REFLECTIONS OF INTERNAL GRAVITY WAVES OFF BUMPY SURFACES

by

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ABSTRACT

Waves in a density stratified fluid, termed internal gravity waves, reflected by a sloping plane make an angle with the horizontal equal to the angle between the incident wave and the horizontal. We add a perturbation to the plane, small in size when compared to the amplitude of the wave motion. An asymptotic expansion is used and problems are derived up to two orders $O(1)$ and $O(\epsilon)$. The leading order problem is the same as the reflection problem by a flat plane. The problem on $O(\epsilon)$ is a radiation problem that we solve using Fourier transforms.

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CHAPTER 1
INTRODUCTION

Oceans and other bodies of water may contain a continuous density stratification due to changes in temperature or salinity, such as at the mouth of a fresh water river hitting a salty sea. Turbulent mixing of these fluids can cause this density stratification, and oscillations may occur within the fluid with buoyancy and gravity as restoring forces. This type of wave motion is called an internal gravity wave, is anisotropic, and occurs in inviscid, incompressible fluids.

In chapters 2 and 3 we start with Euler's equations for invscid, incompressible fluid motion. Assuming time-harmonic motion and using the Boussinesq approximation we derive a hyperbolic second order, linear, three dimensional partial differential equation for the reduced perturbed pressure of the fluid (perturbed pressure in the sense that it is the pressure change from a hydrostatic equilibrium and reduced, since we divide by the density). We then assume plane wave solutions and derive some basic properties such as phase and group velocity.

In chapter 4 we examine reflections of acoustic waves, waves generated in compressible fluids, to gain an understanding of the technique used to solve reflection problems. We assumed plane wave solutions. Acoustic waves were chosen because the reflections behave more intuitively than the internal gravity waves. The reflections of acoustic waves are more intuitive in that the angle between the incident wave and the normal vector to the surface is the same as the angle between the reflected wave and the normal vector. This is more intuitive as this is how a solid object, such as a billiard ball, will reflect off a physical boundary and how light reflects off a mirror.

Chapter 5 is where we look at the internal wave reflection problem off plane boundaries, again assuming plane wave solutions. This problem was first discussed by Phillips [6] and we use a similar method. This problem is a stepping stone to analysis of internal wave reflections off perturbed, or bumpy, surfaces, chapter 6. The two dimensional perturbed problem is examined by Baines [2]. He derives and solves a Fredholm integral equation to describe the reflected wave. We use an alternate method where we examine the differential equation using an asymptotic expansion for the pressure and Fourier transforms.

CHAPTER 2
MATHEMATICAL FORMULATION

Before we are able to look at reflections of internal gravity waves we need to derive the governing equations that model such systems and to gain some understanding of what these equations mean physically and how solutions of these equations behave. We do that here by deriving the governing partial differential equation for wave motion in an incompressible, inviscid, continuously stratified fluid, looking for plane wave solutions and comparing these solutions to plane wave solutions of the three dimensional wave equation, the governing partial differential equation for wave motion in a compressible, inviscid fluid. The latter are more familiar as they represent sound waves.

2.1 DERIVATION OF GOVERNING EQUATIONS FOR INTERNAL WAVES

We first need to derive the governing equations for the wave motion in an inviscid, incompressible, and continuously stratified fluid. In this we begin with the same process as outlined by Akylas and Mei [1] and break off at the end to derive an equation for pressure as opposed to an equation for the vertical velocity component. In the following we use $\mathbf{v} = \langle u, v, w \rangle$ to represent the fluid velocity, p for the pressure, and ρ for density and we use standard Cartesian coordinates with z pointing upwards. Consider a material element of the fluid. This element must satisfy the continuity equation

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.1}$$

and Euler's equations for incompressible, inviscid fluid motion

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial x} \tag{2.3}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial y} \tag{2.4}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial z} + g, \tag{2.5}$$

where $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ represents the total or material derivative and g is the acceleration due to gravity.

Substituting (2.2) into (2.1) gives the following equation for density

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0, \tag{2.6}$$

so ρ must be constant for a material element.

We assume small amplitude motion. That is, the scale of the motion, the scale of the pressure or density perturbation is much much smaller than the scale of the equilibrium pressure or equilibrium density. Assuming the velocities are small, that is $\mathbf{v} = \mathrm{O}(\epsilon)$ where $\epsilon \ll 1$, and linearize the momentum equations (2.3)-(2.5), by ignoring any term of order ϵ^2 or smaller to get

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \tag{2.7}$$

$$\rho\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} \tag{2.8}$$

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - g\rho, \tag{2.9}$$

We assume density and pressure are initially in a state of hydrostatic equilibrium and the wave motion is a perturbation of that equilibrium, so

$$p = \bar{p}(z) + p'(x, y, z, t)$$

$$\rho = \bar{\rho}(z) + \rho'(x, y, z, t),$$

where $\bar{p}$ and $\bar{\rho}$ are the equilibrium pressure and density, respectively, which depend on depth z, and $p'(x, y, z, t)$ and $\rho'(x, y, z, t)$ are the pressure and density perturbations, respectively. Put the above into (2.6) and using the fact that $\bar{\rho}$ does not depend on x, y, and t we arrive at

$$0 = \frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{\partial \bar{\rho}}{\partial z} + w\frac{\partial \rho'}{\partial z}.$$

For small amplitude motion, the following terms are negligible: $u\frac{\partial \rho'}{\partial x}$, $v\frac{\partial \rho'}{\partial y}$, $w\frac{\partial \rho'}{\partial z}$, and the above becomes

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \bar{\rho}}{\partial z} = 0. \tag{2.10}$$

This tells us that the perturbation of the density is related to the advection of the background density in the vertical direction.

Taking the wave motion to be a perturbation from a hydrostatic equilibrium allows us to simplify (2.7)-(2.9). First, in the equilibrium state, the fluid is not moving (i.e., $\mathbf{v} = \mathbf{0}$), so (2.7)-(2.9) give

$$\frac{\partial \bar{p}}{\partial x} = 0$$
$$\frac{\partial \bar{p}}{\partial y} = 0$$
$$\frac{\partial \bar{p}}{\partial z} = -g\bar{\rho}.$$

Putting this into (2.7)-(2.9) for both the equilibrium and perturbation states and using the fact that $\rho' \frac{\partial \mathbf{v}}{\partial t}$ is negligible gives

$$\bar{\rho}(z)\frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial x} \tag{2.11}$$
$$\bar{\rho}(z)\frac{\partial v}{\partial t} = -\frac{\partial p'}{\partial y} \tag{2.12}$$
$$\bar{\rho}(z)\frac{\partial w}{\partial t} = -\frac{\partial p'}{\partial z} - g\rho'. \tag{2.13}$$

Now our system of equations is (2.2), (2.10), and (2.11)-(2.13). We will now simplify this system of partial differential equations. First, take a time derivative of (2.2):

$$\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 v}{\partial t \partial y} + \frac{\partial^2 w}{\partial t \partial z} = 0, \tag{2.14}$$

and x, y, and t derivatives of (2.11), (2.12), and (2.13), respectively:

$$\bar{\rho}\frac{\partial^2 u}{\partial x \partial t} = -\frac{\partial^2 p'}{\partial x^2} \tag{2.15}$$
$$\bar{\rho}\frac{\partial^2 v}{\partial y \partial t} = -\frac{\partial^2 p'}{\partial y^2} \tag{2.16}$$
$$\bar{\rho}\frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2 p'}{\partial t \partial z} - g\frac{\partial \rho'}{\partial t}. \tag{2.17}$$

We put (2.15) and (2.16) into (2.14) and get the equation

$$\frac{-1}{\bar{\rho}}\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}\right) + \frac{\partial^2 w}{\partial t \partial z} = 0. \tag{2.18}$$

Eliminate ρ' from (2.17) by using (2.10):

$$\bar{\rho}\frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2 p'}{\partial t \partial z} + g\frac{\partial \bar{\rho}}{\partial z}w.$$

Put $N^2(z) = \dfrac{-g}{\bar{\rho}}\dfrac{\partial \bar{\rho}}{\partial z}$ to simplify notation. This is called the Brunt-Väisälä frequency or buoyancy frequency. Notice that $N(z)$ is a property of the fluid related to the density stratification. In the next section we discuss why this is a frequency. Making this substitution gives

$$\frac{\partial^2 w}{\partial t^2} + N^2(z)w = \frac{-1}{\bar{\rho}}\frac{\partial^2 p'}{\partial t \partial z}. \tag{2.19}$$

Our system of equations is now (2.18) and (2.19). We wish to remove the $\bar{\rho}$ dependence. To do so we use the Boussinesq approximation. Essentially we take full account of the density when it gives rise to buoyancy forces, but treat it as constant in computing rates of change of momentum from accelerations. This approximation is applicable when the vertical scale of the motion is small compared to the scale of the background density, which we have already assumed. With this assumption, the density changes will make a large difference in our model in terms of buoyancy, so the density is kept around in the equations for buoyancy forces. This assumption also tells us that the density won't make much of a difference in the small scale because the density changes exist on the large scale. In terms of the math we keep $\bar{\rho}$ if it multiplies g and otherwise we replace $\bar{\rho}$ with the constant ρ_0 and, in this case, the only place where $\bar{\rho}$ multiplies g is inside $N^2(z)$. Applying the Boussinesq approximation gives the governing equations

$$\frac{\partial^2 w}{\partial t \partial z} = \frac{1}{\rho_0}\nabla_H^2 p' \tag{2.20}$$

$$\frac{\partial^2 w}{\partial t^2} + N^2(z)w = \frac{-1}{\rho_0}\frac{\partial^2 p'}{\partial t \partial z}, \tag{2.21}$$

where $\nabla_H^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the 2D, horizontal Laplacian operator. It is possible to get a single equation for vertical velocity component w by putting (2.20) into ∇_H^2(2.21), but this will not be necessary.

The governing equation for small amplitude acoustic waves is

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0, \tag{2.22}$$

where ϕ is the velocity potential of the wave motion and c is a physical property of the fluid. We use our method for examining reflections for this problem first as this problem is easier and more intuitive before we look at the more complicated case of internal gravity waves.

2.2 A CLOSER LOOK AT N

In this section we elucidate why $N(z)$ is a frequency. Our approach is similar to the one laid out by Akylas and Mei [1]. Consider a fluid in a hydrostatic equilibrium with density distribution $\bar{\rho}(z)$, where $\bar{\rho}$ increases as z decreases. Move a parcel of fluid upward a distance ζ. Here, the fluid parcel is surrounded by a lighter fluid of density $\bar{\rho}(z+\zeta)$. Thu upward buoyancy force per unit volume is given by the negative value

$$g\{\bar{\rho}(z+\zeta)-\bar{\rho}(z)\} \approx g\frac{\partial\bar{\rho}}{\partial z}\zeta.$$

We apply Newton's Law (Force = mass × acceleration) to a fluid parcel of unit volume, so we get

$$g\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}\zeta = \bar{\rho}\frac{\partial^2\zeta}{\partial t^2},$$

or, using $N^2(z) = -\frac{g}{\bar{\rho}}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}$, we get

$$\frac{\partial^2\zeta}{\partial t^2} + N^2(z)\zeta = 0,$$

which is a differential equation that describes simple harmonic oscillations at the Brunt-Väisälä frequency $N(z)$. Thus, we see that gravity and density gradient provide restoring force that allows for oscillations of fluid parcels displaced from equilibrium.

2.3 TIME HARMONICS

We consider time harmonic solutions to our system of equations. We put $\frac{1}{\rho_0}p'(x,y,z,t) = \mathrm{Re}\left\{p(x,y,z)\mathrm{e}^{-\mathrm{i}\omega t}\right\}$, with similar expressions for w, u, and v, where ω is the frequency of the wave motion. Putting this into (2.20) and (2.21) gives

$$-\mathrm{i}\omega\frac{\partial w}{\partial z} = \nabla_H^2 p \tag{2.23}$$

$$(N^2(z)-\omega^2)w = \mathrm{i}\omega\frac{\partial p}{\partial z}. \tag{2.24}$$

We make the simplification $N(z) = N$, a constant. This standard simplification is necessary for analytic progress. Now put $\frac{\partial}{\partial z}$(2.24) into (2.23) to remove the w dependence and arrive at the partial differential equation for $p(x, y, z)$:

$$\nabla_H^2 p - \frac{\omega^2}{N^2 - \omega^2}\frac{\partial^2 p}{\partial z^2} = 0. \tag{2.25}$$

Notice that if $\omega^2 > N^2$, or, since both terms are frequencies and must be positive, $0 < N < \omega$, we get an elliptic equation that is exactly Laplace's equation if we rescale z. This problem has been much discussed elsewhere, so we consider the case $0 < \omega < N$. This inequality allows us to write $\omega = N\cos(\theta_c)$, for $0 < \theta_c < \frac{\pi}{2}$. Put $\frac{N^2 - \omega^2}{\omega^2} = \tan^2(\theta_c) = \gamma^2$, say, and (2.25) becomes

$$\nabla_H^2 p - \frac{1}{\gamma^2}\frac{\partial^2 p}{\partial z^2} = 0. \tag{2.26}$$

This equation is clearly hyperbolic and is exactly a 2D wave equation where the subtracted piece has derivatives with respect to z as opposed to t, as in the acoustic problem. We can find the fluid velocity from the pressure by putting the boussinesq approximation and time harmonic solutions into (2.11), (2.12), and (2.24). Doing so gives

$$u(x, y, z) = \frac{1}{\mathrm{i}\omega}\frac{\partial p}{\partial x} \tag{2.27}$$

$$v(x, y, z) = \frac{1}{\mathrm{i}\omega}\frac{\partial p}{\partial y} \tag{2.28}$$

$$w(x, y, z) = \frac{\mathrm{i}}{\gamma^2\omega}\frac{\partial p}{\partial z} \tag{2.29}$$

Notice how this differs from the acoustic problem where, assuming time harmonic solutions, the hyperbolic equation becomes the elliptic Helmholtz equation

$$\nabla^2\phi + \frac{\omega^2}{c^2}\phi = 0, \tag{2.30}$$

where $\phi(x, y, z, t) = \mathrm{Re}\left\{\mathrm{e}^{-\mathrm{i}\omega t}\phi(x, y, z)\right\}$.

