Extrapolated Crank-Nicolson Orthogonal Spline Collocation Methods for Burgers' Equation

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Introduction

Burgers' equation is given by the non-linear PDE

$$\frac{\partial \mathbf{u}}{\partial t} - \kappa \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = 0, \quad \mathbf{x} \in \Omega, \quad t \in (T_0, T_1], \tag{1}$$

where κ is a positive constant. The solution $\mathbf{u}$ is subject to the following initial and boundary conditions

$$\mathbf{u}(\mathbf{x}, T_0) = \mathbf{g}(\mathbf{x}), \quad \mathbf{x} \in \overline{\Omega}, \tag{2}$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{h}(t), \quad t \in (T_0, T_1], \quad \mathbf{x} \in \partial\Omega. \tag{3}$$

Throughout, we define,

$$d_t U^n = \frac{U^{n+1} - U^n}{\tau}, \quad U^{n+1/2} = \frac{U^n + U^{n+1}}{2},$$

$$\widetilde{U}^n = \frac{3U^n - U^{n-1}}{2}.$$

and for P_r, the set of polynomials of degree $\leq r$,

$$\mathcal{M}_r = \{v \in C^1[a,b] : v|_{[x_{i-1},x_i]} \in P_r, i = 1, \dots, N\}.$$

Let $\{\zeta_k\}_{k=1}^{r-1}$ be the nodes of the $(r-1)$-point Gauss-Legendre quadrature for $[0,1]$, and let $\mathcal{G}$ be the set of collocation points

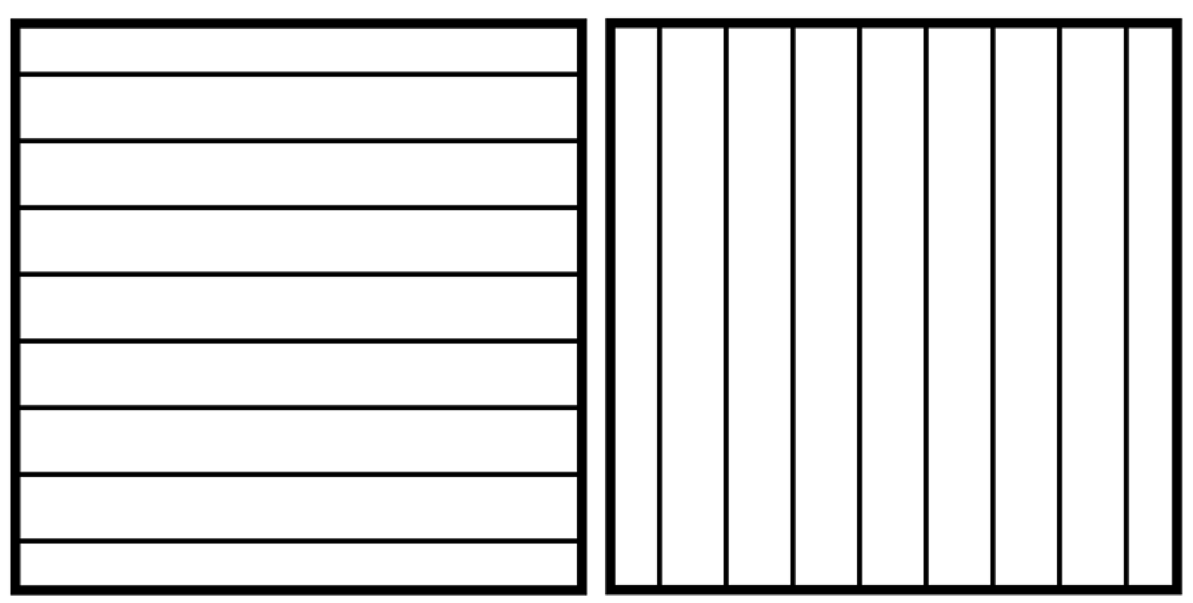
$$\xi_{2i+k} = x_i + h_i \zeta_k, \quad i = 0, \dots, N-1, \quad k = 1, \dots, r-1.$$

Let $d = \dim \mathcal{M}_r = 2N + 2$. If $\{\phi_j\}_{j=1}^{d}$ is a basis for $\mathcal{M}_r$, then $U^n = \sum_{j=1}^{d} U_j^n \phi_j, \quad n = 0, \dots, N_t$. We introduce the $(d-2) \times d$ matrices

$$A = (\phi_j''(\xi_i^x))_{i,j=1}^{d-2,d}, \quad B = (\phi_j(\xi_i^x))_{i,j=1}^{d-2,d}, \quad C = (\phi_j'(\xi_i^x))_{i,j=1}^{d-2,d}.$$

Alternating Direction Implicit Method

ADI methods reduce higher dimensional problems to a series of two-point boundary value problems along each direction.



