the increase of temperature in the cement causes pressure to increase due to thermal pressurization. This temperature and pressure change can cause non-linear mechanical responses, that are dependent on the thermal expansion (Rutqvist et al. 2018).
CHAPTER 3
MODEL FORMULATION

In this study, a coupled model is proposed to model the variation of the stress around the wellbore accounting for not only mechanical mechanisms but also the variation of formation mechanical properties and transport properties. The coupled model is formulated from conservation of mass, momentum, and energy. First, we formulate the mechanical model to describe the deformation of formation using conservation of momentum. Then, we developed the total pressure equation simulating the transport of two phases, oil and water, using conservation of mass. Finally, we present the heat transfer equation in the wellbore using conservation of energy.

3.1 Geomechanical Model

The total strain tensor \( \varepsilon_t \) is the summation of elastic strain tensor \( \varepsilon_e \) and the thermal strain tensor \( \varepsilon_T \):

\[
\varepsilon_t = \varepsilon_e + \varepsilon_T \tag{3.1}
\]

In the cylindrical coordinates, the elastic strain tensor can be written as

\[
\varepsilon_e = \begin{bmatrix}
\frac{\partial u_r}{\partial r} & \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) & \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_\theta}{\partial z} \right) \\
\frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) & \frac{u_r}{r} + \frac{1}{2} \frac{\partial u_\theta}{\partial \theta} & \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) \\
\frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_\theta}{\partial z} \right) & \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) & \frac{\partial u_z}{\partial z}
\end{bmatrix} \tag{3.2}
\]

and the linear isotropic thermal strain tensor has the following form:

\[
\varepsilon_T = \beta \Delta T \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \tag{3.3}
\]
where \( u_r, u_\theta \) and \( u_z \) are the radial, tangential and axial displacement; \( \Delta T \) is the change of temperature; \( \beta \) is the coefficient of thermal expansion.

Considering a 2-D problem, planar stress, the elastic and thermal strain tensors can be written as

\[
\varepsilon_e = \begin{bmatrix}
\frac{\partial u_r}{\partial r} & \frac{1}{2} \left( \frac{1}{\rho} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{\rho} \right) & \frac{u_r}{\rho} + \frac{1}{\rho} \frac{\partial u_\theta}{\partial \theta} \\
\frac{1}{2} \left( \frac{1}{\rho} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{\rho} \right) & \frac{\partial u_r}{\partial \theta} & \frac{1}{\rho} \frac{\partial u_\theta}{\partial \theta} \\
0 & 0 & 0
\end{bmatrix}
\]  

(3.4)

and

\[
\varepsilon_T = \begin{bmatrix}
\beta \Delta T & 0 \\
0 & \beta \Delta T
\end{bmatrix}
\]  

(3.5)

Hence, the total strain tensor can be written as

\[
\varepsilon_t = \begin{bmatrix}
\frac{\partial u_r}{\partial r} + \frac{\beta \Delta T}{\rho} & \frac{1}{2} \left( \frac{1}{\rho} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{\rho} \right) & \frac{u_r}{\rho} + \frac{1}{\rho} \frac{\partial u_\theta}{\partial \theta} \\
\frac{1}{2} \left( \frac{1}{\rho} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{\rho} \right) & \frac{\partial u_r}{\partial \theta} + \frac{\beta \Delta T}{\rho} & \frac{1}{\rho} \frac{\partial u_\theta}{\partial \theta} + \beta \Delta T \\
0 & 0 & 0
\end{bmatrix}
\]  

(3.6)

Substituting Equation 3.6 into the constitutive model

\[
\begin{bmatrix}
\sigma'_{rr} \\
\sigma'_{\theta\theta} \\
\sigma'_{r\theta}
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{21} & C_{22} & 0 \\
0 & 0 & C_{44}
\end{bmatrix}\begin{bmatrix}
\varepsilon_{rr} \\
\varepsilon_{\theta\theta} \\
\varepsilon_{r\theta}
\end{bmatrix}
\]  

(3.7)

we obtain the equation to determine effective stress from the displacement and temperature.

\[
\begin{bmatrix}
\sigma'_{rr} \\
\sigma'_{\theta\theta} \\
\sigma'_{r\theta}
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{21} & C_{22} & 0 \\
0 & 0 & C_{44}
\end{bmatrix}\begin{bmatrix}
\frac{\partial u_r}{\partial r} + \frac{\beta \Delta T}{\rho} \\
\frac{u_r}{\rho} + \frac{1}{\rho} \frac{\partial u_\theta}{\partial \theta} + \beta \Delta T \\
\frac{1}{2} \left( \frac{1}{\rho} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{\rho} \right)
\end{bmatrix}
\]  

(3.8)

where \( C_{ij} \) is the elastic coefficient and \( \sigma'_{rr}, \sigma'_{\theta\theta}, \) and \( \sigma'_{r\theta} \) are effective stresses in the cylindrical coordinates.
In this thesis, it is considered that the rock formation contains two phases, oil and water. Because of two phases, oil and water, are presented in the rock matrix, the effective stress in the rock matrix is calculated from the total stress as (Bui and Tutuncu 2018b)

\[ \sigma' = \sigma - \alpha (S_w p_w + S_o p_o) = \sigma - \alpha [S_w p_w + S_o (p_w + p_{cwo})] = \sigma - \alpha (p_w + S_o p_{cwo}) \]  

(3.9)

where \( p_{cwo} \) is the water-oil capillary pressure; \( S_w \) and \( S_o \) are water and oil saturation of the rock; \( \alpha \) is the Biot tensor. The Biot coefficient \((\alpha)\) is often direction dependent. Hence, using the tensor form helps to account for the anisotropy of the formation. Using this concept, the effective stress tensor can be written as

\[
\begin{align*}
\sigma_{rr} &= C_{11} \left( \frac{\partial u_r}{\partial r} + \beta \Delta T \right) + C_{12} \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \beta \Delta T \right) + \alpha_{rr} (p_w + S_o p_{cwo}) \\
\sigma_{\theta\theta} &= C_{21} \left( \frac{\partial u_r}{\partial r} + \beta \Delta T \right) + C_{22} \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \beta \Delta T \right) + \alpha_{\theta\theta} (p_w + S_o p_{cwo}) \\
\sigma_{r\theta} &= \frac{C_{44}}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) + \alpha_{r\theta} (p_w + S_o p_{cwo})
\end{align*}
\]

(3.10)

For wellbore integrity analysis, the deformation is a slow process comparing to the change of stress, hence, the acceleration term can be ignored. Also, no body force is acting on the rock. The governing equations for stress around the wellbore can be written in the cylindrical coordinates as

In \( r \)-direction

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0
\]

(3.11)

In \( \theta \)-direction

\[
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = 0
\]

(3.12)

Substituting Equation 3.10 into Equations 3.11 and 3.12, we obtain:
Expanding Equation 3.13 we obtain:

\[
\frac{\partial}{\partial r} \left[ C_{11} \left( \frac{\partial u_r}{\partial r} + \beta \Delta T \right) + C_{12} \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \beta \Delta T \right) + \alpha_{rr} \left( p_w + S_{o_p_{cuo}} \right) \right] + \frac{1}{r} \frac{\partial}{\partial \theta} \left[ C_{44} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) + \alpha_{r\theta} \left( p_w + S_{o_p_{cuo}} \right) \right] = 0
\]

and

\[
\frac{1}{r} \frac{\partial}{\partial \theta} \left[ C_{21} \left( \frac{\partial u_r}{\partial r} + \beta \Delta T \right) + C_{22} \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \beta \Delta T \right) + \alpha_{\theta \theta} \left( p_w + S_{o_p_{cuo}} \right) \right] = 0
\]

Expanding Equation 3.13 we obtain:

\[
C_{11} \frac{\partial^2 u_r}{\partial r^2} + C_{12} \frac{\partial u_r}{\partial r} \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} \right) + \alpha_{rr} \frac{\partial}{\partial r} \left( p_w + S_{o_p_{cuo}} \right) + \frac{C_{44}}{2r} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) + \frac{\alpha_{r\theta}}{r} \frac{\partial}{\partial \theta} \left( p_w + S_{o_p_{cuo}} \right) + \beta \left( C_{11} + C_{12} \right) \frac{\partial \Delta T}{\partial r} + C_{11} \frac{\partial u_r}{\partial r} + C_{12} \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} \right) + \frac{\alpha_{r\theta}}{r} \left( p_w + S_{o_p_{cuo}} \right) - C_{21} \frac{\partial u_r}{\partial r} - C_{22} \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} \right) - \frac{\alpha_{\theta \theta}}{r} \left( p_w + S_{o_p_{cuo}} \right) + \frac{\beta \Delta T}{r} \left( C_{11} + C_{12} - C_{21} - C_{22} \right) = 0
\]

or

\[
C_{11} \frac{\partial^2 u_r}{\partial r^2} + \frac{C_{12}}{r} \frac{\partial u_r}{\partial r} - \frac{C_{12}}{r^2} u_r + \frac{C_{12}}{r} \frac{\partial^2 u_\theta}{\partial r^2} - \frac{C_{12}}{r} \frac{\partial u_\theta}{\partial r} - \frac{C_{12}}{r^2} \frac{\partial u_\theta}{\partial r} + \frac{C_{44}}{2r} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{C_{44}}{2r} \frac{\partial u_\theta}{\partial r} + C_{21} \frac{\partial u_r}{\partial r} + C_{22} \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - C_{21} \frac{\partial u_r}{\partial r} - C_{22} \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \alpha_{rr} \frac{\partial}{\partial r} \left( p_w + S_{o_p_{cuo}} \right) + \frac{\alpha_{r\theta}}{r} \frac{\partial}{\partial \theta} \left( p_w + S_{o_p_{cuo}} \right) + \frac{\alpha_{rr} - \alpha_{\theta \theta}}{r} \left( p_w + S_{o_p_{cuo}} \right)
\]
Further simplifying Equation 3.16 yields the geomechanical governing equation in \( r \)-direction:

\[
\begin{align*}
&+ \beta (C_{11} + C_{12}) \frac{\partial \Delta T}{\partial r} + \frac{\beta \Delta T}{r} (C_{11} + C_{12} - C_{21} - C_{22}) = 0 \quad (3.16)
\end{align*}
\]

It should be noted that \( \frac{\partial u_\theta}{\partial z} \approx 0 \), in other words, \( u_\theta \) is only a function of \( r \) and \( \theta \) not \( z \) or the wellbore is deformed but not twisted.

Similarly, after expanding Equation 3.13, we obtain:

\[
\begin{align*}
&\frac{C_{44}}{2} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) + \frac{C_{21}}{r} \frac{\partial}{\partial \theta} \frac{\partial u_r}{\partial r} \\
&+ \frac{C_{22}}{r} \frac{\partial}{\partial \theta} \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} \right) + \frac{C_{44}}{r} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) + \frac{\alpha_r}{r} \frac{\partial}{\partial r} (p_w + S_o P_{cw0}) + \frac{2 \alpha_r}{r} (p_w + S_o P_{cw0}) + \frac{1}{r} \frac{\partial}{\partial \theta} \alpha_\theta (p_w + S_o P_{cw0}) = 0 \quad (3.18)
\end{align*}
\]

or

\[
\begin{align*}
&\frac{2 C_{21}}{2 r} \frac{\partial^2 u_r}{\partial r \partial \theta} - \frac{2 C_{44}}{2 r} \frac{\partial u_r}{\partial r} + \frac{2 C_{44}}{2 r} \frac{\partial^2 u_\theta}{\partial \theta^2} - \frac{C_{44}}{2 r} \frac{\partial u_\theta}{\partial r} + \frac{C_{44}}{2 r} \frac{\partial^2 u_\theta}{\partial \theta^2} \\
&+ \frac{C_{22}}{r} \frac{\partial^2 u_r}{\partial r \partial \theta} + \frac{C_{22}}{2 r} \frac{\partial u_r}{\partial \theta} + \frac{C_{44}}{2 r} \frac{\partial^2 u_\theta}{\partial r \partial \theta} + \frac{C_{44}}{r} \frac{\partial u_r}{\partial \theta} + \frac{C_{44}}{r} \frac{\partial u_\theta}{\partial r} - \frac{C_{44}}{2 r} \frac{\partial^2 u_\theta}{\partial \theta^2} \\
&+ \frac{\alpha_r}{r} \frac{\partial}{\partial r} (p_w + S_o P_{cw0}) + \frac{2 \alpha_r}{r} (p_w + S_o P_{cw0}) + \frac{1}{r} \frac{\partial}{\partial \theta} \alpha_\theta (p_w + S_o P_{cw0}) = 0 \quad (3.19)
\end{align*}
\]

Rearranging Equation 3.19 yields the geomechanical governing equation in \( \theta \)-direction

\[
\begin{align*}
&\frac{C_{44}}{2 r} + \frac{2 C_{21}}{2 r} \frac{\partial^2 u_r}{\partial r \partial \theta} + \frac{2 C_{44}}{2 r} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{C_{22}}{2 r} \frac{\partial^2 u_\theta}{\partial r \partial \theta} + \frac{C_{44}}{2 r} \frac{\partial u_r}{\partial \theta} + \frac{2 C_{22}}{2 r} \frac{\partial u_r}{\partial r} - \frac{C_{44}}{2 r} \frac{\partial^2 u_\theta}{\partial \theta^2} \\
&+ \frac{\alpha_r}{r} \frac{\partial}{\partial r} (p_w + S_o P_{cw0}) + \frac{2 \alpha_r}{r} (p_w + S_o P_{cw0}) + \frac{1}{r} \frac{\partial}{\partial \theta} \alpha_\theta (p_w + S_o P_{cw0}) = 0 \quad (3.20)
\end{align*}
\]
For simplicity, we assume that the Biot’s coefficient is not directional dependent

\[
\begin{cases}
\alpha_{rr} = \alpha_{\theta\theta} = \alpha \\
\alpha_{r\theta} = \alpha_{\theta r} = 0
\end{cases}
\] (3.21)

Equations 3.19 and 3.20 can be simplified as

\[
C_{11} \frac{\partial^2 u_r}{\partial r^2} + \frac{C_{44}}{2r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{2C_{12} + C_{44}}{2r} \frac{\partial^2 u_\theta}{\partial r \partial \theta} + \frac{C_{11} + C_{12} - C_{21}}{r} \frac{\partial u_r}{\partial r} - \frac{2C_{22} + C_{44}}{2r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{C_{22}}{r^2} u_r + \alpha \frac{\partial}{\partial r} (p_w + S_o p_{cw0}) + \\
\beta \left(C_{11} + C_{12}\right) \frac{\partial \Delta T}{\partial r} + \frac{\beta \Delta T}{r} \left(C_{11} + C_{12} - C_{21} - C_{22}\right) = 0
\] (3.22)

and

\[
\frac{C_{44} + 2C_{21}}{2r} \frac{\partial^2 u_r}{\partial r \partial \theta} + \frac{C_{44}}{2} \frac{\partial^2 u_\theta}{\partial r^2} + \frac{C_{22}}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{C_{44}}{2r} \frac{\partial u_\theta}{\partial r} + \frac{2C_{22} + C_{44}}{2r^2} \frac{\partial u_r}{\partial \theta} - \frac{C_{44}}{2r^2} u_\theta + \alpha \frac{1}{r} \frac{\partial}{\partial \theta} \left(p_w + S_o p_{cw0}\right) = 0
\] (3.23)

Equations 3.22 and 3.23 are solved numerically along with the fluid flow and heat transfer equations.

