BUILDING A CONNECTION THROUGH OBSTRUCTION; RELATING GAUGE GRAVITY AND STRING THEORY.

by

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ABSTRACT

Gauge theories of internal symmetries, e.g. the strong and electroweak forces of the Standard Model, have a geometric description in terms of standard fiber bundles. It is tempting then to ask if the remaining force, gravitation, has a description as a gauge theory. The answer is yes, however unlike the internal symmetries of the Standard Model, the story is not so simple. There are dozens of renditions of gravitational gauge theory and no standard fiber bundle description. The main issue in the construction of gravitational gauge theory is the inclusion of translational symmetry. While the Lorentz group, like internal symmetries, acts only at each point, the translational symmetry shifts points in spacetime. For this reason a gauge theory of gravity requires a somewhat more sophisticated fiber bundle known as a composite fiber bundle. When constructing gauge theories of internal symmetries it is easy to take certain topological conditions for granted, like orientability or the ability to define spinors. However it is known that there exist spaces which do not have the properties required to define sensible field theories. Although we may take these topological properties for granted when constructing gauge theories of internal symmetries we haven't had evidence yet to expect we can do the same for gravitational gauge theory. By studying the geometry of the composite bundle formalism underlying viable gauge theories of gravity we have found previously unappreciated subbundles of the primary bundle. We were able to identify these subbundles as the spacetime bundles we would expect to be created by a gauge theory of gravity. Remarkably, the origin of these subbundles leads to the natural inclusion of expected, and unexpected, topological conditions.

While the overall bundle used for gravitation is $P(M, ISO(1,3))$, i.e. a principal Poincaré bundle over a space M, the Poincaré group ($ISO(1,3)$) can be viewed as a bundle in its own right $ISO(\mathbb{R}^4, SO(1,3))$. Thinking of the fiber space itself as yet another bundle leads to consideration of two primary bundles $P(E, SO(1,3))$ and $E(M, \mathbb{R}^4, ISO(1,3), P(M, ISO(1,3)))$.

The split of the total bundle $P(M, ISO(1,3))$ into the two bundles E and $P(E, SO(1,3))$ however requires the existence of a global section of the bundle E. Such a global section is guaranteed to exist by a theorem of Kobayashi and Nomizu. However it is interesting to investigate the topology of the bundle space E and hence of $P(E, SO(1,3))$. The requirement of the global section leads to the definition of a bundle $Q(M, SO(1,3)) \subset P(M, ISO(1,3))$ which can be identified as the frame bundle of spacetime. Its associated bundle, $Q(M, SO(1,3)) \times_{SO(1,3)} \mathbb{R}^4$ where $\times_{SO(1,3)}$ denotes a specific quotient of the product space $Q(M, SO(1,3)) \times \mathbb{R}^4$ by the group $SO(1,3)$, can then be identified as the tangent bundle.

The existence of a global section of E leads to topological conditions on the induced spacetime bundles. Using cohomology with compact support one can show that global sections of E descend to global sections of Q and force the Stiefel-Whitney, Euler and first fractional Pontryagin classes of the spacetime bundles to be trivial. Furthermore the triviality of these characteristic classes is equivalent to the condition that the base space M admit a string structure. Each characteristic class has an interpretation as an obstruction to the creation of a global structure or a topological attribute of the bundle. For the composite bundle formulation the obstructions are to orientablility, parallelizablility, global sections, and conditions related to stable causality and string structures. Similar to the case of a supersymmetric point particle, where the parallelizability of the base manifold determines whether there will be a global anomaly encountered during quantization, whether a manifold admits a string structure will determine if a global anomaly will be encountered in the process of quantization of extended degrees of freedom. This implies that the topological aspects of gravitational gauge theories automatically accommodate the consistent introduction of extended degrees of freedom. This path to structures associated with extended degrees of freedom is in contrast to the typical route, i.e. demanding a consistent quantum theory of gravitation. Here the need for such structures arises classically from demanding that gravitation be realized from a geometrically supported gauge principle.

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As part of an ongoing effort to visit stars beyond ours.

CHAPTER 1

INTRODUCTION

In 1919 Herman Weyl was working on unifying electromagnetism with general relativity. Although not successful, this was the first time he wrote about local symmetry. In 1929 Weyl discovered that the electromagnetic interaction could be explained by taking the free Dirac action which involves a complex (spinor) field ψ and demanding that the global invariance of that action under spacetime-independent phase rotations, $\psi \to e^{i\theta}\psi$, be promoted to a local or spacetime-dependent symmetry, i.e. $\psi \to e^{i\theta(x)}\psi$ [1]. To extend the invariance to the local case, the free Dirac action had to be supplemented by the addition of a new gauge field with a transformation specified by the desired local invariance and which also interacted with the original field, thus the notion that "local invariance requires interactions."

Technically, Weyl's theory was based on a local $U(1)$ invariance where $U(1)$ refers to the abelian (or commuting) group of complex unitary 1×1 matrices. In 1954 Chen Yang and Robert Mills introduce a non-abelian generalization of Weyl's abelian gauge theory. They used the dormant theoretical concept of local symmetry in an attempt to describe the strong interactions between what were then deemed fundamental neutrons and protons in terms of the exchange of pions. Recognizing that the presence of the neutral and two charged pion states reflected the structure of the generators of the group $SU(2)$, they then generalized Weyl's work on abelian $U(1)$ to the non-abelian case of $SU(2)$ [2]. Their construction led to many interesting features including the self-interaction of the gauge fields (not present in the abelian case), but also posed serious problems since the condition of local invariance required the gauge fields to be massless and experimental observations had already established the pions to have nonzero mass. In time the work of Yang and Mills was superseded by the advent of QCD, but it had inspired physicists to revisit the old idea of obtaining interactions from a principle of local gauge invariance.

In 1955 Ryoyu Utiyama tried to apply the principal of gauge invariance to describe Einstein's theory of general relativity [3]. Utiyama's idea was to localize the global Lorentz symmetry, i.e. three boosts and three rotations, of flat spacetime for which the effects of gravitation are absent. Unfortunately Utiyama's work was beset with several difficulties. While the standard formulation of general relativity centered on gravitation realized as curvature sourced by a conserved spacetime energy-momentum tensor, Utiyama's theory instead exhibited curvature sourced by the intrinsic spin angular momentum of matter and did not include any conservation of spacetime energy and momentum.

In 1961 Kibble expanded Utiyama's approach by gauging not just the Lorentz group, but the full Poincaré symmetry of flat spacetime which includes not only the rotations and boosts of the Lorentz group, but spacetime translations as well [4]. Kibble's approach was well motivated since the generators of translation in space and time are well known to generate the Noether current of spatial momentum and energy respectively. Indeed, one may wonder why Utiyama started with only the Lorentz group. As we will eventually see the notion of gauging translations is considerably more abstract and dissociated from the usual notion of gauge transformations which act pointwise in spacetime. Kibble's formalism, now referred to as Poincaré gauge theory, addressed most of the issues with Utiyama's work, but still presented several outstanding problems. Among these was the unavoidable inclusion of torsion, i.e. an antisymmetric component to the specification of the underlying geometry absent in Einstein's original formulation of general relativity, as well as the difficulty in the interpretation of some of the gauge fields. These problems notwithstanding, Kibble's construction remains relatively intact as the most acceptable gauge formulation of gravity, at least as constructed in terms of an action functional.

In the 50 years of development following Kibble's Poincaré gauge theory the issue of gravitation's status as a gauge theory remains an open question [3–26]. Meanwhile internal gauge symmetry continued to enjoy immense success in determining the form of the other fundamental interactions. In 1968 after many years of development ([27–29]) the Glashow-

Salam-Weinberg model of the electroweak interactions based on a spontaneously broken $SU(2)_L \times U(1)_Y$ gauge symmetry not only provided a first step towards unification of the fundamental forces, but also incorporated the recently developed Higgs mechanism for mass generation [30]. Moreover, with mounting experimental evidence for nucleon substructure, steps were being taken that would eventually lead to the formulation of quantum chromodynamics (QCD) as the correct strong interaction between fundamental quarks. QCD was ultimately realized as resulting from the local invariance under $SU(3)$ rotations on a Dirac field of three quark color states, with strong advocation for this model first given by Fritzsch, Gell-Mann and Leutwyler in 1973 [31].

With the overwhelming success of the gauge principle in explaining the structure of what came to be called the Standard Model of particle physics, it may seem surprising that the focus on gauge theories of gravity waned. However it was during the run up to the final development of QCD that another discovery was made that distracted the attention of high-energy physicists interested in adding gravity to the mix. In an early attempt to explain certain symmetries observed in the scattering amplitudes of strongly interacting particles, a model was proposed by Nambu, Nielsen and Susskind that envisioned the strong force as confined to narrow flux tubes connecting the interacting particles [32–34]. Quantization of these flux tubes reproduced the symmetries observed in scattering amplitudes, but unfortunately included a host of problems including tachyons, the need for extra spacetime dimensions as well as a pesky spin-2 particle that didn't quite seem to fit in with the strong interaction story at length scales 10^{-15}m. QCD eventually replaced the flux tube model, however in 1974 the bold leap was made by Scherk and Schwarz to reinterpret this "string theory" as a model of fundamental particles themselves, a step which shrunk the relevant length scale down to 10^{-35}m [35]. With a mechanism in hand for removing the tachyon, willingness to accept the problem of extra dimensions (so long as they were small) and most importantly an interpretation of the spin-2 particle as the graviton, a consistent quantum theory of gravity was born and as a result the focus on (perhaps more mundane) gauge formulations of gravity was set aside.

With the advent of string theory, and in particular its incorporation of fundamental two-dimensional world-sheets as well as the need for small extra spacetime dimensions, the mathematics of topology entered full force into high energy physics. As more physicists began using the tools of topology, the interpretation of some earlier ideas deepened. In particular, it became clear that formulating gauge theories on topologically non-trivial spacetimes required an understanding of the degrees of freedom in terms of fiber bundles. Fiber bundles are essentially the idea of attaching additional dimensions to each point of spacetime to create a larger geometry. What sets fiber bundles apart from the simpler idea of higher dimensional spacetimes is that the fibers do not have to correspond to a spatial direction, for example they can be something as abstract as a group of transformations. Figure 1.1 illustrates the essential elements of a fiber bundle.

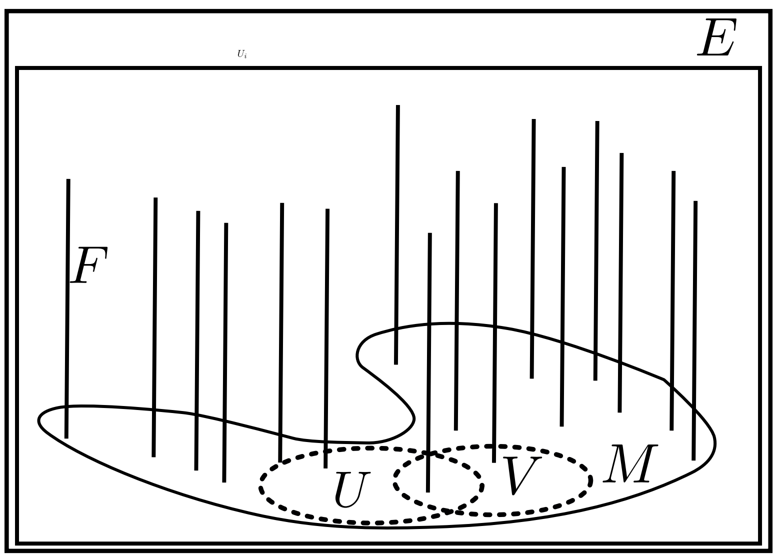

Figure 1.1: A fiber bundle E is the collection of fibers F attached to each point in the base space M. Also shown are local regions in the base space U and V.

Fiber bundles are a way to generalize the typical Cartesian product of two spaces X and Y defined as $X \times Y \equiv \{(x, y)|x \in X, y \in Y\}$. In a Cartesian product space the entire space can be realized as $X \times Y$, however with fiber bundles the Cartesian structure is only required to hold locally. Globally the space need not be so simple. To facilitate the idea of a global deviation from a Cartesian product we define local structures and then require transition functions to sew them together. More precisely, in a small region of the bundle space we have a local "trivialization" in terms of a neighborhood, e.g. U in Figure 1.1, for which the bundle space is a Cartesian product $U \times F$. The transition functions are then used to move from one local trivialization $U \times F$ to a neighboring trivialization $V \times F$. A natural conclusion is that a fiber bundle is trivial if it can be globally realized as a Cartesian product. i.e. all transition functions can be taken to be the identity.

Many familiar geometric spaces can be envisioned as fiber bundles as shown in Figure 1.2. A cylinder can be thought of as a circle where at each point of the circle we fiber a line interval. The same can be said of the torus where at each point of the circle we instead fiber another circle. Both of these spaces are globally Cartesian products and hence trivial fiber bundles. The Möbius band on the other hand, though locally the same as the cylinder, is globally distinct since the transition function required when sewing together the two ends involves a reflection.

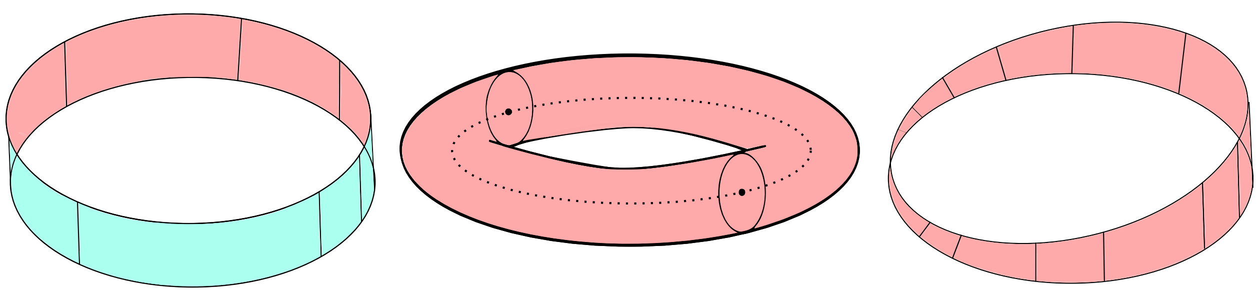

Figure 1.2: Three familiar spaces, the cylinder $S^1 \times [-1, 1]$, the torus $S^1 \times S^1$ and the Möbius band, are displayed. Each can be realized as a fiber bundle. The cylinder and torus are examples of trivial bundles, while the Möbius band is an example of a non-trivial or "twisted" bundle.

In application to physics the base space M is usually taken to be spacetime. Among the fiber spaces in gauge theories are Lie groups which represent the underlying gauge symmetries. Many of the common elements of gauge theory arise from geometric considerations on the total fiber bundle space E. As an example, the gauge field we are familiar with introducing to mediate local invariance of fields arises as a consequence of defining a splitting of the tangent space in the total bundle into two spaces: one which lies parallel to the fiber space and the other parallel to the base space. Such a splitting is formally accomplished via the introduction of a bundle connection ω. The bundle connection can then be "pulled back" to a neighborhood of the base space to form a (local) connection or gauge field.

Weyl's original gauge theory did not make use of fiber bundles, nor did the original gauge formulations of the electroweak or strong interactions. So why then might we want to work with a more mathematically sophisticated theory if the original theory sufficed to explain these interactions? A simple example of the power of the fiber bundle method comes from the electromagnetic gauge theory. Local gauge invariance dictates the form of the Lagrangian from which we are only able to derive half of Maxwell's equations, i.e. those with source terms,

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} J^\mu \Rightarrow \begin{cases} \nabla \cdot \vec{E} = 4\pi\rho \\ \nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \end{cases} \tag{1.1}$$

on the left written in a tensor formalism and on the right translated into a vector formalism. Equation 1.1 is the result of varying the action with respect to the gauge field A_μ which represents the electromagnetic 4-potential. The second half of Maxwell's equations, those without source terms,

$$\mathrm{d}F = 0 \Rightarrow \begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = \frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \end{cases} \tag{1.2}$$

on the left written in a exterior geometric formalism and on the right translated into a vector formalism, have no such interpretation. They are typically asserted as a byproduct of the formalism, i.e. from defining the physical fields in terms of potentials. However with the right tools one can assess that they actually contain information about the topology of field configurations and spacetime. In the fiber bundle formalism the second half of the Maxwell equations arise as a result of the Bianchi identity. This identity states that $D\Omega = 0$ in the bundle space where $\Omega = D\omega$ is the curvature built from the connection ω and D is the covariant derivative. This expression can be pulled back to a neighborhood of the base space resulting in a local expression that takes the form in Equation 1.2. The Bianchi identity describes the topology of the bundle, i.e. fields and spacetime, and is satisfied even before the introduction of an action functional on the bundle. Furthermore magnetic monopoles have a natural interpretation in the fiber bundle formalism. A Dirac monopole is the result of constructing an electromagnetic gauge theory based on $U(1)$ fibered over a sphere S^2. We see as mentioned earlier that fiber bundles provide a means of extending gauge theories to topologically non-trivial spaces such as S^2 [36].