2.4 PLANE WAVE SOLUTIONS

Though the equations are fundamentally different, we can look for plane wave solutions in both, that is solutions of the form $p(x, y, z) = A\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$ or $\phi(x, y, z) = A\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$, where A is the amplitude of the wave motion, $\mathbf{k} = \langle k_1, k_2, k_3 \rangle$ is the wave vector and $\mathbf{r} = \langle x, y, z \rangle$ is the position vector. Solutions of this form are called plane wave solutions as they behave like a plane front propagating through the fluid (see appendix A for a more detailed discussion on plane waves). This is a valid assumption as Mowbray and Rarity show that the behavior predicted by the linear theory for these internal gravity waves closely match with experimental observations of such waves [5]. In fact they show that reflections of internal gravity waves off sloping flat surfaces behave as predicted in Chapter 5.2.2. We will now substitute this guess into our governing equations with time dependence removed (2.30) and (2.26) to get the dispersion relations, which relate the frequencies ω to the wave vectors $\mathbf{k}$.

First, we examine the acoustic problem as that one is easier. Put $\phi(x, y, z) = A\mathrm{e}^{\mathbf{ik \cdot r}}$ into (2.30):

$$A\mathrm{e}^{\mathbf{ik \cdot r}}(-k_1^2 - k_2^2 - k_3^2 + \frac{\omega^2}{c^2}) = 0.$$

We use the fact that $A\mathrm{e}^{\mathbf{ik \cdot r}} \neq 0$ for all x, y, and z and the assumption that ω is positive to get the dispersion relation

$$\omega = ck, \tag{2.31}$$

where $k = |\mathbf{k}|$. This tells us that the frequency of the disturbance depends only upon the magnitude of the wave vector and a physical property of the fluid. Now substitute the same quantity into (2.26):

$$A\mathrm{e}^{\mathbf{ik \cdot r}}(-k_1^2 - k_2^2 + \frac{k_3^2}{\gamma^2}) = 0.$$

Again use the fact that $\mathrm{e}^{\mathbf{ik \cdot r}} \neq 0$ for all x, y, and z to get

$$\frac{k_1^2 + k_2^2}{k_3^2} = \frac{1}{\gamma^2}, \tag{2.32}$$

or we plug in $\gamma^2 = \frac{\omega^2 - N^2}{\omega^2}$ and solve for ω^2 we can relate ω and $\mathbf{k}$ through the dispersion relation

$$\omega^2 = \left(\frac{{k_1}^2 + {k_2}^2}{{k_1}^2 + {k_2}^2 + {k_3}^2} \right) N^2. \tag{2.33}$$

Using $\dfrac{\omega^2}{N^2} = \cos^2(\theta_c)$, $k = \sqrt{k_1^2 + k_2^2 + k_3^2}$, (2.32), and (2.33), get that

$$k_1^2 + k_2^2 = k^2 \cos^2(\theta_c) \tag{2.34}$$

and

$$k_3^2 = k^2 \sin^2(\theta_c). \tag{2.35}$$

Thus, we see that θ_c is the angle between the wave number vector and the k_1k_2-plane. Now we see that the relation $\omega = N\cos(\theta_c)$ tells us that, for the internal gravity waves, the frequency of the disturbance, ω, depends only on the angle the wave vector makes with the horizontal, θ_c and the density stratification of the fluid, N. Given that ω and N are physical properties of the wave and fluid, we see here that these internal waves can only exist and propagate in certain directions, namely on the cone with angle $\theta_c = \arccos\left(\frac{\omega}{N}\right)$ from the horizontal. This is interesting! Both equations have plane wave solutions, but the dispersion relations are completely different. In the acoustic problem the wave frequency is related solely to the magnitude of the wave number vector and for the internal-wave problem the frequency is related solely to the direction of the wave-number vector.

CHAPTER 3
PHASE AND GROUP VELOCITIES

The phase velocity $\mathbf{c}_p$ is the velocity at which the wave crests propagate and the group velocity $\mathbf{c}_g$ is the velocity of energy propagation. By examining these we can gain some physical understanding of these waves and some of the differences between them.

The phase velocity is defined as $\mathbf{c}_p = \dfrac{\omega}{k}\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is the unit vector in the direction of the wave vector $\mathbf{k}$. This makes sense, as we know that the plane wave propagates in the direction of $\mathbf{k}$, and clearly $\mathbf{c}_p$ points in the direction of $\mathbf{k}$ with magnitude $\dfrac{\omega}{k}$. Let us now apply this to the acoustic problem. We know from (2.31) that $\dfrac{\omega}{k} = c$, so putting this into the formula for phase velocity we get

$$\mathbf{c}_p = c\,\hat{\mathbf{k}}, \tag{3.1}$$

and thus, for acoustic waves, the wave crests propagate at speed c in the direction of wave vector $\mathbf{k}$, where c is a physical property of the fluid. Now we see that c is the speed of sound in the fluid.

Now, consider the internal wave problem, where $\dfrac{\omega}{k} = \dfrac{N\cos\theta_c}{k}$. Substituting this into the equation for phase velocity gives

$$\mathbf{c}_p = \frac{\omega}{k}\,\hat{\mathbf{k}} = \frac{N\cos\theta_c}{k}\,\hat{\mathbf{k}}, \tag{3.2}$$

which illustrates that the speed at which the wave crests propagate depends on the density stratification of the fluid and the direction the wave propagates in.

The group velocity is defined as $\mathbf{c}_g = \nabla_{\mathbf{k}}\omega$, where $\nabla_{\mathbf{k}} =< \dfrac{\partial}{\partial k_1}, \dfrac{\partial}{\partial k_2}, \dfrac{\partial}{\partial k_3} >$. Use this to examine the energy propagation of the acoustic problem. First, we notice that $\dfrac{\partial\omega}{\partial k_1} = c\dfrac{k_1}{k}$, and then we use symmetry to find the other two derivatives and obtain

$$\mathbf{c}_g = c\,\hat{\mathbf{k}} = \mathbf{c}_p. \tag{3.3}$$

Thus, energy will propagate in the same direction and with the same speed as the wave crests.

Applying this to the internal-wave problem is a bit more difficult. We first consider $\frac{\partial\omega}{\partial k_1}$. Use implicit differentiation on (2.33) to get

$$2\omega\frac{\partial\omega}{\partial k_1} = \frac{(k_1^2+k_2^2+k_3^2)(2k_1)-(k_1^2+k_2^2)(2k_1)}{(k_1^2+k_2^2+k_3^2)^2}N^2,$$

and if we solve for $\dfrac{\partial\omega}{\partial k_1}$,

$$\begin{aligned}\frac{\partial\omega}{\partial k_1} &= \frac{1}{\omega}\frac{k_3^2k_1N^2}{(k_1^2+k_2^2+k_3^2)^2}\\ &= \frac{N^2}{k^2\omega}\sin^2(\theta_c)k_1\end{aligned}$$

By symmetry of k_1 and k_2 we have

$$\frac{\partial\omega}{\partial k_2} = \frac{N^2}{k^2\omega}\sin^2(\theta_c)k_2.$$

We follow a similar process to obtain $\dfrac{\partial\omega}{\partial k_3}$:

$$\frac{\partial\omega}{\partial k_3} = -\frac{N^2}{k^2\omega}\cos^2(\theta_c)k_3.$$

Thus, we have

$$\mathbf{c}_g = \frac{N^2}{k^2\omega}\left\langle\sin^2(\theta_c)k_1,\ \sin^2(\theta_c)k_2,\ -\cos^2(\theta_c)k_3\right\rangle. \tag{3.4}$$

In the acoustic problem it is clear that $\mathbf{c}_p$ and $\mathbf{c}_g$ point in the same direction, but how are $\mathbf{c}_p$ and $\mathbf{c}_g$ related in the internal problem. For that let us examine the dot product:

$$\begin{aligned}\mathbf{c}_p\cdot\mathbf{c}_g &= \frac{\omega}{k^3}\langle k_1,\ k_2,\ k_3\rangle\cdot\frac{N^2}{k^2\omega}\langle\sin^2(\theta_c)k_1,\ \sin^2(\theta_c)k_2,\ -\cos^2(\theta_c)k_3\rangle\\ &= \frac{N^2}{k^5}(k_1^2\sin^2(\theta_c)+k_2^2\sin^2(\theta_c)-\cos^2(\theta_c)k_3^2)\\ &= \frac{N^2}{k^3}(\sin^2(\theta_c)\cos^2(\theta_c)-\cos^2(\theta_c)\sin^2(\theta_c))\\ &= 0.\end{aligned}$$

The energy propagates perpendicular to wave crest propagation! This is completely unintuitive. If the same were true of surface gravity waves, then getting hit by a wave at the beach would knock you along the shore line as opposed to back into the land. So how does this affect the interaction of these internal gravity waves with the sea floor? That's the aim of the rest of this thesis.

CHAPTER 4
REFLECTIONS OF ACOUSTIC WAVES OFF FLAT SURFACES

Before we examine the interactions of internal gravity waves with a solid boundary, we first look at the easier acoustic problem. Acoustic waves are more intuitive, so we build a mathematical method of examining reflections of waves with this problem before we move onto the unintuitive, yet more engrossing problem of the internal waves. In looking at the acoustic problem we consider the case of a boundary at $z = 0$, and examine how an incident and reflected wave interact at such a boundary. After this we look at the case of a sloping surface as this is more indicative of what we will see in the ocean with the internal gravity waves. In both cases we use a boundary condition that says the velocity of the fluid normal to the surface is zero, mathematically, that is

$$0 = \mathbf{v} \cdot \mathbf{n}, \tag{4.1}$$

or, in the acoustic case,

$$0 = \nabla\phi \cdot \mathbf{n},$$

where $\mathbf{n}$ is a normal vector to the surface. Though in most of what follows, we take $\mathbf{n} = \hat{\mathbf{n}}$, a unit vector, notice that $\mathbf{n}$ does not need to be a normal vector, since the right hand side of (4.1) is 0.

In this section we look only at time harmonic solutions of the three dimensional wave equation (2.22), or, once you remove the time dependence, we look at solutions of the Helmholtz equation (2.30) of the form $\phi(x, y, z) = \mathrm{e}^{\mathbf{ik}\cdot\mathbf{r}}$. To look at incident and reflected waves we say the total velocity potential in the fluid is

$$\phi(x, y, z) = \phi_i(x, y, z) + \phi_r(x, y, z),$$

where

$$\phi_i(x, y, z) = A\mathrm{e}^{\mathbf{ik}\cdot\mathbf{r}}$$

and

$$\phi_r(x,y,z) = B\mathrm{e}^{\mathrm{i}\mathbf{l}\cdot\mathbf{r}}$$

are the velocity potentials of the incident and reflected waves, respectively, $\mathbf{k} = \langle k_1, k_2, k_3\rangle$ is the wave vector of the incident wave, $\mathbf{l} = \langle l_1, l_2, l_3\rangle$ is the wave vector of the reflected wave, A is the amplitude of the incident wave, and B is the amplitude of the reflected wave. The knowns here are $\mathbf{k}$ and A and the aim throughout this section is to find $\mathbf{l}$ and B in terms of known quantities.