### 3.2 Fluid Transport Model

When using high drilling fluid density, the drilling fluid can invade into the formation increasing the pore pressure of the formation around the wellbore, this decreases the principal effective stresses acting on the rock grain pushing the Mohr’s circle toward the failure envelope making the rock easier to fail. While using low drilling fluid density, the higher pore pressure of the formation pushes the formation fluid toward the wellbore reducing the pore pressure of the rock nearby the wellbore pushing the Mohr’s circle away from the failure envelope, hence, making the rock appear to be stronger. Therefore, the determination of the change of the pore pressure is crucially important in determining the effective stresses of the rock around the wellbore. The pore pressure around the wellbore can be obtained by solving the continuity equations, or conservation of mass, for each phase presented in the reservoir.
In this research, we consider two phase flow oil and water. The governing equations for mass transport of water and oil phases are:

For oil phase:

\[
\rho_o \hat{q}_o - \nabla \cdot (\rho_o \mathbf{v}_o) = (1 - \varepsilon_v) \frac{\partial}{\partial t} (\rho_o S_o \phi) \tag{3.24}
\]

For water phase:

\[
\rho_w \hat{q}_w - \nabla \cdot (\rho_w \mathbf{v}_w) = (1 - \varepsilon_v) \frac{\partial}{\partial t} (\rho_w S_w \phi) \tag{3.25}
\]

where \( \hat{q}_w \) and \( \hat{q}_o \) are the specific volume of water and oil phases; \( \mathbf{v}_w \) and \( \mathbf{v}_o \) are velocity of water and oil phases; \( S_w \) and \( S_o \) are saturation of water and oil phases; \( \rho_w \) and \( \rho_o \) are density of water and oil phases; \( \phi \) is formation porosity; \( \varepsilon_v \) is the volumetric strain can be determined from the geomechanical model as

\[
\varepsilon_v = \varepsilon_{rr} + \varepsilon_{\theta \theta} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + 2 \beta \Delta T \tag{3.26}
\]

Using Darcy’s Law, Equations 3.24 and 3.25 can be re written as

\[
\nabla \cdot \left[ \rho_o \frac{k_o}{\mu_o} \nabla (p_o - \gamma_o D) \right] + \rho_o \hat{q}_o = (1 - \varepsilon_v) \frac{\partial}{\partial t} (\rho_o S_o \phi) \tag{3.27}
\]

and

\[
\nabla \cdot \left[ \rho_w \frac{k_w}{\mu_w} \nabla (p_w - \gamma_w D) \right] + \rho_w \hat{q}_w = (1 - \varepsilon_v) \frac{\partial}{\partial t} (\rho_w S_w \phi) \tag{3.28}
\]

where \( k_w \) and \( k_o \) are water and oil permeability tensors; \( D \) is elevation; \( \gamma_w \) and \( \gamma_o \) are specific gravity of water and oil phases; \( p_w \) and \( p_o \) are pressure of water and oil phases.

The last term in the right hand-side of the equation of the transport equation of the oil can be simplified as

\[
\frac{\partial}{\partial t} (\rho_o S_o \phi_o) = \phi \rho_o \frac{\partial S_o}{\partial t} + \phi S_o \rho_o \phi \frac{\partial p_o}{\partial t} + S_o \rho_o \phi \frac{\partial p_o}{\partial t} = \phi \rho_o \frac{\partial S_o}{\partial t} + \phi \rho_o (S_o c_o + S_o c_\phi) \frac{\partial p_o}{\partial t} \tag{3.29}
\]

Hence, the governing equation for oil phase can be written as following:
\[ \nabla \cdot \left[ \frac{k_o}{\mu_o} \nabla (p_o - \gamma_o D) \right] + \rho_o \dot{q}_o = (1 - \varepsilon_o) \rho_o \left[ \phi \frac{\partial S_o}{\partial t} + \phi (S_o c_o + S_o c_\phi) \frac{\partial p_o}{\partial t} \right] \] (3.30)

Similarly, the transport equation for water phase is

\[ \nabla \cdot \left[ \frac{k_w}{\mu_w} \nabla (p_w - \gamma_w D) \right] + \rho_w \dot{q}_w = (1 - \varepsilon_v) \rho_w \left[ \phi \frac{\partial S_w}{\partial t} + \phi (S_w c_w + S_w c_\phi) \frac{\partial p_w}{\partial t} \right] \] (3.31)

Assuming homogeneous permeability formation, the transport equation for oil and water phase can be written as

\[ \nabla \cdot \left[ \frac{k_o}{\mu_o} \nabla (p_o - \gamma_o D) \right] + \rho_o \dot{q}_o = (1 - \varepsilon_v) \rho_o \left[ \phi \frac{\partial S_o}{\partial t} + \phi (S_o c_o + S_o c_\phi) \frac{\partial p_o}{\partial t} \right] \] (3.32)

and

\[ \nabla \cdot \left[ \frac{k_w}{\mu_w} \nabla (p_w - \gamma_w D) \right] + \rho_w \dot{q}_w = (1 - \varepsilon_v) \rho_w \left[ \phi \frac{\partial S_w}{\partial t} + \phi (S_w c_w + S_w c_\phi) \frac{\partial p_w}{\partial t} \right] \] (3.33)

Since

\[ \nabla \cdot \left[ \frac{k_o}{\mu_o} \nabla (p_o - \gamma_o D) \right] = \rho_o \frac{k_o}{\mu_o} \nabla \cdot [\nabla (p_o - \gamma_o D)] + \nabla \left( \rho_o \frac{k_o}{\mu_o} \right) \cdot \nabla (p_o - \gamma_o D) \approx \rho_o \frac{k_o}{\mu_o} \nabla \cdot [\nabla (p_o - \gamma_o D)] \] (3.34)

the transport equation for oil and water phase can be further simplified as

\[ \frac{k_o}{\mu_o} \nabla^2 (p_o - \gamma_o D) + \dot{q}_o = (1 - \varepsilon_v) \left[ \phi \frac{\partial S_o}{\partial t} + \phi (S_o c_o + S_o c_\phi) \frac{\partial p_o}{\partial t} \right] \] (3.35)

and

\[ \frac{k_w}{\mu_w} \nabla^2 (p_w - \gamma_w D) + \dot{q}_w = (1 - \varepsilon_v) \left[ \phi \frac{\partial S_w}{\partial t} + \phi (S_w c_w + S_w c_\phi) \frac{\partial p_w}{\partial t} \right] \] (3.36)

The summation of Equations 3.35 and 3.36 give:

\[ \frac{k_o}{\mu_o} \nabla^2 (p_o - \gamma_o D) + \frac{k_w}{\mu_w} \nabla^2 (p_w - \gamma_w D) + \dot{q}_o + \dot{q}_w = \]
\[(1 - \varepsilon_v) \left[ \phi \frac{\partial}{\partial t} (S_o + S_w) + \phi (S_w c_w + S_w c_\phi) \frac{\partial p_w}{\partial t} + \phi (S_o c_o + S_o c_\phi) \frac{\partial p_o}{\partial t} \right] \tag{3.37} \]

The main objective of adding two equation is to obtain equation to solve for phase pressure. Equation contains pressures of two phases and can be simplified further to reduce the number of variable to one, water phase pressure.

Substituting

\[
\begin{align*}
  p_o & = p_w + p_{cwo} \\
  k_w & = \lambda_w k \\
  \mu_w & = k_w \\
  \mu_o & = \lambda_o k
\end{align*}
\tag{3.38}
\]

into Equation 3.37 yields

\[
\lambda_o k \nabla^2 (p_w + p_{cwo} - \gamma_o D) + \lambda_w k \nabla^2 (p_w - \gamma_w D) + \dot{q}_o + \dot{q}_w =
\]

\[
(1 - \varepsilon_v) \left[ \phi \frac{\partial}{\partial t} (S_w + S_o) + \phi (S_w c_w + S_w c_\phi) \frac{\partial p_w}{\partial t} + \phi (S_o c_o + S_o c_\phi) \frac{\partial p_o}{\partial t} \right] (p_w + p_{cwo}) \tag{3.39}
\]

where \( k \) is absolute permeability, \( \lambda_o = \frac{k_{ro}}{\mu_o} \) and \( \lambda_w = \frac{k_{rw}}{\mu_w} \) are mobility of oil and water; \( p_{cwo} = p_o - p_w \) is capillary pressure. The capillary pressure \( (p_{cwo}) \) can be obtained from saturation as

\[
\begin{align*}
  p_{cwo}(S_w) & = p_{cwo}(S_{wr}), \text{ if } S_w < S_{wr} \\
  p_{cwo}(S_w) & = \alpha_1 \ln \left( \frac{1 - S_{ox} - S_w}{1 - S_w - S_{or} + 0.0001} \right), \text{ if } S_{wx} < S_w < 1 - S_{or} \tag{3.40} \\
  p_{cwo}(S_w) & = \alpha_2 \ln \left( \frac{1 - S_{ox} - S_w}{S_w - S_{or} + 0.0001} \right), \text{ if } S_{wx} > S_w > S_{wr}
\end{align*}
\]

where \( S_{ox} \) and \( S_{wx} \) are coefficients, \( \alpha_1 \) and \( \alpha_2 \) are coefficients, which can be determined as

\[
\begin{align*}
  S_{wx} & = 1 - S_{ox} \\
  \alpha_2 & = \frac{S_{wx} - S_w}{1 - S_{wx} - S_{or}} \alpha_1 \tag{3.41}
\end{align*}
\]
The relative permeability of water and oil can be determined from the following equations as

\[
\begin{align*}
    k_{rw} &= 0, \text{ if } S_w < S_{wr} \\
    k_{rw} &= k_{rw}^* \left( \frac{S_w - S_{wr}}{1 - S_{or} - S_{wr}} \right)^{n_w}, \text{ if } S_{wr} < S_w < 1 - S_{or} \\
    k_{rw} &= k_{rw}^*, \text{ if } S_w > 1 - S_{or} \\
    k_{ro} &= 0, \text{ if } S_o < S_{or} \\
    k_{ro} &= k_{ro}^* \left( \frac{S_o - S_{or}}{1 - S_{or} - S_{wr}} \right)^{n_o}, \text{ if } S_{or} < S_o < 1 - S_{or} \\
    k_{ro} &= k_{ro}^*, \text{ if } S_o > 1 - S_{or}
\end{align*}
\]

where \( S_{wr} \) and \( S_{or} \) are irreducible water and oil saturations, \( n_w \) and \( n_o \) exponents for water and oil.

Because

\[
\begin{align*}
    \phi (S_w c_w + S_w c_\phi) \frac{\partial}{\partial t} (p_{cw}) &\approx 0 \\
    \phi \frac{\partial}{\partial t} (S_w + S_o) &= \phi \frac{\partial}{\partial t} 1 = 0
\end{align*}
\]

Equation 3.39 can be written as

\[
(\lambda_o + \lambda_w) k \nabla^2 (p_w) + \lambda_o k \nabla^2 p_{cw} - \lambda_w k \nabla^2 (\gamma_w D) - \lambda_o k \nabla^2 (\gamma_o D) + \dot{q}_o + \dot{q}_w \\
= (1 - \varepsilon_v) \phi [S_w c_w + (S_w + S_o) c_\phi + S_o c_o] \frac{\partial p_w}{\partial t}
\]

We define the total compressibility (\( c_t \)), total mobility (\( \lambda_t \)), and total specific flow rate (\( \dot{q}_t \)) as

\[
\begin{align*}
    c_t &= S_o c_o + c_\phi + S_w c_w \\
    \lambda_t &= \lambda_o + \lambda_w \\
    \dot{q}_t &= \dot{q}_o + \dot{q}_w
\end{align*}
\]

The total pressure equation can be written as

\[
\lambda_t k \nabla^2 p_w + \lambda_o k \nabla^2 p_{cw} - \lambda_w k \nabla^2 (\gamma_w D) - \lambda_o k \nabla^2 (\gamma_o D) + \dot{q}_t = (1 - \varepsilon_v) \phi c_t \frac{\partial p_w}{\partial t}
\]
In 2-D problem, ignoring the elevation term, the total pressure equation can be written as

\[ \lambda_t k \nabla^2 p_w + \lambda_o k \nabla^2 p_{cwo} - \lambda_w k \nabla^2 (\gamma_w D) - \lambda_o k \nabla^2 (\gamma_o D) + \dot{q}_t = (1 - \varepsilon_v) \phi \epsilon_t \frac{\partial p_w}{\partial t} \quad (3.48) \]

In cylindrical coordinates equation above can be written as

\[ \frac{\lambda_t k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p_w}{\partial r} \right) + \frac{\lambda_o k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p_{cwo}}{\partial r} \right) + \dot{q}_t = (1 - \varepsilon_v) \phi \epsilon_t \frac{\partial p_w}{\partial t} \quad (3.49) \]

The total pressure equation is solved numerically for water phase pressure as presented in Section 4.3.

### 3.3 Heat Transfer Model

When the formation is in contact with a drilling fluid, the temperature of the rock around the wellbore changes inducing stress, or thermal-induced stress. To obtain the temperature change around the wellbore, we need to solve the heat transfer equation, the conservation of energy. The heat transfer equation in wellbore coordinates can be written as

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{z} \frac{\partial T}{\partial z} = \frac{1}{\alpha_T} \frac{\partial T}{\partial t} \quad (3.50) \]

In this thesis, we consider 2-D case, Equation 3.50 can be written as

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = \frac{1}{\alpha_T} \frac{\partial T}{\partial t} \quad (3.51) \]

where \( \alpha_T \) is thermal diffusivity defined as

\[ \alpha_T = \frac{K}{\rho_r c_r} \quad (3.52) \]

where \( c_r \) is specific heat of rock formation, \( K \) is the formation thermal conductivity.

The solution of Equation 3.51 yields the temperature of the formation around the wellbore.
In this chapter, I provide the numerical solutions for the developed model. After that, the model is validated with analytical solutions to prove the numerical accuracy of the model before using it for case studies in the next chapter.

4.1 Solution of Geomechanical Equation

In this section, the solution of momentum equation in radial and tangential directions is presented. Also, boundary conditions are summarized. The inner boundary has constant wellbore pressure, while outer boundary is constant formation in-situ stresses.

4.1.1 Solution of radial momentum equation

In this section, the solution of momentum equation in radial direction is provided.

4.1.1.1 Algebraic equation

In this subsection, momentum equation in radial direction is solved. Also, after discretization all terms are collected to algebraic equation. Using regular node size in $\theta$ and $z$ directions, each term of Equation 3.22 can be discretized as

\[
C_{11} \frac{\partial^2 u_r}{\partial r^2} = \left( \frac{C_{11}}{\Delta r^2} \right)_i^n \left( u_{r_{i+1,j}}^{n+1} - 2u_{r_{i,j}}^{n+1} + u_{r_{i-1,j}}^{n+1} \right) \tag{4.1}
\]

\[
\frac{C_{44}}{2r^2} \frac{\partial^2 u_r}{\partial \theta^2} = \left( \frac{C_{44}}{2r^2 \Delta \theta^2} \right)_i^n \left( u_{r_{i,j+1}}^{n+1} - 2u_{r_{i,j}}^{n+1} + u_{r_{i,j-1}}^{n+1} \right) \tag{4.2}
\]

\[
\frac{2C_{12} + C_{44}}{2r} \frac{\partial^2 u_\theta}{\partial r \partial \theta} = \left( \frac{2C_{12} + C_{44}}{8r \Delta \theta \Delta r} \right)_i^n \left( u_{\theta_{i+1,j+1}}^{n+1} - u_{\theta_{i,j+1}}^{n+1} - u_{\theta_{i+1,j-1}}^{n+1} + u_{\theta_{i-1,j-1}}^{n+1} \right) \tag{4.3}
\]

\[
\frac{C_{11} + C_{12} - C_{21}}{r} \frac{\partial u_r}{\partial r} = \left( \frac{C_{11} + C_{12} - C_{21}}{2r \Delta r} \right)_i^n \left( u_{r_{i+1,j}}^{n+1} - u_{r_{i-1,j}}^{n+1} \right) \tag{4.4}
\]
\[- \frac{2C_{22} + C_{44}}{2r^2} \frac{\partial u_\theta}{\partial \theta} = - \left( \frac{2C_{22} + C_{44}}{4r^2 \Delta \theta} \right)^n_i \left( u_{\theta, i+1}^{n+1} - u_{\theta, i}^{n+1} \right) \tag{4.5} \]

\[- \frac{C_{22}}{r^2} u_r = - \left( \frac{C_{22}}{r^2} \right)^n_i u_r^{n+1} \tag{4.6} \]

\[\alpha \frac{\partial}{\partial r} (p_w + S_0 p_{cw0}) = \alpha \frac{p_{w, i+\frac{1}{2}} - p_{w, i-\frac{1}{2}} + (S_0 p_{cw0})_{i+\frac{1}{2}} - (S_0 p_{cw0})_{i-\frac{1}{2}}}{\Delta r} \tag{4.7} \]

\[\beta (C_{11} + C_{12}) \frac{\partial T}{\partial r} = \left[ \frac{\beta (C_{11} + C_{12})}{\Delta r} \right]^n_i \left( T_{i+\frac{1}{2}}^n - T_{i-\frac{1}{2}}^n \right) \tag{4.8} \]

\[\frac{\beta \Delta T}{r} (C_{11} + C_{12} - C_{21} - C_{22}) = (C_{11} + C_{12} - C_{21} - C_{22})^n_i \left( \frac{\beta \Delta T}{r} \right)^n \tag{4.9} \]

where \( n \) is the step; \( \Delta r \) is the grid size in radial direction; \( \Delta \theta \) is grid size in tangential direction; subscripts \( i \) and \( j \) indicate node index in \( r \)-direction and \( \theta \)-direction.