However there are things that typical fiber bundles cannot do. In particular gauge transformations have an interpretation as vertical fiber automorphisms. These are isomorphisms of the fiber space (here a Lie group) onto itself for which the projection of the total space to base space is unaffected. This means that gauge transformations move us only "vertically" through the fiber space, see Figure 1.3. However translations are vital to gauge theories of

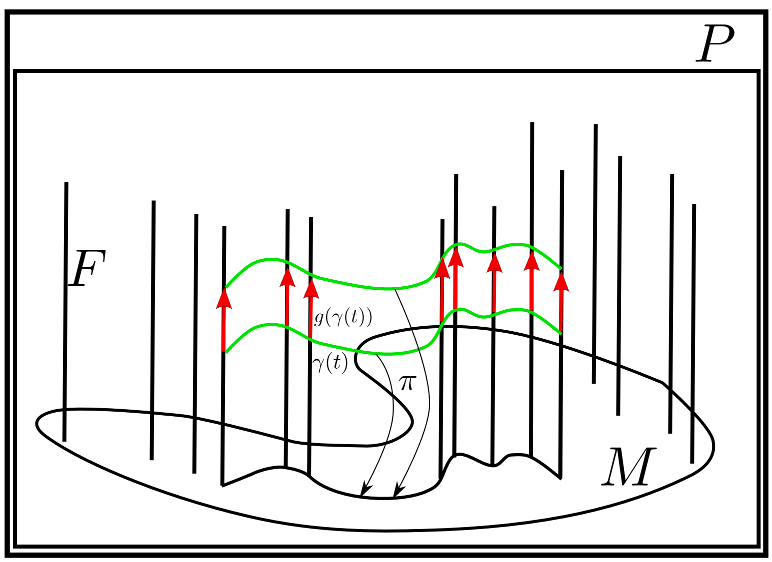

Figure 1.3: A gauge transformation of a curve γ running through the total bundle is displayed. The projection π is shown taking both curves down to the base space. The projection of the gauge transformed curve is identical to that of the untransformed curve.

gravitation, and translations move between points in the base space! This calls for a more sophisticated fiber structure which allows the shifting of points in the base. The construction of such bundles, known as composite fiber bundles [14], is described in detail in chapter 3.

A crucial aspect of the composite fiber bundles used for gravitation is the requirement of a certain global mapping from the base space to the total bundle space called a section, see Figure 1.4. It is the requirement of a global section which leads us to consider the topology

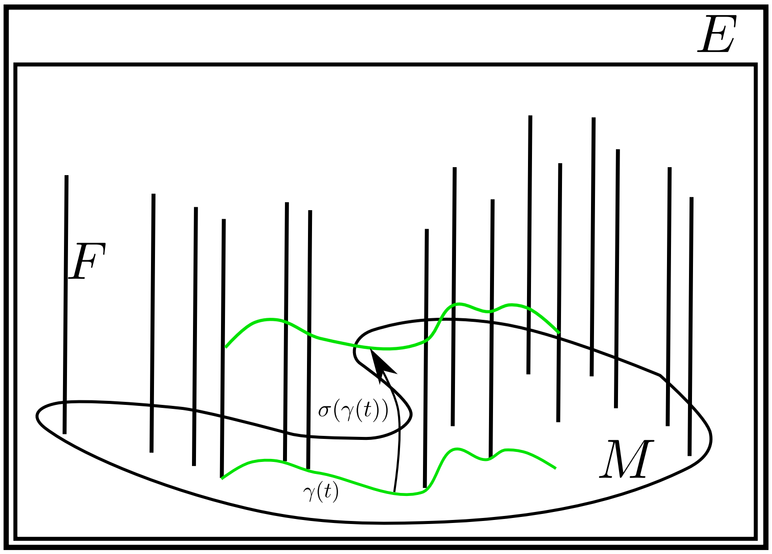

Figure 1.4: A section is a mapping $\sigma : M \to E$ that assigns a value of the fiber for every point in the base space M.

of the bundle space. For it is the topology of the base space which can prevent us from being able to define a section globally over the base space. This leads to the notion of "topological obstructions" which are properties of a space that determine if we can consistently form certain topological structures over it.

Since these obstructions arise from the topology of a space and are independent of the particular metric geometry with which it is endowed, it makes sense to identify them in terms of topological invariants, i.e. quantities which are the same for a given topology regardless of metric geometry. These are quantities which are invariant under continuous bijections known as homeomorphisms. Homeomorphisms embody the notion of topology as "rubber band geometry" and topological invariants are unchanged by the stretching and distortion of homeomorphisms. A familiar use of topological invariants is to distinguish spaces which may look the same locally, but differ only in their global structure. This is the case in Figure 1.2, the cylinder and Möbius band are both locally a line segment fibered over an open interval.

However globally they are distinct due to the twisting from the transition function used to complete the base circle in the Möbius band case. One means of distinguishing them is to compute all of the topological invariants of the two spaces and see if they agree. If not, they are topologically distinct. For fiber bundles an important set of topological invariants are known as characteristic classes. In the case of the cylinder and the Möbius band the first Stiefel-Whitney class, w_1, of the two spaces are not the same. For the cylinder $w_1 = 1$ and for the Möbius strip $w_1 = -1$, therefore these spaces cannot be the same globally.

Another important use of characteristic classes, and particularly relevant for this thesis, is that these invariants can pose obstructions to the creation of certain global structures. As an example consider the question of whether an everywhere non-zero section of the tangent bundle to a space exists. For S^2 (see Figure 1.5) this question is colloquially stated as "can you comb the hair on a sphere?" The answer is no and is rooted in the non-

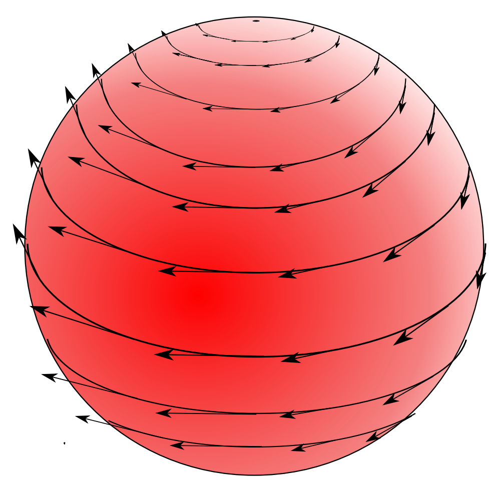

Figure 1.5: An example of a vector field over the sphere. The point at the north pole is a singular point in the vector field, i.e. a point at which the vector is zero in the tangent space.

triviality of the Euler class of the sphere. From this example we see that characteristic classes contain important information with regards to topological obstructions. In fact if any of the characteristic classes of a bundle are non-trivial then certain structures, such as mappings, cannot be defined globally. The first Stiefel-Whitney class is an obstruction to the bundle being orientable, where the orientability of a space describes the ability to consistently define a notion of volume throughout the bundle. Looking back at the cylinder and Möbius band example, we might have anticipated the distinction to arise from the first Stiefel-Whitney class since indeed while the cylinder is orientable, the Möbius band is not.

The primary result of my work is an analysis of the characteristic classes (and the topological obstructions they imply) of the composite bundle formalism underlying gauge gravitation. In my investigation I have identified the characteristic classes needed to describe the composite bundle formalism as the first four Stiefel-Whitney classes, the Euler class and the first fractional Pontryagin class. Looking to the interpretation of each class I found that the composite bundle formalism requires the space be orientable, admit a spin structure and to admit a string structure. The latter requirement is perhaps the most surprising. Just as the admittance of a spin structure allows the consistent introduction of spinor degrees of freedom on the base space, the admittance of a string structure allows us to consistently define extended degrees of freedom or strings. This is in stark contrast to the usual motivation to work with extended degrees of freedom. Typically extended degrees of freedom are invoked in order to develop a consistent quantum theory of gravity. In contrast my research gives a classical (albeit topological) motivation for the consideration of extended degrees of freedom. Again this all arises from the natural insistence that gravitation, like the other forces of the Standard Model, be formulated using the principal of local gauge invariance.

The remainder of this thesis is dedicated to filling in the details and providing evidence to support these results. The second chapter is devoted to the relevant background material needed to understand what a gauge theory of gravity should accomplish. Chapter 3 discusses the gauge theory of gravity in the composite bundle formalism. In chapter 4 we develop the necessary tools of characteristic classes and apply them to composite bundles to arrive at the conclusion that the base manifold must admit (among other things) a string structure. We will end with conclusions and an outlook to future directions for this line of research.

CHAPTER 2

BACKGROUND

There are many topics we must cover if we are to understand topological obstructions in composite gauge theories of gravitation. First and foremost we must be familiar with two general topics: the basic aspects of gauge theory and general relativity (both as Einstein envisioned it and in more sophisticated formulations). To this end section 2.1 of this chapter is dedicated to a functional approach to gauge theory and section 2.2 is dedicated to the basic content of Einstein's theory of general relativity. Along with general relativity comes some mathematical formalism and in order to appreciate the connections between general relativity and fiber bundle formalisms we will also need to develop further topics in Riemannian geometry. To accommodate these additional topics in Riemannian geometry the section on general relativity is broken into subsections covering manifolds (section 2.2.1), vectors and differential forms defined on manifolds (section 2.2.2 and section 2.2.3), maps between manifolds (section 2.2.4) and non-coordinate bases (section 2.2.5). We will need these finer points of differential geometry to understand the natural definitions of quantities on fiber bundles. Following section 2.2 on general relativity section 2.3 is devoted to the basics of fiber bundles. The main goal of this section will be to reproduce all of the key concepts introduced in the functional approach of section 2.1. These tools will be essential in our later discussion of the more sophisticated bundle formulations used in gravitation.

2.1 Functional Gauge Theory

The simplest setting to gain an appreciation of the gauging procedure is the functional approach. I call this the functional approach because it stems from the requirement of invariance of a functional, namely the action integral of a free field. This is the approach used to determine the form of the electroweak and strong interactions of the Standard Model. In addition the first attempts at gravitational gauge theory by Utiyama and Kibble used this method. As an example of this approach let us consider the action for a complex scalar field $\phi : \mathbb{R}^4 \to \mathbb{C}$ given by[1],

$$S = \int \frac{1}{2} \partial_\mu \phi^* \partial^\mu \phi \mathrm{d}^4 x. \tag{2.1}$$

Equation 2.1 enjoys a global phase invariance,

$$\phi \to e^{i\frac{q}{\hbar c}\alpha}\phi, \qquad \phi^* \to e^{-i\frac{q}{\hbar c}\alpha}\phi^*, \tag{2.2}$$

where α is a parameter which determines the amount of phase and q is a coupling constant which can be identified with the electric charge. This transformation will only be a symmetry of the action above provided the transformation is global, i.e. if the parameter α does not depend on spacetime. If α depends on spacetime ($\alpha(x)$) the action will pick up extra terms due to the derivative of the parameter,

$$\partial_\mu \phi \to e^{i\frac{q}{\hbar c}\alpha}\partial_\mu \phi + i\frac{q}{\hbar c}(\partial_\mu \alpha) e^{i\frac{q}{\hbar c}\alpha}\phi. \tag{2.3}$$

However we can promote the global symmetry of the action to a local symmetry by modifying the derivative operator to a covariant form by the addition of a compensating (gauge) field,

$$\partial_\mu \to \nabla_\mu = \partial_\mu + i\frac{q}{\hbar c}A_\mu. \tag{2.4}$$

The invariance of the action is now guaranteed so long as the gauge field transforms as follows,

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \alpha. \tag{2.5}$$

Expanding Equation 2.5 in components we can identify $A^0 = V/c$ as the electric potential and A^j as the vector potential. Equation 2.5 reproduces the gauge transformation seen in electromagnetism,

$$V \to V - \frac{\partial \alpha}{\partial t}, \qquad \vec{A} \to \vec{A} + \vec{\nabla}\alpha. \tag{2.6}$$

[1] I will employ summation notation throughout. The index will always take values $0 \to n-1$ for $\dim(M) = n$ for some space M unless otherwise stated. As an example $V_\mu V^\mu \equiv \sum_{\mu=0}^{n-1} V_\mu V^\mu$

If we now make a local gauge transformation the transformed derivative of the field will be homogeneous,

$$(\nabla_\mu \phi)' = e^{i\frac{q}{\hbar c}\alpha} \nabla_\mu \phi, \tag{2.7}$$

exactly as our original derivative for global gauge symmetries. In the next chapter we will see this same notion of covariant differentiation in general relativity.

Making the replacement of ∂_μ by ∇_μ in Equation 2.1 we will have succeeded in fixing the action,

$$S = \int \nabla_\mu \phi^* \nabla^\mu \phi \mathrm{d}^4x = \int \partial_\mu \phi^* \partial^\mu \phi + i\frac{q}{2\hbar c}(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi) A^\mu + \frac{q^2}{2\hbar^2 c^2} A_\mu A^\mu \phi^* \phi \mathrm{d}^4x, \tag{2.8}$$

to be invariant under what are known as local $U(1)$ gauge transformations. At this point the gauge field A_μ represents a background in our theory, i.e. a quantity which must be supplied by hand before calculation. The removal and interpretation of backgrounds is a subtle topic that is not often appreciated in the literature on gauge theories of gravity [37]. To remove the background we introduce a gauge invariant kinetic term for the gauge field of the form,

$$L_0 = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}, \text{ where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.9}$$

Including this term in our action, Equation 2.8, we then have,

$$S = \int \frac{1}{2} \nabla_\mu \phi^* \nabla^\mu \phi - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \mathrm{d}^4x. \tag{2.10}$$

A variation of Equation 2.10 with respect to the gauge field now results in differential equations which determine the gauge field. The equations of motion which result are half the Maxwell equations (those with source terms),

$$\partial_\mu F^{\nu\mu} = \frac{4\pi}{c} J^\nu = \frac{4\pi}{c}(i\frac{q}{2\hbar}\eta^{\mu\nu}[\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi] + \frac{q^2}{\hbar^2 c}\phi^* \phi A^\nu). \tag{2.11}$$

Incredibly we have introduced the electromagnetic interaction into our free theory by the demand that the action be invariant under local $U(1)$ transformations. This method demonstrates for us the common pieces we will expect of any gauge theory; symmetry transformations such as Equation 2.2 and Equation 2.5, covariant differentiation as in Equation 2.4, and an introduction of a field strength (Equation 2.9). If we used a non-abelian symmetry we would have a Yang-Mills type theory. An excellent review of Yang-Mills theories and their geometry has been given by Daniel and Viallet [38].

2.2 General Relativity

The fundamental object of interest in general relativity is the metric tensor $g_{\mu\nu}$. The metric is a multi-linear map which assigns to every two vectors a corresponding real number. Unlike Newtonian physics which presupposes a fixed spacetime and metric over which a gravitational field is defined, general relativity describes gravity through a dynamical metric, i.e. a dynamical spacetime geometry. Gravitational effects result from the curvature of spacetime. In both electromagnetism and general relativity there are field equations which describe how sources create fields and separate equations of motion describing the behavior of test particles in the background fields. For electromagnetism these are the Maxwell equations and the Lorentz force equation in conjunction with Newton's laws. In general relativity we use the Einstein field equations and the geodesic equation. In what follows we will be concerned with generating the Einstein field equations. Many of the elements that we saw in the previous chapter apply to general relativity. Suppose we have some vector $V = V^\mu \partial_\mu$ and we wish to express its components V^μ in another basis. The transformed components can be expressed in terms of the components in the old basis as,

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu. \tag{2.12}$$

If the components of the transformation $\frac{\partial x'^\mu}{\partial x^\nu}$ are constants then the transformation of the derivative is homogeneous,

$$\partial_\lambda V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \partial_\lambda V^\nu, \tag{2.13}$$

similar to the case of global $U(1)$ transformations of the previous section. If instead the transformation has spacetime dependence (local transform) then we find additional terms,

$$\partial_\lambda V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\partial_\lambda V^\nu + \partial_\lambda \frac{\partial x'^\mu}{\partial x^\nu} V^\nu. \tag{2.14}$$

Equation 2.12 describes what is called the tensor transformation law. If an entity transforms as Equation 2.12 we can identify it as a tensor. To be specific this would be a $(1,0)$ tensor[2]. Looking now at Equation 2.14 we see the derivative of a vector V is not a tensor as it does not transform like Equation 2.12. To remedy the non-tensorial transformation of the derivative we do as in the previous section, introduce a covariant derivative $\nabla_\mu = \partial_\mu + \Gamma^\lambda{}_{\mu\nu}$. If we demand that a form of Equation 2.7 hold for our vectors we can deduce the transformation of the "gauge" field Γ [39],

$$\Gamma'^\lambda{}_{\mu\nu} = \frac{\partial x'^\beta}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\alpha}\Gamma^\alpha{}_{\beta\gamma} - \frac{\partial x'^\beta}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\nu}\frac{\partial^2 x'^\nu}{\partial x^\beta x^\gamma}. \tag{2.15}$$

Equation 2.15 is the definition of how a connection (gauge field) transforms. We see that Γ is not a tensor since it does not have a homogeneous transformation. But this is okay since the connection was built to compensate for the non homogeneous transformation of the derivative. Now the derivative will obey a form of Equation 2.7 for coordinate transformations.