4.1 BOUNDARY AT $z = 0$

Put a plane at $z = 0$. We will examine reflections of acoustic waves off this boundary. To do this we say the fluid is in $z \geq 0$ and enforce that the group velocity of the incident wave points in the negative z direction, that is put $k_3 < 0$. Now apply the boundary condition (4.1) at $z = 0$ using $\hat{\mathbf{n}} = \langle 0, 0, 1\rangle$:

$$\begin{aligned} 0 &= \nabla\phi|_{z=0} \cdot \hat{\mathbf{n}} \\ &= \mathrm{i}Ak_3\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\langle x,y,0\rangle} + \mathrm{i}Bl_3\mathrm{e}^{\mathrm{i}\mathbf{l}\cdot\langle x,y,0\rangle}. \end{aligned} \tag{4.2}$$

For this equation to be satisfied the exponents must balance, that is,

$$\mathbf{k}\cdot\langle x,y,0\rangle = \mathbf{l}\cdot\langle x,y,0\rangle,$$

or, since the x and y coefficients must balance separately, we get $l_1 = k_1$ and $l_2 = k_2$. We use the dispersion relation (2.31) to find k_3. From there we know

$$k = \frac{\omega}{c} = l,$$

where k is as before and similarly $l = |\mathbf{l}|$. Using the fact that the magnitudes are equal and canceling k_1 and k_2 with l_1 and l_2 gives

$$l_3^2 = k_3^2.$$

We require physically that the energy of the reflected wave travels away from the boundary, or that $l_3 > 0$ and since $k_3 < 0$, that means we take

$$l_3 = -k_3.$$

Intuition says that the angle made between the incident wave and the normal vector should be same as the angle between the reflected wave and the normal vector, like a billiard ball bouncing off the wall of the pool table. We can examine the angles by looking at the dot products of the wave vectors with the normal, where we take the negative of $\mathbf{k}$ to get the appropriate angle as opposed to $\frac{\pi}{2}$ radians more than the appropriate angle. This gives

$$-\mathbf{k} \cdot \hat{\mathbf{n}} = -k_3$$
$$\mathbf{l} \cdot \hat{\mathbf{n}} = l_3 = -k_3,$$

which is exactly what our intuition says, that the angle of incidence is equal to the angle of reflection. To find B we substitute the above into (4.2) giving

$$0 = \mathrm{i}k_3 \mathrm{e}^{\mathrm{i}(k_1 x + k_2 y)}(A - B),$$

or

$$A = B.$$

Thus, there is no change of amplitude between the incident and reflected waves, which is good as it says we are not transferring any energy into the boundary. This also tell us that there is no phase change upon reflection, which is what we expect as we are not considering any energy transmission into the boundary. Putting all this together gives the reflected wave

$$\phi_r(x, y, z) = A\mathrm{e}^{\mathrm{i}(k_1 x + k_2 y - k_3 z)},$$

a wave that behaves exactly as we would expect.

4.2 SLOPING BOUNDARY

In the ocean the sea floor is often sloped and in the internal problem our z-axis is fixed due to the effects of gravity. Because of this we consider a sloping boundary in the acoustic case as a stepping stone to the internal problem. To model this we assume a boundary at $\mathbf{r} \cdot \hat{\mathbf{n}} = d$ for some constant d and unit normal vector $\hat{\mathbf{n}}$. Now consider orthonormal vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ in this plane. Notice that $\{\mathbf{c}_1, \mathbf{c}_2, \hat{\mathbf{n}}\}$ is an orthonormal basis for $\mathbb{R}^3$. Thus, we can write

$$\begin{aligned}
\mathbf{r} &= \lambda \mathbf{c}_1 + \mu \mathbf{c}_2 + \nu \hat{\mathbf{n}} \\
\mathbf{k} &= k^{(1)} \mathbf{c}_1 + k^{(2)} \mathbf{c}_2 + k^{(3)} \hat{\mathbf{n}} \\
\mathbf{l} &= l^{(1)} \mathbf{c}_1 + l^{(2)} \mathbf{c}_2 + l^{(3)} \hat{\mathbf{n}},
\end{aligned}$$

for λ, μ, ν, depending on x, y, and z, and for some $k^{(1)}$, $k^{(2)}$, $k^{(3)}$, $l^{(1)}$, $l^{(2)}$, and $l^{(3)}$. We use this change of variables to make the following simplifications

$$\begin{aligned}
\mathbf{r} \cdot \hat{\mathbf{n}} &= \nu \\
\mathbf{k} \cdot \mathbf{r} &= \lambda k^{(1)} + \mu k^{(2)} + \nu k^{(3)} \\
\mathbf{k} \cdot \hat{\mathbf{n}} &= k^{(3)} \\
\mathbf{l} \cdot \mathbf{r} &= \lambda l^{(1)} + \mu l^{(2)} + \nu l^{(3)} \\
\mathbf{l} \cdot \hat{\mathbf{n}} &= l^{(3)}.
\end{aligned}$$

This change of variables will allow us to perform calculations that nearly parallel those from the previous subsection. Now substitute these into the (4.1) at $\mathbf{r} \cdot \hat{\mathbf{n}} = d$:

$$\begin{aligned}
0 &= \nabla \phi|_{\nu=d} \cdot \hat{\mathbf{n}} \\
&= \mathrm{i} A (\mathbf{k} \cdot \hat{\mathbf{n}}) \mathrm{e}^{\mathrm{i}(\lambda k^{(1)} + \mu k^{(2)} - d k^{(3)})} + \mathrm{i} B (\mathbf{l} \cdot \hat{\mathbf{n}}) \mathrm{e}^{\mathrm{i}(\lambda l^{(1)} + \mu l^{(2)} - d l^{(3)})} \qquad (4.3)
\end{aligned}$$

To satisfy this equation for all λ and μ we must have that $l^{(1)} = k^{(1)}$ and $l^{(2)} = k^{(2)}$. Following a similar process as in the previous subsection we arrive at the conclusion

$$l^{(3)^2} = k^{(3)^2}.$$

Physically we require the reflected wave to propagate energy away from the boundary, so we get

$$l^{(3)} = -k^{(3)}.$$

Again we examine the angles made between the incident/reflected wave and the normal to check against our intuition:

$$-\mathbf{k} \cdot \hat{\mathbf{n}} = -k^{(3)}$$
$$\mathbf{l} \cdot \hat{\mathbf{n}} = l^{(3)} = -k^{(3)}.$$

Now we find the amplitude of the reflected wave, B, by substituting the above into (4.3) giving

$$0 = \mathrm{i}Ak^{(3)}\mathrm{e}^{\mathrm{i}dk^{(3)}} + \mathrm{i}B(-k^{(3)})\mathrm{e}^{\mathrm{i}d(-k^{(3)})}$$

or

$$B = A\mathrm{e}^{2\mathrm{i}dk^{(3)}}.$$

Thus, the amplitude of the reflected wave is exactly equal to the amplitude of the incident wave and the reflected wave will have a change of phase. We put everything together to get the total reflected wave

$$\phi_r(x, y, z) = A\mathrm{e}^{2\mathrm{i}dk^{(3)}(\lambda k^{(1)}+\mu k^{(2)}-\nu k^{(3)})},$$

which is exactly what we expect. The wave reflects across the normal vector to the surface (see Figure 4.1), which is what we arrived at in the previous subsection excepting a phase change.

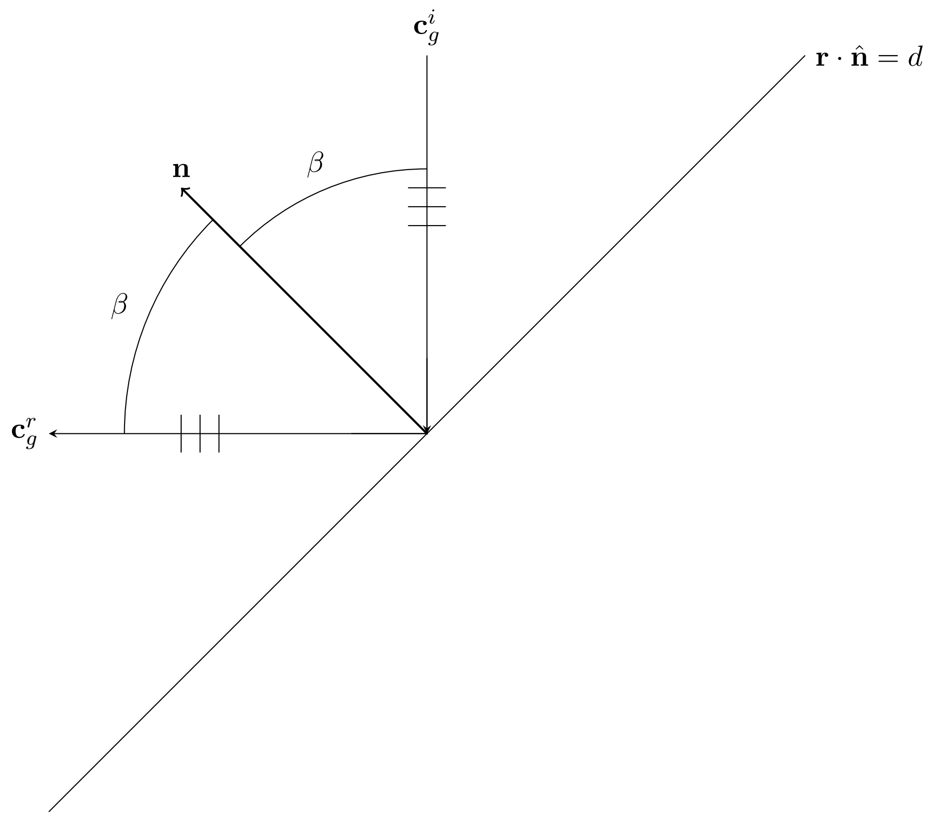


Figure 4.1: For acoustic waves, angle between the incident wave and the normal vector to the surface, call it β, say, will be the same as the angle between the reflected wave and the normal vector.

CHAPTER 5

REFLECTIONS OF INTERNAL WAVES OFF FLAT SURFACES

Before we look at reflections of internal gravity waves off a perturbed surface, it will behoove us to examine the case of reflections of such waves off a flat surface. To do this we will again use the boundary condition (4.1) applied to a plane boundary. To do this we first need to derive all of the components of the velocity vector $\mathbf{v}$.

We know from the partial differential equation that the pressure p is such that

$$p(x, y, z) = A\mathrm{e}^{\mathbf{ik}\cdot\mathbf{r}},$$

given the dispersion relation (2.33) is satisfied. We need to derive the velocity components from this. Substitute this into (2.27)-(2.29):

$$\begin{aligned} u(x, y, z) &= \frac{Ak_1}{\omega}\mathrm{e}^{\mathbf{ik}\cdot\mathbf{r}} \\ v(x, y, z) &= \frac{Ak_2}{\omega}\mathrm{e}^{\mathbf{ik}\cdot\mathbf{r}} \\ w(x, y, z) &= \frac{-Ak_3}{\gamma^2\omega}\mathrm{e}^{\mathbf{ik}\cdot\mathbf{r}} \end{aligned}$$

Just as in the acoustic case, we consider both an incident wave $\mathbf{v}_i$ and a reflected wave $\mathbf{v}_r$ where the total disturbance in the fluid is given by $\mathbf{v} = \mathbf{v}_i + \mathbf{v}_r$, where

$$\mathbf{v}_i(x, y, z) = \frac{A}{\omega}\mathrm{e}^{\mathbf{ik}\cdot\mathbf{r}}\left\langle k_1,\ k_2,\ -k_3\gamma^{-2}\right\rangle$$

and

$$\mathbf{v}_r(x, y, z) = \frac{B}{\omega}\mathrm{e}^{\mathbf{il}\cdot\mathbf{r}}\left\langle l_1,\ l_2,\ -l_3\gamma^{-2}\right\rangle.$$

We take the fluid to be above the plane and we impose the physical condition that energy should radiate down towards the plane, that is the z-component of the group velocity (3.4) must be negative, or

$$0 > -\frac{N^2}{k^2\omega}\cos^2(\theta_c)k_3.$$

This enforces the condition $k_3 > 0$. This runs counter to intuition, since the incident wave must be moving upwards, away from the boundary, for energy to propagate down towards the boundary and is a result of the energy flowing perpendicular to the wave front. For the reflected wave we necessitate that the energy propagate away from the boundary, this could be upwards or downwards depending on the boundary.

5.1 BOUNDARY AT $z = 0$

We place a boundary at $z = 0$ and examine the reflections of internal waves with respect to this boundary. Apply (4.1) with $\hat{\mathbf{n}} = \langle 0, 0, 1 \rangle$:

$$\begin{aligned} 0 &= (\mathbf{v}_i + \mathbf{v}_r)|_{z=0} \cdot \hat{\mathbf{n}} \\ &= -k_3\gamma^{-2}A\omega^{-1}\mathrm{e}^{\mathrm{i}(k_1x+k_2y)} - l_3\gamma^{-2}B\omega^{-1}\mathrm{e}^{\mathrm{i}(l_1x+l_2y)} \\ &= A\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\langle x,y,0\rangle} + B\mathrm{e}^{\mathrm{i}\mathbf{l}\cdot\langle x,y,0\rangle} \\ &= (A\mathrm{e}^{\mathrm{i}(k_1x+k_2y)} + B\mathrm{e}^{\mathrm{i}(l_1x+l_2y)}). \end{aligned}$$

To satisfy this equation for all x and y we find that $l_1 = k_1$ and $l_2 = k_2$. In order to find l_3 we use the dispersion relation (2.32) to get

$$\frac{k_1^2 + k_2^2}{k_3^2} = \frac{1}{\gamma^2} = \frac{l_1^2 + l_2^2}{l_3^2}, \tag{5.1}$$

or by applying what we found above and the fact that energy must propagate away from the boundary ($l_3 < 0 < k_3$) we get $l_3 = -k_3$, and hence, $B = A$. Thus, the reflected wave pressure is given by

$$p_r(x, y, z) = A\mathrm{e}^{\mathrm{i}(k_1x+k_2y-k_3z)}.$$

This result is similar to the result in the acoustic case.