$U_1^{n+1/2}(\cdot, y),\ y \in \mathcal{G}_y$ $\qquad$ $U_1^{n+1}(x, \cdot),\ x \in \mathcal{G}_x$

ADI-ECN-OSC in two Space Variables

- **ADI-ECN-OSC Scheme:**
  Let $\mathcal{M} = \mathcal{M}_r \otimes \mathcal{M}_r$ and

$$\mathcal{G} = \{(x,y) : x \in \mathcal{G}, y \in \mathcal{G}\}$$

The ADI-ECN-OSC scheme for solving (1)–(3) in two spatial dimensions involves finding $\mathbf{U}^n = [U_1^n, U_2^n]^T \in \mathcal{M} \times \mathcal{M}$, $n = 2, \dots, N_t$, such that, for $n = 1, \dots, N_t - 1$,

$$\left[\frac{\mathbf{U}^{n+1/2} - \mathbf{U}^n}{0.5\tau} - \kappa\left(\mathbf{U}_{xx}^{n+1/2} + \mathbf{U}_{yy}^n\right) + \left(\widetilde{\mathbf{U}}^n \cdot \nabla\right) \widetilde{\mathbf{U}}^n\right](x,y) = 0, \quad (x,y) \in \mathcal{G}$$

$$\left[\frac{\mathbf{U}^{n+1} - \mathbf{U}^{n+1/2}}{0.5\tau} - \kappa\left(\mathbf{U}_{xx}^{n+1/2} + \mathbf{U}_{yy}^{n+1}\right) + \left(\widetilde{\mathbf{U}}^n \cdot \nabla\right) \widetilde{\mathbf{U}}^n\right](x,y) = 0, \quad (x,y) \in \mathcal{G}.$$

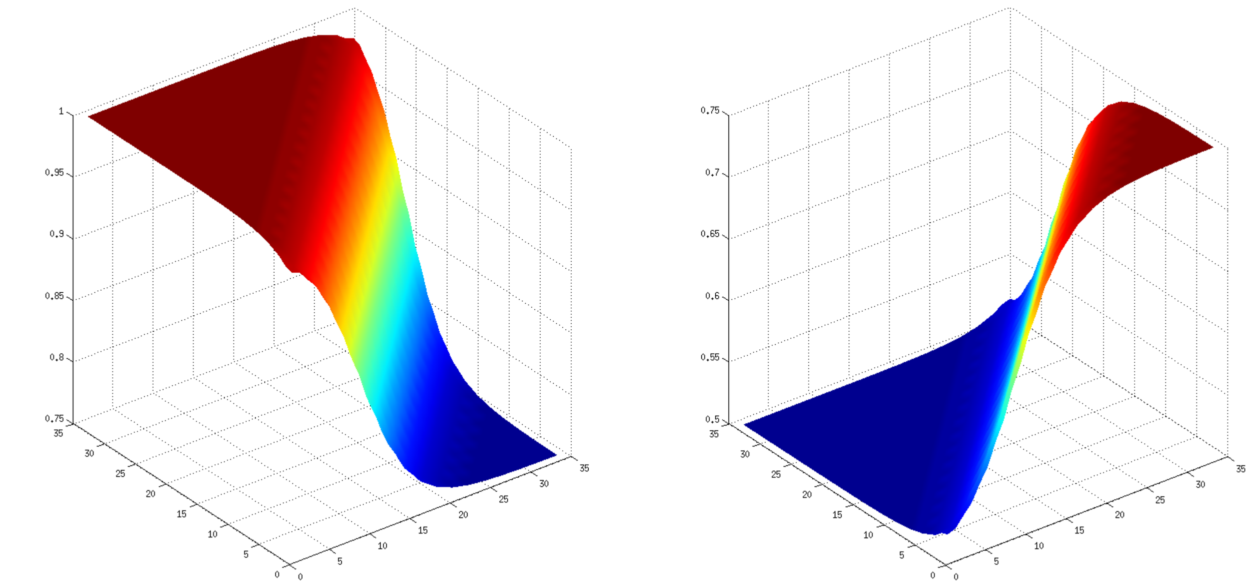
where $\mathbf{U}^0$ is given. The scheme is initialized using a predictor/corrector method.