In general relativity we further restrict the connection with two important conditions: it must be a metric connection and it must have vanishing torsion. A metric connection obeys $\nabla_\lambda g_{\mu\nu} = 0$ and a torsion-less connection obeys $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$. In special relativity the metric can be moved past derivatives $\partial_\mu(V_\nu \eta^{\nu\alpha}) = \eta^{\nu\alpha}\partial_\mu(V_\nu)$, this is the motive behind introducing the metricity condition $\nabla_\lambda g_{\mu\nu} = 0$ in curved space. The vanishing of torsion is an assumption which was first made by Einstein and has been used since. Of course there is an additional choice one could make, instead one could work with a non-vanishing torsion and the vanishing of the symmetric part of the connection. This choice of connection decomposition leads to teleparallel theories of gravitation which we will not detail in this thesis.

[2] The notation (p,q) denotes the number of vector (p) and dual vector (q) indexes.

Imposing the metric connection condition and using the vanishing torsion condition we can solve for the connection in terms of the metric giving [39],

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\rho}(\partial_{\rho}g_{\mu\nu} - \partial_{\mu}g_{\nu\rho} - \partial_{\nu}g_{\rho\mu}). \tag{2.16}$$

Although we have seen many similarities between general relativity and the gauge theory described in the previous section, Equation 2.16 is the first strong deviation. The connection is written in terms of the dynamical quantity of the theory. It has “internal” structure provided by the metric. This is in contrast to an electromagnetic gauge theory where the connection (gauge field) has no a priori definition in terms of other dynamic quantities in the theory.

The measure of the curvature of spacetime is given by the Riemann curvature tensor. In the previous section the electromagnetic field strength can be built as $[\nabla_{\mu}, \nabla_{\nu}]\phi = -F_{\mu,\nu}\phi$. We can do the same for general relativity and the result is the Riemann curvature tensor [39],

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\lambda}_{\nu\beta}\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\lambda}_{\mu\beta}\Gamma^{\alpha}_{\nu\lambda}. \tag{2.17}$$

To obtain the Einstein field equations we need a “gauge” invariant field strength to use in the action. Unlike electromagnetism we can build this quantity by contracting (summing over) various indices of the Riemann tensor. First we contract to obtain the Ricci tensor $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$, and then we contract to obtain the Ricci scalar or scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$. The scalar curvature is used in what is called the Einstein-Hilbert action [39],

$$S_{EH} = \int \sqrt{-\det(g_{\mu\nu})}R\mathrm{d}^4x, \tag{2.18}$$

where the factor of $\sqrt{-\det(g_{\mu\nu})} = \sqrt{-g}$ is needed to maintain general coordinate or diffeomorphism invariance. The Einstein field equations in vacuum result from a variation of Equation 2.18 with respect to the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$,

$$\delta S_{EH} = \int \sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}\mathrm{d}^4x = 0. \tag{2.19}$$

If we include a matter action S_M with the Einstein-Hilbert action S_{EM} a variation of the total action,

$$S = \frac{1}{16\pi G} S_M + S_{EM}, \tag{2.20}$$

with respect to the metric yields the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{2.21}$$

where the energy-momentum tensor $T_{\mu\nu}$ is obtained by $-2\frac{\delta S_M}{\delta g_{\mu\nu}}$ [39]. Solutions to Equation 2.21 are metric tensors which describe the spacetime geometry. These are the basic elements of Einstein's original formulation of general relativity. However for the purposes of this thesis we will need to go into further detail and gain an appreciation of Riemannian geometry. In the following subsections we will detail the supplemental information on manifolds, maps between manifolds and non-coordinate bases needed in section 2.3 to discuss fiber bundles.

2.2.1 Manifolds

The Einstein field equations represent the curving of spacetime in the presence of matter. The solutions to Equation 2.21 are spacetime metrics on a Pseudo-Riemannian manifold. Pseudo refers to the presence of negative signs in the diagonal terms of the metric, e.g. the Minkowski metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. This is also known as a pseudo Riemannian structure on a manifold. To be precise an n-dimensional C^∞ manifold M is a topological space[3] endowed with the following [36],

1. M has a family of pairs $\{(U_i, \phi_i)\}$
2. The collection U_i is an open cover of M and φ_i are a collection of homeomorphism from M onto R^n.
3. On each intersection $U_i \cap U_j \neq \emptyset$ the map $\phi_{ij} = \varphi_i \circ \varphi_j^{-1}$ called the transition function is infinitely differentiable.

[3] [40] is good text if you need to review point-set topology.

In essence a manifold is a space which, when viewed over a small region, looks identical to Euclidean space $\mathbb{R}^n$. This is basically the equivalence principal at work. The equivalence principal states that in small enough region of space and a short enough interval in time, physics is indistinguishable from that of flat spacetime. This is due to the observation that any gravitational force can be effectively removed by using a freely falling reference frame.

In this thesis we will be particularly interested in topologically non-trivial manifolds. General relativity is consistent on these spaces as well, and indeed this is where the power of the formal definition above becomes relevant. As a simple example of the construction of a topologically non-trivial manifold consider the circle S^1 as a subset of $\mathbb{R}^2$ given by $S^1 = \{(x, y)|x^2 + y^2 = 1\} \subset \mathbb{R}^2$, see Figure 2.1. We must have a minimum of two charts

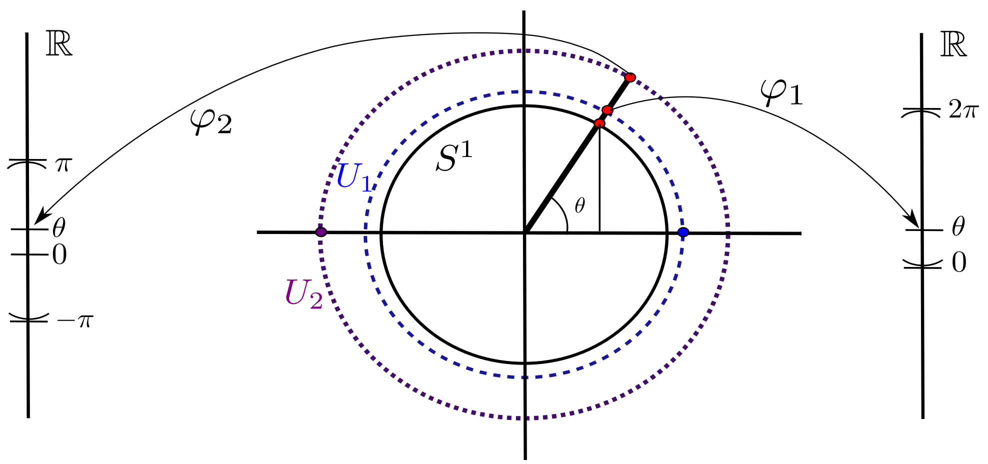

Figure 2.1: The manifold S^1 shown with one choice of the two charts needed to cover it displayed as larger circles for clarity. The black dots denote the start and end of each cover while the red dots denote a position in S^1 and relative position in chart one and two.

to provide a cover of S^1. We will take $U_1 = (0, 2\pi)$ and $U_2 = (-\pi, \pi)$ as our cover. On each chart we need a function $\varphi_i : U_i \to \mathbb{R}^2$ called the local trivialization. For simplicity we choose [36],

$$\varphi_1(\theta) =(\cos\theta, \sin\theta) \in \mathbb{R}^2 \tag{2.22a}$$

$$\varphi_2(\theta) =(\cos\theta, \sin\theta) \in \mathbb{R}^2. \tag{2.22b}$$

If we look at the image of our sets U_1 and U_2 under their respective trivializations we find $\varphi_1(U_1) = S - \{(1,0)\}$ and $\varphi_2(U_1) = S - \{(-1,0)\}$ and that $S^1 - \{(1,0)\} \cup S^1 - \{(-1,0)\} = S^1$. For points in the overlap, $p \in U_1 \cap U_2$, the transition function ϕ_{12} takes us between copies of $\mathbb{R}^2$ as $\varphi_1 = \phi_{12} \circ \varphi_2$. For the circle we have constructed our transition function is simple, $\phi_{12} = \varphi_1 \circ \varphi_2^{-1} = \varphi_1(\varphi_2^{-1}) = \theta$. In general the transition functions will not be so simple. They encode the information for stitching together pieces of Euclidean spaces to create more complicated spaces.

2.2.2 Vectors

Over a curved manifold we will need a more refined description of vectors. It is necessary to affix to each point $p \in M$ a tangent space for vectors to "live", see Figure 2.2. To construct the tangent space let $\gamma : \mathbb{R} \to M$ be a curve into the manifold M such that $\gamma(0) = p \in M$ and $\dot{\gamma}(0) = X$ where X is the vector tangent to γ at $t = 0$. Let $\Gamma = \{\gamma_i\}$ denote the collection of all such curves with $\gamma_i(0) = p$. The corresponding collection of all the vectors of the curves γ_i is a n-dimensional vector space at $p \in M$ called the tangent space denoted $T_pM = \frac{\mathrm{d}}{\mathrm{d}t}\Gamma|_{t=0}$[36, 39, 41]. We typically have no need to introduce the tangent space when

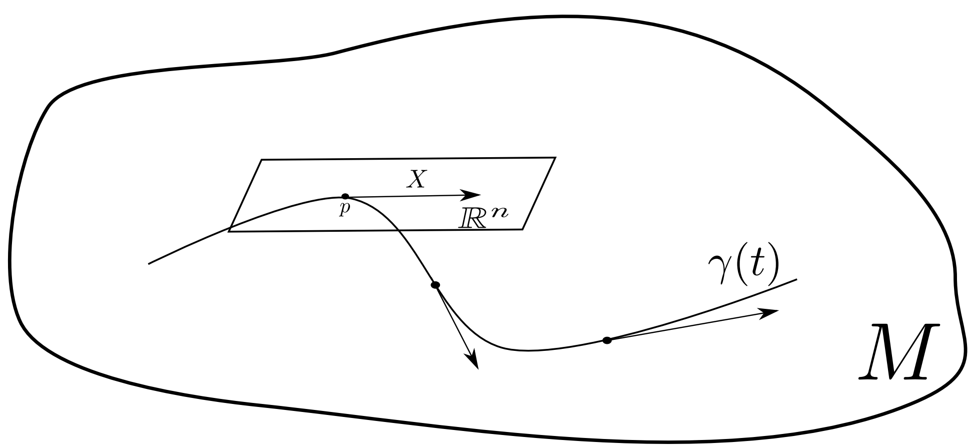

Figure 2.2: A manifold M is shown with a curve $\gamma(t)$ running through it. At each point along the curve the exists a vector. Each vector is defined at the point in a tangent space. The vector X is one such vector.

working with flat space in Cartesian coordinates since the tangent space at each point is isomorphic to the flat Cartesian space itself. This often leads to confusion since in flat spacetime dimensions we often draw a vector originating at a point p and terminating at q see Figure 2.3. Without stating it we have implicitly created a copy of $\mathbb{R}^2$ centered at p.

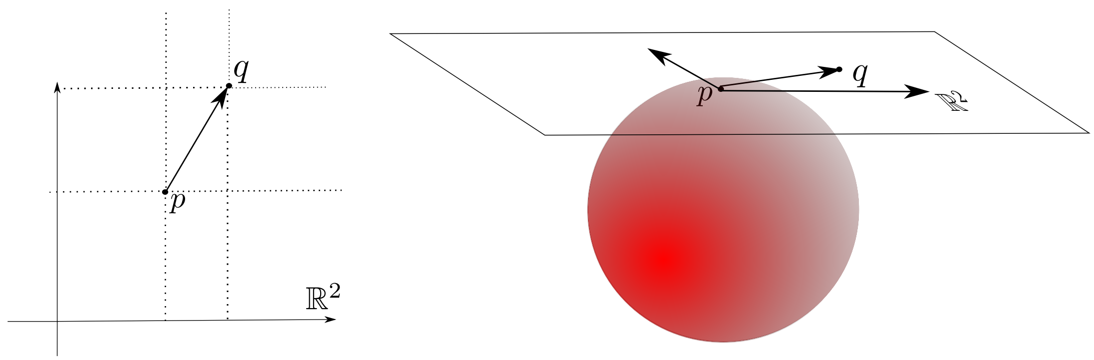

Figure 2.3: A vector in $\mathbb{R}^2$ is shown originating at p. Although it seems to point towards q it is important to realize that it lives not in the space itself but in the tangent space, i.e. $q \in T_p\mathbb{R}^2$, which happens to be parallel to itself at each point and is degenerate with the underlying space. In contrast, a vector on a two-sphere also lives in the tangent space which is clearly distinct from the underlying space and varies from point to point.

So it seems as though the vector lives in the plane rather then in its tangent space. The distinction is clearer in curved spaces as shown in Figure 2.3.

2.2.3 Forms

A 1-form is an element of the dual tangent space, $\omega \in T_p^*M$. When we take a dot product in linear algebra of a vector, say V, with itself what we are doing is associating an element of the dual basis to V, say V^T. The element V^T of the dual basis is a map $V^T : V \to \mathbb{R}$. A 1-form is exactly this operation, the 1-form ω is a map $\omega : T_pM \to \mathbb{R}$. A pullback can then be viewed as a map $f^* : T_f^*(p)M \to T_p^*N$, this is displayed in Figure 2.4.

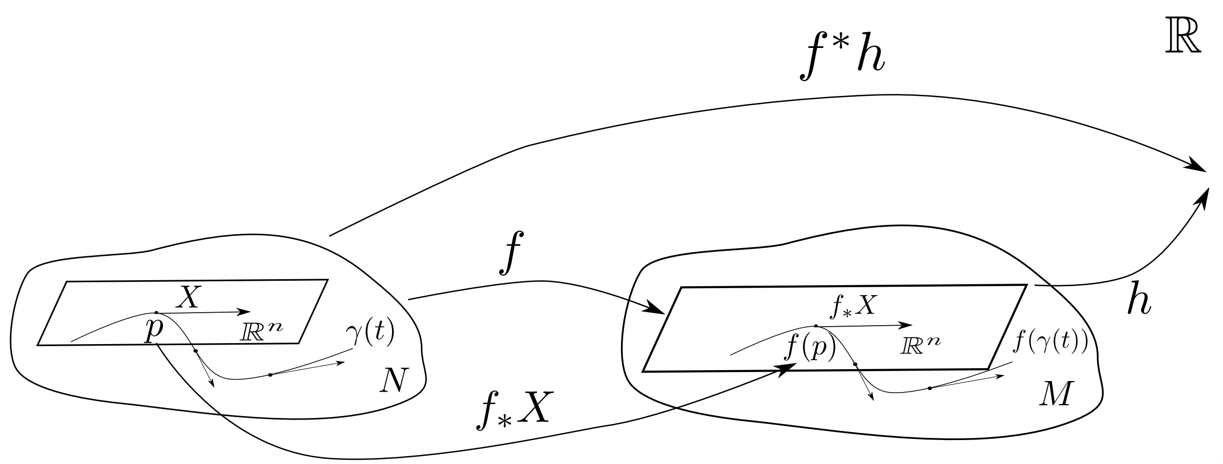

Figure 2.4: Two manifolds M and N are shown along with the tangle of maps relating them.