Put all terms together yields the finite approximation of Equation 3.22.

\[\left( \frac{C_{11}}{\Delta r^2} \right)^n_i \left( u_{r, i+1, j}^{n+1} - 2u_{r, i, j}^{n+1} + u_{r, i-1, j}^{n+1} \right) + \left( \frac{C_{44}}{2r^2 \Delta \theta^2} \right)^n_i \left( u_{\theta, i+1, j}^{n+1} - 2u_{\theta, i, j}^{n+1} + u_{\theta, i-1, j}^{n+1} \right) \]

\[+ \left( \frac{2C_{12} + C_{44}}{8r \Delta \theta \Delta r} \right)^n_i \left( u_{\theta, i+1, j+1}^{n+1} - u_{\theta, i-1, j+1}^{n+1} - u_{\theta, i+1, j-1}^{n+1} + u_{\theta, i-1, j-1}^{n+1} \right) + \]

\[\left( \frac{C_{11} + C_{12} - C_{21}}{2r \Delta r} \right)^n_i \left( u_{r, i+1, j}^{n+1} - u_{r, i-1, j}^{n+1} \right) - \left( \frac{2C_{22} + C_{44}}{4r^2 \Delta \theta} \right)^n_i \left( u_{\theta, i+1, j}^{n+1} - u_{\theta, i-1, j}^{n+1} \right) \]

\[- \left( \frac{C_{22}}{r^2} \right)^n_i u_r^{n+1} + \alpha \frac{p_{w, i+\frac{1}{2}, j} - p_{w, i-\frac{1}{2}, j} + (S_0 p_{cw0})_{i+\frac{1}{2}, j} - (S_0 p_{cw0})_{i-\frac{1}{2}, j}}{\Delta r} + \]

\[\left[ \frac{\beta (C_{11} + C_{12})}{\Delta r} \right]^n_i \left( T_{i+\frac{1}{2}}^n - T_{i-\frac{1}{2}}^n \right) + (C_{11} + C_{12} - C_{21} - C_{22})^n_i \left( \frac{\beta \Delta T}{r} \right)^n \]

Reorganizing Equation 4.10, we obtain:

\[\left[ \left( \frac{C_{11}}{\Delta r^2} \right)^n_i - \left( \frac{C_{11} + C_{12} - C_{21}}{2r \Delta r} \right)^n_i \right] u_{r, i+1, j}^{n+1} + \left( \frac{C_{44}}{2r^2 \Delta \theta^2} \right)^n_i u_{r, i-1, j}^{n+1} \]

\[- 2 \left( \frac{C_{11}}{\Delta r^2} \right)^n_i + 2 \left( \frac{C_{44}}{2r^2 \Delta \theta^2} \right)^n_i + \left( \frac{C_{22}}{r^2} \right)^n_i u_r^{n+1} + \]

\[\left( \frac{C_{11}}{\Delta r^2} \right)^n_i + \left( \frac{C_{11} + C_{12} - C_{21}}{2r \Delta r} \right)^n_i \right] u_{r, i+1, j}^{n+1} + \left( \frac{C_{44}}{2r^2 \Delta \theta^2} \right)^n_i u_{r, i-1, j}^{n+1} \]

\[\]
we obtain the algebraic equation

\[ A_{i,j} u_{r_{i-1,j}}^{n+1} + B_{i,j} u_{r_{i,j}}^{n+1} + C_{i,j} u_{r_{i+1,j}}^{n+1} + D_{i,j} u_{r_{i,j+1}}^{n+1} + B_{i,j} u_{r_{i,j+1}}^{n+1} + E_{i,j} u_{t_{i-1,j-1}}^{n+1} + F_{i,j} u_{t_{i,j-1}}^{n+1} - E_{i,j} u_{t_{i,j+1}}^{n+1} - F_{i,j} u_{t_{i+1,j}}^{n+1} + E_{i,j} u_{t_{i+1,j+1}}^{n+1} = R_{i,j} \]
4.1.1.2 Boundary conditions

*Outer boundary condition*

Equation 4.19 can be written for the node at the outer boundary ($i = N$) as

\[
\begin{align*}
A_{N,j}u_{n+1}^{r_{N-1,j}} + B_{N,j}u_{r_{N,j}}^{n+1} + C_{N,j}u_{N+1,j}^{r_{N,j}} + D_{N,j}u_{r_{N,j+1}}^{n+1} + E_{N,j}u_{θ_{N,j-1}}^{n+1} + \\
F_{N,j}u_{θ_{N,j+1}}^{n+1} - E_{N,j}u_{θ_{N,j-1}}^{n+1} = R_{N,j}
\end{align*}
\]

(4.20)

At the boundary of the simulation grid, the stress is equal to the in-situ stresses. Since the given stress is given in the cartesian coordinates, we first need to transform this stress to cylindrical coordinates. Transforming a stress tensor from cartesian to cylindrical coordinates can be done as followed.

\[
\begin{bmatrix}
\sigma_{rr} & \sigma_{rθ} & \sigma_{rz} \\
\sigma_{θr} & \sigma_{θθ} & \sigma_{θz} \\
\sigma_{zr} & \sigma_{zθ} & \bar{σ}_{zz}
\end{bmatrix}
= \begin{bmatrix}
cosθ & sinθ & 0 \\
-sinθ & cosθ & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} & σ_{xy} & σ_{xz} \\
σ_{yx} & σ_{yy} & σ_{yz} \\
σ_{zx} & σ_{zy} & σ_{zz}
\end{bmatrix}
\begin{bmatrix}
cosθ & -sinθ & 0 \\
sinθ & cosθ & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  

(4.21)

For planar stress with the given in-situ stress tensor is in the principal direction, Equation 4.21 can be written as

\[
\begin{bmatrix}
σ^o_{rr} & σ^o_{rθ} \\
σ^o_{θr} & σ^o_{θθ}
\end{bmatrix}
= \begin{bmatrix}
cosθ & sinθ \\
-sinθ & cosθ
\end{bmatrix}
\begin{bmatrix}
σ_H & 0 \\
0 & σ_h
\end{bmatrix}
\begin{bmatrix}
cosθ & -sinθ \\
sinθ & cosθ
\end{bmatrix}
\]  

(4.22)

or

\[
\begin{align*}
σ^o_{rr} &= σ_Hcos^2θ + σ_hsin^2θ \\
σ^o_{θθ} &= σ_Hsin^2θ + σ_hcos^2θ \\
σ^o_{rθ} &= σ^o_{θr} = -σ_Hsinθcosθ + σ_hsinθcosθ = \frac{σ_h - σ_H}{2}sin2θ
\end{align*}
\]

(4.23)

where $θ$ is the direction from the maximum horizontal stress and the superscript $o$ indicates the outer boundary.

Using Equation 3.10, stress at the outer boundary nodes are related to displacement as
We obtain:

\[
\left\{ \begin{array}{l}
\sigma_{rr}^n = C_{11} \left( \frac{u_r}{r} + \beta \Delta T \right) + C_{12} \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \beta \Delta T \right) + \alpha_{rr} (p_w + S_\text{oPcw}) \\
\sigma_{\theta\theta}^n = C_{21} \left( \frac{\partial u_r}{\partial r} + \beta \Delta T \right) + C_{22} \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \beta \Delta T \right) + \alpha_{\theta\theta} (p_w + S_\text{oPcw}) \\
\sigma_{r\theta}^n = \frac{C_{44}}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) + \alpha_{r\theta} (p_w + S_\text{oPcw})
\end{array} \right.
\] (4.24)

or in the numerical form:

\[
\left\{ \begin{array}{l}
\sigma_{rr}^n = C_{11N}^n \left[ \frac{u_r^{n+1}}{r} - u_r^{n-1} + \frac{1}{2\Delta r} \frac{\partial u_\theta}{\partial \theta} \right] + \alpha_{rr} (p_w + S_\text{oPcw})_N^n \\
\sigma_{\theta\theta}^n = C_{21N}^n \left[ \frac{u_r^{n+1}}{r} - u_r^{n-1} + \frac{1}{2\Delta \theta} \frac{\partial u_\theta}{\partial r} \right] + \alpha_{\theta\theta} (p_w + S_\text{oPcw})_N^n \\
\sigma_{r\theta}^n = \frac{C_{44}}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) + \alpha_{r\theta} (p_w + S_\text{oPcw})_N^n
\end{array} \right.
\] (4.25)

We obtain:

\[
\frac{2\Delta r}{C_{11N}^n} \left\{ \sigma_{rr}^n - C_{12N}^m \left[ \frac{u_r^{n+1}}{r} - u_r^{n-1} + \frac{1}{2\Delta \theta} \frac{\partial u_\theta}{\partial r} \right] + \alpha_{rr} (p_w + S_\text{oPcw})_N^n \right\}
\]

(4.26)

and

\[
u_{\theta r}^{n+1} = 2\Delta r \left[ \left( \frac{2}{C_{44}} \right)_N^n \sigma_{r\theta}^n + \frac{u_\theta^{n+1}}{r} - \frac{1}{r} \left( \frac{u_r^{n+1}}{r} - u_{\theta r}^{n+1} \right) \right] + u_{\theta r}^{n-1}
\]

(4.27)

Since Equation 4.27 is applicable for any \( j \), \( u_{\theta r}^{n+1} \) and \( u_{\theta r}^{n+1} \) can be obtained as
\[ u_{\theta_{N+1,j-1}}^{n+1} = 2\Delta r \left( \frac{2}{C_{44}} \right)_N \sigma_{\theta_{N,j-1}}^o + \left( \frac{u_{\theta_{N,j-1}}^{n+1}}{T_N} - \frac{1}{r_N} \frac{u_{\theta_{N,j-1}}^{n+1} - u_{\theta_{N,j-2}}^{n+1}}{2\Delta \theta} \right) + u_{\theta_{N-1,j-1}}^{n+1} \] (4.28)

and

\[ u_{\theta_{N+1,j+1}}^{n+1} = 2\Delta r \left( \frac{2}{C_{44}} \right)_N \sigma_{\theta_{N,j+1}}^o + \left( \frac{u_{\theta_{N,j+1}}^{n+1}}{T_N} - \frac{1}{r_N} \frac{u_{\theta_{N,j+1}}^{n+1} - u_{\theta_{N,j}}^{n+1}}{2\Delta \theta} \right) + u_{\theta_{N-1,j+1}}^{n+1} \] (4.29)

Substituting \( u_{\theta_{N+1,j}}^{n+1}, u_{\theta_{N+1,j-1}}^{n+1} \) and \( u_{\theta_{N+1,j+1}}^{n+1} \) into Equation 4.20 we obtain:

\[
(A_{N,j} + D_{N,j}) u_{r_{N+1,j}}^{n+1} + B_{N,j} u_{r_{N,j}}^{n+1} + \left( C_{N,j} - D_{N,j} \frac{2\Delta r}{C_{12}^m} C_{12}^m \frac{1}{r_N} \frac{1}{\Delta \theta} + 2\Delta r E_{N,j} \frac{1}{r_N} \frac{1}{\Delta \theta} \right) u_{r_{N+1,j}}^{n+1} + B_{N,j} u_{r_{N,j+1}}^{n+1} + \left( F_{N,j} + D_{N,j} \frac{\Delta r}{C_{12}^m} \frac{1}{r_N} \frac{1}{\Delta \theta} - 2\Delta r E_{N,j} \frac{1}{r_N} \frac{1}{\Delta \theta} \right) u_{\theta_{N,j+1}}^{n+1}
\]

\[
- \left( F_{N,j} + D_{N,j} \frac{\Delta r}{C_{12}^m} \frac{1}{r_N} \frac{1}{\Delta \theta} - 2\Delta r E_{N,j} \frac{1}{r_N} \frac{1}{\Delta \theta} \right) u_{\theta_{N,j+1}}^{n+1} - \Delta r E_{N,j} \frac{1}{r_N} \frac{1}{\Delta \theta} u_{r_{N,j+2}}^{n+1} = R_{N,j} + 2\Delta r E_{N,j} \left( \frac{2}{C_{44}} \right)_N \left( \sigma_{\theta_{N,j-1}}^o - \sigma_{\theta_{N,j+1}}^o \right) - D_{N,j} \frac{2\Delta r}{C_{12}^m} \sigma_{r_{N,j}}^o
\]

\[
+ D_{N,j} 2\Delta r (\beta \Delta T)_N^n + D_{N,j} \frac{2\Delta r}{C_{12}^m} C_{12}^m (\beta \Delta T)_N^n + D_{N,j} \frac{2\Delta r}{C_{12}^m} \left[ \alpha (p_w + S_{oPcw}) \right]_N^n
\]

Set

\[ C'_{N,j} = \left( C_{N,j} - D_{N,j} \frac{2\Delta r}{C_{12}^m} \frac{1}{r_N} \frac{1}{\Delta \theta} + 2\Delta r E_{N,j} \frac{1}{r_N} \frac{1}{\Delta \theta} \right) \] (4.31)

\[ E'_{N,j} = \Delta r E_{N,j} \frac{1}{r_N} \frac{1}{\Delta \theta} \] (4.32)

\[ F'_{N,j} = F_{N,j} + D_{N,j} \frac{\Delta r}{C_{12}^m} \frac{1}{r_N} \frac{1}{\Delta \theta} - 2\Delta r E_{N,j} \frac{1}{r_N} \] (4.33)

\[ R'_{N,j} = R_{N,j} + 2\Delta r E_{N,j} \left( \frac{2}{C_{44}} \right)_N \left( \sigma_{\theta_{N,j-1}}^o - \sigma_{\theta_{N,j+1}}^o \right) - D_{N,j} \frac{2\Delta r}{C_{12}^m} \sigma_{r_{N,j}}^o + D_{N,j} 2\Delta r (\beta \Delta T)_N^n + D_{N,j} \frac{2\Delta r}{C_{12}^m} C_{12}^m (\beta \Delta T)_N^n + D_{N,j} \frac{2\Delta r}{C_{12}^m} \left[ \alpha (p_w + S_{oPcw}) \right]_N^n \] (4.34)

the algebraic equation for nodes at the outer boundary can be written as
\[(A_{N,j} + D_{N,j}) u_{r_{N-1,j}}^{n+1} + Bu_{r_{N,j}}^{n+1} + C_{N,j} u_{N,j}^{n+1} + B_{N,j} u_{N,j+1}^{n+1} + F_{N,j} u_{\theta_{N,j-1}}^{n+1} - F_{N,j} u_{\theta_{N,j+1}}^{n+1} - E_{N,j} u_{r_{N,j-2}}^{n+1} - E_{N,j} u_{r_{N,j+2}}^{n+1} = R'_{N,j}\]  

(4.35)

**Inner boundary condition**

Equation 4.19 can be written for the nodes at the inner boundary \((i = 1)\) as

\[
A_{1,j} u_{r_{0,j}}^{n+1} + B_{1,j} u_{r_{1,j}}^{n+1} + C_{1,j} u_{r_{1,j}}^{n+1} + D_{1,j} u_{r_{2,j}}^{n+1} + B_{1,j} u_{r_{1,j+1}}^{n+1} + E_{1,j} u_{\theta_{0,j-1}}^{n+1} + F_{1,j} u_{\theta_{1,j-1}}^{n+1} - E_{1,j} u_{\theta_{2,j-1}}^{n+1} - F_{1,j} u_{\theta_{1,j+1}}^{n+1} + E_{1,j} u_{\theta_{2,j+1}}^{n+1} = R_{1,j}
\]  