- **The Matrix Vector Form:**
  The two-point boundary value problems to be solved along each direction take the form

$$\begin{bmatrix} \phi_1(a), \dots, \phi_d(a) \\ B - \frac{\tau}{2}\kappa A \\ \phi_1(b), \dots, \phi_d(b) \end{bmatrix} \mathbf{U}_1^{n+1} = \begin{bmatrix} h_1(t_{n+1}) \\ (B + \frac{\tau}{2}\kappa A)\, \mathbf{U}_1^{n+1/2} - \tau\left(B\widetilde{\mathbf{U}}_1^n\right) \circ \left(C\widetilde{\mathbf{U}}_1^n\right) \\ h_1(t_{n+1}) \end{bmatrix},$$

where $\mathbf{U}_1^n = [U_{1,1}^n, \dots, U_{1,d}^n]^T$ and, $\circ$, denotes the element-wise product of two vectors.

- **Sample Solutions:**
  For $\kappa = 1/80$ and $T_1 = 1$ we obtain the ADI-ECN-OSC solution of U_1 (left) and U_2 (right):



Convergence Analysis

If h and τ are sufficiently small and u, the solution to (1)–(3), is sufficiently smooth, then we expect,

$$\|u^n - U^n\|_\infty \leq C(u)(h^{r+1} + \tau^2).$$

Numerical Results

We selected $N_t = N^{(r+1)/2}$ so that

$$O(h^{r+1}) + O(\tau^2) = O(N^{-(r+1)}).$$

For $r = 3, 4, 5, 6$, at the final time, $T_1 = 1$, we computed the maximum norm error on uniform partitions of $\Omega = [0,1] \times [0,1]$,

$$\|u(\cdot, T_1) - U^{N_t}\|_\infty$$

- Error and convergence rates for $\kappa = 1/10$:

$r = 3$			$r = 4$		
N	error	rate	N	error	rate
4	1.1509e-06	-	4	4.0261e-07	-
9	5.2610e-08	3.8047	9	7.7300e-09	4.8745
16	5.6653e-09	3.8733	16	4.4450e-10	4.9637
25	9.8488e-10	3.9204	25	4.8090e-11	4.9831
$r = 5$			$r = 6$		
N	error	rate	N	error	rate
4	1.1554e-07	-	4	2.9336e-08	-
9	8.9069e-10	5.9997	9	9.9434e-11	7.0130
16	2.8157e-11	6.0035	16	1.7609e-12	7.0106
25	1.9358e-12	5.9990	25	2.7645e-13	4.1489

- Error and convergence rates for $\kappa = 1/80$:

$r = 3$			$r = 4$		
N	error	rate	N	error	rate
4	3.0845e-02	-	4	1.3195e-03	-
9	1.4019e-04	6.6513	9	1.0875e-05	5.9173
16	1.3151e-05	4.1131	16	6.2177e-07	4.9736
25	1.7190e-06	4.5593	25	6.4249e-08	5.0860
$r = 5$			$r = 6$		
N	error	rate	N	error	rate
4	6.7440e-05	-	4	1.1591e-05	-
9	7.1948e-07	5.5991	9	5.5052e-08	6.5970
16	1.9044e-08	6.3121	16	8.4839e-10	7.2523
25	1.1696e-09	6.2519	25	3.3607e-11	7.2344

Conclusions/Future Work

- ADI-ECN-OSN scheme can be used to construct methods of arbitrarily high order acuracy.
- The ADI-ECN-OSN method shows excelent convergence results for small Reynolds numbers.
- Convergence analysis of ADI-ECN-OSC scheme for Burgers' Equation in 2 space variables is underway.
- The method can be extended to solve the Navier-Stokes equations.

References

1 D. Peaceman, H. Rachford, *The numerical solution of parabolic and elliptic differential equations*, J. SIAM, 3 (1955), 28–41.