5.2 SLOPING BOUNDARY

For the sloping boundary case we take the boundary to be $z = x\tan(\alpha)$, where α is the angle the boundary makes with the horizontal. Without loss of generality take $0 \leq \alpha \leq \frac{\pi}{2}$, as the case $\frac{-\pi}{2} \leq \alpha < 0$ can be considered with a different choice of axes. This gives unit normal vector

$\hat{\mathbf{n}} = \langle -\sin(\alpha), 0, \cos(\alpha)\rangle$. We now use (4.1) on $z = x\tan(\alpha)$ to get

$$\begin{aligned} 0 &= (\mathbf{v}_i + \mathbf{v}_r)|_{z=x\tan(\alpha)} \cdot \hat{\mathbf{n}} \\ &= \left(k_1 s_\alpha - k_3\gamma^{-2}c_\alpha\right)\frac{A}{\omega}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\langle x,y,xt_\alpha\rangle} + \left(l_1 s_\alpha - l_3\gamma^{-2}c_\alpha\right)\frac{B}{\omega}\mathrm{e}^{\mathrm{i}\mathbf{l}\cdot\langle x,y,xt_\alpha\rangle}, \end{aligned} \tag{5.2}$$

where $s_\alpha = \sin(\alpha)$, $c_\alpha = \cos(\alpha)$, and $t_\alpha = \tan(\alpha)$. To satisfy this equation for all x and y we require that

$$l_2 = k_2 \tag{5.3}$$

and

$$k_1 + k_3 t_\alpha = l_1 + l_3 t_\alpha. \tag{5.4}$$

We use these two equations along with (5.1) to solve for $\mathbf{l}$ in terms of $\mathbf{k}$. We consider two cases here, the two dimensional case, that is $k_2 = 0$, and the three dimensional case.

5.2.1 TWO DIMENSIONS ($k_2 = 0$)

We start off with the case $k_2 = 0$ as it is easier to visualize and simplifies calculations. Using $k_2 = 0$ and (5.3) we simplify (5.1) to get $\frac{l_1^2}{l_3^2} = \frac{k_1^2}{k_3^2}$, or, if we use the restriction that energy must propagate away from the boundary, we find that

$$\frac{-l_3}{l_1} = \frac{k_3}{k_1}.$$

This tells us that the slope of the reflected wave depends only on the slope of the incident wave, and not the angle the boundary makes with the horizontal. In fact we can conclude that the direction of the reflected wave will be a reflection across the z-axis or x-axis as opposed to reflecting across the normal vector to the surface (the latter is how acoustic waves reflect).

In order to solve for the exact direction the reflected wave travels in, we rearrange the above to get

$$\frac{l_1}{k_1} = \frac{-l_3}{k_3}.$$

Divide (5.4) by k_1 to get

$$\begin{aligned} 1 + \frac{k_3}{k_1}\tan\alpha &= \frac{l_1}{k_1} + \frac{l_3}{k_1}\tan\alpha \\ &= \frac{l_1}{k_1} + \frac{k_3}{k_3}\frac{l_3}{k_1}\tan\alpha \\ &= \frac{l_1}{k_1} - \frac{l_1}{k_1}\frac{k_3}{k_1}\tan\alpha. \end{aligned}$$

Solving for $\frac{l_1}{k_1}$ gives

$$\frac{l_1}{k_1} = \frac{k_1 + k_3\tan\alpha}{k_1 - k_3\tan\alpha}.$$

Since $k_3 > 0$ and $k_2 = 0$, we have $k_3 = k\sin(\theta_c)$ and $k_1 = \pm\cos(\theta_c)$. Making these substitutions gives two cases: k_1 is positive and k_1 is negative. In the positive case we get

$$\begin{aligned} \frac{l_1}{k_1} &= \frac{\cos(\theta_c)\cos(\alpha) + \sin(\theta_c)\sin(\alpha)}{\cos(\theta_c)\cos(\alpha) - \sin(\theta_c)\sin(\alpha)} \\ &= \frac{\cos(\theta_c - \alpha)}{\cos(\theta_c + \alpha)}. \end{aligned}$$

Similarly, if k_1 is negative,

$$\frac{l_1}{k_1} = \frac{\cos(\theta_c + \alpha)}{\cos(\theta_c - \alpha)}.$$

Solving for l_1 and l_3 gives

$$\begin{aligned} l_1 &= k_1\frac{\cos(\theta_c - \alpha)}{\cos(\theta_c + \alpha)} \\ l_3 &= -k_3\frac{\cos(\theta_c - \alpha)}{\cos(\theta_c + \alpha)}, \end{aligned}$$

if $k_1 > 0$, or

$$\begin{aligned} l_1 &= k_1\frac{\cos(\theta_c + \alpha)}{\cos(\theta_c - \alpha)} \\ l_3 &= -k_3\frac{\cos(\theta_c + \alpha)}{\cos(\theta_c - \alpha)}, \end{aligned}$$

if $k_1 < 0$. We use the above result and (5.2) to find the reflected wave amplitude is

$$B = \begin{cases} -\dfrac{(k_1 s_\alpha - k_3\gamma^{-2}c_\alpha)\cos(\theta_c+\alpha)}{(k_1 s_\alpha + k_3\gamma^{-2}c_\alpha)\cos(\theta_c-\alpha)}A, & \text{for } k_1 > 0, \\ -\dfrac{(k_1 s_\alpha - k_3\gamma^{-2}c_\alpha)\cos(\theta_c-\alpha)}{(k_1 s_\alpha + k_3\gamma^{-2}c_\alpha)\cos(\theta_c+\alpha)}A, & \text{for } k_1 < 0. \end{cases}$$

Notice that the magnitude of the reflected wave amplitude is not in general equal to the magnitude of the incident wave amplitude. This does not correspond to a gain or loss of energy, since the energy density changes, becoming more focused or more spread out [4].

There are two ways that the wave can reflect, either the wave can reflect upwards (i.e., l_3 and k_3 have opposite signs) as in Figure 5.1 or backwards (i.e. l_1 and k_1 have opposite signs) as in Figure 5.2. In order for the wave to reflect upwards we get $l_3 < 0 < k_3$, or that $\dfrac{\cos(\theta_c-\alpha)}{\cos(\theta_c+\alpha)}, \dfrac{\cos(\theta_c+\alpha)}{\cos(\theta_c-\alpha)} > 0$. This also tells us that l_1 has the same sign as k_1. For this to be satisfied, both cosine terms must be positive or both terms must be negative. For the positive case we require

$$-\frac{\pi}{2} < \theta_c - \alpha < \frac{\pi}{2}$$

and

$$-\frac{\pi}{2} < \theta_c + \alpha < \frac{\pi}{2}.$$

We can isolate α to get

$$\theta_c - \frac{\pi}{2} < \alpha < \theta_c + \frac{\pi}{2},$$
$$-\theta_c - \frac{\pi}{2} < \alpha < \frac{\pi}{2} - \theta_c,$$

and our original assumption about α,

$$0 \le \alpha \le \frac{\pi}{2}.$$

To satisfy all of the above inequalities we find that $0 < \alpha < \dfrac{\pi}{2} - \theta_c$. The case where both cosine terms are negative can be treated the same way and it yields no possible values for α. In order

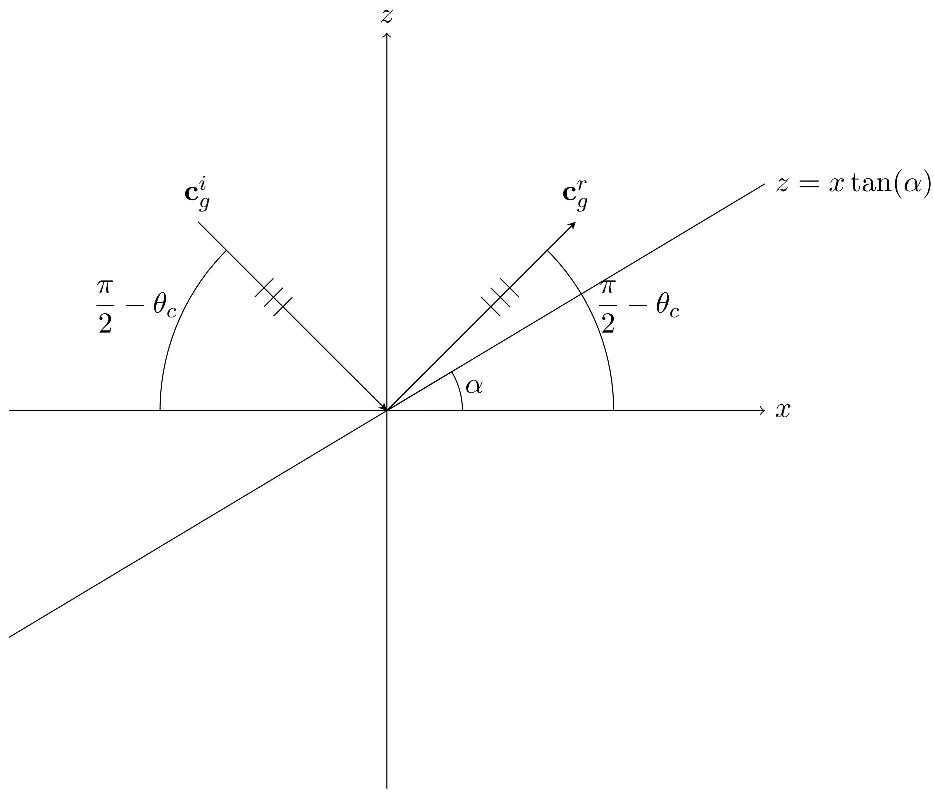


Figure 5.1: If the angle of the slope, α, is smaller than the angle the group velocity makes with the horizontal, $\dfrac{\pi}{2} - \theta_c$, then the wave will reflect upwards.

for the wave to reflect backwards we get $l_1 < 0 < k_1$, or that $\dfrac{\cos(\theta_c - \alpha)}{\cos(\theta_c + \alpha)} < 0$. Following a similar process as before we find this backwards reflection to happen when $\frac{\pi}{2} - \theta_c < \alpha < \frac{\pi}{2}$. In the case $\alpha = \dfrac{\pi}{2} - \theta_c$, the model breaks down. The problem for α near $\dfrac{\pi}{2} - \theta_c$ is discussed in more detail by Thorpe [7] and by Dauxois and Young [3].

5.2.2 THREE DIMENSIONS

Above we saw for $\alpha > \dfrac{\pi}{2} - \theta_c$ the reflected wave would propagate down and away from the boundary. Analysis can be done on the three dimensional problem discussed here to arrive at a similar result, just with a dependence on the angle between the wave number vector and the x-axis [7]. In this section we consider only the case $0 \leq \alpha < \frac{\pi}{2} - \theta_c$, so there will be no downward reflected waves.

We examine this case in a different way from above; we use a change of variables (A process

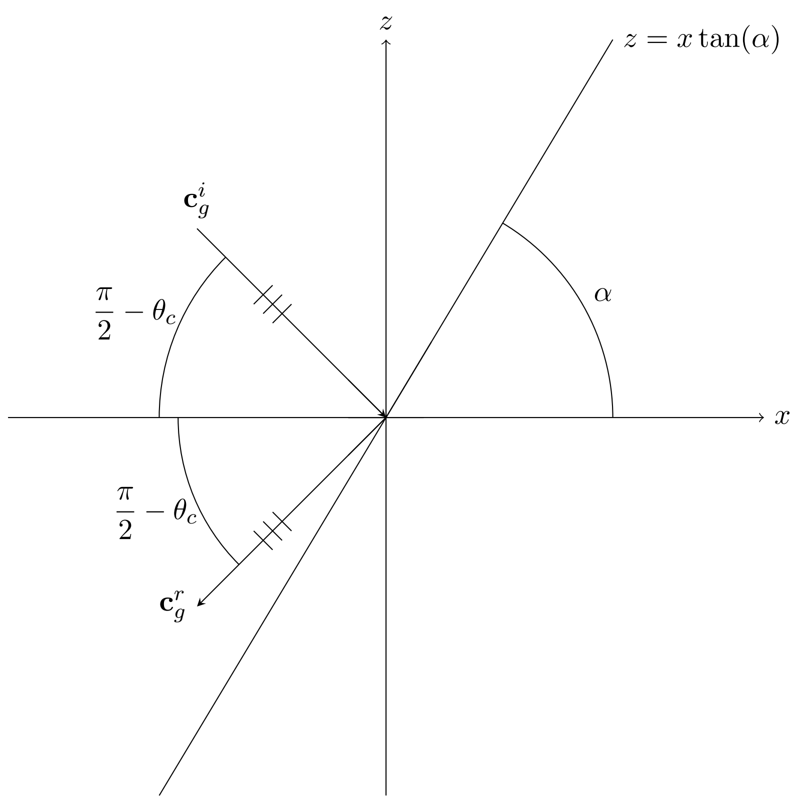


Figure 5.2: If the angle of the slope, α, is larger than the angle the group velocity makes with the horizontal, $\frac{\pi}{2} - \theta_c$, then the wave will reflect backwards.