As the name 1-form suggests, there are higher order forms created by taking tensor products of the dual basis. A differential r-form is a total antisymmetric (0,r) tensor, and we denote the space of r-forms on a manifold by $\Omega^r(M)$,

$$\Omega^r(M) = T^*M \wedge T^*M \wedge \cdots \wedge T^*M. \tag{2.23}$$

To facilitate the antisymmetry of an r-form we introduce the wedge product $\wedge$. As an example consider the wedge product of two basis 1-forms $\mathrm{d}x$ and $\mathrm{d}y$,

$$\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}x \otimes \mathrm{d}y - \mathrm{d}y \otimes \mathrm{d}x. \tag{2.24}$$

We see that the wedge product of the two basis 1-forms is a 2-form and by definition is the totally antisymmetric tensor product. From Equation 2.24 we see that the wedge product of a form with itself is identically zero and switching the order $\mathrm{d}y \wedge \mathrm{d}x$ produces a sign difference $\mathrm{d}y \wedge \mathrm{d}x = -\mathrm{d}x \wedge \mathrm{d}y$. For a general p-form ξ and q-form ζ the sign changes compound and with a little thought we have $\xi \wedge \zeta = (-1)^{pq}\zeta \wedge \xi$. A general r-form can be expressed as $\omega = \frac{1}{r!}\omega_{\mu_1\cdots\mu_r}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}$, its expansion is similar to the expansion of a vector in terms of components and basis elements. Just as in $\mathbb{R}^3$ we can express a vector V as $V = V^1\hat{e}_1 + V^2\hat{e}_2 + V^3\hat{e}_3$, we can express a form by its components and dual basis elements.

A useful operation on forms is the exterior derivative d. The exterior derivative is map $\mathrm{d} : \Omega^r(M) \to \Omega^{r+1}(M)$, and in general is defined as,

$$d\omega = \frac{1}{r!} \left(\partial_\nu \omega_{\mu_1 \cdots \mu_r}\right) \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}, \tag{2.25}$$

for an r-form ω. As an example consider the exterior derivative of the 1-form $\xi = \xi_\mu \mathrm{d}x^\mu$ in three dimensions,

$$\begin{aligned} d\xi &= \partial_\nu \xi_\mu \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \\ &= \partial_1 \xi_2 \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \partial_1 \xi_3 \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \partial_2 \xi_1 \mathrm{d}x^2 \wedge \mathrm{d}x^1 + \partial_2 \xi_3 \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &+ \partial_3 \xi_1 \mathrm{d}x^3 \wedge \mathrm{d}x^1 + \partial_3 \xi_2 \mathrm{d}x^3 \wedge \mathrm{d}x^2 \qquad (2.26) \\ &= \left(\frac{\partial \xi_2}{\partial x^1} - \frac{\partial \xi_1}{\partial x^2}\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\frac{\partial \xi_3}{\partial x^1} - \frac{\partial \xi_1}{\partial x^3}\right) \mathrm{d}x^1 \wedge \mathrm{d}x^3 \\ &\quad + \left(\frac{\partial \xi_3}{\partial x^2} - \frac{\partial \xi_2}{\partial x^3}\right) \mathrm{d}x^2 \wedge \mathrm{d}x^3. \qquad (2.27) \end{aligned}$$

Identify $x^1 = x, x^2 = y, x^3 = z$ and notice that to specify a basis vector, say for the x direction, we can either use $\hat{i}$ or give the area it is perpendicular to $\mathrm{d}y\mathrm{d}z \equiv \mathrm{d}y \wedge \mathrm{d}z$. Equation 2.27 is then simply the curl $\vec{\nabla}\times$ of a vector of multivariate functions.

2.2.4 Maps Between Manifolds

It will often be the case that we need to consider maps from a manifold N to M. A smooth map $f : N \to M$ induces a map on the tangent space. Consider manifolds M and N of same the dimension for simplicity. Let $f : N \to M$ be a smooth map and let $h : M \to \mathbb{R}$. We can use the map f to send points in N to M and then we can use h to take points in M into the reals. Or we can compose h and f to map directly to $\mathbb{R}$ as $h \circ f$. As above let $\gamma : \mathbb{R} \to N$ such that $\gamma(0) = p \in N$ and $\dot{\gamma}(0) = X \in T_pN$. Then we define the pushforward of X by f as,

$$f_* X = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\bigg|_{t=0}. \tag{2.28}$$

The pushforward is a map between tangent spaces $f_* : T_pN \to T_{f(p)}M$, Figure 2.4 shows a pictorial representation of the above discussion. We have the ability to pushforward a vector, but can we pull it back? Unfortunately we will not be able to move vectors back but we can pull back a function. We define the pullback $f^*h = h \circ f$ of the function h by f by its action on a vector,

$$f^*hX = h(f_*X) = h\left(\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\bigg|_{t=0}\right). \tag{2.29}$$

In words "the pullback of a function on a vector is the function of the pushforward of the vector". This operation will be especially useful when applied to a differential form.

2.2.5 Non-coordinate Basis

A common feature of gauge theories of gravitation is the adoption of a non-coordinate basis. Often tauted as an additional layer of formalism we will see that they offer a dramatic simplification even when not working with gauge gravity. The idea is to fix a Minkowski frame at each point in spacetime. We do this by a rotation of the basis in the tangent space of the manifold, TM, at each point,

$$\hat{e}_i = e^\mu{}_i \frac{\partial}{\partial x^\mu}, \tag{2.30}$$

$\hat{e}_i$ is called a frame and $e^\mu{}_i$ a tetrad. I will use Latin letters to denote a non-coordinate index and Greek to denote a coordinate index. If we require that the frames be orthonormal with respect to the metric we have,

$$g(\hat{e}_i, \hat{e}_j) = e^\mu{}_i e^\nu{}_j g(\partial_\mu, \partial_\nu) = e^\mu{}_i e^\nu{}_j g_{\mu\nu} = \eta_{ij}. \tag{2.31}$$

Requiring the tetrad be invertible we can reverse Equation 2.31 to express the metric in terms of the tetrad,

$$e^i{}_\mu e^j{}_\nu \eta_{ij} = g_{\mu\nu}. \tag{2.32}$$

In terms of the inverse tetrad we can define a dual basis which we require to be orthonormal to $\hat{e}_i$,

$$\hat{\theta}^i = e^i{}_\mu \mathrm{d}x^\mu. \tag{2.33}$$

We can define a non-coordinate connection as well. Let,

$$\omega^j{}_k = \Gamma^j_{ki}\hat{\theta}^i, \tag{2.34}$$

where the connection Γ is represented in terms of the tetrad as $\Gamma^k{}_{j\ell} = e^k{}_\mu e_j{}^\nu \nabla_\nu e_\ell{}^\mu$ [36]. The connection and dual frame satisfy the following relations (defining the covariant exterior derivative $D = \mathrm{d} + \omega\wedge$),

$$D\omega = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j = R^i{}_j, \tag{2.35a}$$
$$D\hat{\theta} = \mathrm{d}\hat{\theta}^i + \omega^i{}_j \wedge \hat{\theta}^j = T^i. \tag{2.35b}$$

Equation 2.35a and Equation 2.35b are known as the Cartan structure equations and describe the curvature and torsion of the space. Here R is the curvature 2-form and T is the torsion 2-form [36, 41, 42]. They make calculating the Riemann tensor much easier as demonstrated in Appendix A. Aside from convenience in calculation, there is another reason to work in a non-coordinate basis. The only way to incorporate spinors in general relativity is by defining them in a local Minkowski frame. In fact the connection we introduced, Equation 2.34, is often called the spin connection. With it we can define covariant differentiation of spinors. However we will only be able to define spinors over certain manifolds. We will see in chapter 4 that the Stiefel-Whitney classes give a description of when we can define spinors on a given manifold.

2.3 Fiber Bundles

Fiber bundles are ubiquitous in physics. They have been used to describe quasi-particle excitations in superfluid helium as described in [43]. They have found application by Guichardet in [44] to study the rotational and vibrational modes of diatomic molecules. They are also a natural mathematical framework for studying the configuration space of a physical system. A playful example is in the work of Montgomery [45] who used fiber bundles to study a question on the minds of great scientist for decades, "Why does a cat always land on its feet?" Fiber bundles are also center stage when studying anomalies in quantum field theories and play a prominent role in the index theorems of elliptic operators [36, 46]. Perhaps the most well known application of fiber bundles to physics comes from field theories with a gauge symmetry, a topic to which we now turn.

Fiber bundles allow us the more satisfying approach, in contrast to the functional formalism, of developing gauge theories as additional topological structure on spacetime. In the fiber bundle formalism the symmetry groups form "fibers" which "protrude" from each point of spacetime, see Figure 2.5. Subject to certain stipulations, curves are then free to

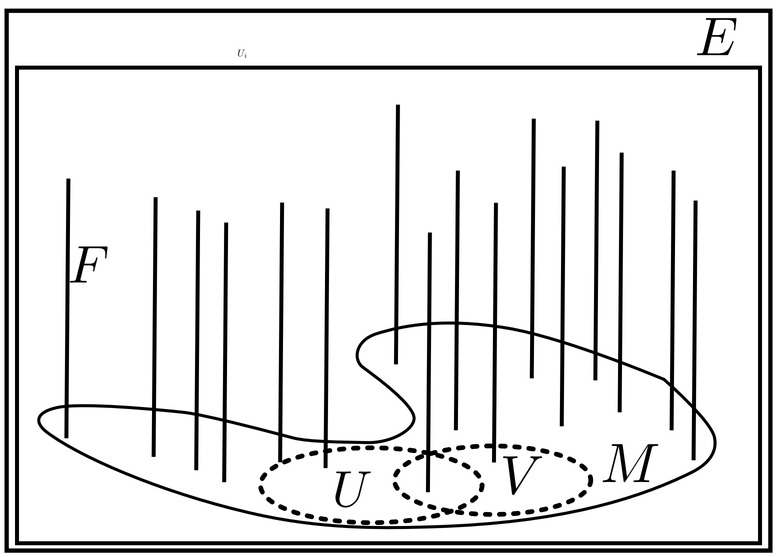

Figure 2.5: A depiction of a fiber bundle is shown. The base space M has fiber spaces F attached to each point $p \in M$. The total space E is the base space and the collection of fiber spaces F. Two open neighborhoods U and V are shown intersecting and a fiber over a point $q \in U \cup V$ is displayed. It is over these intersecting neighborhoods that we must use transition functions to move from one open neighborhood to the next.

move not only through spacetime but, via "lifts", through the total fiber space as well. The conditions on "lifting" a quantity through the bundle or higher into a further topologically restricted space will form the basis of much of our analysis. Familiar quantities like the covariant derivative will emerge from considering curves and quantities in the total space rather then purely along the base space. A simple way to think of fiber bundles is as a means of creating new topological spaces using existing spaces. Many familiar topological spaces can be realized as a fiber bundle. For instance a cylinder can be thought of as a circle, S^1, for which at each point $p \in S^1$ we attach a finite interval $[-1, 1] \in \mathbb{R}$ as seen in Figure 2.6. In this way a cylinder can be thought of as $S^1 \times [-1, 1] = \{(\theta, z)|\theta \in S^1, z \in [-1, 1]\}$.

A differentiable fiber bundle is denoted by either $E \xrightarrow{\pi} M$ or $E(M, F, G, \pi)$ and consists of the following data [36, 41],

1. Differentiable manifolds E,M and F, called the total, base and fiber space respectively.

2. A surjection $\pi : E \to M$ called the projection.

3. A Lie group G called the structure group such that G has left action on the fiber.

4. An open cover $\{U_i\}$ of the base with diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p \in M$.

5. On every non empty overlapping set of neighborhoods $U_i \cap U_j$ we require a G valued transition function $t_{ij} = \phi_i \circ \phi_j^{-1}$ such that $\phi_i = t_{ij}\phi_j$.

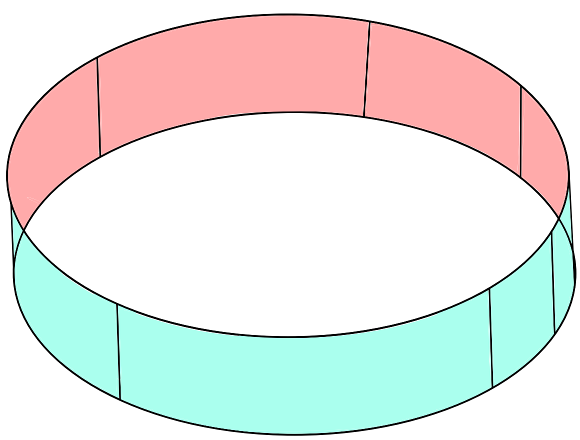

Figure 2.6: A cylinder depicted as a fiber bundle $S^1 \times [-1, 1]$.

In the example of the cylinder the fiber bundle we constructed was trivial. All of the transition functions can be taken to be the identity and the bundle can be written globally as $S^1 \times [-1,1]$. If instead we choose to twist the bundle by choosing one trivialization of a point u in the total space as $\phi_1^{-1}(u) = (\theta, z)$ and the other as $\phi_2^{-1}(u) = (\theta, -z)$ we arrive instead at the Möbius strip shown in Figure 2.7

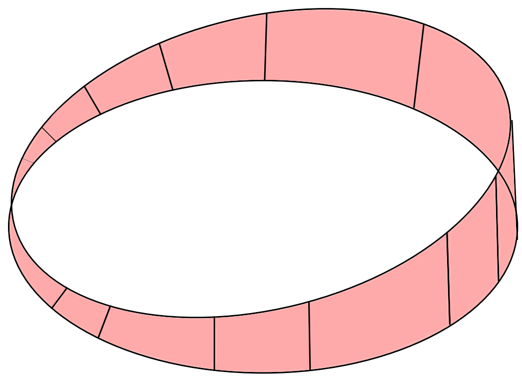

Figure 2.7: A Möbius strip is locally $S^1 \times [-1,1]$. However the global topology is not that of a cylinder.

An important class of fiber bundles is obtained if we take the fiber space F to be the structure group G itself. A differentiable fiber bundle with a Lie group as the fiber space is called a principal bundle and is denoted $P \xrightarrow{\pi} G$ or $P(M,G)$. Principal bundles play a central role in gauge theory. In addition to the left group action of G on the fibers we can also define a right action R_g such that if $u \in P$ we have $R_g u = ug$. As an example take the principal bundle as $P(\mathbb{R}^{1,3}, U(1))$. $U(1)$ is the unitary group of 1×1 matrices, the group of complex numbers whose magnitude is one. The group operation is multiplication and an arbitrary element of $U(1)$ can be expressed as $e^{i\psi}$. $\mathbb{R}^{1,3}$ is Minkowski space and is a trivial base space so we need just a single coordinate chart $(U, \varphi = x^\mu)$. We can write the local trivialization of a point $u \in P$ as $\phi^{-1}(u) = (p, e)$ where $p \in \mathbb{R}^{1,3}$ and $e \in U(1)$ is the identity element of the group. This particular trivialization is known as the canonical local trivialization [36]. Given two trivializations ϕ, ϕ' there exists an element of G such that $\phi' = \phi g = \phi(ug)$. Therefore we have,

$$\phi'(p, e^{i\psi}) = \phi(p, e)e^{i\psi} = \phi(p, ee^{i\psi}) = \phi(p, e^{i\psi}). \tag{2.36}$$

By repeated application of the right action we move ourselves through the fiber G. This leads us to define the whole fiber as $\pi^{-1}(p) = \{ug|g \in G\}$ [36]. This property will hold in any principal bundle. Naturally we will always choose to represent our sections as right multiplication of the canonical local trivialization.

We have seen in the definition of a fiber bundle that we have a way of moving from the total space P down to the base with the projection π. An important related concept will be of a section of a fiber bundle. A section $\sigma : M \to E$ is a smooth mapping such that $\pi \circ \sigma(p) = id_M$. If a section is defined only on a chart U_i then we say σ_i is a local section [36]. Sections are assignments of points in the base space with points in the total space P and induce mappings of vectors in the base space to vectors in the total space. An example of a section of a principal bundle $P(M, G)$ for $p \in M$, $e \in G$ is given by $\sigma(p) = \phi(p, e) = u \in P$.

A bundle will not always admit a global section, we typically only work with local sections defined on subsets of the space M. In fact we have the following theorem, "A principal bundle $P(M, G)$ is said to be trivial iff P admits global sections." [36, 41, 42]. We will also want to know whether a vector bundle is trivial. A problem however is that every vector bundle admits a global zero section. For a vector bundle to be trivial we instead have that "A vector bundle $E(M, F, G, \pi)$ is said to be trivial iff E admits an everywhere non-zero global section." [36, 41, 42]. We will use this theorem in chapter 4.

We saw earlier that maps between manifolds induce differential maps which act on vectors and forms. Now that we have some maps at our disposal we will want to know when a vector in the bundle lays parallel with the base or with the fiber. This can be accomplished by defining a splitting of the tangent space T_uP into vertical and horizontal subspaces V_uP and H_uP respectively. The requirements of the split are [36, 41],

1. $T_uP = H_uP \oplus V_uP$.

2. A smooth vector field X in P can be written as $X = X^H + X^V$ for $X^H \in H_uP$ and $X^V \in V_uP$.

3. $R_g^* H_u P = H_{ug} P$ where R_g^* is differential map induced by the right action of g.