(4.36)

The inner boundary has constant pressure which is the pressure of the drilling fluid. Similar to the outer boundary case (Equation 4.23), the component of stress tensor in cylindrical coordinates at the inner boundary node as

\[
\begin{align*}
\sigma_{rr}^i &= p_{\text{well}} \cos^2 \theta + p_{\text{well}} \sin^2 \theta = p_{\text{well}} \\
\sigma_{\theta \theta}^i &= p_{\text{well}} \sin^2 \theta + p_{\text{well}} \cos^2 \theta = p_{\text{well}} \\
\sigma_{r \theta}^i &= \sigma_{\theta r}^i = -p_{\text{well}} \sin \theta \cos \theta + p_{\text{well}} \sin \theta \cos \theta = 0
\end{align*}
\]

Using Equation 3.10, stress at the outer boundary nodes are related to displacement as

\[
\begin{align*}
p_{\text{well}} &= C_{11} \left( \frac{u_r}{\partial r} + \beta \Delta T \right) + C_{12} \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \beta \Delta T \right) + \alpha_{rr} \left( p_w + S_o p_{\text{cwo}} \right) \\
p_{\text{well}} &= C_{21} \left( \frac{\partial u_r}{\partial r} + \beta \Delta T \right) + C_{22} \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \beta \Delta T \right) + \alpha_{\theta \theta} \left( p_w + S_o p_{\text{cwo}} \right) \\
0 &= \frac{C_{44}}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) + \alpha_{r \theta} \left( p_w + S_o p_{\text{cwo}} \right)
\end{align*}
\]  

(4.38)

or in the numerical form:

\[
p_{\text{well}} = C_{111}^{n} \left[ \frac{u_{r_{2,j}}^{n+1} - u_{r_{0,j}}^{n+1}}{2 \Delta r} + (\beta \Delta T)^{n}_{1} \right] + C_{121}^{n} \left[ \frac{u_{r_{1,j}}^{n+1} - u_{r_{1,j}}^{n+1}}{r_{1}} + \frac{1}{r_{1}} \frac{u_{r_{1,j+1}}^{n+1} - u_{r_{1,j-1}}^{n+1}}{2 \Delta \theta} + (\beta \Delta T)^{n}_{1} \right] + [\alpha \left( p_w + S_o p_{\text{cwo}} \right)]^{n}_{1}
\]

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Further simplifying, we obtain:

\[
p_{\text{well}} = C_{211}^m \left[ \frac{u_{r_{2,j}}^{n+1} - u_{r_{0,j}}^{n+1}}{2\Delta r} + (\beta \Delta T)^n \right] + \\
C_{221}^m \left[ \frac{u_{r_{1,j}}^{n+1}}{r_1} + \frac{1}{r_1} \frac{u_{\theta_{1,j+1}}^{n+1} - u_{\theta_{1,j-1}}^{n+1}}{2\Delta \theta} + (\beta \Delta T)^n \right] + \left[ \alpha (p_w + S_o p_{\text{cwo}}) \right]^n_1 
\]

(4.39)

and

\[
0 = \left( \frac{C_{44}}{2} \right)^n_1 \left( \frac{1}{r_1} \frac{u_{r_{1,j+1}}^{n+1} - u_{r_{1,j-1}}^{n+1}}{2\Delta \theta} + \frac{u_{\theta_{2,j}}^{n+1} - u_{\theta_{0,j}}^{n+1} - u_{\theta_{1,j}}^{n+1}}{r_1} \right)
\]

Further simplifying, we obtain:

\[
- \frac{2\Delta r}{C_{111}^m} \left\{ p_{\text{well}} - C_{121}^m \left[ \frac{u_{r_{0,j}}^{n+1}}{r_1} + \frac{1}{r_1} \frac{u_{\theta_{1,j+1}}^{n+1} - u_{\theta_{1,j-1}}^{n+1}}{2\Delta \theta} + (\beta \Delta T)^n \right] - \left[ \alpha (p_w + S_o p_{\text{cwo}}) \right]^n_1 \right\}
\]

(4.40)

and

\[
u_{\theta_{0,j}}^{n+1} = u_{\theta_{2,j}}^{n+1} - 2\Delta r \left( \frac{u_{\theta_{1,j}}^{n+1}}{r_1} - \frac{1}{r_1} \frac{u_{r_{1,j+1}}^{n+1} - u_{r_{1,j-1}}^{n+1}}{2\Delta \theta} \right)
\]

(4.41)

Since Equation 4.41 is applicable for any \( j \), \( u_{\theta_{0,j+1}}^{n+1} \) and \( u_{\theta_{0,j-1}}^{n+1} \) can be obtained as

\[
u_{\theta_{0,j+1}}^{n+1} = u_{\theta_{2,j+1}}^{n+1} - 2\Delta r \left( \frac{u_{\theta_{1,j+1}}^{n+1}}{r_1} - \frac{1}{r_1} \frac{u_{r_{1,j+2}}^{n+1} - u_{r_{1,j}}^{n+1}}{2\Delta \theta} \right)
\]

(4.42)

and

\[
u_{\theta_{0,j-1}}^{n+1} = u_{\theta_{2,j-1}}^{n+1} - 2\Delta r \left( \frac{u_{\theta_{1,j-1}}^{n+1}}{r_1} - \frac{1}{r_1} \frac{u_{r_{1,j}}^{n+1} - u_{r_{1,j-2}}^{n+1}}{2\Delta \theta} \right)
\]

(4.43)

Substituting \( u_{r_{0,j}}^{n+1} \), \( u_{\theta_{0,j+1}}^{n+1} \), and \( u_{\theta_{0,j-1}}^{n+1} \) into Equation 4.36 yields

\[
(A_{1,j} + D_{1,j}) u_{r_{2,j}}^{n+1} + \left( C_{1,j} + A_{1,j} \frac{2\Delta r}{C_{111}^m} C_{121}^m \frac{1}{r_1} + 4\Delta r E_{1,j} \frac{1}{r_1} \frac{1}{2\Delta \theta} \right) u_{r_{1,j}}^{n+1}
\]

\[
+ B_{1,j} u_{r_{1,j-1}}^{n+1} + B_{1,j} u_{r_{1,j+1}}^{n+1} - 2\Delta r E_{1,j} \frac{1}{r_1} \frac{u_{r_{1,j-2}}^{n+1} - u_{r_{1,j}}^{n+1}}{2\Delta \theta} - 2\Delta r E_{1,j} \frac{1}{r_1} \frac{u_{r_{1,j+2}}^{n+1} - u_{r_{1,j}}^{n+1}}{2\Delta \theta}
\]

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\begin{align*}
+ \left( F_{1,j} u_{\theta_1,j-1}^{n+1} - A_{1,j} \frac{2\Delta r}{C_{111}^n} u_{C_{121}^n r_1} \frac{1}{\Delta \theta} - 2\Delta r E_{1,j} \frac{1}{r_1} \right) u_{\theta_1,j+1}^{n+1} = \\
\left( F_{1,j} - A_{1,j} \frac{2\Delta r}{C_{111}^n} u_{\theta_1,j+1}^{n+1} \right) u_{\theta_1,j+1}^{n+1} = R_{1,j} - 2\Delta r (\beta \Delta T)^n A_{1,j} + \\
A_{1,j} \frac{2\Delta r}{C_{111}^n} p_{well} - A_{1,j} \frac{2\Delta r}{C_{121}^n} (\beta \Delta T)^n - \frac{2\Delta r}{C_{111}^n} A_{1,j} [\alpha (p_w + S_o p_{cwo})]_1^n \tag{4.44}
\end{align*}

Set

\begin{align*}
C_{1, j}' &= \left( C_{1, j} + A_{1, j} \frac{2\Delta r}{C_{111}^n} (\beta \Delta T)^n - \frac{2\Delta r}{C_{111}^n} A_{1, j} [\alpha (p_w + S_o p_{cwo})]_1^n \right) \tag{4.45} \\
E_{1, j}' &= \left( 2\Delta r E_{1,j} \frac{1}{r_1} \frac{1}{\Delta \theta} \right) \tag{4.46} \\
F_{1, j}' &= \left( F_{1,j} u_{\theta_1,j-1}^{n+1} - A_{1,j} \frac{2\Delta r}{C_{121}^n} \frac{1}{r_1} \frac{1}{\Delta \theta} - 2\Delta r E_{1,j} \frac{1}{r_1} \right) \tag{4.47}
\end{align*}

and

\begin{align*}
R_{1,j}' &= R_{1,j} - 2\Delta r (\beta \Delta T)^n A_{1,j} + A_{1,j} \frac{2\Delta r}{C_{111}^n} p_{well} - \\
&+ A_{1,j} \frac{2\Delta r}{C_{121}^n} (\beta \Delta T)^n - \frac{2\Delta r}{C_{111}^n} A_{1,j} [\alpha (p_w + S_o p_{cwo})]_1^n \tag{4.48}
\end{align*}

the algebraic equation for inner boundary nodes can be written as

\begin{align*}
(A_{1,j} + D_{1,j}) u_{r_{1,j-2}}^{n+1} + C_{1,j}' u_{r_{1,j-1}}^{n+1} + B_{1,j} u_{r_{1,j-1}}^{n+1} + B_{1,j} u_{r_{1,j-1}}^{n+1} - E_{1,j}' u_{r_{1,j-2}}^{n+1} - E_{1,j}' u_{r_{1,j+2}}^{n+1} + F_{1,j} u_{\theta_1,j-1}^{n+1} - F_{1,j} u_{\theta_1,j+1}^{n+1} = R_{1,j}' \tag{4.49}
\end{align*}

### 4.1.2 Solution of tangential momentum equation

In this section, the solution of tangential momentum equation is presented.

#### 4.1.2.1 Algebraic equation

In this subsection, the solution of tangential momentum equation is provided. Also, the algebraic equation is presented. The finite difference approximations of each term in Equation 3.20 are:

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\[
\frac{C_{44} + 2C_{21}}{2r} \frac{\partial^2 u_r}{\partial r \partial \theta} = \left( \frac{C_{44} + 2C_{21}}{8r \Delta \theta \Delta r} \right)_i^n \left( u_{r_{i+1,j+1}}^{n+1} - u_{r_{i-1,j+1}}^{n+1} - u_{r_{i+1,j-1}}^{n+1} + u_{r_{i-1,j-1}}^{n+1} \right) \quad (4.50)
\]

\[
\frac{C_{44}}{2} \frac{\partial^2 u_{\theta}}{\partial r^2} = \left( \frac{C_{44}}{2r^2} \right)_i^n \left( u_{\theta_{i+1,j}}^{n+1} - 2u_{\theta_{i,j}}^{n+1} + u_{\theta_{i-1,j}}^{n+1} \right) \quad (4.51)
\]

\[
\frac{C_{22}}{r^2} \frac{\partial^2 u_{\theta}}{\partial \theta^2} = \left( \frac{C_{22}}{r^2 \Delta \theta^2} \right)_i^n \left( u_{\theta_{i+1,j+1}}^{n+1} - 2u_{\theta_{i,j}}^{n+1} + u_{\theta_{i-1,j-1}}^{n+1} \right) \quad (4.52)
\]

\[
\frac{C_{44}}{2r} \frac{\partial u_{\theta}}{\partial r} = \left( \frac{C_{44}}{4r \Delta r} \right)_i^n \left( u_{\theta_{i+1,j+1}}^{n+1} - u_{\theta_{i-1,j}}^{n+1} \right) \quad (4.53)
\]

\[
\frac{2C_{22} + C_{44}}{r^2} \frac{\partial u_r}{\partial \theta} = \left( \frac{2C_{22} + C_{44}}{2r^2 \Delta \theta} \right)_i^n \left( u_{r_{i,j+1}}^{n+1} - u_{r_{i,j-1}}^{n+1} \right) \quad (4.54)
\]

\[
\frac{1}{r} \frac{\partial}{\partial \theta} \left( p_w + S_o p_{cwo} \right) = \left( \frac{\alpha}{r} \right)_i^n \frac{p_{w_{j+\frac{1}{2}}}^n - p_{w_{j-\frac{1}{2}}}^n + (S_o p_{cwo})_{j+\frac{1}{2}}^n - (S_o p_{cwo})_{j-\frac{1}{2}}^n}{\Delta \theta} \quad (4.56)
\]

Put all terms together, we have the finite difference approximation of Equation 3.20 as

\[
\left( \frac{C_{44} + 2C_{21}}{8r \Delta \theta \Delta r} \right)_i^n \left( u_{r_{i+1,j+1}}^{n+1} - u_{r_{i-1,j+1}}^{n+1} - u_{r_{i+1,j-1}}^{n+1} + u_{r_{i-1,j-1}}^{n+1} \right) \\
+ \left( \frac{C_{44}}{2r^2} \right)_i^n \left( u_{\theta_{i+1,j+1}}^{n+1} - 2u_{\theta_{i,j}}^{n+1} + u_{\theta_{i-1,j-1}}^{n+1} \right) + \\
\left( \frac{C_{22}}{r^2 \Delta \theta^2} \right)_i^n \left( u_{\theta_{i+1,j+1}}^{n+1} - 2u_{\theta_{i,j}}^{n+1} + u_{\theta_{i-1,j-1}}^{n+1} \right) + \\
\left( \frac{C_{44}}{4r \Delta r} \right)_i^n \left( u_{\theta_{i+1,j+1}}^{n+1} - u_{\theta_{i-1,j}}^{n+1} \right) + \\
\left( \frac{2C_{22} + C_{44}}{2r^2 \Delta \theta} \right)_i^n \left( u_{r_{i,j+1}}^{n+1} - u_{r_{i,j-1}}^{n+1} \right) - \\
\left( \frac{1}{r} \right)_i^n \frac{p_{w_{j+\frac{1}{2}}}^n - p_{w_{j-\frac{1}{2}}}^n + (S_o p_{cwo})_{j+\frac{1}{2}}^n - (S_o p_{cwo})_{j-\frac{1}{2}}^n}{\Delta \theta}
\]

Rearranging Equation 4.57, we have

\[
\left( \frac{C_{44} + 2C_{21}}{8r \Delta \theta \Delta r} \right)_i^n u_{r_{i-1,j+1}}^{n+1} - \\
\left( \frac{2C_{22} + C_{44}}{2r^2 \Delta \theta} \right)_i^n u_{r_{i,j-1}}^{n+1} - \\
\left( \frac{C_{44} + 2C_{21}}{8r \Delta \theta \Delta r} \right)_i^n u_{r_{i-1,j-1}}^{n+1} - \\
\left( \frac{1}{r} \right)_i^n \frac{p_{w_{j+\frac{1}{2}}}^n - p_{w_{j-\frac{1}{2}}}^n + (S_o p_{cwo})_{j+\frac{1}{2}}^n - (S_o p_{cwo})_{j-\frac{1}{2}}^n}{\Delta \theta}
\]

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\[
\frac{(C_{44} + 2C_{21})^n}{8r \Delta \theta \Delta r} u_{r_{i+1,j-1}}^{n+1} + \frac{(2C_{22} + C_{44})^n}{2r^2 \Delta \theta} u_{r_{i,j+1}}^{n+1} + \frac{(C_{44} + 2C_{21})^n}{8r \Delta \theta \Delta r} u_{r_{i+1,j+1}}^{n+1} + \\
\left[ \left( \frac{C_{44}}{2\Delta r^2} \right)_i^n - \left( \frac{C_{44}}{4r \Delta r} \right)_i^n \right] v_{\theta_{i-1,j}}^{n+1} - \left[ \left( \frac{C_{44}}{\Delta r^2} \right)_i^n + 2 \left( \frac{C_{22}}{r^2 \Delta \theta^2} \right)_i^n + \left( \frac{C_{44}}{2r^2} \right)_i^n \right] v_{\theta_{i,j}}^{n+1} \\
+ \left[ \left( \frac{C_{44}}{2\Delta r^2} \right)_i^n + \left( \frac{C_{44}}{4r \Delta r} \right)_i^n \right] u_{\theta_{i+1,j}}^{n+1} + \left( \frac{C_{22}}{r^2 \Delta \theta^2} \right)_i^n v_{\theta_{i,j-1}}^{n+1} + \left( \frac{C_{22}}{r^2 \Delta \theta^2} \right)_i^n v_{\theta_{i,j+1}}^{n+1} = \\
- \left( \frac{\alpha}{r} \right)_i^n p_{w_{j+\frac{1}{2}}}^n - p_{w_{j-\frac{1}{2}}}^n + (S_{opcw0})_{j+\frac{1}{2}}^n - (S_{opcw0})_{j-\frac{1}{2}}^n \\
\frac{\Delta \theta}{\Delta \theta} \Delta \theta
\]