similar to the above was initially used to solve the three dimensional problem; it can be found in Appendix B). Following Thorpe [7] we put

$$X = xc_\alpha + zs_\alpha, \qquad Y = y, \qquad Z = -xs_\alpha + zc_\alpha, \tag{5.5}$$

where $c_\alpha = \cos(\alpha)$ and $s_\alpha = \sin(\alpha)$. Here X points up the slope and Z is normal to the surface (see Figure 5.3). This change of variables gives

$$\begin{aligned}
\frac{\partial p}{\partial x} &= \frac{\partial p}{\partial X}c_\alpha - \frac{\partial p}{\partial Z}s_\alpha, & \frac{\partial^2 p}{\partial x^2} &= \frac{\partial^2 p}{\partial X^2}c_\alpha^2 - 2\frac{\partial^2 p}{\partial X \partial Z}c_\alpha s_\alpha + \frac{\partial^2 p}{\partial Z^2}s_\alpha^2,\\
\frac{\partial p}{\partial y} &= \frac{\partial p}{\partial Y}, & \frac{\partial^2 p}{\partial y^2} &= \frac{\partial^2 p}{\partial Y^2},\\
\frac{\partial p}{\partial z} &= \frac{\partial p}{\partial X}s_\alpha + \frac{\partial p}{\partial Z}c_\alpha, & \frac{\partial^2 p}{\partial z^2} &= \frac{\partial^2 p}{\partial X^2}s_\alpha^2 + 2\frac{\partial^2 p}{\partial X \partial Z}c_\alpha s_\alpha + \frac{\partial^2 p}{\partial Z^2}c_\alpha^2.
\end{aligned}$$

We substitute these equations into (2.25) and (4.1) to get the differential equation

$$0 = \frac{\omega^2}{N^2}\left(\frac{\partial^2 p}{\partial X^2} + \frac{\partial^2 p}{\partial Y^2} + \frac{\partial^2 p}{\partial Z^2}\right) - c_\alpha^2\frac{\partial^2 p}{\partial X^2} + 2c_\alpha s_\alpha \frac{\partial^2 p}{\partial X \partial Z} - s_\alpha^2\frac{\partial^2 p}{\partial Z^2} - \frac{\partial^2 p}{\partial Y^2}, \tag{5.6}$$

with boundary condition $\mathbf{v}(X,Y,Z)\cdot\mathbf{n}(X,Y,Z) = 0$ on $Z = 0$, where $\mathbf{n}(X,Y,Z) = \langle 0,0,1\rangle$.

As before, we assume plane wave solutions, $p(X,Y,Z) = A\mathrm{e}^{\mathrm{i}(K_1X + K_2Y + K_3Z)}$. Substituting this into the transformed equation (2.25) gives the dispersion relation

$$\omega^2 = \frac{N^2}{K^2}[(c_\alpha K_1 - s_\alpha K_3)^2 + K_2^2], \tag{5.7}$$

where $K^2 = K_1^2 + K_2^2 + K_3^2$. Put $\omega = N\cos(\theta_c)$ to get a quadratic equation for K_3 :

$$\begin{aligned}
0 &= N^2c_c^2 - \frac{N^2}{K^2}[(c_\alpha K_1 - s_\alpha K_3)^2 + K_2^2]\\
&= c_c^2K_1^2 + c_c^2K_2^2 + c_c^2K_3^2 - c_\alpha^2K_1^2 + 2c_\alpha s_\alpha K_1K_3 - s_\alpha^2K_1^2 - K_2^2\\
&= (c_c^2 - s_\alpha^2)K_3^2 + 2(s_\alpha c_\alpha K_1)K_3 + (c_c^2 - c_\alpha^2)K_1^2 - (1 - c_c^2)K_2^2\\
&= aK_3^2 + bK_3 + c,
\end{aligned}$$

where

$$a = c_c^2 - s_\alpha^2, \tag{5.8}$$

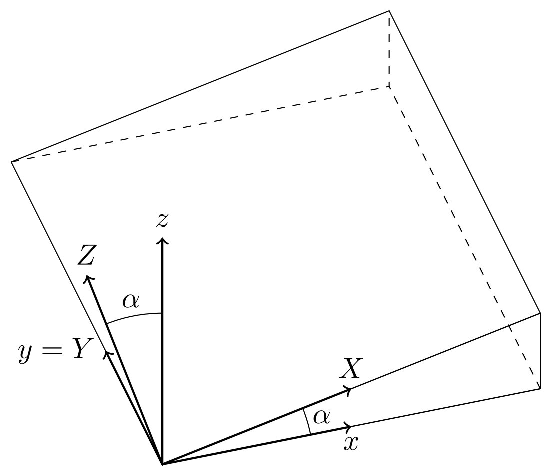


Figure 5.3: A visual representation of the change of variables. X points up the slope and Z points perpendicular to the slope.

$$b(K_1) = s_\alpha c_\alpha K_1, \tag{5.9}$$

$$c(K_1, K_2) = (c_c^2 - c_\alpha^2)K_1^2 - s_c^2 K_2^2, \tag{5.10}$$

$c_c = \cos(\theta_c)$, and $s_c = \sin(\theta_c)$. Thus, $aK_3 = -b \pm \Delta$, where

$$\begin{aligned}
\Delta &= \sqrt{b^2 - ac} \\
&= \sqrt{s_\alpha^2 c_\alpha^2 K_1^2 - (c_c^2 - s_\alpha^2)K_1^2 c_c^2 + (c_c^2 - s_\alpha^2)K_1^2 c_\alpha^2 + (c_c^2 - s_\alpha^2)K_2^2 s_c^2} \\
&= \sqrt{(c_\alpha^2 + s_\alpha^2 - c_c^2)c_c^2 K_1^2 + s_c^2(c_c^2 - s_\alpha^2)K_2^2} \\
&= s_c\sqrt{c_c^2 K_1^2 + aK_2^2}.
\end{aligned} \tag{5.11}$$

Recall the assumptions $0 < \alpha < \dfrac{\pi}{2} - \theta_c$ and $0 < \theta_c < \dfrac{\pi}{2}$. This tells us that $\sin(0) < \sin(\alpha) < \sin(\frac{\pi}{2} - \theta_c)$ or, equivalently, $0 < \sin(\alpha) < \cos(\theta_c)$. Thus, $a = c_c^2 - s_\alpha^2 > 0$ and hence, Δ is real.

Now we consider the $Z-$component of the group velocity, so we can later throw away solutions that do not make physical sense (i.e., the incident wave must have group velocity down toward the surface and the reflected wave must have group velocity away from the surface). From (5.7):

$$\begin{aligned}
\omega\frac{\partial\omega}{\partial K_3} &= N^2\left\{\frac{K^2(c_\alpha K_1 - s_\alpha K_3)(-s_\alpha) - (c_\alpha K_1 - s_\alpha K_3)^2 K_3 - K_3 K_2^2}{K^4}\right\} \\
&= \frac{-N^2}{K^2}\left\{(c_\alpha K_1 - s_\alpha K_3)s_\alpha + \underbrace{\frac{(c_\alpha K_1 - s_\alpha K_3)^2 + K_2^2}{K^2}}_{=c_c^2 \text{ from (5.7)}} K_3\right\}
\end{aligned}$$

$$= \frac{-N^2}{K^2}\{(c_c^2 - s_\alpha^2)K_3 + c_\alpha s_\alpha K_1\}$$
$$= \frac{-N^2}{K^2}\{aK_3 + b\}$$
$$= \mp\frac{N^2}{K^2}\Delta$$

So, the incident wave, $p_i(X,Y,Z) = A\mathrm{e}^{\mathrm{i}(K_1X+K_2Y+K_3Z)}$, which has group velocity toward the surface, will have wave number vector given by $\left\langle K_1,\ K_2,\ \dfrac{-b(K_1)+\Delta(K_1,K_2)}{a}\right\rangle$ and the reflected wave, $p_r(X,Y,Z) = B\mathrm{e}^{\mathrm{i}(L_1X+L_2Y+L_3Z)}$, which has group velocity away from the surface, will have wave number vector given by $\left\langle L_1,\ L_2,\ \dfrac{-b(L_1)-\Delta(L_1,L_2)}{a}\right\rangle$.

To find the fluid velocity we must apply the change of variables to (2.27)-(2.29). Doing so gives

$$\mathbf{v}_i = A\omega^{-1}\mathrm{e}^{\mathrm{i}(K_1X+K_2Y+K_3Z)}\langle K_1c_\alpha - K_3s_\alpha,\ K_2,\ -(K_1s_\alpha + K_3c_\alpha)\gamma^{-2}\rangle$$
$$\mathbf{v}_r = B\omega^{-1}\mathrm{e}^{\mathrm{i}(L_1X+L_2Y+L_3Z)}\langle L_1c_\alpha - L_3s_\alpha,\ L_2,\ -(L_1s_\alpha + L_3c_\alpha)\gamma^{-2}\rangle$$

We apply the boundary condition to solve for the reflected wave components in terms of the incident wave components:

$$0 = A\mathrm{e}^{\mathrm{i}(K_1X+K_2Y)}(K_1s_\alpha + K_3c_\alpha) + B\mathrm{e}^{\mathrm{i}(L_1X+L_2Y)}(L_1s_\alpha + L_3c_\alpha).$$

Equating the exponential terms for all X and Y tells us that $L_1 = K_1$ and $L_2 = K_2$ and, hence, $L_3 = \dfrac{-b(K_1)-\Delta(K_1,K_2)}{a}$. The reflected amplitude is

$$B = \frac{-\{as_\alpha K_1 + c_\alpha[-b(K_1)+\Delta(K_1,K_2)]\}}{as_\alpha K_1 + c_\alpha[-b(K_1)-\Delta(K_1,K_2)]}A.$$

The magnitude of the reflected amplitude is not necessarily equal to the magnitude of the incident amplitude, but, as discussed before, this is resultant from a change in energy density, not a change in total energy. Put L_1, L_2, L_3, and B into $\mathbf{v}_r$ above to find the reflected velocity. As in the two dimensional case there is a focusing of energy as the angle of the slope with the horizontal nears the critical angle of $\alpha = \dfrac{\pi}{2} - \theta_c$, where, at that critical angle, the denominator of the amplitude B vanishes and the equations breakdown. Refer to Thorpe [7] for analysis of these reflections off slopes near this critical angle.

CHAPTER 6
REFLECTIONS OF INTERNAL WAVES OFF BUMPY SURFACES

We now examine what happens when the internal waves reflect off of bumpy surfaces, with the bumps local to some area. This will be similar to our previous problem, however, the boundary will have a small perturbation, $O(\epsilon)$ from the flat case. In this instance we say the height of the perturbation is small when compared to the amplitude of the wave motion. Note that this is a different ϵ than the one implicitly used when assuming small amplitude wave motion. In this section we consider only perturbations that are relatively flat, that is the slope of the surface always makes an angle with the xy-plane shallower than $\frac{\pi}{2} - \theta_c$. This is to guarantee the reflected wave will always have energy propagate upwards, as allowing for downward reflections is much more difficult. This case is what Baines calls flat bumps [2]. Allowing for reflections in both directions is termed steep bumps. A graphical representation can be seen in Figure 6.1.