The first two conditions state that we would like the total space to look like a generalization of Euclidean space, i.e. like R^2. The third condition comes from the idea that these fibers extend over the manifold, see Figure 2.8. We would like every horizontal subspace to

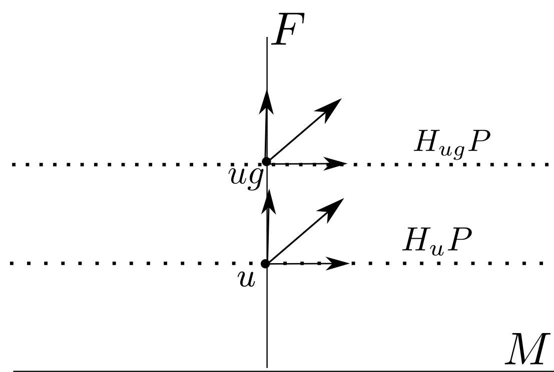

Figure 2.8: A vector in the total space P shown decomposed into its vertical and horizontal components. The vector is additionally shown right translated "up" through the bundle.

lay parallel above the base space. Just as we can write any section as a right translation of the canonical local trivialization we require horizontal subspaces be related by a right translation. Splitting the tangent space in this way is called specifying a connection on P. As we shall see this connection is closely related to the gauge fields used in chapter 2 section 2.1 to represent the electromagnetic field. However here the information of the connection has been encoded geometrically. In order to exact the more familiar notion of connection we first define the fundamental vector field, $A^\#$, at a point $u \in P$ generated by $A \in \mathfrak{g}$ as [36, 41],

$$A^\# f(u) = \frac{\mathrm{d}}{\mathrm{d}t} f(ue^{At})|_{t=0}, \tag{2.37}$$

for any function $f : P \to \mathbb{R}$. The vector $A^\#$ is directed entirely along the fiber. This can be seen by pushing forward a vector $X \in V_u P$ by the projection π as $\pi_* X$. With this in mind the connection 1-form $\omega \in \mathfrak{g} \otimes T_u^* P$ is defined so that [36, 41],

1. $\omega(A^\#) = A$ for $A \in \mathfrak{g}$,

2. $R_g^* \omega = g^{-1} \omega g$.

The horizontal subspace can then be defined as the kernel of ω, that is [36, 41],

$$H_u P = \{X \in T_u P | \omega(X) = 0\}. \tag{2.38}$$

Now that we have a connection on the total space we can use a section σ to pullback the connection form ω to the base. It can be shown [36] that the connection form on the base can be written,

$$\mathcal{A}_i = \sigma_i^* \omega. \tag{2.39}$$

However it is useful to think of what information we as observers have access to. The connection 1-form ω is a global quantity defined in the total bundle. As observers we are only able to make local measurements, say of the strength of the electromagnetic field in a local region of space. It is then more useful to start with a local connection 1-form $\mathcal{A}$ on $U_i \subset M$ on the base and define the connection 1-form ω as [36],

$$\omega = g_i^{-1} \pi^* \mathcal{A}_i g_i + g_i^{-1} \mathrm{d} g_i. \tag{2.40}$$

In Equation 2.5 we saw that the gauge field A_μ needed to transform in a unique way for the action to be invariant under a $U(1)$ gauge transformation. In the language of bundles a gauge transformation of a principal fiber bundle is a base preserving fiber automorphism [9, 14]. That is $f : G \to G$ such that $R_g \circ f = f \circ R_g$ and $f \circ \pi = \pi$. The first condition says that f must commute with all right actions of G and so is a left action of G, L_g [9]. The second condition means that the transformation is directed vertically along the fiber and ensures that the gauge transformation does not induce a diffeomorphism of the base. If we write $f(u) = u\eta(u)$ for $u \in P$ and $g \in G$ and $\eta : G \to G$ defined as $\eta(u) = u^{-1} g u$ we can determine the effect of a gauge transformation on a vector $X \in T_u P$. Let $\gamma : \mathbb{R} \to P$ such that $\gamma(0) = u$ and $\dot{\gamma}(0) = X$ (where the dot represents a parameter derivative) using Equation 2.28 we find,

$$f_*X = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\bigg|_{t=0} \tag{2.41a}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\eta(\gamma(t))\bigg|_{t=0} \tag{2.41b}$$

$$= \left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\eta + \gamma\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)\bigg|_{t=0} \tag{2.41c}$$

$$= R_{g*}X + f(\gamma)\eta^{-1}(\gamma)\frac{\mathrm{d}\eta}{\mathrm{d}t}\bigg|_{t=0} \tag{2.41d}$$

$$f_*X = R_{g*}X + (\eta^{-1}d\eta)^{\#}(X). \tag{2.41e}$$

This string of equations deserves some explanation. Equation 2.41a is the definition of the pullback of a vector. In moving to the next line we have simply used the definition of f. In the next line, Equation 2.41c, we have used the chain rule of differentiation. In Equation 2.41d we have replaced the first term by noticing it is the pushforward of the vector X. In the second term of Equation 2.41d we replaced $\gamma = f(\gamma)\eta^{-1}(\gamma)$. In the final step the second term can be seen to represent the fundamental vector field at $f(u)$ via Equation 2.37,

$$\begin{aligned}(\eta^{-1}d\eta)^{\#} f(u) &= \frac{\mathrm{d}}{\mathrm{d}t} f(ue^{\eta^{-1}\mathrm{d}\eta})\bigg|_{t=0} \\ &= f(u)\frac{\mathrm{d}}{\mathrm{d}t}e^{\eta^{-1}\mathrm{d}\eta}\bigg|_{t=0} = f(u)\eta^{-1}\mathrm{d}\eta = f(u)\eta^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\eta(\gamma)\bigg|_{t=0}.\end{aligned} \tag{2.42}$$

Once we have the action of a gauge transformation on a vector in the total space, we can apply the connection 1-form ω and pullback the result to the base space. By doing this we will have obtained the transformation properties of the local connection 1-form $\mathcal{A}$. Using Equation 2.41e let $\sigma_i : M \to P$ be a local section over the subset $U_i \subset M$. Applying ω to Equation 2.41e we find,

$$\omega(f_*X) = \omega(R_{g*}X) + \omega((\eta^{-1}d\eta)^{\#}(X)) \tag{2.43}$$

$$\Rightarrow f^*\omega(X) = R_g^*\omega(X) + \omega((\eta^{-1}d\eta)^{\#}(X)). \tag{2.44}$$

Now we must apply σ_i^* to pullback the gauge transformation of the connection 1-form ω to the gauge transformation of the local connection 1-form $\mathcal{A}$,

$$\sigma_i^* f^* \omega(X) = g^{-1} \sigma_i \omega(X) g + \sigma_i \omega((\eta^{-1} d\eta)^{\#}(X)) \tag{2.45}$$
$$\Rightarrow f^* \mathcal{A}_i = g^{-1} \mathcal{A}_i(X) g + \eta^{-1} d\eta(X). \tag{2.46}$$

If we look at the specific case of $M = \mathbb{R}^{1,3}$ and $G = U(1)$, we can parameterize an element of G by $g = e^{i\theta}$. Coordinatizing the space we can expand the local connection in the dual basis as $\mathcal{A} = \mathcal{A}_\mu \mathrm{d}x^\mu$. The exterior derivative then takes the familiar form when acting on functions(say $\theta \in \mathscr{F}(M)$), $\mathrm{d}\theta = \partial_\mu \theta \mathrm{d}x^\mu$,

$$f^* \mathcal{A}_i = \mathcal{A}'_i = \mathcal{A}_i + e^{-i\theta} d(e^{i\theta}) = (\mathcal{A}_{i\mu} + i\partial_\mu \theta)\mathrm{d}x^\mu$$
$$\mathcal{A}'_{i\mu} = \mathcal{A}_{i\mu} + i\partial_\mu \theta. \tag{2.47}$$

Equation 2.47 is identical to Equation 2.5, so vertical bundle automorphisms are indeed the correct notion of gauge transformations for internal symmetry groups.

In the example of complex scalar electromagnetism we introduced a field strength tensor $F^{\mu\nu}$ to the action to remove the background dependence of A_μ from the action. The field strength tensor in the fiber bundle language is the pullback of the curvature 2-form Ω. To define this quantity we first define the covariant derivative of an r-form $\xi \in \Omega^r(P)$ as [36, 41],

$$D\xi(X_1 \cdots X_r) = \mathrm{d}\xi(X_1^H \cdots X_r^H), \tag{2.48}$$

where $X_1 \cdots X_r \in T_u P$. The curvature 2-form can then be defined as the covariant derivative of the connection 1-form,

$$\Omega(X, Y) = D\omega(X, Y) = \mathrm{d}\omega(X, Y) + [\omega(X), \omega(Y)]. \tag{2.49}$$

Investigating cases of X, Y as vectors in $H_u P$ or $V_u P$ it is easy to see the second equivalence of Equation 2.49, as shown in [36]. The commutator of a $\mathfrak{g}$ valued p-form ξ and q-form η is given as,

$$[\xi, \eta] = \xi \wedge \eta - (-1)^{pq} \eta \wedge \xi = [w_\alpha, w_\beta] \otimes \xi^\alpha \wedge \eta^\beta, \tag{2.50}$$

where we have decomposed the forms in the Lie algebra as $\xi = \xi^\alpha \otimes w_\alpha$. If we put $\eta = \xi$ we have,

$$[\xi, \xi] = 2\xi \wedge \xi = [w_\alpha, w_\beta]\xi^\alpha \wedge \xi^\beta. \tag{2.51}$$

We can use Equation 2.51 in Equation 2.49 to write the curvature in a more useful way. First we calculate the second term in Equation 2.49 [36],

$$\begin{aligned}
[\omega, \omega](X, Y) &= [T_\alpha, T_\beta]\omega^\alpha \wedge \omega^\beta(X, Y) && \text{(2.52a)}\\
&= [T_\alpha, T_\beta](\omega^\alpha(X)\omega^\beta(Y) - \omega^\alpha(Y)\omega^\beta(X)) && \text{(2.52b)}\\
&= 2[T_\alpha, T_\beta]\omega^\alpha(X) \wedge \omega^\beta(Y) = 2[\omega(X), \omega(Y)]. && \text{(2.52c)}
\end{aligned}$$

To move from Equation 2.52b to Equation 2.52c we have simply relabeled indices and used properties of the commutator. Using Equation 2.52c and Equation 2.51 in Equation 2.49 we can re-express the curvature as [36, 41],

$$D\omega(X, Y) = \mathrm{d}\omega(X, Y) + [\omega(X), \omega(Y)] = (\mathrm{d}\omega + \omega \wedge \omega)(X, Y). \tag{2.53}$$

The curvature, like the connection, of the bundle is a global quantity in the total space. In a subset of the base space a local section can be chosen and used to pullback the curvature to the base space,

$$\mathcal{F}_i = \sigma_i^* \Omega. \tag{2.54}$$

Using the fact that $\sigma_i^*(\omega \wedge \omega) = \sigma_i^*\omega \wedge \sigma_i^*\omega$ we find the local form of the bundle curvature,

$$\mathcal{F}_i = \mathrm{d}\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i. \tag{2.55}$$

In the case of electromagnetism the symmetry is abelian, so the wedge product drops out leaving,

$$\mathcal{F} = \mathrm{d}\mathcal{A} = (\partial_\mu A_\nu - \partial_\nu A_\mu)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{2.56}$$

The components of Equation 2.56 are identical to Equation 2.9. Having reproduced the electromagnetic field strength tensor we can see it as a manifestation of the bundle curvature. In Riemannian geometry the curvature represents the failure of a vector to return to its initial orientation after parallel transport around a closed loop. The celebrated Singer-Ambrose theorem tells us the curvature 2-form represents the failure of a vector to return to its initial orientation under horizontal lift around a closed curve [36], see Figure 2.9.

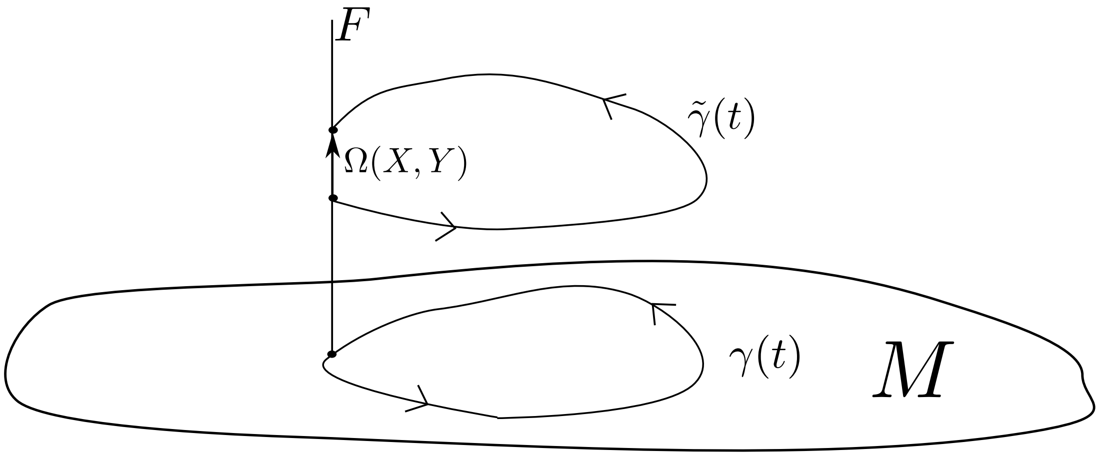

Figure 2.9: Around a closed curve $\gamma(t)$ in the base space the failure of the lifted curve $\tilde{\gamma}(t)$ to close in the total space is measured by the curvature $\Omega(X,Y)$.

The remaining element of the fiber bundle formalism we need to reproduce our example of electromagnetism is the introduction of matter fields. Matter fields are often accommodated by the introduction of the vector bundle associated with $P(M,G)$. Imagine we have a principal bundle $P(M,G)$ and a vector space V such that G has left action on V under a representation $\rho : G \to GL(n,K)$ with K the field[4] underlying V. If we assign an action on the product space $P \times V$ as,

$$(u,\nu) \to (ug, \rho(g^{-1})\nu), \tag{2.57}$$

the associated vector bundle is the equivalence class of points where we identify all elements of the form $(u,\nu) \sim (ug, \rho(g^{-1})\nu)$, [36, 41]. We denote the vector bundle associated to P by $E = P \times_G V$ or $E(M,F,G,\pi_E,P)$. Although this definition will serve us well, an identical notion of the space of equivariant vector valued functions on P denoted by $C(P,V)$ is much simpler to work with. If we take $\tau \in C(P,V)$ the condition of equivariance is $\tau(ug) =$

[4]Field here is used in the mathematical sense of the word rather the physical notion of a field. In mathematics a field is essentially a ring with additional structure.

$\rho(g^{-1})\tau(u)$, exactly the condition by which we quotient $P \times V$ to construct $E = P \times_G V$ [47].

In terms of the associated bundles a matter field is a section $\xi : M \to E$ of the bundle $E = P \times_G V$. Or it is simply an element of the space $C(P, V)$ [47]. To obtain the definition of the covariant derivative on one of our matter fields we consider a curve that has been horizontally lifted into the bundle. A curve $\gamma' : \mathbb{R} \to P$ is considered horizontally lifted if $\pi \circ \gamma' = \gamma$ is a curve in the base space and the tangent vector to γ' always belongs to the horizontal subspace of P [36, 41]. Picking a curve which trivializes as $\phi^{-1}(\gamma') = (\gamma(t), e)$ we can represent a lifted curve by $\tilde{\gamma} = \gamma'(t) g(\gamma'(t))$ for some $g \in G$. For some $\psi \in C(P, V)$ its covariant derivative in the direction of X is given by,

$$\nabla_X \psi = \frac{\mathrm{d}}{\mathrm{d}t} \psi(\tilde{\gamma}(t)) \bigg|_{t-=0} . \tag{2.58}$$

If we insert the definition of $\tilde{\gamma}$ into Equation 2.58 we arrive at the covariant derivative as follows,

$$= \frac{\mathrm{d}}{\mathrm{d}t} \psi(\gamma'(t) g(\gamma')) \bigg|_{t=0} \tag{2.59a}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} g^{-1}(\gamma') \psi(\gamma') \bigg|_{t=0} \tag{2.59b}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} g^{-1} \psi(\gamma') + g^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \psi \bigg|_{t=0} \tag{2.59c}$$

$$= -g^{-1} \frac{\mathrm{d}g}{\mathrm{d}t} g^{-1} \psi(\gamma') + g^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \psi \bigg|_{t=0} \tag{2.59d}$$

$$= -g^{-1} \left(\mathcal{A}(X)\psi + \frac{\mathrm{d}\psi(\gamma')}{\mathrm{d}t} \right) \bigg|_{t=0} . \tag{2.59e}$$

At this point we have all of the material needed to discuss the gauge constructions of gravitation. Although more complicated, the fiber bundle formalism offers valuable information about gauge theories, particularly when formulated on topologically non-trivial spaces. Indeed, it is through this formalism that we arrive at conditions that have to be satisfied in order for certain gauge constructions to make sense. These conditions, as they arise in the composite bundle gauge formulation of gravitation, are the primary focus of this work.