Set
\[
F_{i,j} = \left( \frac{C_{44} + 2C_{21}}{8r \Delta \theta \Delta r} \right)_i^n
\]
\[
G_{i,j} = \left( \frac{2C_{22} + C_{44}}{2r^2 \Delta \theta^2} \right)_i^n
\]
\[
H_{i,j} = \left[ \left( \frac{C_{44}}{2\Delta r^2} \right)_i^n - \left( \frac{C_{44}}{4r \Delta r} \right)_i^n \right]
\]
\[
I_{i,j} = - \left[ \left( \frac{C_{44}}{\Delta r^2} \right)_i^n + 2 \left( \frac{C_{22}}{r^2 \Delta \theta^2} \right)_i^n + \left( \frac{C_{44}}{2r^2} \right)_i^n \right]
\]
\[
J_{i,j} = \left[ \left( \frac{C_{44}}{2\Delta r^2} \right)_i^n + \left( \frac{C_{44}}{4r \Delta r} \right)_i^n \right]
\]
\[
K_{i,j} = \left( \frac{C_{22}}{r^2 \Delta \theta^2} \right)_i^n
\]
\[
R_{i,j} = - \left( \frac{\alpha}{r} \right)_i^n p_{w_{j+\frac{1}{2}}}^n - p_{w_{j-\frac{1}{2}}}^n + (S_{opcw0})_{j+\frac{1}{2}}^n - (S_{opcw0})_{j-\frac{1}{2}}^n \\
\frac{\Delta \theta}{\Delta \theta} \Delta \theta
\]

we obtain the following algebraic equation:

\[
F_{i,j} u_{r_{i-1,j-1}}^{n+1} - G_{i,j} u_{r_{i,j-1}}^{n+1} - F_{i,j} u_{r_{i+1,j+1}}^{n+1} + G_{i,j} u_{r_{i+1,j-1}}^{n+1} + F_{i,j} u_{r_{i+1,j+1}}^{n+1} + \\
+ H_{i,j} v_{\theta_{i-1,j}}^{n+1} + I_{i,j} v_{\theta_{i,j}}^{n+1} + J_{i,j} v_{\theta_{i+1,j}}^{n+1} + K_{i,j} v_{\theta_{i,j-1}}^{n+1} + K_{i,j} v_{\theta_{i,j+1}}^{n+1} = R_{i,j}
\]

4.1.2.2 Boundary conditions

**Outer boundary condition**
The algebraic equation (Equation 4.66) for the nodes at the outer boundary \((i = N)\) can be written as

\[
F_{N,j}u_{N,j+1}^{n+1} - G_{N,j}u_{N,j-1}^{n+1} - F_{N,j}u_{N,j+1}^{n+1} - F_{N,j}u_{N,j+1}^{n+1} - F_{N,j}u_{N,j+1}^{n+1} + G_{N,j}u_{N,j+1}^{n+1} + H_{N,j}u_{N,j-1}^{n+1} + I_{N,j}u_{N,j+1}^{n+1} + J_{N,j}u_{N,j+1}^{n+1} + K_{N,j}u_{N,j+1}^{n+1} + K_{N,j}u_{N,j+1}^{n+1} = R_{N,j}
\]

(4.67)

Since Equation 4.26 is applicable for any \(j\), \(u_{N,j+1}^{n+1}\) and \(u_{N,j+1}^{n+1}\) can be obtained as

\[
u_{N,j+1}^{n+1} = -2\Delta r (\beta \Delta T)^n_N + u_{N,j}^{n+1} + \frac{2\Delta r}{C_{11N}^m} \left\{ \frac{\sigma_{rr N,j}^o}{C_{12N}^m} \left[ \frac{u_{N,j+1}^{n+1}}{r_N} + \frac{1}{r_N} \frac{u_{N,j}^{n+1} - u_{N,j+2}^{n+1}}{2\Delta \theta} + (\beta \Delta T)^n_N \right] \right\}
\]

(4.68)

and

\[
u_{N,j+1}^{n+1} = -2\Delta r (\beta \Delta T)^n_N + u_{N,j}^{n+1} + \frac{2\Delta r}{C_{11N}^m} \left\{ \frac{\sigma_{rr N,j+1}^o}{C_{12N}^m} \left[ \frac{u_{N,j+1}^{n+1}}{r_N} + \frac{1}{r_N} \frac{u_{N,j+1}^{n+1} - u_{N,j+2}^{n+1}}{2\Delta \theta} + (\beta \Delta T)^n_N \right] \right\}
\]

(4.69)

Substituting \(u_{N,j+1}^{n+1}\) (Equation 4.68), \(u_{N,j+1}^{n+1}\) (Equation 4.69), and \(u_{N,j+1}^{n+1}\) (Equation 4.27) into the algebraic equation for the node at the outer boundary (Equation 4.67) we obtain:

\[
\left(-G_{N,j} + F_{N,j} \frac{2\Delta r}{C_{11N}^m} C_{12N}^m \frac{1}{r_N} \frac{\Delta r}{r_N} J_{N,j} \frac{1}{\Delta \theta} \right) u_{N,j-1}^{n+1} + \left(I_{N,j} + J_{N,j} \frac{2\Delta r}{r_N} + F_{N,j} \frac{2\Delta r}{H_{11N}^m} C_{12N}^m \frac{1}{\Delta \theta r_N} \right) u_{N,j}^{n+1} + \left(-F_{N,j} \frac{\Delta r}{C_{11N}^m} C_{12N}^m \frac{1}{r_N} \frac{1}{\Delta \theta} u_{N,j+2}^{n+1} + \left(G_{N,j} - F_{N,j} \frac{2\Delta r}{C_{11N}^m} C_{12N}^m \frac{1}{r_N} - J_{N,j} \frac{2\Delta r}{2\Delta \theta r_N} \right) u_{N,j+1}^{n+1}
\]
the algebraic equation for the node at the outer boundary can be written as

\[-F_{N,j} \frac{\Delta r}{C_{11N}^m} C_{12N}^m \frac{1}{r_N} \frac{1}{\Delta \theta} u_{\theta_{N,j+2}}^{n+1} + (H_{N,j} + J_{N,j}) u_{\theta_{N-1,j}}^{n+1} + K_{N,j} u_{\theta_{N,j-1}}^{n+1} + K_{N,j} u_{\theta_{N,j+1}}^{n+1} = R_{N,j} - \frac{2 \Delta r}{C_{11N}^m} r_N \frac{1}{\Delta \theta} \sigma_{\theta_{N,j-1}}^\rho - \frac{2 \Delta r}{C_{11N}^m} r_N \frac{1}{\Delta \theta} \sigma_{\theta_{N,j+1}}^\rho - 2 \Delta r J_{N,j} \left( \frac{2}{C_{44}^m} \right)_N \sigma_{\theta_{N,j}}^\rho \]  

(4.70)

Set

\[G'_{N,j} = -G_{N,j} + F_{N,j} \frac{2 \Delta r}{C_{11N}^m} C_{12N}^m \frac{1}{r_N} + \frac{\Delta r}{r_N} J_{N,j} \frac{1}{\Delta \theta} \]

(4.71)

\[I'_{N,j} = I_{N,j} + J_{N,j} \frac{2 \Delta r}{r_N} + F_{N,j} \frac{2 \Delta r}{C_{11N}^m} C_{12N}^m \frac{1}{r_N} \Delta \theta r_N \]

(4.72)

\[F'_{N,j} = F_{N,j} \frac{\Delta r}{C_{11N}^m} C_{12N}^m \frac{1}{r_N} \frac{1}{\Delta \theta} \]

(4.73)

\[R_{\theta_{N,j}} = R_{N,j} - \frac{2 \Delta r}{C_{11N}^m} r_N \sigma_{\theta_{N,j-1}}^\rho - \frac{2 \Delta r}{C_{11N}^m} r_N \sigma_{\theta_{N,j+1}}^\rho - 2 \Delta r J_{N,j} \left( \frac{2}{C_{44}^m} \right)_N \sigma_{\theta_{N,j}}^\rho \]

(4.74)

the algebraic equation for the node at the outer boundary can be written as

\[G'_{N,j} u_{\theta_{N,j-1}}^{n+1} + I'_{N,j} u_{\theta_{N,j}}^{n+1} - F'_{N,j} u_{\theta_{N,j+1}}^{n+1} - G_{N,j} u_{\theta_{N,j-2}}^{n+1} - F_{N,j} u_{\theta_{N,j+2}}^{n+1} + (H_{N,j} + J_{N,j}) u_{\theta_{N-1,j}}^{n+1} + K_{N,j} u_{\theta_{N,j-1}}^{n+1} + K_{N,j} u_{\theta_{N,j+1}}^{n+1} = R_{\theta_{N,j}} \]

(4.75)

**Inner boundary condition**

Equation 4.66 can be written for the nodes at the inner boundary \((i = 1)\) as

\[F_{1,j} u_{\theta_{0,j}}^{n+1} - G_{1,j} u_{\theta_{1,j}}^{n+1} - F_{1,j} u_{\theta_{0,j+1}}^{n+1} - F_{1,j} u_{\theta_{2,j}}^{n+1} + G_{1,j} u_{\theta_{1,j+1}}^{n+1} + F_{1,j} u_{\theta_{2,j+1}}^{n+1} + H_{1,j} u_{\theta_{0,j}}^{n+1} + I_{1,j} u_{\theta_{1,j}}^{n+1} + J_{1,j} u_{\theta_{2,j}}^{n+1} + K_{1,j} u_{\theta_{1,j-1}}^{n+1} + K_{1,j} u_{\theta_{1,j+1}}^{n+1} = R_{1,j} \]

(4.76)

Substituting \(u_{\theta_{r0,j-1}}^{n+1}\) (Equation 4.42), \(u_{\theta_{r0,j+1}}^{n+1}\) (Equation 4.43), and \(u_{\theta_{0,j}}^{n+1}\) (Equation 4.41) into Equation 4.76, we obtain:

\[- \left( F_{1,j} \frac{2 \Delta r}{C_{111}^m} C_{120}^m \frac{1}{r_1} + H_{1,j} \Delta r \frac{1}{r_1} \frac{1}{\Delta \theta} + G_{1,j} \right) u_{\theta_{r0,j-1}}^{n+1} -

(50)
\[
\begin{align*}
&\left( F_{1,j} \frac{2\Delta r}{C_{n11}} \frac{1}{r_1} \frac{1}{\Delta \theta} + H_{1,j} \Delta r \frac{1}{r_1} \frac{1}{\Delta \theta} - I_{1,j} \right) u_{\theta_{1,j}}^{n+1} \\
&- F_{1,j} \frac{\Delta r}{C_{n11}} \frac{1}{r_1} \frac{1}{\Delta \theta} u_{\theta_{1,j+2}}^{n+1} + F_{1,j} \frac{\Delta r}{C_{n11}} \frac{1}{r_1} \frac{1}{\Delta \theta} u_{\theta_{1,j+2}}^{n+1} \\
&+ (H_{1,j} + J_{1,j}) u_{\theta_{2,j}}^{n+1} + K_{1,j} u_{\theta_{1,j-1}}^{n+1} + K_{1,j} u_{\theta_{1,j+1}}^{n+1} = R_{1,j}
\end{align*}
\]

Set
\[
F'_{1,j} = \left( F_{1,j} \frac{2\Delta r}{C_{n11}} \frac{1}{r_1} + H_{1,j} \Delta r \frac{1}{r_1} \frac{1}{\Delta \theta} \right)
\]
\[
I'_{1,j} = \left( F_{1,j} \frac{2\Delta r}{C_{n11}} \frac{1}{r_1} + H_{1,j} \Delta r \frac{1}{r_1} \frac{1}{\Delta \theta} - I_{1,j} \right)
\]
\[
F''_{1,j} = F_{1,j} \frac{\Delta r}{C_{n11}} \frac{1}{r_1} \frac{1}{\Delta \theta}
\]

and the algebraic equation for the node at the inner boundary \( i = 1 \) can be written as
\[
- \left( F'_{1,j} + G_{1,j} \right) u_{r_{1,j-1}}^{n+1} - I'_{1,j} u_{\theta_{1,j}}^{n+1} - F''_{1,j} u_{\theta_{1,j-2}}^{n+1} + \left( F'_{1,j} + G_{1,j} \right) u_{r_{1,j+1}}^{n+1} + \\
F''_{1,j} u_{\theta_{1,j+2}}^{n+1} + (H_{1,j} + J_{1,j}) u_{\theta_{2,j}}^{n+1} + K_{1,j} u_{\theta_{1,j-1}}^{n+1} + K_{1,j} u_{\theta_{1,j+1}}^{n+1} = R_{1,j}
\]

### 4.1.3 Determination of stress

The component of stress tensor at node \( i,j \) can be numerically approximated from Equation 3.10 as

\[
\sigma^{n+1}_{rr_{i,j}} = C^{n+1}_{n11_{i,j}} \left[ \frac{u^{n+1}_{r_{i+1,j}} - u^{n+1}_{r_{i-1,j}}}{2\Delta r} + (\beta \Delta T)_{i}^{n+1} \right] + \\
C^{n+1}_{n12_{i,j}} \left[ \frac{u^{n+1}_{r_{i,j+1}} - u^{n+1}_{r_{i,j-1}}}{r_i} + \frac{1}{2\Delta \theta} \right] + (\beta \Delta T)_{i}^{n+1} + [\alpha (p_w + S_{\alpha p_{cwo}})]_{i}^{n+1} \\
\sigma^{n+1}_{\theta_{i,j}} = C^{n+1}_{21_{i,j}} \left[ \frac{u^{n+1}_{r_{i,j+1}} - u^{n+1}_{r_{i,j-1}}}{2\Delta r} + (\beta \Delta T)_{i}^{n+1} \right] +
\]

51
4.1.4 Determination of volumetric strain

The volumetric strain at nodes \( i \) can be obtained numerically from Equation 3.26 as

\[
\varepsilon_{vi}^{n+1} = \frac{u_{r_i}^{n+1} - u_{r_i}^{n+1}}{\Delta r} + \frac{u_{\theta i}^{n+1} - u_{\theta i}^{n+1}}{\Delta \theta} + \frac{\Delta \theta_0}{\Delta r} + \frac{\Delta \theta_0}{\Delta \theta} + 2\beta_{n+1}^i \Delta T_{n+1}^{n+1}
\]  (4.83)

4.1.5 Determination failure status

The principal stresses \( (\sigma_e) \) are determined from the following characteristics equation

\[
\text{det} \begin{bmatrix} \sigma_{rr} - \sigma_e & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} - \sigma_e \end{bmatrix} = 0 \Rightarrow \sigma_e^2 - (\sigma_{rr} + \sigma_{\theta\theta}) \sigma_e - \sigma_{rr} \sigma_{\theta\theta} + \sigma_{r\theta}^2 = 0
\]  (4.84)

Equation 4.84 has two solutions:

\[
\begin{cases}
\sigma_1 = \frac{(\sigma_{rr} + \sigma_{\theta\theta}) + \sqrt{(\sigma_{rr} - \sigma_{\theta\theta})^2 - 4\sigma_{r\theta}^2}}{2} \\
\sigma_3 = \frac{(\sigma_{rr} + \sigma_{\theta\theta}) - \sqrt{(\sigma_{rr} - \sigma_{\theta\theta})^2 - 4\sigma_{r\theta}^2}}{2}
\end{cases}
\]  (4.85)

Hence, the principal stresses of the node \((i, j)\) at time step \( n + 1 \) can be determined as

\[
\begin{cases}
\sigma_{1i,j}^{n+1} = \frac{(\sigma_{rr} + \sigma_{\theta\theta})_{i,j}^{n+1} + \sqrt{[(\sigma_{rr} - \sigma_{\theta\theta})_{i,j}^{n+1}]^2 - 4(\sigma_{r\theta_{i,j}}^{n+1})^2}}{2} \\
\sigma_{3i,j}^{n+1} = \frac{(\sigma_{rr} + \sigma_{\theta\theta})_{i,j}^{n+1} - \sqrt{[(\sigma_{rr} - \sigma_{\theta\theta})_{i,j}^{n+1}]^2 - 4(\sigma_{r\theta_{i,j}}^{n+1})^2}}{2}
\end{cases}
\]  (4.86)

Then failure criteria are used to check if the rock fails or not.