Consider $p(x, y, z) = p_0(x, y, z) + \epsilon p_1(x, y, z) + \epsilon^2 p_2(x, y, z) + \cdots$. Putting this into (2.25) and using linearity gives $p_j(x, y, z)$ satisfies

$$\nabla_H^2 p_j - \frac{1}{\gamma^2}\frac{\partial^2 p_j}{\partial z^2} = 0 \tag{6.1}$$

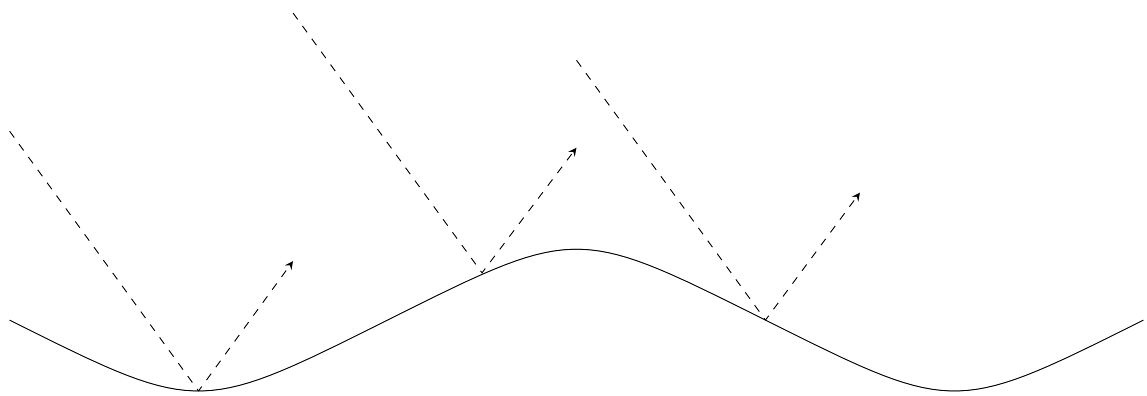
for each $j = 0, 1, 2, \ldots$. The velocity $\mathbf{v}_j$ at each order of ϵ is given by

$$\mathbf{v}_j(x, y, z) = \left\langle \frac{1}{\mathrm{i}\omega}\frac{\partial p_j}{\partial x},\ \frac{1}{\mathrm{i}\omega}\frac{\partial p_j}{\partial y},\ \frac{\mathrm{i}}{\gamma^2\omega}\frac{\partial p_j}{\partial z} \right\rangle \tag{6.2}$$

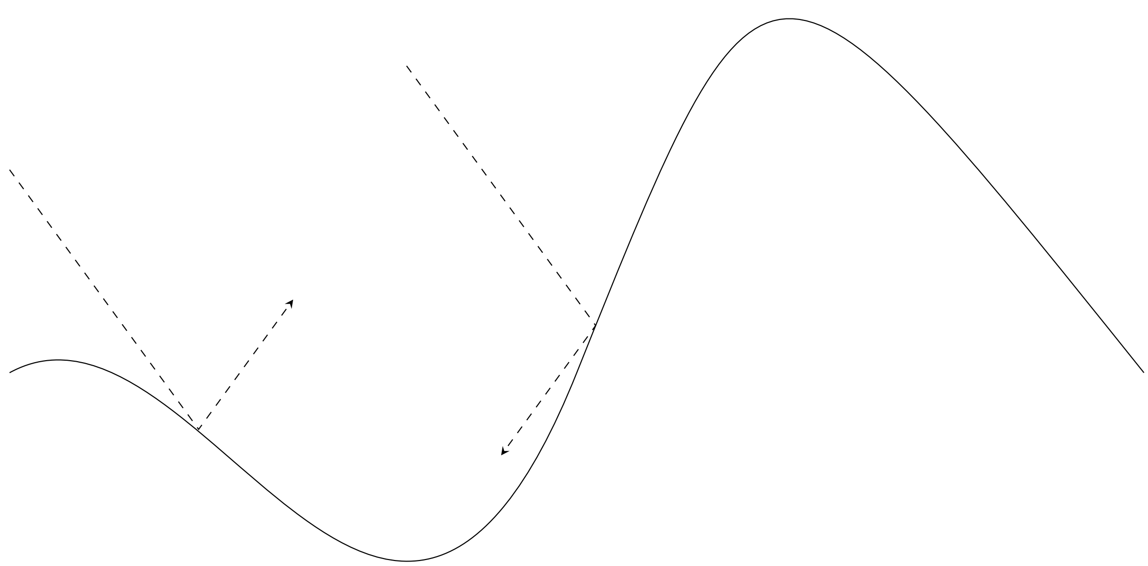
for $j = 0, 1, 2, \ldots$, where the total velocity is given by $\mathbf{v} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2 + \cdots$. Recall Taylor's theorem states that we can approximate an n times differentiable function in z, $F(x, y, z)$, at a point near $z = 0$ with the polynomial:

$$F(x, y, z) \approx F(x, y, 0) + z\frac{\partial F}{\partial z}\bigg|_{(x,y,0)} + \frac{z^2}{2}\frac{\partial^2 F}{\partial z^2}\bigg|_{(x,y,0)} + \cdots + \frac{z^n}{n!}\frac{\partial^n F}{\partial z^n}\bigg|_{(x,y,0)}.$$

We use both of the above along with boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is any normal vector to the surface (the magnitude does not matter, since the right hand side is 0), to derive problems on each order of ϵ and solve each problem separately.



(a)



(b)

Figure 6.1: (a) Flat bump topography, with the wave characteristics of the incident wave and their reflections shown, and (b) steep bump topography.

6.1 BOUNDARY AT $z = \epsilon f(x, y)$

Assume the bottom topography has form $z = \epsilon f(x, y)$ for some f. We look at perturbations local to some area, so we say that $f(x, y) = 0$ outside a circle of radius R. This gives normal vector $\mathbf{n} = \left\langle -\epsilon \frac{\partial f}{\partial x},\ -\epsilon \frac{\partial f}{\partial y},\ 1 \right\rangle$. Apply Taylor's theorem to find

$$\begin{aligned}
\mathbf{v}(x, y, \epsilon f(x, y)) &= \mathbf{v}(x, y, 0) + \epsilon f \frac{\partial \mathbf{v}}{\partial z}\bigg|_{(x,y,0)} + \frac{\epsilon^2 f^2}{2} \frac{\partial^2 \mathbf{v}}{\partial z^2}\bigg|_{(x,y,0)} + \cdots \\
&= \mathbf{v}_0(x, y, 0) + \epsilon \left[\mathbf{v}_1 + f \frac{\partial \mathbf{v}_0}{\partial z} \right]_{(x,y,0)} + \epsilon^2 \left[\mathbf{v}_2 + f \frac{\partial \mathbf{v}_1}{\partial z} + \frac{f^2}{2} \frac{\partial^2 \mathbf{v}_0}{\partial z^2} \right]_{(x,y,0)} + \cdots .
\end{aligned}$$

Substitute this into the boundary condition:

$$0 = [\mathbf{v}_0 \cdot \langle 0,\ 0,\ 1 \rangle]_{(x,y,0)} + \epsilon \left[\mathbf{v}_1 \cdot \langle 0,\ 0,\ 1 \rangle + \mathbf{v}_0 \cdot \left\langle -\frac{\partial f}{\partial x},\ -\frac{\partial f}{\partial y},\ 0 \right\rangle + f \frac{\partial \mathbf{v}_0}{\partial z} \cdot \langle 0,\ 0,\ 1 \rangle \right]_{(x,y,0)} + \cdots .$$

Hence, the boundary conditions for the first two orders are

$$\begin{aligned}
O(1): &\qquad \mathbf{v}_0(x, y, 0) \cdot \langle 0,\ 0,\ 1 \rangle = 0 \\
O(\epsilon): &\qquad \mathbf{v}_1(x, y, 0) \cdot \langle 0,\ 0,\ 1 \rangle = -f \frac{\partial \mathbf{v}_0}{\partial z}\bigg|_{(x,y,0)} \cdot \langle 0,\ 0,\ 1 \rangle + \mathbf{v}_0(x, y, 0) \cdot \left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ 0 \right\rangle
\end{aligned}$$

The leading order, $O(1)$, problem is to find $p_0(x, y, z)$ such that

$$\nabla_H^2 p_0 - \frac{1}{\gamma^2} \frac{\partial^2 p_0}{\partial z^2} = 0$$

subject to the boundary condition

$$\mathbf{v}_0(x, y, 0) \cdot \langle 0,\ 0,\ 1 \rangle = 0.$$

We say $p_0(x, y, z) = p_0^i(x, y, z) + p_0^r(x, y, z)$, where $p_0^i(x, y, z) = A\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$ is known and $p_0^r(x, y, z) = B\mathrm{e}^{i\mathbf{l}\cdot\mathbf{r}}$ is to be found. This is the same as the reflection problem we solved in Chapter 5. We found $\mathbf{l} = \langle k_1, k_2, -k_3 \rangle$, $B = A$, and hence, $p_0^r(x, y, z) = A\mathrm{e}^{i(k_1 x + k_2 y - k_3 z)}$. We use this to define the problem on $O(\epsilon)$.

The more interesting, $O(\epsilon)$, problem is to find $p_1(x, y, z)$ such that

$$\nabla_H^2 p_1 - \frac{1}{\gamma^2}\frac{\partial^2 p_1}{\partial z^2} = 0 \tag{6.3}$$

subject to the boundary condition

$$\begin{aligned}\frac{\partial p_1}{\partial z} &= -\mathrm{i}\omega\gamma^2 \left[-f\frac{\partial \mathbf{v}_0}{\partial z}\cdot\langle 0,\ 0,\ 1\rangle + \mathbf{v}_0 \cdot \left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ 0\right\rangle\right]_{(x,y,0)} \\ &= -\mathrm{i}\gamma^2 A\mathrm{e}^{\mathrm{i}(k_1 x + k_2 y)}\left(2\mathrm{i}fk_3^2\gamma^{-2} + \frac{\partial f}{\partial x}k_1 + \frac{\partial f}{\partial y}k_2\right) \\ &= g(x, y),\end{aligned} \tag{6.4}$$

where $g(x, y) = -\mathrm{i}\gamma^2 A\mathrm{e}^{\mathrm{i}(k_1 x + k_2 y)}\left(2\mathrm{i}fk_3^2\gamma^{-2} + \frac{\partial f}{\partial x}k_1 + \frac{\partial f}{\partial y}k_2\right)$. Notice this is not a reflection problem, but a radiation problem with the prescribed velocity $g(x, y)$ in the direction $\langle 0,\ 0,\ 1\rangle$. We employ the use of the two dimensional Fourier transform

$$\widetilde{f}(\xi, \eta) = \mathcal{F}\{f(x, y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\mathrm{e}^{-\mathrm{i}(\xi x + \eta y)}\mathrm{d}x\mathrm{d}y, \tag{6.5}$$

with the inverse transform

$$f(x, y) = \mathcal{F}^{-1}\{\widetilde{f}(\xi, \eta)\} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{f}(\xi, \eta)\mathrm{e}^{\mathrm{i}(\xi x + \eta y)}\mathrm{d}\xi\mathrm{d}\eta. \tag{6.6}$$

Multiply (6.3) by $\mathrm{e}^{-\mathrm{i}(\xi x + \eta y)}$ and integrate with respect x and y to get the transformed differential equation

$$\xi^2\widetilde{p_1} + \eta^2\widetilde{p_1} + \frac{1}{\gamma^2}\frac{\partial^2\widetilde{p_1}}{\partial z^2} = 0 \tag{6.7}$$

using $\mathcal{F}\left\{\frac{\partial^n p_1}{\partial x^n}\right\} = (\mathrm{i}\xi)^n\widetilde{p_1}$ and $\mathcal{F}\left\{\frac{\partial^n p_1}{\partial y^n}\right\} = (\mathrm{i}\eta)^n\widetilde{p_1}$ with $n = 2$. Solving for $\widetilde{p_1}(\xi, \eta, z)$ gives

$$\widetilde{p_1}(\xi, \eta, z) = C(\xi, \eta)\mathrm{e}^{\mathrm{i}\sigma z} + D(\xi, \eta)\mathrm{e}^{-\mathrm{i}\sigma z},$$

where $\sigma = \sqrt{\gamma^2(\xi^2 + \eta^2)}$, C and D are an unknown functions of ξ and η. This is a radiation problem, so we apply the radiation condition, energy must flow away from the surface. In Chapter 5 we saw that, for energy to propagate upwards, the constant in front of z should be negative, so we find $C = 0$, and hence,

$$\widetilde{p_1}(\xi, \eta, z) = D(\xi, \eta)\mathrm{e}^{-\mathrm{i}\sigma z}.$$

We use the boundary condition to determine $D(\xi, \eta)$.

Multiply (6.4) by $e^{-i(\xi x + \eta y)}$ and integrate with respect x and y to get the boundary condition in Fourier space:

$$\frac{\partial \widetilde{p_1}}{\partial z} = \widetilde{g}(\xi, \eta).$$

Putting $\widetilde{p_1}$ into the above allows us to find

$$D(\xi, \eta) = \frac{i}{\sigma}\widetilde{g}.$$

Hence,

$$\widetilde{p_1}(\xi, \eta, z) = \frac{i}{\sigma}\widetilde{g}(\xi, \eta)e^{-i\sigma z},$$

and the pressure on $O(\epsilon)$ is given by

$$p_1(x, y, z) = \mathcal{F}^{-1}\left\{\frac{i}{\sqrt{\gamma^2(\xi^2 + \eta^2)}}\widetilde{g}(\xi, \eta)e^{-i\sqrt{\gamma^2(\xi^2+\eta^2)}z}\right\}.$$

The result obtained has the form of an integral with integrand $\mathcal{A}(\xi, \eta)e^{iE}$, where $\mathcal{A}(\xi, \eta) = \frac{i\widetilde{g}(\xi, \eta)}{\gamma\sqrt{\xi^2 + \eta^2}}$ and $E = \xi x + \eta y - \gamma\sqrt{\xi^2 + \eta^2}$. We use cylindrical polar coordinates for $\langle x, y\rangle$ and $\langle \xi, \eta\rangle$ given by

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad \xi = \kappa\cos\beta, \qquad \text{and} \qquad \eta = \kappa\sin\beta.$$

Thus, $E = \kappa[r(\cos\beta\cos\phi + \sin\beta\sin\phi) - \gamma z]$. Now, we can introduce conical polar coordinates (recall characteristic surfaces are cones of angle $\theta_c = \arccos(\frac{\omega}{N})$ with the xy-plane), $r = \psi c_c + \zeta s_c$ and $z = -\psi s_c + \zeta c_c$, so that in the far field within the wave beams, ψ is finite and $\zeta \to \infty$. This allows us to rewrite E as

$$E = \zeta\kappa[s_c(\cos\beta\cos\phi + \sin\beta\sin\phi) - \gamma c_c] + \psi\kappa[c_c(\cos\beta\cos\phi + \sin\beta\sin\phi) + \gamma s_c].$$

Hence the integral becomes

$$\int_0^\infty \int_0^{2\pi} \mathcal{B}(\kappa, \beta)e^{i\zeta\Theta(\kappa,\beta)}d\beta d\kappa,$$

where $\Theta(\kappa, \beta) = \kappa[s_c(\cos\beta\cos\phi + \sin\beta\sin\phi) - \gamma c_c]$ and

$$\mathcal{B}(\kappa, \beta) = \gamma^{-1}\widetilde{g}(\kappa, \beta)\mathrm{e}^{\mathrm{i}\psi\kappa[c_c(\cos\beta\cos\phi+\sin\beta\sin\phi)+\gamma s_c]}.$$

In the far field ζ is large, so the above integral can be approximated with the method of stationary phase, with phase $\Theta(\kappa, \beta)$.