CHAPTER 3

COMPOSITE GAUGE THEORY OF GRAVITY

Composite fiber bundles add to the uses of fiber bundles in physics. They were initially introduced by Sardanashvily as the mathematical formalism for gauge theories of broken symmetries [48–51]. However they have also been used as the geometric framework behind the field theories introduced by Wess, Coleman and Zumino [52–55]. The most recent use of composite fiber bundles, and the most important for our purposes, are as a geometric framework for gravitational gauge theory [14, 56, 57]. The use of composite fiber bundles as a geometric realization of gauge gravity has been extended to a network of gauge theories based on ever larger symmetry groups [58, 59].

There are a variety of interesting topics to address in composite gauge gravity, however for the purposes of this thesis we narrow our focus to two. We first focus on the gauge theory aspects to make connections to the standard story of internal fiber bundles discussed in the previous chapter. Following the work of Tresguerres ([14]) the primary focus of the discussion in section 3.1 will be on fixing what is seen as problems in the standard Poincaré gauge theory. Once satisfied by the resolution of these issues in section 3.2 we turn to the geometry of the composite fiber bundles where we offer a new interpretation of the set of bundles available in a composite gauge theory of gravity. This new interpretation is the foundation for the discussion to follow in chapter 4.

3.1 Gauge Theory Based on Composite Bundles

In standard Poincaré gauge theory (developed in the functional approach or the typical bundle approach) we work with the Poincaré group as our symmetry group $ISO(1,3) = SO(1,3) \rtimes \mathbb{R}^{1,3}$, where $SO(1,3)$ is the Lorentz group and $\mathbb{R}^{1,3}$ is the group of translations. We can split the connection into a direct sum of two components [42], the spin connection ω and the coframe field θ. The spin connection arises from the Lorentz symmetry and the coframe field from the translational symmetry. As a result there are two conserved quantities, the energy-momentum and spin-angular momentum currents, which arise from variations of the action with respect to the gauge degrees of freedom. The energy-momentum current couples to the curvature of the Lorentz connection and the spin-angular momentum current couples to the curvature of the translational connection. Both connections transform as proper connections (see Equation 2.41e) under Poincaré transformations. At first glance, the torsion part aside, this appears to be a satisfactory gauge theory containing Einstein's general relativity. However there are two significant criticisms of this approach. First we saw in chapter 2 section 2.2.5 that the dual coframe has one contravariant (upper) Lorentz index. One of the niceties of this notation is that the transformation properties of objects are easily seen. With one contravariant Lorentz index the dual frame transforms as a contravariant vector under Lorentz rotations. This can be determined by recalling for $\Lambda \in SO(1,3)$ the Lorentz metric is invariant ${\Lambda^i}_a {\Lambda^j}_b \eta_{ij} = \eta_{ab}$,

$$ds^2 = \hat{\theta}^i \hat{\theta}^j \eta_{ij} = \hat{\theta}^a \hat{\theta}^b {\Lambda^i}_a {\Lambda^j}_b \eta_{ij} = \hat{\theta}^a \hat{\theta}^b \eta_{ab}. \tag{3.1}$$

This is not the case in standard Poincaré gauge theory where the coframe is a connection and transforms as such. This is a problem with the standard Poincaré gauge theory as developed by Kibble [4]. Either the coframe field is not a connection or the standard approach is not sensitive enough to detect the proper transformation properties.

The second criticism has to do with the actions of the symmetry groups. In general relativity the symmetry groups involve spacetime transformations and in particular include translations which move between points in spacetime. In chapter 2 we defined gauge transformations as vertical bundle automorphisms, where we made sure they did not induce changes in the base spacetime. This was noticed by Lord in [9], where he proposed that spacetime gauge theory be based on the bundle $P(G/H, H)$ and the verticality condition $f \circ \pi = \pi$ should be relaxed. Without the verticality condition Lord could have gauge transformations which induce transformations on the base spacetime. Gravitational gauge theory is primarily based on the Poincaré group $SO(1,3) \rtimes \mathbb{R}^4$, where the translations genuinely move points in the base. Therefore we will see that dropping the verticality condition seems to be a step in the proper direction.

In [14], Tresguerres noticed that although Lord's construction allowed transformations in the base, his bundle $P(G/H, H)$ lacked true translational connections. The translational connections in [9] are pure gauge, i.e. they have no contribution to the curvature. Tresguerres remedied the lack of translational connections by considering $P(M, G)$ to be split into a chain of bundles $P \xrightarrow{\pi_{PE}} E \xrightarrow{\pi_{EM}} M$ known as a composite bundle, see Figure 3.1. The splitting of a bundle into a chain is the basis of modern composite gauge theory. A simple way to envision this is to imagine a football field. Over each point of the field there is a blade of grass growing. The field corresponds to the base space and the grass corresponds to the fiber space. If we now imagine that the grass has hairs growing out of it the hair on the grass corresponds to the additional fiber space we require of a composite bundle, just as is depicted in Figure 3.1.

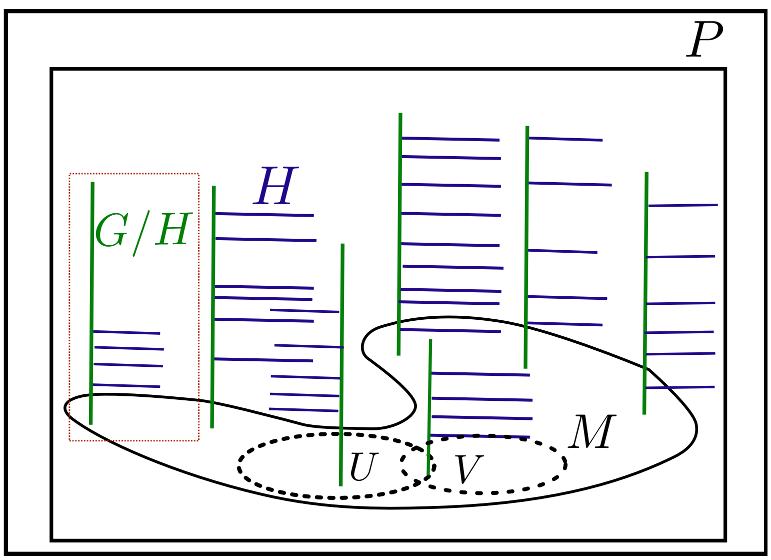

Figure 3.1: A depiction of a composite bundle is shown. Like ordinary fiber bundles there is a base space M and fiber spaces G/H for each point of M. A point $u \in E$ in the total space E (an example of which is boxed in red) created by the base M and fiber space F is decomposed locally as $u = (x, a) \in M \times G/H$. However for a composite bundle we have additional fiber spaces attached at each point $u \in E$. The total space of the composite bundle is then locally $P \cong E \times H$ and can be decomposed further as $(x, a, h) \in M \times G/H \times H$.

Much of the analysis of composite bundles is very similar to what I showed in chapter 2. However now we will have two bundles over the base space M which will be related. Suppose that we have some bundle $P(M, G)$ such that $H \subset G$ is a closed Lie subgroup of G. Proposition 5.6 of [42] says that provided $E = P \times_G G/H$ admits global sections we can split the bundle as $P \xrightarrow{\pi_{PE}} E \xrightarrow{\pi_{EM}} M$. In each bundle sector there is a projection, $\pi_{PE} : P \to E$ and $\pi_{EM} : E \to M$ such that $\pi = \pi_{EM} \circ \pi_{PE} : P \to M$. As with typical bundles we can take a local trivialization. Locally $\pi_{PE} : P \to E$ has coordinates (u_E, h) for $u_E \in E$ and $h \in H$. Likewise the sector $\pi_{EM} : E \to M$ has local coordinates (p, a) for $p \in M$ and $a \in G/H$.

We can take sections as before only now there are three possible choices, one for each sector ($\sigma_{ME} : M \to E$ and $\sigma_{EP} : E \to P$), and one for the total space ($\sigma : M \to P$). If we want our description to be consistent then we need to require that $\sigma = \sigma_{EP} \circ \sigma_{ME}$. Sections can be written as before in terms of the local canonical trivialization,

$$\sigma_{ME}(x) = \phi_{ME}(p, e_{G/H})a(\xi), \qquad \sigma_{EP} = \phi_{EP}(u_E, e_H)h, \tag{3.2}$$

where we have the identity elements $e_{G/H} \in G/H$ and $e_H \in H$ and the elements $u_E \in E$, $a \in G/H$ and $h \in H$. Formally we write u_E for the input to the section $\sigma_{EP} : E \to P$. However we have in mind that we are using a local section. Locally the bundle $E \to M$ can be seen as $M \times G/H$ and so we might as well write a point $u_E \in E$ in Equation 3.2 as (p, ξ).

Gauge transformations in the total bundle P respect the conditions we imposed in chapter 2, i.e. they commute with the right action and are vertical with respect to the fiber. However in the sector $P \to E$ the verticality condition is relaxed. The gauge transformations in $P \to E$ are allowed to affect the base space E. Essentially relaxing vertically means a gauge transformation in the bundle $P \to E$ is allowed to move fibers to fibers. Even though the points in the base space do not change, the fibers associated with the points in the base do change. This is just as if we shifted from some point p to some point q in the base i.e. a translation from p to q in the base space. We can write the gauge transformation as some

$f = L_g$ for $g \in G$,

$$f(\sigma_{EP}(x, \xi)) = \sigma_{EP}(p, \xi')b. \tag{3.3}$$

Following the procedure in chapter 2 section 2.3 we can consider the gauge transformations of a vector X tangent to a curve passing through the point $u = \sigma_{EP}(p, \xi)$ to find an equation analogous to Equation 2.41e,

$$f_* X = Q_*(R_{b*}X + (b^{-1}\mathrm{d}b)^{\#}(X)). \tag{3.4}$$

There are two differences between Equation 2.41e and Equation 3.4. First although we used an element $g \in G$ as our left action, only the component $b \in H$ survives in the expression for the transformation. Second there is an additional pullback by $Q = R_b^{-1} \circ L_g$ present in the transformation. These differences will conspire to ensure the coframe transforms as a contravariant Lorentz vector. To see this we first need to introduce the connections. There are two bundles on which we can put a connection, i.e. $P \xrightarrow{\pi_{PE}} E$ and $E \xrightarrow{\pi_{EM}} M$. But we must be careful to remember that overall the bundle is $P \to M$, so in truth there is only one connection 1-form ω. The two connection 1-forms are the "shadow" of this connection pulled back to their respective base spaces,

$$\mathcal{A}_M = \sigma^*_{MP}\omega, \qquad \mathcal{A}_E = \sigma^*_{EP}\omega. \tag{3.5}$$

The connection on E can be further pulled back to the base space M so that the two "shadow" connection 1-forms satisfy [14],

$$A_M = \sigma^*_{ME}A_E = \sigma^*_{ME}\sigma^*_{EP}\omega. \tag{3.6}$$

Over the bundle sector $P \to E$ we can split the total connection into a sum of two components $\omega = \omega_R + \omega_T$, where the subscript R denotes the Lorentz connection and the subscript T denotes the translational connection. Analogous to Equation 2.40 we have,

$$\omega_R = h^{-1}(\mathrm{d} + \pi^*_{PE}\mathcal{A}_R)h, \tag{3.7}$$

where I have put A_R as the local form of the Lorentz connection on the base space E. Additionally we have,

$$\omega_T = h^{-1}\pi^*_{PE}\mathcal{A}_T h. \tag{3.8}$$

Where $\mathcal{A}_T$ denotes the local form of the translational connection on E. Applying the total connection ($\omega = \omega_R + \omega_T$) to Equation 3.4, pulling the result to the base space E and equating terms based on their expansions in the Lie algebras of the translations and Lorentz rotations we arrive at the gauge transformation of the gauge fields,

$$h^{-1}\mathrm{d}h + h^{-1}\mathcal{A}_R h = \mathcal{A}_R, \qquad h^{-1}\mathcal{A}_T h = \mathcal{A}_T. \tag{3.9}$$

The infinitesimal variation can be computed from $\delta A = A - A'$ where A' is the gauge transformed form of A. We expand the group transformations $h = e^{i\Lambda_{\alpha\beta}\epsilon^{\alpha\beta}} \approx I + i\Lambda_{\alpha\beta}\epsilon^{\alpha\beta}$ to arrive at,

$$\delta\mathcal{A}^i_T = -\mathcal{A}^j_T\epsilon^i{}_j. \tag{3.10}$$

This is exactly the infinitesimal variation of a Lorentz transformation. Breaking the bundle $P \to M$ to $P \to E \to M$ gives us the right transformation properties of $\mathcal{A}_T$ while leaving it to still be identified as a gauge potential [14]. This gauge potential ($\mathcal{A}_T$) when pulled back to the base by a canonical local trivialization $\sigma_{ME}(p) = \phi(p, e)$ is identified as $\sigma^*_{ME}\mathcal{A}_T = \hat{\theta}$, i.e. the coframe. As we saw in chapter 2, we can decompose the coframe as $\hat{\theta}^i = e^i{}_\mu \mathrm{d}x^\mu$ (Equation 2.33) and so we can write the metric as,

$$g_{\mu\nu} = e^i{}_\mu e^j{}_\nu \eta_{ij}. \tag{3.11}$$

A further indication of composite gauge theories ability to describe gravitation comes from expanding the tetrad in terms of the spacetime connection [14],

$$e_\mu{}^i = \partial_\mu\xi^i + \mathcal{A}_{R\mu j}{}^i\xi^j + \mathcal{A}_{T\mu}{}^i. \tag{3.12}$$

We can see that the tetrad has internal structure in terms of the dynamic gauge fields. This is just as I mentioned in chapter 2 section 2.2 where we saw that the Christoffel connection was defined in terms of the metric.

The discussion of composite gauge gravity in the literature has been focused on developing the gauge theory aspects of the formalism [14, 57, 58]. The discussion of the gauge theory aspects of composite gauge gravity above followed closely the work of Tresguerres [14]. However the geometric and topological aspects of the bundles discussed above has been absent in the literature. In the following section we develop some of the geometric aspects of the bundles used in composite gauge theories of gravitation. It is in first understanding the geometry of the bundles that the underlying topological conditions discussed in chapter 4 become clear.

3.2 Geometry of Composite Bundles

To determine whether composite bundles provide a consistent formulation of gravitation, we should stipulate what sort of objects we expect the construction to reproduce. This point has been slighted in the literature. The requirements arise from considering what bundles are relevant to general relativity. To formulate general relativity via a gauge principal we must show how the tangent bundle and the frame bundle arise as a consequence of local symmetry. The tangent bundle is a vector bundle over a manifold M denoted $TM \equiv E(M, \mathbb{R}^n, Gl(n, \mathbb{R}))$. Its structure group is the general linear group and its fiber space is just n dimensional Euclidean space. Associated with the tangent bundle is the frame bundle, a principal fiber bundle denoted $\mathcal{F}M \equiv P(M, Gl(n, \mathbb{R}))$. So the tangent bundle is the associated vector bundle to the frame bundle. A Lorentzian metric on a space M is an inner product operation on the tangent bundle $\eta : TM \otimes TM \to \mathbb{R}$. Introducing a Lorentzian metric on the tangent bundle reduces the structure group of the tangent bundle and the frame bundle from the general linear group to the orthogonal group, $Gl(n, \mathbb{R}) \to O(1, n-1)$ (supposing that there is one timelike dimension and $n-1$ spatial dimensions). So we are left with $TM = E(M, \mathbb{R}^n, O(1, n-1))$ and $\mathcal{F}M = P(M, O(1, n-1))$. We can further reduce the symmetry group and subsequently extend it as we continue to remove topological obstructions. This will be the topic of the following section. However we are still left with the question "how do the frame bundle and its associated vector bundle arise in the context of composite gauge theories of gravitation?"