The rock would fail under tension if

\[
\sigma_{3i,j}^{n+1} + |\sigma_T| < 0
\]  (4.87)
Using Mohr’s Coulomb failure criterion, the rock would fails if

\[
\sigma_{i,j}^{n+1} - (p_w + S_{o,p_{cwo}})_i^{n+1} \left(1 + \sin \phi \right)_{i,j}^{n+1} \left[\sigma_{i,j}^{n+1} - (p_w + S_{o,p_{cwo}})_i^{n+1}\right] > UCS_{i,j}^{n+1}
\] (4.88)

4.2 Solution of Heat Transfer Equation

In this section, heat transfer equation is solved by finite-difference method. Also, boundary conditions are applied, inner boundary is temperature of well, while outer boundary is formation temperature.

4.2.1 Algebraic equation

Equation 3.51 can be discretized as

\[
\frac{1}{r_i} \frac{1}{\Delta r_i} \left( \frac{T_{i+1}^{n+1} - T_i^{n+1}}{\Delta r_{i+\frac{1}{2}}} - \frac{T_i^{n+1} - T_{i-1}^{n+1}}{\Delta r_{i-\frac{1}{2}}} \right) = \frac{\alpha T}{\Delta t} T_i^{n+1} - T_i^n
\] (4.89)

or

\[
\frac{1}{r_i} \frac{r_{i-\frac{1}{2}}}{\Delta r_i} \frac{T_{i-\frac{1}{2}}^{n+1}}{\Delta r_{i-\frac{1}{2}}} T_{i-\frac{1}{2}}^{n+1} = \left[ \frac{1}{r_i} \frac{r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}}}{\Delta r_{i-\frac{1}{2}} + \Delta r_{i+\frac{1}{2}}} + \frac{\alpha T}{\Delta t} \right] T_i^{n+1} - \alpha T_i^n T_i^n
\] (4.90)

Set

\[
\begin{align*}
A_T &= \frac{1}{r_i} \frac{r_{i-\frac{1}{2}}}{\Delta r_i} \\
B_T &= -\left[ \frac{1}{r_i} \frac{r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}}}{\Delta r_{i-\frac{1}{2}} + \Delta r_{i+\frac{1}{2}}} + \frac{\alpha T}{\Delta t} \right] \\
C_T &= \frac{1}{r_i} \frac{r_{i+\frac{1}{2}}}{\Delta r_i} \\
R_T &= -\frac{\alpha T_i^n}{\Delta t}
\end{align*}
\] (4.91)

the algebraic equation for heat transfer around the wellbore can be written as
\[ A_{T_i} T_{i-1}^{n+1} + B_{T_i} T_i^{n+1} + C_{T_i} T_{i+1}^{n+1} = R_{T_i} \] (4.92)

### 4.2.2 Boundary conditions

In this section, the solution of inner and boundary conditions are presented.

#### 4.2.2.1 Inner boundary condition

The inner boundary condition of the heat transfer equation is \( T |_{r=r_w} = T_{\text{well}} \). Hence, the algebraic equation for node 1 can be written as

\[ A_{T_1} T_{\text{well}} + B_{T_1} T_1^{n+1} + C_{T_1} T_2^{n+1} = R_{T_1} \] (4.93)

or

\[ B_{T_1} T_1^{n+1} + C_{T_1} T_2^{n+1} = R_{T_1} - A_{T_1} T_{\text{well}} \] (4.94)

where \( T_{\text{well}} \) is the temperature at the wellbore or wellbore temperature.

#### 4.2.2.2 Outer boundary condition

The outer boundary condition of the heat transfer equation is \( T |_{r=\infty} = T_i \). The simulation area is selected to be sufficient enough so that the boundary nodes have the temperature of the formation. Hence, the algebraic equation for node \( N \) can be written as

\[ A_{T_N} T_{N-1}^{n+1} + B_{T_N} T_N^{n+1} + C_{T_N} T_i = R_{T_N} \] (4.95)

or

\[ A_{T_N} T_{N-1}^{n+1} + B_{T_N} T_N^{n+1} = R_{T_N} - C_{T_N} T_i \] (4.96)

where \( T_i \) is initial reservoir temperature.
4.3 Solution of Fluid Flow Equation

In this section, the discretization of fluid flow equation is provided.

4.3.1 Algebraic equation

The total pressure equation (Equation 3.49) is discretized as

\[
\left( \frac{\lambda t k}{r} \right)_i \frac{1}{\Delta r} \left( r_{i+\frac{1}{2}} \frac{p^m_{n+1} - p_{n_i}}{\Delta r} - r_{i-\frac{1}{2}} \frac{p^m_{n+1} - p_{n_i}}{\Delta r} \right) + \\
\left( \frac{\lambda o k}{r} \right)_i \frac{1}{\Delta r} \left( r_{i+\frac{1}{2}} \frac{p^n_{wco_i+1} - p^n_{wco_i}}{\Delta r} - r_{i-\frac{1}{2}} \frac{p^n_{wco_i} - p^n_{wco_i-1}}{\Delta r} \right) + \hat{q}_t \tag{4.97}
\]

\[
\frac{1}{\Delta t} \left[ (1 - \varepsilon_v) \phi c_t \frac{p^m_{n+1} - p^n_{w_i}}{\Delta t} \right]
\]

Rearranging Equation 4.97, we obtain:

\[
\left( \frac{\lambda t k}{r} \right)_i \frac{1}{\Delta r} \left[ r_{i+\frac{1}{2}} \frac{p^n_{wco_i+1} - p^n_{wco_i}}{\Delta r} - r_{i-\frac{1}{2}} \frac{p^n_{wco_i} - p^n_{wco_i-1}}{\Delta r} \right] + \frac{(1 - \varepsilon_v) \phi c_t}{\Delta t} p^n_{w_i} + \\
\frac{(1 - \varepsilon_v) \phi c_t}{\Delta t} p^n_{w_i} = \frac{(1 - \varepsilon_v) \phi c_t}{\Delta t} \frac{p^n_{w_i} - p^n_{w_i}}{\Delta t}
\]

\[
B_{p_i} = - \left[ \left( \frac{\lambda t k}{r} \right)_i \frac{1}{\Delta r^2} \left( r_{i+\frac{1}{2}} + r_{i-\frac{1}{2}} \right) + \frac{(1 - \varepsilon_v) \phi c_t}{\Delta t} \right] = \left[ A_{p_i} + C_{p_i} + \frac{(1 - \varepsilon_v) \phi c_t}{\Delta t} \right] \tag{4.101}
\]

\[
R_{p_i} = - \left( \frac{\lambda t k}{r} \right)_i \frac{1}{\Delta r} \left( r_{i+\frac{1}{2}} \frac{p^n_{wco_i+1} - p^n_{wco_i}}{\Delta r} - r_{i-\frac{1}{2}} \frac{p^n_{wco_i} - p^n_{wco_i-1}}{\Delta r} \right) - (1 - \varepsilon_v) \phi c_t \frac{p^n_{w_i}}{\Delta t}
\]

The algebraic equation for fluid flow equation can be written as
\[ A_{p_i} p_{i+1}^{n+1} + B_{p_i} p_{i+1}^{n+1} + C_{p_i} p_{i+1}^{n+1} = R_{p_i} \]  
\( 4.103 \)

4.3.2 Boundary conditions

In this section, the solution of inner and outer boundary conditions are provided.

4.3.2.1 Inner boundary condition

At the wellbore, the pressure is constant and equal to the pressure of the drilling fluid, \( p \big|_{r=r_w} = p_{well} \). Hence, the algebraic equation for node 1 can be written as

\[ A_{p1} p_{well} + B_{p1} p_{w1}^{n+1} + C_{p1} p_{w2}^{n+1} = R_{p1} \]  
\( 4.104 \)

or

\[ B_{p1} p_{w1}^{n+1} + C_{p1} p_{w2}^{n+1} = R_{p1} - A_{p1} p_{well} = R^*_{p1} \]  
\( 4.105 \)

4.3.2.2 Outer boundary condition

The outer boundary equation for fluid flow equation is \( p \big|_{r=\infty} = p_i \), where \( p_i \) is the initial formation pressure. By selecting the simulation area sufficiently large, the node at the boundary node \( n \) has the pressure of the formation. Hence, the algebraic equation for node \( N \) can be written as

\[ A_{pN} p_{wN-1}^{n+1} + B_{pN} p_{wN}^{n+1} + C_{pN} p_i = R_{pN-2} \]  
\( 4.106 \)

or

\[ A_{pN} p_{wN-1}^{n+1} + B_{pN} p_{wN}^{n+1} = R_{pN-2} - C_{pN} p_i = R^*_{pN-2} \]  
\( 4.107 \)

4.3.3 Determination of saturation and oil phase pressure

Equation 3.36 can be written in cylindrical coordinates as

\[ \frac{k_w}{\mu_w} \frac{\partial^2 p_w}{\partial r^2} + \frac{1}{r} \frac{k_w}{\mu_w} \frac{\partial p_w}{\partial r} + \hat{q}_w = (1 - \varepsilon_v) \left[ \phi \frac{\partial S_w}{\partial t} + \phi (S_w c_w + S_w c_{\phi}) \frac{\partial p_w}{\partial t} \right] \]  
\( 4.108 \)

The water saturation can be obtained from Equation 4.108 as
we obtain:

\[ q_{wi} = \phi (1 - \varepsilon_v) i \left( \frac{S_{wi+1} - S_{wi}}{\Delta t} + (S_w c_w + S_w c_\phi) \frac{p_{wi+1} - p_{wi}}{\Delta t} \right) \]

Further simplifying equation above, with \( q_{wi} = 0 \) (no source term) and regular grid size, we obtain:

\[ \left( \frac{k_w}{\mu_w} \right)_i \frac{1}{\Delta r} \left[ \frac{p_{wi+1}^{n+1} - p_{wi}^{n+1}}{\Delta r_{i+\frac{1}{2}}} - \frac{p_{wi}^{n+1} - p_{wi-1}^{n+1}}{\Delta r_{i-\frac{1}{2}}} \right] + \left( \frac{k_w}{\mu_w} \right)_i \frac{1}{\Delta r} \frac{p_{wi+1}^{n+1} - p_{wi}^{n+1}}{\Delta r_{i+\frac{1}{2}}} \]

\[ + \phi (1 - \varepsilon_v) i \left( \frac{S_{wi+1}^{n+1} - S_{wi}^{n+1}}{\Delta t} + (S_w c_w + S_w c_\phi) \frac{p_{wi+1}^{n+1} - p_{wi}^{n+1}}{\Delta t} \right) \]

The water saturation can be found as

\[ S_{wi}^{n+1} = - (S_w c_w + S_w c_\phi) \frac{p_{wi+1}^{n+1} - p_{wi}^{n+1}}{\Delta r} + S_{wi}^{n+1} \]

\[ \frac{\Delta t}{\phi (1 - \varepsilon_v) i} \left[ \left( \frac{k_w}{\mu_w} \right)_i \frac{1}{\Delta r} \left( \frac{p_{wi+1}^{n+1} - 2p_{wi}^{n+1} + p_{wi-1}^{n+1}}{\Delta r^2} \right) + \left( \frac{k_w}{\mu_w} \right)_i \frac{1}{\Delta r} \frac{p_{wi+1}^{n+1} - p_{wi}^{n+1}}{\Delta r} \right] \]

The water saturation for node 1 can be determined as

\[ S_{w1}^{n+1} = - (S_w c_w + S_w c_\phi) \frac{p_{w1+1}^{n+1} - p_{w1}^{n+1}}{\Delta r} + S_{w1}^{n+1} \]

\[ \frac{\Delta t}{\phi (1 - \varepsilon_v)_1} \left[ \left( \frac{k_w}{\mu_w} \right)_1 \frac{1}{\Delta r} \left( \frac{p_{w2}^{n+1} - 2p_{w1}^{n+1} + p_{well}^{n+1}}{\Delta r^2} \right) + \left( \frac{k_w}{\mu_w} \right)_1 \frac{1}{\Delta r} \frac{p_{w2}^{n+1} - p_{w1}^{n+1}}{\Delta r} \right] \]

The water saturation for node \( N \) can be determined as

\[ S_{wN}^{n+1} = - (S_w c_w + S_w c_\phi) \frac{p_{wN+1}^{n+1} - p_{wN}^{n+1}}{\Delta r} + S_{wN}^{n+1} \]

\[ \frac{\Delta t}{\phi (1 - \varepsilon_v)_N} \left[ \left( \frac{k_w}{\mu_w} \right)_N \frac{1}{\Delta r} \left( \frac{p_i - 2p_{wN}^{n+1} + p_{wN-1}^{n+1}}{\Delta r^2} \right) + \left( \frac{k_w}{\mu_w} \right)_N \frac{1}{\Delta r} \frac{p_i - p_{wN}^{n+1}}{\Delta r} \right] \]

The oil saturation can be determined from water saturation as \( S_o^{n+1} = 1 - S_w^{n+1} \). Having water and oil saturation, oil phase pressure can be calculated from water pressure as

\[ p_{o}^{n+1} = p_w^{n+1} + p_{cwo}^{n+1} \]
4.4 Gridding Scheme

In this thesis, it is considered geomechanical unknowns as \( u_{i,j} \) at the edge of the node. Fluid-flow and thermal unknowns \( (p_{i,j}, T_{i,j}) \) are located in the middle of the grid system in Figure 4.1.

![Figure 4.1: Wellbore gridding scheme.](image)

4.5 Model Validation

In this section, we present the validation of the numerical solution with the analytical solution to validate the accuracy of the computer program before conducting the case studies. The numerical solution of geomechanical model is validated with the analytical solution for planar stress around wellbore in isotropic stress field with constant pore pressure. The numerical solution of heat transfer equation is validated with the analytical solution of steady-state heat transfer around wellbore. The numerical of fluid flow equation is validated with the analytical solution of steady-state pressure drop around wellbore.
4.5.1 Planar stress around wellbore in isotropic stress field with constant pore pressure

In this problem, we consider a vertical wellbore in the isotropic stress field with constant pore pressure as shown in Figure 4.2. The displacement and strains can be obtained analytically as

\[ u_r = C_1 r + \frac{C_2}{r} = \frac{\sigma_h}{2(\lambda + G)} r + \frac{\sigma_h - p_w r_w^2}{2G} \frac{r}{r} \tag{4.115} \]

\[ \varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{\sigma_h}{2(\lambda + G)} - \frac{\sigma_h - p_w r_w^2}{2G} \frac{1}{r^2} \]
\[ \varepsilon_{\theta\theta} = \frac{u_r}{r} = \frac{\sigma_h}{2(\lambda + G)} + \frac{\sigma_h - p_w r_w^2}{2G} \frac{1}{r^2} \tag{4.116} \]

Hence, the analytical solution for radial and hoop stresses around the wellbore are:

\[ \begin{cases} 
\sigma_{rr} = \sigma_h - (\sigma_h - p_w) \frac{r_w^2}{r^2} + \alpha p_p = \sigma_h \left( 1 - \frac{r_w^2}{r^2} \right) + \frac{r_w^2}{r^2} p_w + \alpha p_p \\
\sigma_{\theta\theta} = \sigma_h + (\sigma_h - p_w) \frac{r_w^2}{r^2} + \alpha p_p = \sigma_h \left( 1 + \frac{r_w^2}{r^2} \right) - \frac{r_w^2}{r^2} p_w + \alpha p_p 
\end{cases} \tag{4.117} \]

where \( p_p \) is pore pressure, \( p_w \) is wellbore pressure, \( G \) and \( \lambda \) is shear modulus and Lame’s constant.