6.2 BOUNDARY AT $z = x\tan(\alpha) + \epsilon f(x, y)$

Assume the bottom topography has form $z = x\tan(\alpha) + \epsilon f(x, y)$ for some f. Applying the change of variables given by (5.5) to (6.1) tells us that $p_j(X, Y, Z)$ solves

$$0 = \frac{\omega^2}{N^2}\left(\frac{\partial^2 p_j}{\partial X^2} + \frac{\partial^2 p_j}{\partial Y^2} + \frac{\partial^2 p_j}{\partial Z^2}\right) - c_\alpha^2\frac{\partial^2 p_j}{\partial X^2} + 2c_\alpha s_\alpha\frac{\partial^2 p_j}{\partial X\partial Z} - s_\alpha^2\frac{\partial^2 p_j}{\partial Z^2} - \frac{\partial^2 p_j}{\partial Y^2},$$

for each $j = 0, 1, 2, \ldots$, and

$$\mathbf{v}_j(X, Y, Z) = \frac{1}{\mathrm{i}\omega}\left\langle \frac{\partial p}{\partial X}c_\alpha - \frac{\partial p}{\partial Z}s_\alpha,\ \frac{\partial p}{\partial Y},\ -\gamma^{-2}\left(\frac{\partial p}{\partial X}s_\alpha + \frac{\partial p}{\partial Z}c_\alpha\right)\right\rangle.$$

gives the velocity. Applying the same change of variables to the boundary gives $Z = \epsilon F(X, Y)$, where $F(X, Y) = f(x, y)$. Thus, $\mathbf{n}(X, Y, Z) = \left\langle -\epsilon\frac{\partial F}{\partial X},\ -\epsilon\frac{\partial F}{\partial Y},\ 1\right\rangle$. Taylor's theorem gives

$$\begin{aligned}\mathbf{v}(X, Y, \epsilon F(X, Y)) &= \mathbf{v}(X, Y, 0) + \epsilon F\left.\frac{\partial \mathbf{v}}{\partial Z}\right|_{(X,Y,0)} + \frac{\epsilon^2 F^2}{2}\left.\frac{\partial^2 \mathbf{v}}{\partial Z^2}\right|_{(X,Y,0)} + \cdots \\ &= \mathbf{v}_0(X, Y, 0) + \epsilon\left[\mathbf{v}_1 + F\frac{\partial \mathbf{v}_0}{\partial Z}\right]_{(X,Y,0)} + \epsilon^2\left[\mathbf{v}_2 + F\frac{\partial \mathbf{v}_1}{\partial Z} + \frac{F^2}{2}\frac{\partial^2 \mathbf{v}_0}{\partial Z^2}\right]_{(X,Y,0)} + \cdots.\end{aligned}$$

Substitute this into the boundary condition:

$$0 = [\mathbf{v}_0\cdot\langle 0,\ 0,\ 1\rangle]_{(X,Y,0)} + \epsilon\left[\mathbf{v}_1\cdot\langle 0,\ 0,\ 1\rangle + \mathbf{v}_0\cdot\left\langle -\frac{\partial F}{\partial X},\ -\frac{\partial F}{\partial Y},\ 0\right\rangle + F\frac{\partial \mathbf{v}_0}{\partial Z}\cdot\langle 0,\ 0,\ 1\rangle\right]_{(X,Y,0)} + \cdots.$$

Hence, the boundary conditions for the first two orders are

$$\begin{aligned} O(1): &\quad \mathbf{v}_0(X, Y, 0)\cdot\langle 0,\ 0,\ 1\rangle = 0 \\ O(\epsilon): &\quad \mathbf{v}_1(X, Y, 0)\cdot\langle 0,\ 0,\ 1\rangle = -F\left.\frac{\partial \mathbf{v}_0}{\partial Z}\right|_{(X,Y,0)}\cdot\langle 0,\ 0,\ 1\rangle + \mathbf{v}_0(X, Y, 0)\cdot\left\langle \frac{\partial F}{\partial X},\ \frac{\partial F}{\partial Y},\ 0\right\rangle\end{aligned}$$

The leading order, $O(1)$, problem is to find $p_0(x,y,z)$ such that

$$0 = \frac{\omega^2}{N^2}\left(\frac{\partial^2 p_0}{\partial X^2} + \frac{\partial^2 p_0}{\partial Y^2} + \frac{\partial^2 p_0}{\partial Z^2}\right) - c_\alpha^2\frac{\partial^2 p_0}{\partial X^2} + 2c_\alpha s_\alpha\frac{\partial^2 p_0}{\partial X\partial Z} - s_\alpha^2\frac{\partial^2 p_0}{\partial Z^2} - \frac{\partial^2 p_0}{\partial Y^2}$$

subject to the boundary condition

$$\mathbf{v}_0(X,Y,0)\cdot\langle 0,\ 0,\ 1\rangle = 0.$$

We say $p_0(X,Y,Z) = p_0^i(X,Y,Z) + p_0^r(X,Y,Z)$, where $p_0^i(X,Y,Z) = A\mathrm{e}^{i(K_1X+K_2Y+K_3Z)}$ is known and $p_0^r(X,Y,Z) = B\mathrm{e}^{i(L_1X+L_2Y+L_3Z)}$ is to be found. This is the same as the reflection problem we solved in Chapter 5. We found

$$L_1 = K_1, \qquad L_2 = K_2, \qquad L_3 = \frac{-b(K_1) - \Delta(K_1,K_2)}{a},$$

and

$$B = \frac{-\{as_\alpha K_1 + c_\alpha[-b(K_1) + \Delta(K_1,K_2)]\}}{as_\alpha K_1 + c_\alpha[-b(K_1) - \Delta(K_1,K_2)]}A,$$

and hence,

$$p_0^r(x,y,z) = \frac{-\{as_\alpha K_1 + c_\alpha[-b(K_1) + \Delta(K_1,K_2)]\}}{as_\alpha K_1 + c_\alpha[-b(K_1) - \Delta(K_1,K_2)]}A\mathrm{e}^{\mathrm{i}[K_1X+K_2Y-(b(K_1)+\Delta(K_1,K_2)]/aZ]},$$

where a, $b(K_1)$, and $\Delta(K_1,K_2)$ are as defined by (5.8), (5.9), and (5.11). We use this to define the problem on $O(\epsilon)$.

The more interesting, $O(\epsilon)$, problem is to find $p_1(x,y,z)$ such that

$$0 = \frac{\omega^2}{N^2}\left(\frac{\partial^2 p_1}{\partial X^2} + \frac{\partial^2 p_1}{\partial Y^2} + \frac{\partial^2 p_1}{\partial Z^2}\right) - c_\alpha^2\frac{\partial^2 p_1}{\partial X^2} + 2c_\alpha s_\alpha\frac{\partial^2 p_1}{\partial X\partial Z} - s_\alpha^2\frac{\partial^2 p_1}{\partial Z^2} - \frac{\partial^2 p_1}{\partial Y^2} \tag{6.8}$$

subject to the boundary condition

$$\begin{aligned}\mathbf{v}_1(X,Y,0)\cdot\langle 0,\ 0,\ 1\rangle &= -F\frac{\partial\mathbf{v}_0}{\partial Z}\bigg|_{(X,Y,0)}\cdot\langle 0,\ 0,\ 1\rangle + \mathbf{v}_0(X,Y,0)\cdot\left\langle\frac{\partial F}{\partial X},\ \frac{\partial F}{\partial Y},\ 0\right\rangle. \\ &= \frac{1}{\omega}\mathrm{e}^{i(K_1X+K_2Y)}\bigg\{A(c_\alpha K_1 - s_\alpha K_3)\frac{\partial F}{\partial X} + AK_2\frac{\partial F}{\partial Y} + B(c_\alpha K_1 - s_\alpha L_3)\frac{\partial F}{\partial X} \\ &\quad + BK_2\frac{\partial F}{\partial Y} + \mathrm{i}AK_3(s_\alpha K_1 + c_\alpha K_3)\gamma^{-2}F + \mathrm{i}BL_3(s_\alpha K_1 + c_\alpha L_3)\gamma^{-2}F\bigg\} \\ &= \frac{1}{\omega}g(X,Y),\ \text{say}.\end{aligned} \tag{6.9}$$

To solve this problem we again use the Fourier transform pair given by (6.5) and (6.6). Multiplying (6.8) by $\mathrm{e}^{-\mathrm{i}(\xi X+\eta Y)}$ and integrate with respect to X and Y gives the equation for $\widetilde{p_1}$:

$$\frac{\omega^2}{N^2}\left(-\xi^2\widetilde{p_1}-\eta^2\widetilde{p_1}+\frac{\partial^2\widetilde{p_1}}{\partial Z^2}\right)+c_\alpha^2\xi^2\widetilde{p_1}+2ic_\alpha s_\alpha\xi\frac{\partial\widetilde{p_1}}{\partial Z}-s_\alpha^2\frac{\partial^2\widetilde{p_1}}{\partial Z^2}+\eta^2\widetilde{p_1}=0,$$

or, equivalently

$$\left(c_c^2-s_\alpha^2\right)\frac{\partial^2\widetilde{p_1}}{\partial Z^2}+\left(2ic_\alpha s_\alpha\xi\right)\frac{\partial\widetilde{p_1}}{\partial Z}-\left((c_c^2-c_\alpha^2)\xi^2-s_c^2\eta^2\right)\widetilde{p_1}=0. \tag{6.10}$$

Solving this equation gives $\widetilde{p_1}(\xi,\eta,Z)=D(\xi,\eta)\mathrm{e}^{\mathrm{i}\sigma Z}$, where $D(\xi,\eta)$ is unknown and σ is given by

$$a\sigma^2+2b(\xi)\sigma+c(\xi,\eta)=0,$$

where a, $b(\xi)$, and $c(\xi,\eta)$ are as defined by (5.8)-(5.10). Hence,

$$a\sigma=-b(\xi)\pm\Delta(\xi,\eta),$$

with $\Delta(\xi,\eta)$ given by (5.11). Since this is a radiation problem, and the radiation condition says that energy must propagate away from the boundary we have, due to the analysis done in Chapter 5, $a\sigma=-b(\xi)-\Delta(\xi,\eta)$.

Multiply (6.9) by $\mathrm{e}^{-\mathrm{i}(\xi X+\eta Y)}$ and integrate with respect to X and Y to get the boundary condition in Fourier space:

$$\frac{\mathrm{i}}{\omega\gamma^2}\left(\mathrm{i}\xi D(\xi,\eta)s_\alpha+\mathrm{i}\sigma D(\xi,\eta)c_\alpha\right)=\frac{1}{\omega}\widetilde{g}(\xi,\eta)$$

Solve for $D(\xi,\eta)$:

$$D(\xi,\eta)=-\frac{\gamma^2\widetilde{g}(\xi,\eta)}{\xi s_\alpha+\sigma c_\alpha}.$$

Thus the transformed pressure on order ϵ is

$$\widetilde{p_1}(\xi,\eta,Z)=D(\xi,\eta)\mathrm{e}^{\mathrm{i}\sigma Z},$$

and

$$p_1(x,y,z)=\mathcal{F}^{-1}\{D(\xi,\eta)\mathrm{e}^{-\mathrm{i}[b(\xi)+\Delta(\xi,\eta)]/aZ}\}.$$

Taking a closer look at $D(\xi, \eta)$ shows that the denominator is the same as the denominator of B in the sloping three dimensional flat reflections in Chapter5, replacing K_1 and K_2 with ξ and η, respectively. Hence, the denominator will vanish when $\alpha = \dfrac{\pi}{2} - \theta_c$. We cannot say what happens in this case as we do not have an exact form for the numerator without choosing a specific bump equation $F(X, Y)$. However, in general, we expect to have a singularity when α is exactly this critical angle. This is why we take α to be less than this critical angle.