To answer this question we need to go to back to, how and when, can we split a principal bundle into a "tower" of bundles. The basic idea, as Tresguerres applies it, comes from proposition 5.5 and 5.6 of Kobayashi and Nomizu. These propositions are where we started from earlier in the chapter where I assumed most importantly that proposition 5.6 held. Proposition 5.6 states "The structure group G of $P(M,G)$ is reducible to a closed subgroup H if and only if the associated bundle $E(M,G/H,G,P)$ admits a cross section $\sigma : M \to E = P/H$" [42]. We are able to identify $E = P/H$ due to proposition 5.5 which states, "The bundle $E = P \times_G G/H$ associated with P with standard fiber G/H can be identified with P/H as follows. An element of E represented by $(u, a\xi_0) \in P \times G/H$ is mapped into the element of P/H represented by $ua \in P$ where $a \in G$ and ξ_0 is the origin of G/H, i.e, the coset H." [42]. This proposition assures us that the associated bundle E can be seen to be exactly the total bundle P quotient-ed by the closed subgroup H[5]. When reading these propositions it is easy to use an everyday reading of the word reducible. However reducible in these propositions has a very specific mathematical meaning. Kobayashi and Nomizu say that provided there exists an embedding $f : P'(M,G') \to P(M,G)$ then the image $f(P)$ is a sub-bundle and we say G is reducible to G' [42]. An understanding of what reduced means is crucial for the identification of the topological structures of classical spacetime. The global section of E leads to a subbundle of $P(M,G)$ given by,

$$Q(M,H) \equiv \{u \in P(M,G) | \pi_{PE}(u) = \sigma(x)\}. \tag{3.13}$$

Since the section σ is global and so too is π_{PE} we can define a new global section given by $q \equiv \pi_{PE}^{-1} \circ \sigma : M \to Q$. We will use this section in chapter 4 to understand the topology of spacetime induced by the composite bundle structure. There is a further bundle we will need, as we know from chapter 2, for every principal bundle there is an associated vector bundle. So we can also construct $Q' = Q \times_H \mathbb{R}^n$. As a result of requiring we can split the bundle we have a collection bundles to work with which are displayed in Table 3.1. In this collection of

[5] More on cosets can be found in the Appendix B

Table 3.1: The collection of bundles formed during a composite bundle construction and their projections.

Bundle	Projection
$P(E, H)$	π_{PE}
$E(M, G/H, G, P)$	π_{EM}
$Q(M, H)$	μ
$Q(M, H) \times_H \mathbb{R}^n$	μ_A

bundles we find the appearance of the frame and tangent bundle in composite gauge theory. For $G = ISO(1,3)$ the Poincaré group, $H = SO(1,3)$ the Lorentz group and $G/H = \mathbb{R}^4$ the bundle Q is diffeomorphic to the frame bundle $\mathcal{F}M \cong Q(M, SO(1,3))$. The associated vector bundle to Q is diffeomorphic to the tangent bundle $Q' = Q(M, SO(1,3)) \times_{SO(1,3)} \mathbb{R}^4 \cong TM$. The global sections of the bundle E, which assure us a subbundle $Q \subset P$, also act to connect the translational and rotational gauge degrees of freedom of $P(E, SO(1,3))$ and $E \cong P/SO(1,3)$ to the spacetime bundles $\mathcal{F}M$ and TM.

Another natural question to ask is, "over which manifolds can we find global sections of E?" Theorem 5.7 of Kobayashi and Nomizu gives that if the base space M is paracompact and the fiber space, here $F = G/H$, is diffeomorphic to an Euclidean space any section defined over a closed subset of M can be extended to the entire space M [42]. In chapter 4 we will discuss why this theorem will hold for the composite bundle we have constructed for gravitation. Additionally using all of the bundles in Table 3.1 we will see in chapter 4 that a consequence of demanding that we can break the bundle $P \to M$ to $P \to E \to M$ is that the induced tangent and frame bundles of M must have a set of trivial characteristic classes.

CHAPTER 4
TOPOLOGICAL OBSTRUCTIONS

When considering the more complicated constructions of composite fiber bundles the chain of bundles $P \to E \to M$ the bundle E was required to admit a global section σ. We know that these sections exist but the question now is "what does this condition imply for the subbundle Q?" The way we can answer this question is by understanding the cohomology groups used to characterize the topology of the bundle. The characteristic classes discussed in the introduction, such as the Stiefel-Whitney classes, are elements of cohomology groups. It is beyond the scope of this thesis to give a in depth treatment of cohomology so in the following I will introduce only what is necessary to detail the conclusion of this research; the composite gauge theory of gravitation requires M admit a string structure. A simple place to start will be to develop the de Rham cohomology built from differential forms of a manifold. Following this is section 4.2 which briefly details Čech cohomology, a theory built from sets and transition functions. Section 4.3 uses the Čech cohomology to discuss the basic characteristic classes relevant to the tangent and frame bundle. Finally we detail the characteristic classes of the composite bundle gauge theory of gravitation in section 4.4. We will see that the composite gauge theory of gravity requires a more refined cohomology theory.

4.1 de Rham Cohomology

The simplest example of cohomology is the de Rham cohomology. It is built from the following observation,

$$\Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^r(M). \tag{4.1}$$

The exterior derivative creates a chain of mappings from each exterior power of the dual tangent space to the next, or in other words it takes a form of a given degree and returns a form of one degree higher. Notice that for $\alpha \in \Omega^r$ two applications of the exterior derivative gives,

$$d^2\alpha = \frac{1}{r!}\left(\partial_\gamma \partial_\nu \omega_{\mu_1\cdots\mu_r}\right) dx^\gamma \wedge dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} = 0. \tag{4.2}$$

That is, the exterior derivative is a nilpotent operator. The quantity $\partial_\gamma \partial_\nu \omega_{\mu_1\cdots\mu_r}$ is symmetric and is summed over an antisymmetric quantity and therefore vanishes. This leads us to define two sets, the cocycle group $Z^r(M,\mathbb{R}) = \{\alpha \in \Omega^r(M) | d\alpha = 0\}$, any $\alpha \in Z^r$ has vanishing exterior derivative by definition (called a closed form), and the coboundary group $B^r(M,\mathbb{R}) = \{\alpha \in \Omega^r(M) | \alpha = d\beta\}$, any $\alpha \in B^r$ has vanishing exterior derivative because it is itself the exterior derivative of a lower degree form (called an exact form) [36, 60]. The de Rham cohomology with real coefficients is then defined as [36, 60],

$$H^r(M;\mathbb{R}) = \frac{Z^r(M,\mathbb{R})}{B^r(M,\mathbb{R})}. \tag{4.3}$$

For instance a given $\alpha \in Z^r(M)$ defines a class $[\alpha] \in H^r(M;\mathbb{R})$ such that $[\alpha] = \{\alpha' \in Z^r(M) | \alpha' = \alpha + d\eta, \eta \in \Omega^{r-1}(M)\}$. The class $[\alpha]$ is the set of all such α that differ by an exact form, e.g. $d\eta$.

4.2 Čech Cohomology

de Rham cohomology is not the only cohomology theory. Another important cohomology theory is built entirely from the transition functions of a bundle. We learned earlier that a bundle is trivial iff we can choose every transition function to be trivial. So it is natural to suspect a cohomology built from the transition functions to be an appropriate tool to understand over which manifolds will $E(M, G/H, G, P)$ have a global section.

To build the Čech cohomology we follow the same general steps as before and follow closely [36]. Let $\mathbb{Z}_2$ be the multiplicative group of elements $\{-1, 1\}$. Analogous to the cocycle group in the de Rham cohomology we define the Čech r-cochain as function $f(t_0, \cdots, t_r) \in \mathbb{Z}_2$ defined on the set $U = \cap_{j=0}^r U_j$ and is a function of the transition functions t_r. Additionally we require f be invariant under arbitrary permutation. Let C^r denote the multiplicative group of Čech r-cochains. Then we also require a nilpotent operator $\delta : C^r(M) \to C^{r+1}(M)$ defined as,

$$\delta(f(t_0, \cdots, t_r)) = \prod_{k=0}^{r+1} f(t_0, \cdots, \hat{t}_k, \cdots, t_{r+1}), \tag{4.4}$$

where the "hat" above the k$^{\text{th}}$ entry denotes removal of that quantity. For example consider a 2-cochain f. If we allow the boundary operator to act on f we find the following 3-cochain,

$$\delta(f(t_0, t_1)) = f(t_1, t_2)f(t_0, t_2)f(t_1, t_2). \tag{4.5}$$

We can see that δ is a nilpotent operation, i.e. applying δ twice we will be removing two elements,

$$\delta^2(f(t_0, \cdots, t_r)) = \prod_{j,k=0}^{r+1} f(t_0, \cdots, \hat{t}_k, \cdots, \hat{t}_j, \cdots, t_{r+1}). \tag{4.6}$$

Note that every combination will appear twice since f is invariant under arbitrary permutations,

$$f(t_0, \cdots, \hat{t}_k, \cdots, \hat{t}_j, \cdots, t_{r+1}) = f(t_0, \cdots, \hat{t}_j, \cdots, \hat{t}_k, \cdots, t_{r+1}). \tag{4.7}$$

Since the values f takes are in $\mathbb{Z}_2$ we have $f^2 = 1$ and so we see δ is nilpotent,

$$\delta^2(f(t_0, \cdots, t_r)) = 1. \tag{4.8}$$

If we let $Z^r(M; \mathbb{Z}_2) = \{f \in C^r(M) | \delta f = 1\}$ be the cocycle group and $B^r(M; \mathbb{Z}_2) = \{f \in C^r(M) | f = \delta h, h \in C^{r-1}\}$ the coboundary group, we define the Čech cohomology as,

$$H^r(M; \mathbb{Z}_2) = \frac{Z^r(M; \mathbb{Z}_2)}{B^r(M; \mathbb{Z}_2)}. \tag{4.9}$$

An element $f \in Z^r(M, \mathbb{Z}_2)$ defines a class $[f] \in H^r(M, \mathbb{Z}_2)$ as $[f] = \{f' \in Z^r(M, \mathbb{Z}_2) | f' = \delta h f, h \in C^r(M, \mathbb{Z}_2)\}$. As an example consider the tangent bundle to S^2, TS^2. The transition functions in the case of the S^2 are,

$$t_{ij} = \begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}. \tag{4.10}$$

Letting f be a Čech 1-cochain defined as $f(i, j) \equiv \det(t_{ij})$ we see using Equation 4.10 that $f(i, j) = 1$. Using the cocycle condition of a fiber bundle ($t_{ij}t_{jk}t_{ki} = I$, [36, 41]) we find that,

$$\delta f(i,j) = \prod_{m=0}^{2} f(0,\hat{m},2) = f(1,2)f(0,2)f(0,1)$$
$$= \det(t_{12})\det(t_{02})\det(t_{01}) = \det(t_{12}t_{02}t_{01}) = I. \qquad (4.11)$$

So the cochain $f \in Z^r(M, \mathbb{Z}_2)$. And we have that the class defined by $f = 1$ is $w_1 = [f] = 1$ is trivial, i.e. the sphere is orientable.

4.3 Characteristic Classes

Stiefel-Whitney classes w_r are elements of the Čech cohomology classes. The non triviality of these classes are the obstructions to the creation of certain structures on bundles. It is well known that a non-vanishing first Stiefel-Whitney class is an obstruction to orientability. Let M be a manifold and $TM \xrightarrow{\pi} M$ be its tangent bundle. The Stiefel-Whitney classes of M are the same as the those of its tangent bundle. Proof of this statement can be found in [60]. And so when we speak of the Stiefel-Whitney classes of a manifold we mean the classes of its tangent bundle.

Let TM have a Riemannian structure provided by $g : \pi^{-1}(p) \otimes \pi^{-1}(p) \to \mathbb{R}$. Recall from chapter 2 that $\pi^{-1}(p) \cong F$, i.e. the Riemannian structure on TM is then a mapping $g : TM \otimes TM :\to \mathbb{R}$. We see a Riemannian structure on the tangent space is the equivalent of a metric[6]. As we described in section 2.2.5 of chapter 2, we could choose a local frame as a rotation of the natural basis in the tangent space. If we consider the set of all of these choices we can build a principal fiber bundle called the frame bundle. The frame bundle has the tangent bundle as an associated bundle and is written $\mathcal{FM}(M, Gl(n, \mathbb{R}))$. In chapter 2 when we rotated our basis of the tangent space, at first we had a choice of ${e^\mu}_i \in Gl(4, \mathbb{R})$. However we required that the frame $\hat{e}$ be orthonormal with respect to the metric. This requirement reduces our set of choices from a $Gl(n, \mathbb{R})$ rotation to an $O(n)$ (or $O(1,3)$ for Lorentzian spacetimes) rotation. The reduction in structure group means on overlapping open neighborhoods $U_i \cap U_j$ that a choice of frame over U_i is related to the

[6]The metric we consider is what is known as a fiber metric in the literature.

choice in frame of U_j by $e_\alpha = t_{ij} e_\beta$ with $t_{ij} \in O(n)$ as the transition function. Let f be the determinant function, we have $f(t_{ij}) = \pm 1$, i.e. f is indeed valued in $\mathbb{Z}_2$. On a triple intersection $U_i \cap U_j \cap U_k$ the transition functions (t_{ij}, t_{jk}, t_{ki}) are required to satisfy the cocycle condition $t_{ij} t_{jk} t_{ki} = I$ as stated in the previous section. Using the cocycle condition we can act the boundary operator on our cochain function to find $\delta f(i,j) = 1$, so $f \in Z^1(M;\mathbb{Z}_2)$ and defines an equivalence class $[f] \in H^1(M;\mathbb{Z}_2)$. This first class $w_1 = [f]$ is the first Stiefel-Whitney class. It follows that M is orientable iff w_1 is trivial. This can be seen quickly since if M is orientable then $O(n) \to SO(n)$ (or $SO(1,3)$ for Lorentzian spacetimes) and we choose either positive or negative orientation $\det(t_{ij}) = \pm 1$. For the sake of argument lets choose $\det(t_{ij}) = 1$. Then $f(t_{ij}) = 1$, hence $w_1 = 1 \in \mathbb{Z}_2$, and so is trivial. If w_1 is trivial then f is a coboundary and can be written $f = \delta f_0$ with $f_0 = 1$. This means we are able to choose our transition functions to all satisfy $\det(t_{ij}) = 1$, hence our bundle is orientable [36].

The second Stiefel-Whitney class w_2 is an obstruction to a bundle admitting a spin structure. Defining spinors on a space requires a lifting to the covering group of $SO(n)$ (or the covering group of $SO(1,3)$ for Lorentzian spacetimes) and the second Stiefel-Whitney class describes the obstruction to defining this lifting. Since we can reconstruct a bundle from its transition functions, consider a manifold M whose tangent bundle is orientable i.e. the first Stiefel-Whitney class is trivial, and let $\{t_{ij}\}$ be the set of transition functions of the associated principal frame bundle $\mathcal{F}M$. If we let $\psi : SPIN(n) \to SO(n)$ be the typical double covering of the group $SO(n)$ we can define a complementary set of transition functions $\{\tilde{t}_{ij}\}$ such that $\psi(\tilde{t}_{ij}) = t_{ij}$ and,

$$\tilde{t}_{ij} \tilde{t}_{jk} \tilde{t}_{ki} = I, \tag{4.12}$$

on a triple intersection $U_i \cap U_j \cap U_k$. The set of $\{\tilde{t}_{ij}\}$ defines a spin bundle over M denoted $SPIN(M)$ and M is said to admit a spin structure. Consequently we have that a manifold M admits a spin structure iff the second Stiefel-Whitney class vanishes [36].

In [61, 62] the authors make a case for the topology of spacetime. Their papers center around the interpretations of the third and fourth Stiefel-Whitney classes. They argue that the interpretations of the remaining classes are related to obstructions to chiral spinors and causality respectively (Table 4.1 displays the interpretations of each Stiefel-Whitney class). It should be noted that although the first and second Stiefel-Whitney classes had been known to represent obstructions to defining consistent volume elements and to defining spinors on a space, the third and fourth classes did not have such an interpretation. It was only recently that Antonsen and Flagga demonstrated the relationship between the remaining Stiefel-Whitney classes and obstructions to spacetime structures [61, 62]. The literature concerning characteristic classes and spacetime focuses on general spaces and then restricts their properties until they meet those familiar to general relativity and the Standard Model. As we discussed in the introduction the fiber bundle approach to gauge theories is needed when discussing gauge theories on topologically non-trivial spaces. By writing a gauge theory of gravity in terms of fiber bundles we have inadvertently accessed a larger collection of possible spaces on which we could write gravitational theory. This is just like the example of the writing down electromagnetic gauge theory on a sphere. However we will see in the next section that the process of constructing a composite gauge theory actually fills in missing topological conditions we might expect of a gravitational theory. Although fiber bundles allow for extensions of familiar theories to topologically non-trivial spacetimes the composite gauge theory construction of gravitation actually helps to restrict the set of possible spaces.

Table 4.1: A list of the Stiefel-Whitney classes and their interpretations as obstructions to topological entities over a manifold M. Entries collected from [60–62].