Figure 4.2: Wellbore in an isotropic stress field.
Consider a case with the input parameters provided in Table 4.1, the comparison of analytical and numerical results of radial and hoop stresses variation with radial distance from wellbore is showed in Figure 4.3. The good match between numerical and analytical solutions confirms the accuracy of the numerical solution.

Table 4.1: Input parameters for geomechanical model validation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_n$</td>
<td>5000</td>
<td>psi</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.22</td>
<td>-</td>
</tr>
<tr>
<td>$E$</td>
<td>$2.5 \times 10^6$</td>
<td>psi</td>
</tr>
<tr>
<td>$r_w$</td>
<td>0.25</td>
<td>ft</td>
</tr>
<tr>
<td>$r_o$</td>
<td>2.5</td>
<td>ft</td>
</tr>
<tr>
<td>$p_w$</td>
<td>2000</td>
<td>psi</td>
</tr>
</tbody>
</table>

4.5.2 Steady-state heat transmission around wellbore

In this problem, we consider a cylinder with the temperature inside is kept at constant temperature $T_1$, and the outside temperature is kept at constant temperature $T_2$. The steady
state heat transfer \( \left( \frac{\partial T}{\partial t} = 0 \right) \) around this cylinder can be obtained by solving Equation 3.51 as following

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = 0 \quad (4.118)
\]

with the two boundary conditions:

\[
\begin{align*}
T \mid_{r=r_1} &= T_1 \\
T \mid_{r=r_2} &= T_2
\end{align*}
\quad (4.119)
\]

where \( r_1 \) and \( r_2 \) are inside and outside radius of the cylinder.

The heat distribution in the cylinder wall can be obtained analytically as

\[
T = \frac{T_2 - T_1}{\ln \frac{r_1}{r_2}} + \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln r_1 \quad (4.120)
\]

Using the input parameters in Table 4.2, we can obtain the analytical and numerical solution of steady-state heat transfer in the cylinder as shown in Figure 4.4. The close match between the analytical and numerical solutions validates the accuracy of the numerical solution.

### Table 4.2: Input parameters for heat transfer model validation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_r )</td>
<td>0.142</td>
<td>btu/(lb.(^\circ)F)</td>
</tr>
<tr>
<td>( \rho_r )</td>
<td>160</td>
<td>lb/ft(^3)</td>
</tr>
<tr>
<td>( K )</td>
<td>17.07</td>
<td>btu/(day*ft(^\circ)F)</td>
</tr>
<tr>
<td>( \Delta t )</td>
<td>10</td>
<td>sec</td>
</tr>
<tr>
<td>( T_1 )</td>
<td>100</td>
<td>(^\circ)F</td>
</tr>
<tr>
<td>( T_2 )</td>
<td>200</td>
<td>(^\circ)F</td>
</tr>
<tr>
<td>( r_1 )</td>
<td>1</td>
<td>ft</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>10</td>
<td>ft</td>
</tr>
</tbody>
</table>
4.5.3 Steady-state pressure drop around wellbore

Consider a wellbore with radius $r_w$ is producing oil from a finite uniform formation with the outer radius $r_o$, when the formation flow reach steady-state ($\frac{\partial p}{\partial t} = 0$), the pressure around the wellbore can be obtained by solving the following governing equation

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} = 0 \tag{4.121}$$

with the two boundary conditions:

$$\begin{align*}
    p |_{r=r_w} &= p_{well} \\
    p |_{r=r_o} &= p_f
\end{align*} \tag{4.122}$$

The pressure as a function of the radial distance to the wellbore can be obtained

$$p = \frac{p_i - p_{well}}{\ln \frac{r_w}{r_o}} \ln r + p_{well} - \frac{p_w - p_i}{\ln \frac{r_w}{r_o}} \ln r_w \tag{4.123}$$

where $p_i$ is the pressure at the outer boundary of the formation.

Using the input parameters in Table 4.3, we can obtain the analytical and numerical solution of steady-state pressure profile around the wellbore as shown in Figure 4.5. It can be observed that the numerical solution matches between the analytical solution. Hence,
this validates the accuracy of the numerical solution of fluid flow equation.

Table 4.3: Input parameters for fluid flow model validation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.6</td>
<td>cp</td>
</tr>
<tr>
<td>$k$</td>
<td>1</td>
<td>md</td>
</tr>
<tr>
<td>$c_t$</td>
<td>$10^{-6}$</td>
<td>1/psi</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.1</td>
<td>-</td>
</tr>
<tr>
<td>$\Delta r$</td>
<td>0.05</td>
<td>ft</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta t$</td>
<td>10</td>
<td>sec</td>
</tr>
<tr>
<td>$p_{well}$</td>
<td>2000</td>
<td>psi</td>
</tr>
<tr>
<td>$p_i$</td>
<td>5000</td>
<td>psi</td>
</tr>
<tr>
<td>$r_w$</td>
<td>1</td>
<td>ft</td>
</tr>
<tr>
<td>$r_o$</td>
<td>30</td>
<td>ft</td>
</tr>
</tbody>
</table>

(a) Pressure around wellbore.

(b) Fluid flow model validation.

Figure 4.5: Fluid flow model validation.
CHAPTER 5
RESULTS AND DISCUSSION

In this chapter, I conducted case studies using the developed model to investigate the effect of different factors, namely the effect of fluid, the effect of temperature, the effect transport properties and the effect of fluid and rock interaction on the integrity of the wellbore. The main objective is to evaluate the impact of these factors on the minimum and maximum mud weight to provide the insight for practical operations.

5.1 Comparison of Coupled and Conventional Models

In this section, I compare the minimum and maximum mud weights estimated by only geomechanical model and coupled model using the data from Table 5.1. The Mohr’s circle at the failure of two cases, only geomechanical model and coupled model, is plotted in Figure 5.1. It can be observed a significant difference in the stress state between two case. The main reason for the difference is the coupled model accounted for thermal and flow induced stresses. It can be observed that the Mohr’s circle at the failure of the coupled model is larger and shifting away from the failure envelope indicating that the minimum mud weight obtained from the coupled model is smaller than that from the conventional model. The minimum mud weight obtained from the conventional model, only geomechanical model, is 7.62 ppg. However, calculations made by coupled model suggest that the minimum mud weight is 7.06 ppg. This is because of the two main reasons. Firstly, when breakout occurs, stresses around the wellbore push the fluid from the formation toward the wellbore resulting in flow-induced stresses. Secondly, the lower temperature at the wellbore reduces the rock volume, shrinkage, hence reducing the stress acting toward the wellbore. It is also observed that the tensile failure occurs at higher mud weight as predicted by geomechanical model. The maximum mud weight for geomechanical model is 22.44 ppg, while the the maximum mud weight for coupled model is 22.07 ppg. The main reason for this difference is because
the build-up of pore pressure has stronger effect on the effective stress than the effect of the temperature. When the drilling fluid penetrates into the formation increasing the pore pressure nearby the wellbore, reducing the required mud weight to create a fracture on the wellbore. While the temperature has counter effect on the maximum mud weight, but the effect of rock shrinkage due to the reduction of temperature is smaller than that of the pore pressure alteration. Hence, the maximum mud weight obtained from the coupled model is smaller than that of the conventional model.

Although the difference between two models is not very significant in this case study, but for the case with a very narrow mud weight, when the maximum mud weight is close to minimum mud weight, this difference is very important in selecting a proper mud weight. Hence, in case of narrow mud weight, for example ultra-deep wells or high pressure and high temperature wells, the coupled model should be used to determine a suitable mud weight for safely drilling the wellbore. Due to the time limit and computational power available for this project, the case studies presented here are conducted for an isotropic stress field, hence the mud weight window obtained here is wider than what is typically observed at the same depth. However, the fundamental physics is remained the same, and the observations should be valid for anisotropic stress field. Although only isotropic stress field is considered, the model and computer program were developed for anisotropic stress field.

5.2 Effect of Temperature Change on Mud Weight Window

In this case study, the thermal effect on the collapse pressure and fracture pressure is analyzed. The maximum and minimum mud weights for seven different wellbore temperatures ranging from 40 to 160 °C as shown in Table 5.2. The calculated mud weight for each case using the coupled model is summarized in Table 5.2. The results show that when wellbore temperature decreases, the minimum and maximum mud weight are decreasing, but the minimum mud weight is decreasing slowly than the maximum mud weight. Therefore, mud weight window become wider. This indicates that the formation is strengthen when the temperature decreases. When temperature decreases, it reduces the rock volume making the
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>TVD</td>
<td>12000</td>
<td>ft</td>
<td>$C_o$</td>
<td>1443</td>
<td>psi</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>10</td>
<td>seconds</td>
<td>$\rho_r$</td>
<td>2560</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>$N_{step}$</td>
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<td>day</td>
<td>$c_r$</td>
<td>594</td>
<td>m/(sec$^2 \cdot$°C)</td>
</tr>
<tr>
<td>$E$</td>
<td>6.8</td>
<td>Mpsi</td>
<td>$K$</td>
<td>7.0842</td>
<td>kg/(sec$^3 \cdot$°C)</td>
</tr>
<tr>
<td>UCS</td>
<td>5000</td>
<td>psi</td>
<td>$\alpha_{dif}$</td>
<td>26.7654</td>
<td>sec/m$^2$</td>
</tr>
<tr>
<td>$\mu_i$</td>
<td>30</td>
<td>deg</td>
<td>$\beta$</td>
<td>$10^{-6}$</td>
<td>C$^{-1}$</td>
</tr>
<tr>
<td>$[\sigma_T]$</td>
<td>100</td>
<td>psi</td>
<td>$\sigma_h$</td>
<td>7000</td>
<td>psi</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.22</td>
<td>-</td>
<td>$\Delta r$</td>
<td>0.05</td>
<td>ft</td>
</tr>
<tr>
<td>$p_i$</td>
<td>5000</td>
<td>psi</td>
<td>$S_{wr}$</td>
<td>0.2</td>
<td>-</td>
</tr>
<tr>
<td>$r_o$</td>
<td>9</td>
<td>ft</td>
<td>$S_{or}$</td>
<td>0.2</td>
<td>-</td>
</tr>
<tr>
<td>$r_w$</td>
<td>0.3</td>
<td>ft</td>
<td>$n_w$</td>
<td>3</td>
<td>-</td>
</tr>
<tr>
<td>$k$</td>
<td>1</td>
<td>md</td>
<td>$n_o$</td>
<td>2</td>
<td>-</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.1</td>
<td>-</td>
<td>$\sigma_H$</td>
<td>7000</td>
<td>psi</td>
</tr>
<tr>
<td>$c_t$</td>
<td>$10^{-6}$</td>
<td>psi$^{-1}$</td>
<td>$t_{max}$</td>
<td>1</td>
<td>day</td>
</tr>
<tr>
<td>$c_{\phi}$</td>
<td>$10^{-6}$</td>
<td>psi$^{-1}$</td>
<td>$\sigma_v$</td>
<td>7000</td>
<td>psi</td>
</tr>
<tr>
<td>$c_o$</td>
<td>$10^{-6}$</td>
<td>psi$^{-1}$</td>
<td>$k^*_r$</td>
<td>0.5</td>
<td>-</td>
</tr>
<tr>
<td>$G$</td>
<td>2.78</td>
<td>Mpsi</td>
<td>$k^*_r$</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>$T_{well}$</td>
<td>80</td>
<td>°C</td>
<td>$\mu_o$</td>
<td>0.6</td>
<td>cp</td>
</tr>
<tr>
<td>$T_i$</td>
<td>100</td>
<td>°C</td>
<td>$\mu_w$</td>
<td>1</td>
<td>cp</td>
</tr>
<tr>
<td>$S_{wx}$</td>
<td>0.35</td>
<td>-</td>
<td>$\alpha_1$</td>
<td>-30.55</td>
<td>-</td>
</tr>
<tr>
<td>$p_i$</td>
<td>5000</td>
<td>psi</td>
<td>$\lambda$</td>
<td>2.189</td>
<td>Mpsi</td>
</tr>
</tbody>
</table>

formation stronger, thus we can use lower mud weight without causing breakout (Figure 5.2). The opposite trend is observed for the maximum mud weight as shown in the Figure 5.2.
Figure 5.1: Fluid flow and temperature effect on the Mohr’s circles at the failure.

Because using hot drilling fluid increases breakdown pressure. That is why the higher mud weight is required to fracture the formation. Hence, in the case of narrow window mud weight, we can control the mud temperature to widen the mud weight window that allows us to drill wellbore safely.

Table 5.2: Calculated mud weight for different temperature

<table>
<thead>
<tr>
<th>Wellbore temperature</th>
<th>Minimum mud weight</th>
<th>Maximum mud weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter Value Unit</td>
<td>Value Unit</td>
<td>Value Unit</td>
</tr>
<tr>
<td>(T_{w_1}) 160°C</td>
<td>7.81 ppg</td>
<td>23.57 ppg</td>
</tr>
<tr>
<td>(T_{w_2}) 140°C</td>
<td>7.62 ppg</td>
<td>23.19 ppg</td>
</tr>
<tr>
<td>(T_{w_3}) 120°C</td>
<td>7.44 ppg</td>
<td>22.82 ppg</td>
</tr>
<tr>
<td>(T_{w_4}) 100°C</td>
<td>7.25 ppg</td>
<td>22.44 ppg</td>
</tr>
<tr>
<td>(T_{w_5}) 80°C</td>
<td>7.06 ppg</td>
<td>22.07 ppg</td>
</tr>
<tr>
<td>(T_{w_6}) 60°C</td>
<td>6.88 ppg</td>
<td>21.69 ppg</td>
</tr>
<tr>
<td>(T_{w_7}) 40°C</td>
<td>6.69 ppg</td>
<td>21.31 ppg</td>
</tr>
</tbody>
</table>
It is also observed that, the maximum and minimum mud weights are linearly dependent on temperature. The main physical reason for this is because thermal-induced stress is linearly dependent on temperature. Similar results are obtained by Yan et al. (2014). They studied effect of temperature on wellbore stability in high-temperature formations. They also investigated the effect of temperature on rock mechanical properties on different formations. Based on thermoelastic theory, they calculated the stress around wellbore accounting for thermal effect. Their result show that the fracture pressure is more sensitive to temperature change. Moreover, when temperature decrease, which leads to formation contraction, the collapse and fracture pressure decrease simultaneously. When temperature increases it causes formation to expand, which results in an increase of both collapse and fracture pressure. Zhang et al. (2019) also have the similar observation when they investigated the effect of temperature on time dependent wellbore stability in geothermal drilling. Finite-difference method, based on the transient heat transfer model, was used to determine the effect of temperature on wellbore stability. Then, the stress distribution around wellbore for transversely isotropic formation was obtained from coupling (1) time-dependent temperature, which was
applied to wellbore boundaries, with (2) temperature-stress by Duhamel principle. Their results show that the collapse pressure also increases with the increase of wellbore temperature. This is because of increase of temperature directly increases the stress difference at the minimum horizontal stress. Thus, shear failure possibility in wellbore is increased.

5.3 Effect of Permeability on Mud Weight Window

This case study intends to study the effect of one of the most important transport property, permeability, on the minimum and maximum mud weights. The mud weight is calculated for different permeability using the coupled model as presented in Table 5.3. It can be observed that from the result presented in Figure 5.3 the mud weight window is widen with the increase of the formation permeability. In other words, the maximum mud weight increases with the increase of the permeability, while the minimum mud weight decreases with the increase of the permeability. It can also be observed that the effect of permeability on the maximum and minimum mud weights is not very significant. The main reason is in this research we study the stability of the wellbore in 1 day so the pore pressure of the nodes next to the wellbore after that long time ($t_{max} = 1$ day) is almost equal to the wellbore pressure. However, the result could be significantly different if we consider the stability of the wellbore in shorter time frame.