As in the previous case, we will approximate the inverse Fourier transform in the far field. The integral will have an exponential term $\mathrm{e}^{\mathrm{i}E}$, with

$$aE = a(\xi X + \eta Y) - (c_\alpha s_\alpha \xi + \Delta)Z.$$

The characteristic surfaces being cones along the z-axis, tell us that we should express E in terms of x, y, and z. This gives

$$aE = x(c_\alpha \xi c_c^2 + s_\alpha \Delta) + A\eta y - z(s_\alpha \xi s_c^2 + c_\alpha \Delta).$$

Using the same coordinate substitutions as in section 5.1 (to get $\zeta \to \infty$ and ψ finite in the far field), we find $E = \kappa(\zeta\tau + \psi\chi)$, where

$$a\tau = (c_\alpha c_c c_\phi - s_\alpha s_c)c_c s_c \cos\beta + a s_c s_\phi \sin\beta + (s_\alpha s_c c_\phi - c_\alpha c_c)s_c\sqrt{c_c^2 - s_\alpha^2 \sin^2\beta},$$

and

$$a\chi = (c_\alpha c_c^3 c_\phi + s_a l s_c^3)\cos\beta + a c_c s_\phi \sin\beta + (s_\alpha c_c c_\phi + c_\alpha s_c)s_c\sqrt{c_c^2 - s_\alpha^2 \sin^2\beta},$$

with $s_\phi = \sin\phi$ and $c_\phi = \cos\phi$. Thus,

$$p = \frac{1}{4\pi^2}\int_0^\infty \int_0^{2\pi} \mathcal{B}(\kappa, \beta)\mathrm{e}^{\mathrm{i}\zeta\kappa\tau}\mathrm{d}\beta\mathrm{d}\kappa,$$

where

$$\mathcal{B}(\kappa, \beta) = D(\kappa, \beta)\mathrm{e}^{\mathrm{i}\psi\kappa\chi}.$$

We can approximate this integral using method of stationary phase to find p_1 in the far field.

CHAPTER 7
CONCLUSION

In this thesis we presented a method for examining the reflections of internal gravity waves off bumpy surfaces using asymptotics and Fourier transforms. We began by deriving a partial differential equation for the time harmonic reduced perturbed pressure of the fluid from Euler's equations for inviscid, incompressible fluid flow and the Boussinesq approximation. We then assumed plane wave solutions and derived the dispersion relation, phase velocity, and group velocity. Doing so showed us that the direction in which the wave propagates depends solely on the frequency of the wave and the stratification of the fluid. We also found that energy propagates perpendicular to the wave front propagation.

We then looked at reflections of acoustic waves to get an understanding for the mathematics behind a reflection problem and then applied those techniques to the internal gravity waves. We used a change of variables to examine reflections off a sloping plane boundary and found that the waves will reflect in only two directions depending on the steepness of the slope of the boundary.

Finally, we found that for perturbed surfaces we get a new problem at each order of ϵ, with the leading order problem being the reflection problem off the flat surface and the smaller order problem are radiation problems with prescribed velocities on the boundary. We transformed the radiation problem to a Fourier space, so that we could solve the problem and apply the radiation condition that energy must propagate away from the boundary. A solution is obtained via an inverse transform.

Additional work can be done by applying the technique to a specific bottom topography. The work here could also be extended upon by allowing for energy transmission into the boundary by assuming not only an incident and reflected wave, but also a transmitted wave. Future work could also include relaxing the flat bump condition, to allow for reflections that propagate downwards as well as upwards.

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APPENDIX A PLANE WAVES

Plane waves are waves of the form

$$\phi(x, y, z, t) = A\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

where O_{xyz} are standard Cartesian coordinates with z pointing upwards, ϕ is the velocity potential of the fluid, ω is the frequency of the wave, A is the amplitude of the wave motion, $\mathbf{k}$ is the wave vector, and $\mathbf{r} = \langle x, y, z\rangle$ is the position vector. Consider the case where $\mathbf{k} = \langle k, 0, 0\rangle$. This gives

$$\phi(x, y, z, t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)},$$

which is clearly a wave that propagates in the direction of positive x, so nothing happens in y or z, or perpendicular to $\mathbf{k}$. Similarly, for $\mathbf{k} = \langle 0, k, 0\rangle$ and $\mathbf{k} = \langle 0, 0, k\rangle$, we get

$$\phi(x, y, z, t) = A\mathrm{e}^{\mathrm{i}(ky-\omega t)}$$

and

$$\phi(x, y, z, t) = A\mathrm{e}^{\mathrm{i}(kz-\omega t)},$$

respectively, which again shows nothing happens perpendicular to the $\mathbf{k}$. Taking a linear combination of these $\mathbf{k}$ gives us wave vector $\mathbf{k} =< k_1, k_2, k_3 >$, for some constants k_1, k_2, and k_3 which gives us velocity potential

$$\phi(x, y, z, t) = \mathrm{e}^{\mathrm{i}(k_1x+k_2y+k_3z-\omega t)}$$

that again has no motion perpendicular to $\mathbf{k}$ which is where we get the plane front of the plane wave that propagates in the direction of $\mathbf{k}$.

APPENDIX B AN ALTERNATE METHOD FOR SOLVING THE THREE DIMENSIONAL REFLECTIONS OF INTERNAL GRAVITY WAVES OFF A FLAT SLOPED SURFACE

This appendix presents an alternate method for solving for the reflected internal gravity wave off a flat sloped surface. This was the original method employed to solve this problem, however the method presented in chapter 5 has a closer relation to the methods used in chapter 6.

Recall, the total fluid velocity is given by $\mathbf{v} = \mathbf{v}_i + \mathbf{v}_r$, where

$$\mathbf{v}_i(x, y, z) = \frac{A}{\omega} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \left\langle k_1,\ k_2,\ -k_3\gamma^{-2} \right\rangle$$

is the incident velocity and

$$\mathbf{v}_r(x, y, z) = \frac{B}{\omega} \mathrm{e}^{\mathrm{i}\mathbf{l}\cdot\mathbf{r}} \left\langle l_1,\ l_2,\ -l_3\gamma^{-2} \right\rangle$$

is the reflected velocity. Also recall the total velocity is subject to the boundary condition $\mathbf{v}\cdot\mathbf{n} = 0$ on the boundary $z = x\tan(\alpha)$, for $0 \le \alpha \le \frac{\pi}{2}$. Applying the boundary condition leads to the equation

$$0 = \left(k_1 s_\alpha - k_3\gamma^{-2} c_\alpha\right) \frac{A}{\omega} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\langle x, y, x t_\alpha\rangle} + \left(l_1 s_\alpha - l_3\gamma^{-2} c_\alpha\right) \frac{B}{\omega} \mathrm{e}^{\mathrm{i}\mathbf{l}\cdot\langle x, y, x t_\alpha\rangle},$$

where $c_\alpha = \cos(\alpha)$, $s_\alpha = \sin(\alpha)$, and $t_\alpha = \tan(\alpha)$. Equating the exponential terms for all x and y leads to $l_2 = k_2$ and $k_1 + k_3\tan(\alpha) = l_1 + l_3\tan(\alpha)$. We put these two equations into the following equation derived from the dispersion relation

$$\frac{k_1^2 + k_2^2}{k_3^2} = \frac{l_1^2 + l_2^2}{l_3^2}$$

to derive a quadratic equation for l_3 :

$$\frac{k_1^2 + k_2^2}{k_3^2} l_3^2 = t_\alpha^2 l_3^2 - 2k_1 t_\alpha l_3 - 2t_\alpha^2 k_3 l_3 + 2t_\alpha k_1 k_3 + t_\alpha^2 k_3^2 + k_1^2 + k_2^2.$$

Putting $\frac{k_1^2 + k_2^2}{k_3^2} = \gamma^{-2}$ and reorganizing gives

$$0 = (\gamma^{-2} - t_\alpha^2) l_3^2 + 2t_\alpha(k_1 + t_\alpha k_3) l_3 - 2t_\alpha k_1 k_3 - t_\alpha^2 k_3^2 - \gamma^{-2} k_3^2.$$

Multiply through by γ^2 to get

$$0 = (1-\gamma^2 t_\alpha^2) l_3^2 + 2\gamma^2 t_\alpha (k_1 + t_\alpha k_3) l_3 - (2\gamma^2 t_\alpha k_1 k_3 + (\gamma^2 t_\alpha^2 + 1) k_3^2).$$

Applying the quadratic formula gives

$$\begin{aligned}
l_3 &= \frac{-2\gamma^2 t_\alpha (k_1 + t_\alpha k_3) \pm \sqrt{4\gamma^4 t_\alpha^2 (k_1 + t_\alpha k_3)^2 + 4(1-\gamma^2 t_\alpha^2)(2\gamma^2 t_\alpha k_1 k_3 + (\gamma^2 t_\alpha^2 + 1) k_3^2)}}{2(1-\gamma^2 t_\alpha^2)} \\
&= \frac{-\gamma^2 t_\alpha (k_1 + t_\alpha k_3) \pm \sqrt{\gamma^4 t_\alpha^2 (k_1 + t_\alpha k_3)^2 + (1-\gamma^2 t_\alpha^2)(2\gamma^2 t_\alpha k_1 k_3 + (\gamma^2 t_\alpha^2 + 1) k_3^2)}}{1-\gamma^2 t_\alpha^2} \\
&= \frac{-\gamma^2 t_\alpha (k_1 + t_\alpha k_3) \pm \sqrt{\gamma^4 t_\alpha^2 (k_1 + t_\alpha k_3)^2 + (1-\gamma^2 t_\alpha^2)[(\gamma^2 t_\alpha^2 k_3^2 + 2\gamma^2 t_\alpha k_1 k_3 + \gamma^2 k_1^2) - \gamma^2 k_1^2 + k_3^2]}}{1-\gamma^2 t_\alpha^2} \\
&= \frac{-\gamma^2 t_\alpha (k_1 + t_\alpha k_3) \pm \sqrt{\gamma^4 t_\alpha^2 (k_1 + t_\alpha k_3)^2 + (1-\gamma^2 t_\alpha^2)[\gamma^2 (t_\alpha k_3 + k_1)^2 + k_3^2 - \gamma^2 k_1^2]}}{1-\gamma^2 t_\alpha^2} \\
&= \frac{-\gamma^2 t_\alpha (k_1 + t_\alpha k_3) \pm \sqrt{\gamma^2 (t_\alpha k_3 + k_1)^2 + (1-\gamma^2 t_\alpha^2)(k_3^2 - \gamma^2 k_1^2)}}{1-\gamma^2 t_\alpha^2} \\
&= \frac{-\gamma^2 t_\alpha (k_1 + t_\alpha k_3) \pm \sqrt{\gamma^2 (t_\alpha k_3 + k_1)^2 + k_3^2 - \gamma^2 k_1^2 - \gamma^2 t_\alpha^2 k_3^2 + \gamma^4 t_\alpha^2 k_1^2}}{1-\gamma^2 t_\alpha^2} \\
&= \frac{-\gamma^2 t_\alpha (k_1 + t_\alpha k_3) \pm \sqrt{\gamma^2 (t_\alpha k_3 + k_1)^2 + (k_3^2 + \gamma^4 t_\alpha^2 k_1^2 + 2\gamma^2 t_\alpha k_1 k_3) - (2\gamma^2 t_\alpha k_1 k_3 + \gamma^2 k_1^2 + \gamma^2 t_\alpha^2 k_3^2)}}{1-\gamma^2 t_\alpha^2} \\
&= \frac{-\gamma^2 t_\alpha (k_1 + t_\alpha k_3) \pm \sqrt{\gamma^2 (t_\alpha k_3 + k_1)^2 + (k_3 + \gamma^2 t_\alpha k_1)^2 - \gamma^2 (t_\alpha k_3 + k_1)^2}}{1-\gamma^2 t_\alpha^2} \\
&= \frac{-\gamma^2 t_\alpha (k_1 + t_\alpha k_3) \pm \sqrt{(k_3 + \gamma^2 t_\alpha k_1)^2}}{1-\gamma^2 t_\alpha^2} \\
&= \frac{-\gamma^2 t_\alpha (k_1 + t_\alpha k_3) \pm k_3 + \gamma^2 t_\alpha k_1}{1-\gamma^2 t_\alpha^2}.
\end{aligned}$$

In the addition case we get $l_3 = k_3$ which tells us $l_1 = k_1$, and thus $\mathbf{l} = \mathbf{k}$. This does not result in a reflected wave, so we throw this result away. Looking at the other case we get

$$l_3 = \frac{-(1+\gamma^2 t_\alpha^2) k_3 - 2\gamma^2 t_\alpha k_1}{1-\gamma^2 t_\alpha^2}$$

and

$$l_1 = \frac{(1+\gamma^2 t_\alpha^2)k_1 + 2t_\alpha k_3}{1-\gamma^2 t_\alpha^2},$$

or

$$\mathbf{l} = \left\langle \frac{(1+\gamma^2 t_\alpha^2)k_1 + 2t_\alpha k_3}{1-\gamma^2 t_\alpha^2}, k_2, \frac{-(1+\gamma^2 t_\alpha^2)k_3 - 2\gamma^2 t_\alpha k_1}{1-\gamma^2 t_\alpha^2} \right\rangle.$$

The reflected amplitude is given by

$$B = -\frac{k_1 s_\alpha - k_3 \gamma^{-2} c_\alpha}{l_1 s_\alpha - l_3 \gamma^{-2} c_\alpha} A$$