Table 5.3: Effect of permeability on mud weight window

<table>
<thead>
<tr>
<th>Permeability</th>
<th>Minimum mud weight</th>
<th>Maximum mud weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_1$</td>
<td>100 md</td>
<td>7.02 ppg</td>
</tr>
<tr>
<td>$k_2$</td>
<td>10 md</td>
<td>7.03 ppg</td>
</tr>
<tr>
<td>$k_3$</td>
<td>1 md</td>
<td>7.04 ppg</td>
</tr>
<tr>
<td>$k_4$</td>
<td>0.1 md</td>
<td>7.06 ppg</td>
</tr>
</tbody>
</table>
5.4 Effect of Wettability on Mud Weight Window

In this thesis, a numerical study was conducted to investigate the effect of wettability on drilling fluid penetration into the formation, and how it affects the selection of the mud weight. In this case, five formations with different wettability are investigated. The wettability of these formations are ranging from highly water-wet to highly oil-wet. Different sets of permeability and capillary pressure parameters were used to draw the relative permeability and capillary pressure curves as shown in Table 5.4. The relative permeability and capillary pressure for these five different cases are plotted in Figure 5.4 and Figure 5.5, respectively.

The wettability index ($W_I$) is calculated using the model by Amott-Harvey as

$$W_I = \frac{\text{Spontaneous Water Imbibition}}{\text{Total Water Imbibition}} - \frac{\text{Spontaneous Oil Imbibition}}{\text{Total Oil Imbibition}}$$

From the results, it is observed that when the formation changes from highly water-wet to highly oil-wet, the minimum and the maximum mud weights are increasing as shown in Figure 5.6. For water-wet formations collapse of the wellbore occurs at lower mud weights.
Figure 5.4: Relative permeability used to study the effect of wettability.

Figure 5.5: Capillary pressure used to study the effect of wettability.
Table 5.4: Input parameters used for obtaining the relative permeabilities and capillary pressures

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
<th>Case 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_w )</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( n_o )</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( k_{ro}^* )</td>
<td>0.8</td>
<td>0.7</td>
<td>0.45</td>
<td>0.6</td>
<td>0.5</td>
</tr>
<tr>
<td>( k_{rw}^* )</td>
<td>0.1</td>
<td>0.2</td>
<td>0.5</td>
<td>0.9</td>
<td>1</td>
</tr>
<tr>
<td>( S_{wr} )</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>( S_{or} )</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>( S_{wx} )</td>
<td>0.65</td>
<td>0.6</td>
<td>0.5</td>
<td>0.4</td>
<td>0.35</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>-30.55</td>
<td>-30.55</td>
<td>-30.55</td>
<td>-30.55</td>
<td>-30.55</td>
</tr>
<tr>
<td>( WI )</td>
<td>0.5</td>
<td>0.33</td>
<td>0.16</td>
<td>-0.33</td>
<td>-0.5</td>
</tr>
<tr>
<td>min MW</td>
<td>6.98</td>
<td>7.02</td>
<td>7.03</td>
<td>7.07</td>
<td>7.09</td>
</tr>
<tr>
<td>max MW</td>
<td>21.74</td>
<td>21.8</td>
<td>21.85</td>
<td>21.98</td>
<td>22.02</td>
</tr>
<tr>
<td>Wettability</td>
<td>Highly water-wet</td>
<td>Water-wet</td>
<td>Mixed-wet</td>
<td>Oil-wet</td>
<td>Highly oil-wet</td>
</tr>
</tbody>
</table>

than in highly oil wet formations. However, higher maximum mud weight is required in order to fracture the oil-wet formations than water-wet formations. The main reason for this is the wettability affects the transport of phases in and out of formation, hence, it affects pore pressure. For highly water-wetted formations, water can easily move toward the wellbore, when using low mud density, decreasing the pore pressure of the rock nearby the wellbore. This pushes the Mohr’s circle away from the failure envelope, hence, making the rock stronger. In other words, we can use lower mud weight to prevent the wellbore from breaking out. Using excessively high mud weight forces the drilling fluid to invade into the formation. If the formation is highly water-wetted, the drilling fluid can easily move into the formation. Therefore, the pore pressure build up nearby the wellbore is smaller than
that of the highly oil-wetted formation. In other words, it requires higher mud pressure to fracture the highly oil-wetted formation. However, the effect of wettability is not significant, because the rock elements on the bore wall have pore pressures quit close to the wellbore pressure. However, for very low permeability formations such as shale formations, the effect of wettability can be significant, because it takes longer time to change the pore pressure of the rock nearby the wellbore. Therefore, several researches have suggested that changing wettability can be an efficient method to improve wellbore stability (Yue et al. 2018).

![Figure 5.6: Effect of wettability change on mud weight window.](image)

5.5 Effect of Fluid and Rock Interaction

Water saturation has a significant impact on the geomechanical properties of the rock due to fluid and rock interaction. Mechanical properties such as Young’s Modulus, Unconfined Compressive Strength (UCS) and friction angle can be affected by the water content. As the drilling fluid invades into the rock formation, it typically weaken the rock, hence changing the mechanical properties of the rock as well as changing the stress distribution around the wellbore. This study intends to investigate the effect of Young’s Modulus, UCS and friction angle on the maximum and minimum mud weight.
5.5.1 Effect of the Young’s Modulus variation

It is often found that in the literature that the Young’s Modulus is linearly dependent with the increase of water saturation (Chenevert 1970).

\[ E = E_i - K_E \Delta S_w \]  \hspace{1cm} (5.2)

where \( E_i \) is the initial Young’s Modulus, \( K_E \) is the Young’s Modulus change coefficient, \( \Delta S_w \) is the change of water saturation.

Lin and Lai (2013) investigated experimentally the effect of water saturation on Barnett’s geomechanical properties. 19 core samples was used and saturated with KCl water to prevent water swelling. As result, water saturation increase lead to the Young’s Modulus and UCS decrease. Also, as water saturation increases to 1%, the Young’s Modulus is increased to 6.1% when compared to dry core samples. They also obtained following correlation for the Young’s Modulus versus water content:

\[ E = -22820S_w + 3758859 \]  \hspace{1cm} (5.3)

where \( E \) is the Young’s Modulus, psi; \( S_w \) is water saturation, %.

In this study, I investigated the effect of Young’s Modulus change coefficient on mud weight window as shown in Table 5.5. During drilling, there is small amount of drilling fluid penetrating into the formation. But when drilling fluid density increases, more drilling fluid penetrating the formation and changing the Young’s Modulus of the formation. This is why the minimum mud weight decreases slowly than the maximum mud weight as shown in Figure 5.7. The higher the Young’s Modulus change coefficient the higher the Young’s Modulus drop when contacting with water. The formation is more sensitive to water saturation when the higher the Young’s Modulus coefficient change. When the water saturation increases, the rock is becoming weaker, this indicates that lower mud weight can be safely used to drill the formation. It should be noted for minimum mud weight, the wellbore pressure is lower than the pore pressure resulting in the flow of phases toward wellbore reducing oil saturation because oil phase has higher mobility at the initial saturation. In other words,
even though the wellbore pressure is lower than the pore pressure the water saturation in
the rock increases resulting in the change of rock mechanical property as captured by the
model.

Table 5.5: Effect of Young’s Modulus change coefficient on mud weight window

<table>
<thead>
<tr>
<th>Young’s Modulus change coefficient</th>
<th>Minimum mud weight</th>
<th>Maximum mud weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter Value Unit</td>
<td>Value Unit</td>
<td>Value Unit</td>
</tr>
<tr>
<td>$K_{E1}$ 282 kpsi</td>
<td>7.05 ppg</td>
<td>22.05 ppg</td>
</tr>
<tr>
<td>$K_{E2}$ 1282 kpsi</td>
<td>7.03 ppg</td>
<td>21.98 ppg</td>
</tr>
<tr>
<td>$K_{E3}$ 2282 kpsi</td>
<td>7.00 ppg</td>
<td>21.92 ppg</td>
</tr>
<tr>
<td>$K_{E4}$ 3282 kpsi</td>
<td>6.97 ppg</td>
<td>21.85 ppg</td>
</tr>
<tr>
<td>$K_{E5}$ 4282 kpsi</td>
<td>6.94 ppg</td>
<td>21.78 ppg</td>
</tr>
</tbody>
</table>

Figure 5.7: Effect of Young’s Modulus change coefficient on mud weight window.
5.5.2 Effect of internal friction coefficient variation

It is observed in the literature that when water saturation increases, it decreases the friction angle of the rock.

\[
\mu_i = \mu_{i_0} - K_\mu \Delta S_w
\]  \hspace{1cm} (5.4)

where \(\mu_{i_0}\) is initial friction coefficient, \(K_\mu\) is the friction angle change coefficient.

Ikari et al. (2007) investigated how water saturation affects the internal friction angle of montmorillonite-based fault gouge. Four different mixtures of Ca-montmorillonite and quartz were used in the experiment with four different water content. They observed that friction angle decreased with the increase of water content. Specifically, friction angle of 50/50 mixtures for dry samples ranging from 0.57 to 0.64 decreased to ranges of 0.21 to 0.55 in hydration cases. Also, the same trend was observed for 100% montmorillonite samples. Its friction angle ranging from 0.41 to 0.62 for dry conditions decreased to a range of 0.03 to 0.29 for wet conditions. From their plot of friction angle versus water saturation the following correlation can be obtained:

\[
\mu_i = -0.019\Delta S_w + 0.5395
\]  \hspace{1cm} (5.5)

where matching coefficient is 0.4899, \(\mu_i\) is internal friction coefficient, dimensionless, \(\Delta S_w\) is the change water saturation, %.

In this case study, sensitivity analysis is conducted to the effect of the change internal friction angle on the stability of the wellbore and how it affects the minimum mud weight estimation. Six cases with different friction angle change coefficient are selected for investigation as shown in Table 5.6. The higher the coefficient \(K_2\) indicates the more sensitive the internal friction coefficient change with water saturation. As water saturation increases, the friction angle decreases. Using \(K_\mu = 0.02\), the internal friction angle decreases from 30 to 24.6 degree, increases the tendency of the rock to fail under compression as shown in Figure 5.8. Hence, it is requires higher mud weight to prevent the wellbore from the breakout as shown in Figure 5.9. However, the maximum mud weight is not affected by friction
angle change, because the friction angle is used to calculate only breakout failure, not tensile failure as in the determination of the maximum mud weight.

Table 5.6: Effect of internal friction angle change coefficient on mud weight window

<table>
<thead>
<tr>
<th>Internal friction angle change coefficient</th>
<th>Minimum mud weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>Value</td>
</tr>
<tr>
<td>$K_{\mu_1}$</td>
<td>0</td>
</tr>
<tr>
<td>$K_{\mu_2}$</td>
<td>0.0025</td>
</tr>
<tr>
<td>$K_{\mu_3}$</td>
<td>0.0050</td>
</tr>
<tr>
<td>$K_{\mu_4}$</td>
<td>0.0100</td>
</tr>
<tr>
<td>$K_{\mu_5}$</td>
<td>0.0150</td>
</tr>
<tr>
<td>$K_{\mu_6}$</td>
<td>0.0200</td>
</tr>
</tbody>
</table>

5.5.3 Effect of unconfined compressive strength variation

The unconfined compressive strength of rocks is often linearly dependent with the increase of water saturation.

$$UCS = UCS_i - K_{UCS} \Delta S_w$$  \hspace{1cm} (5.6)

where $UCS_i$ is initial unconfined compressive strength, $K_{UCS}$ is UCS change coefficient, $\Delta S_w$ is the change of water saturation.

Water saturation could facilitate to deformation and the failure of the rock. Bao et al. (2019) investigated the water saturation effect on failure and the deformation of the rock. In their study, Brazilian test and uniaxial compression test are used. Moreover, six different conditions for water saturation were used and rock samples were submerged into water and then dried. After that, compressive and tensile tests were done. As the result, both compressive and tensile strengths are decreased linearly to the logarithm of water content. Lin and Lai (2013) experimentally investigated the effect of water saturation on the Barnett
Figure 5.8: Mohr’s circles at the failure (with and without the change of internal friction coefficient).

Figure 5.9: Internal friction change coefficient on minimum mud weight.
shale sample geomechanical properties. Their experimental results suggest that both the Young’s Modulus and unconfined compressive strength are decreased with the increase of water saturation. However, the Poisson’s Ratio is less dependent on water saturation. They found that the unconfined compressive strength has linear relationship to water content as follows

\[ \text{UCS} = -43S_w + 4357 \]  \hspace{1cm} (5.7)

where matching coefficient is 0.7627, \( S_w \) is water saturation, %.

In this case study, I investigate the effect of UCS change coefficient on the minimum mud weight. Four different rate of changes of the UCS are considered in this case study as summarized in Table 5.7. It can be observed that when the water saturation increase the rock is weakening making it easier to fail under compression, hence requiring higher mud weight to prevent the wellbore from breakout failure (Figure 5.10). When UCS changes it will affect the cohesive strength of the rock and failure envelope. These parameters used to determine when the rock is likely to fail. When the UCS of the rock is more sensitive to water saturation, formation become weaker, cohesive strength is reduced, thus it is easier to break the formation.

Table 5.7: Effect of UCS change coefficient on mud weight window

<table>
<thead>
<tr>
<th>UCS change coefficient</th>
<th>Minimum mud weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>Value Unit</td>
</tr>
<tr>
<td>( K_{\text{UCS}_1} )</td>
<td>43 psi</td>
</tr>
<tr>
<td>( K_{\text{UCS}_2} )</td>
<td>83 psi</td>
</tr>
<tr>
<td>( K_{\text{UCS}_3} )</td>
<td>166 psi</td>
</tr>
<tr>
<td>( K_{\text{UCS}_4} )</td>
<td>332 psi</td>
</tr>
</tbody>
</table>
Figure 5.10: UCS change coefficient on minimum mud weight.
In this chapter the conclusions and recommendations about this research are presented.

6.1 Summary and Remarks

In this thesis, I developed a numerical coupled model to study several factors affecting the determination of the safe mud window. Based on the result from numerical studies presented here, the following concluding remarks can be drawn:

1. The effect of temperature variation on mud weight window can result in more than 5% change in mud weight. As temperature decreases, the formation becomes stronger resulting in lower minimum and maximum mud weights. This is important because a lot of additives are often needed to increase mud density by 5%. This implies that the changing the temperature of drilling fluid can help to overcome instability issue when drilling through formations with narrow mud weight window.

2. We found that the change of mechanical properties due to the invasion of drilling fluids into the formation is the most important factor affecting the determination of mud weight window. The sensitivity of the Young’s Modulus, UCS and internal friction coefficient with water saturation are the key factors to consider when studying the effect of fluid and rock interaction on the stability of the wellbore.

3. The effect of permeability and wettability have a small effect on the mud weight window. Higher mud weight can be used to drill through highly permeable formation without fracturing the formation. Changing the wettability of the drilling fluid is also considered as one of the method to improve the stability of the wellbore, especially for the formation which is highly sensitive to water saturation.
6.2 Recommendations

This study provides a model to investigate the effect of different factors on the determination of the safe mud window. This tool is particularly important to study the time-dependent wellbore stability and determine a more accurate mud weight window. Based on the results from this study, we recommend the following for further research and practical applications:

1. Coupled modeling is a very useful approach to study the complexity of the physics at the bottom of the wellbore to understand the physical interaction between fluid and rock and estimate suitable mud weight for drilling through complex formations. Hence, future research on the wellbore stability analysis should be focused more on coupled modeling.

2. More effort should be spent on the interaction between fluid and rock to accurately evaluate the long-term stability of the wellbore. More experimental work should be conducted to study the sensitivity of water saturation on formation mechanical properties such as the Young’s Modulus, UCS, and internal friction angle to determine a suitable mud weight for safely drilling the wellbore.

3. The model developed in this thesis can also be used to study the integrity of the cement sheath during hydraulic fracturing operations as well as other complex problems related to the wellbore.
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