Stiefel-Whitney class	What is it obstructing?
w_1	Orientability
w_2	Spin
w_3	Chiral Spinors
w_4	Causality

The reference [61] contains a useful list of equivalent statements for a bundle to admit a spin structure. From their list an equivalent statement to the vanishing of the second

Stiefel-Whitney class is, "an orientable manifold M admits a spin structure iff M is parallelizable" [61]. Parallelizability is the ability to define a global section $\sigma : M \to \mathcal{F}M$ of the frame bundle, i.e. the frame bundle must be trivial. For us to use this fact we must recall the two equivalent ways of deciding the triviality of a vector bundle. We can either find an everywhere non-zero global section of the vector bundle or we can find a global section of the associated principal bundle. This means that showing that we can find a section of $E(M, F, G, P)$ means we can find a section of $P(M, G)$. The obvious next question to ask then is "What is the condition for a global section of a vector bundle?" Looking to Milnor and Stasheff, "if the oriented vector bundle ξ possesses a nowhere zero cross-section, then the Euler class $e(\xi)$ must be zero"[60].

4.4 String Manifolds and Composite Gauge Gravity

Now that we have collected most of the characteristic classes needed we can discuss how this plays out for the composite bundles. If we were to imagine at first that the base manifold we work with is compact it will be easier. Then we start with the total bundle $P(M, G)$ where we interpret the Poincaré group ($G = ISO(1,3)$) as the bundle $G(G/H, H)$ for $G/H \cong \mathbb{R}^4$ the translations and $H = SO(1,3)$ the Lorentz rotations. Then we can construct the bundle $E(M, \mathbb{R}^4, ISO(1,3), P) = P \times_{ISO(1,3)} \mathbb{R}^4 \cong P/SO(1,3)$ associated to $P(M, ISO(1,3))$. From chapter 3 we know that there exists a bundle $P(E, SO(1,3))$ with E as the base space and $SO(1,3)$ as the fiber space. Then provided that there exist a global section $\sigma : M \to E$ the structure group can be reduced from $ISO(1,3)$ to $SO(1,3) \subset ISO(1,3)$ [42]. Additionally from chapter 3 we know that that this means there exists $Q(M, SO(1,3)) \subset P(M, ISO(1,3))$ and an associated bundle $Q(M, SO(1,3)) \times_{SO(1,3)} \mathbb{R}^4$. The bundle $Q(M, SO(1,3))$ can be identified with the principal frame bundle over M and its associated vector bundle $Q(M, SO(1,3)) \times_{SO(1,3)} \mathbb{R}^4$ can be identified as the tangent bundle over M. By creating a principal Poincaré bundle we have indeed succeeded in creating the needed bundles for general relativity, a tangent bundle and a frame bundle. If we did not require a global section of $E \cong P/H$ then we would have no way of connecting the gauge bundle with the frame bundle. Meaning that the gauge bundle would not influence spacetime. It is unclear whether this point is clear to Tresguerres et al. Global sections of E ensure communication with the tangent space of the base spacetime. The theorem used in chapter 3 guarantees global sections of this bundle in the case of paracompact manifolds, in a compact spacetime it is simple to see the extension of the theorem. The question now is what this implies of the topology of the base manifold. In chapter 3 we saw that the bundle Q as $Q(M, SO(1,3)) \equiv \{u \in P(M, ISO(1,3)|\pi_{PE}(u) = \sigma(x)\}$ leading us to define the global section $q = \pi_{PE}^{-1} \circ \sigma : M \to Q(M, SO(1,3)) \cong \mathcal{F}M$. We now have that the frame bundle of M is trivial and as a consequence we can extend this global section to $q' : M \to Q(M, SO(1,3)) \times_{SO(1,3)} \mathbb{R}^4 \cong TM$ leading to a trivial tangent bundle. Putting a Riemannian structure on the tangent bundle we can then verify that the both the first and second Stiefel-Whitney classes are trivial as done above. Then as consequence of the second Stiefel-Whitney class being trivial the third Stiefel-Whitney class is also trivial [61]. Then the tangent bundle is orientable and possesses a global section so its Euler class must be zero. But we need to be a bit more precise, $e(M) = e(TM) \in H^4(M;\mathbb{Z})$. With integer coefficients the Euler class is carried by the natural homomorphism to the top Stiefel-Whitney class and so it to is trivial [60]. If instead we work with real coefficients in the de Rham cohomology the Euler class of a even dimensional manifold squares to the first Pontryagin class p_1, requiring it to vanish.

The condition for a manifold to have a string structure is that the first fractional Pontryagin class vanish. This means that not only must p_1 be trivial but the cohomology group must be torsion free [63]. A cohomology group H^r can be decomposed into two pieces, a free piece and a torsion piece. The vanishing of the first fractional Pontryagin class is the condition that $H^4(M;\mathbb{Z})$ be decomposed as only a free piece and the characteristic class $p_1 \in H^4(M;\mathbb{R})$ be trivial. We already saw that the first Pontryagin class is trivial. What is left to show is that the decomposition of the cohomology group has only a free piece. The idea here is that a finitely generated abelian group is the direct sum of a free abelian group of finite rank and a finite abelian group [64]. A free group has a basis in which we can represent an element in terms of the basis elements and for a group to be finitely generated there must be a finite number of basis elements. The finite abelian subgroup is the torsion subgroup. Loosely the torsion subgroup is made up of all elements with finite order, i.e. it takes finitely many applications to return to the identity. Depending on additional constraints we put on our base space we have two directions to go. Provided that M is a connected and compact manifold, the homology groups are finitely generated. And provided there is an orientation (which there is), we can use a theorem from Hatcher [64] "If M is a compact connected manifold of dimension n then $H_{n-1}(M, \mathbb{Z})$ is trivial if M is orientable." In this case using the universal coefficient theorem and Poincaré duality (a relationship between homology and cohomology groups) we have $H^4(M, \mathbb{Z}) = H_4(M, \mathbb{Z}) \oplus H_3(M, \mathbb{Z})$, but $H_3(M, \mathbb{Z})$ is trivial so we have exact Poincaré duality $H^4(M, \mathbb{Z}) = H_4(M, \mathbb{Z})$. Along with the previous result of the triviality of the integral cohomology class we conclude that the manifold M admits a lifting to STRING(n).

The situation becomes a little more complicated when the manifold M is non-compact. The case for non-compact spacetimes has been made by many authors [65–67]. In the case of compact spacetimes there can exist closed time-like curves which upset the causality of spacetime. As discussed above in [61, 62] the authors make a case for the topology of spacetime and rule out compact manifolds. Without the condition of compactness the creation of cohomology groups becomes more involved, i.e. we need cohomology with compact support. If we were to work with a manifold M and define on some compact subset $U \subset M$ a function $\tilde{f} : M \to \mathbb{R}$ such that $\tilde{f}|_U = f : U \to \mathbb{R}$ and $\tilde{f}|_{M-U} = 0$, then we say that f has compact support. This idea will be the basis for compactly supported cohomology. We will have cochains that vanish outside of a compact set. Using compactly supported cohomology we need to show that the homology groups of the space are finitely generated so that we can use lemma from Hatcher to say that the top cohomology class has no torsion. What we need to show is that the dimension of the compactly supported cohomology groups is finite.

In appendix C we briefly go over the proof that this is true for the paracompact base spaces we are using in composite gauge gravity.

With finite dimensional cohomology groups we can now use the Poincaré duality for compactly supported cohomology to say that the spaces H_c^r are isomorphic to H_{n-r} which is to say the dimension of the homology groups are finite for these spaces [64, 68]. The using lemma 3.27 from Hatcher which states that "$H_i(M, M-A; \mathbb{R}) = 0$ for $i > n$ on a compact subset A of an n dimensional manifold M" we see that there are finitely many homology groups [64]. With a finite number and finite dimension, our homology groups are finitely generated. Then using the universal coefficient theorem $H_r(M; \mathbb{R}) \cong H_r(M; \mathbb{Z}) \otimes \mathbb{R}$ we have the integer homology groups are finitely generated. We can finally use the flavor of corollary 3.28 of [64] showing that $H_{r-1}(M; \mathbb{Z})$ has trivial torsion subgroup. The result of all of this formalism is that in the more realistic case of a paracompact base space we also have the trivialization of first Pontryagin class and the fourth cohomology class has no torsion. From this we can conclude that the condition of having a global section of the bundle $E \cong P(M, ISO(1,3))/SO(1,3)$ leads the base manifold to admit a string structure.

The spin structure like the string structure on a bundle is a topological structure defined on the bundle space. Topological structures have played a role in the consistency of perturbative theories via anomaly cancellation for some time now. An anomaly in a quantum field theory is a general term for a classical symmetry which breaks down when the classical field theory is quantized [46]. As an example we have discussed earlier in this section that if we can lift to the spin group we can define a principal spin bundle $SPIN(M, Sl(2, \mathbb{C}^2) \oplus \overline{\mathbb{C}^2})$. It is well known that if we cannot define a spin bundle on spacetime then we will encounter a global anomaly in the quantization of a supersymmetric point particle [69, 70]. It was Killingback who was the first to point out the relevance of the lifting to the string group to physics [69]. He describes the role the of the 2-form B and its 3-form field strength H in the cancellation of spacetime anomalies. The 3-form obeys,

$$\mathrm{d}H = Tr(F^2) - Tr(R^2) \tag{4.13}$$

for F a Yang-Mills field strength and R the curvature 2-form. Integrating this over a closed subset of base space M results in a consistency condition that the two Pontryagin classes, of the tangent bundle of spacetime and of either $SO(32)$ or $E_8 \times E_8$, be equivalent [69]. It should come as a surprise that the composite gauge theories of gravity require a string structure. We have not started with extended degrees of freedom in our theory but the consistency of composite gauge gravity requires a lifting to the string group. To write down a composite theory of gravity we are required to have satisfied topological conditions related to anomaly cancellation in string theory.

Further works have strengthened the connection between string manifolds and anomalies. In [71] Waldorf considers connections A on the bundle $SPIN$ and defines a string class and further a geometric string structure as a pairing of a string structure and a connection on $SPIN$. Studying the consequences Waldorf finds that under certain conditions there exists a 3 form H on M with the prescribed properties needed for anomaly cancellation as described above. Most importantly the exterior derivative of H "is one-half of the Pontryagin 4-form of A" [71]. Although we did not specify in chapter 3 that we needed a connection of the spin bundle over M, it is a natural construction to add to the bundle. When we do, we find the composite theories of gravitation come naturally equipped with information needed for anomaly cancellation. Seeking a gauge theory formulation of general relativity has naturally led us to an important topic in quantum field theory. In hindsight it should be natural that we have found ourselves within the realm of anomalies. Using fiber bundles has pushed the symmetry to the forefront of our thinking. Continually refining the topological structure of the symmetry groups we were bound to run into anomaly cancellation mechanisms.

CHAPTER 5

CONCLUSIONS

Gauge symmetries have been critical to our understanding of the Standard Model of particle physics. The fiber bundle formalism provides a rigorous mathematical formalism for understanding the internal symmetries of the Standard Model. Due to the success of fiber bundles in describing the gauge symmetries of field theories it is natural to think general relativity can also be written in this way. However naive formulations of general relativity as a gauge theory in the fiber bundle formalism are not sensitive enough to construct the appropriate transformation properties of the coframe or to properly handle the translational symmetry. A more sophisticated construction known as composite fiber bundles, as developed in [14], is capable of providing adequate transformation properties of the coframe and inclusion of translational symmetry. However the literature on composite fiber bundle realizations of gravitation has been lacking an interpretation of the peculiar condition of a global section of the bundle $E = P(M, ISO(1,3))/(SO(1,3))$. This research has shown that the condition of global sections of E defines subbundles which can be interpreted as the frame and tangent bundle of spacetime. While fiber bundle formulations of gauge theories typically allow for the extension of a gauge theory to non-trivial topological spaces, the global sections required in the composite bundle construction of gravitation places restrictions on the allowed spaces. Studying the ensuing topology of the subbundles found in the composite gauge theory of gravity reveals that the composite bundle formulation comes equipped with topological conditions we might expect, and not expect, of a sensible spacetime.

5.1 Main Results

In order to properly include gauged translations in the fiber bundle formalism we needed to look beyond typical fiber bundles. The relevant formalism is in terms of composite bundles where we study the chain of bundles $P \to E \to M$. However the symmetry group of the bundle $P(M, ISO(1,3))$ is reducible only if there exists a global section $\sigma : M \to E$. Up until now the reason for requiring the reducibility of the bundles used in the composite approach has been absent in the literature. We found that the reason to require reducibility is to obtain the subbundle $Q(M, SO(1,3)) \subset P(M, ISO(1,3))$ and its associated vector bundle $Q(M, SO(1,3)) \times_{SO(1,3)} \mathbb{R}^n$ which represent the frame bundle and tangent bundle of general relativity. The interpretation of the subbundles Q and Q' provided have not yet to my knowledge been provided in the literature. When a bundle $P(M, ISO(1,3))$ is reducible we have a mapping which provides a subbundle of the original bundle given by $Q(M, SO(1,3))$. Identifying this subbundle $Q(M, SO(1,3))$ with the frame bundle of spacetime was crucial for my research. It provides a way to see the appearance of the spacetime bundles we would expect to find when gauging a spacetime symmetry group. In addition it shows how to tie together the spacetime degrees of freedom with the gauge degrees of freedom. We might have expected that the spacetime connection is influenced by the connections in the underlying gauge bundle. Since the spacetime bundles are subbundles of the gauge bundles it seems natural that a connection on the spacetime bundles is a descendant of, or is induced by the gauge bundles of which it is a subset. As it turns out this is true. Recall from chapter 2 section 2.3 that a connection on a bundle is a splitting of the tangent space of a bundle into horizontal and vertical subspaces. Then proposition 6.2 of Kobayashi and Nomizu states that given $P(M, ISO(1,3))$ and a reduced subbundle $Q(M, SO(1,3))$ then there is a unique connection on P such that the horizontal subspaces of the connection on P are mapped into horizontal subspaces of $Q(M, SO(1,3))$ [42]. Indeed we see that the connection on the bundle Q is inherited by that of the total bundle P and so by constructing a principal bundle $P(M, ISO(1,3))$ we have succeeded in inducing a nontrivial connection on the spacetime bundles.

With our new interpretation of the subbundles (created in the process of ensuring global sections of E) we investigated the topology of the subbundles. The fiber bundle approach to gauge theory is based on local properties of the base space and fiber spaces. This means that typically the only information available is local information, making global properties of the space difficult to study. This is in contrast to the functional approach where one starts with globally defined quantities and looks for consistent local analogs. However in the case of the composite gauge theory of gravitation we have found global information about the topology is available due to the need for reducibility of the bundle $P(M, ISO(1,3))$. The requirement of a global section of the bundle E descends to a global section of the bundle $Q(M, SO(1,3))$, i.e. our spacetime frame bundle. Global sections of Q imply a trivial frame and tangent bundle and so we found that the Stiefel-Whitney, Euler and first fractional Pontryagin classes of the bundle Q to be trivial. Since the topology of a space is measured by that of its frame and tangent bundle, the set of topological restrictions that must be bypassed to construct composite bundles is consistent with the conclusion that spacetime must be a string manifold. The properties of string manifolds have been and still currently are under investigation. An established fact is that the condition of a manifold admitting a string structure is a basic requirement in order to remove anomalies in the heterotic superstring [69, 71–73]. The condition of global sections, natural to composite bundles, has revealed that in order for the composite bundle theory to be constructed, the base manifold must come equipped with the beginnings of anomaly cancellation mechanisms needed in string theory. This is the first connection, to my knowledge, between anomalies in string theory and gauge theories of gravitation. If after constructing our composite bundle we go further by requiring the string structure admit a connection we are led to the existence of a 3-form on M which facilitates the cancellation of global and perturbative anomalies in the heterotic superstring [71]. This shows a new and alternative reason for the need to accommodate extended degrees of freedom. In contrast to the typical rhetoric that these are required for a consistent quantum theory of gravity, this work gives classical motivation for the consideration of extended degrees of freedom.

5.2 Outlook

There are further topological hurdles we could seek to remove. The next step in the chain is called a fivebrane structure [63]. Although this requires a higher dimensional spacetime in order to be non-trivial there are reasons to believe that this condition is also related to anomalies in string theory [63]. My analysis was based on four-dimensional spacetime, however generalizations to higher dimensions are readily available. It would be interesting to see what effect increasing the number of dimensions of spacetime has on the characteristic classes of the frame subbundle of the composite theory. This is an area I am currently investigating.

In typical string theory accounts of spacetime structure, the requisite extra dimensions (beyond the four we observe) are assumed to be compact with a size small enough to forbid nonzero momentum modes at or below observed energy scales so that they are effectively invisible. Thus we would only experience the non-compact four dimensions of our spacetime. This means that overall spacetime is some type of product of a compact and non-compact manifold. While our discussion has focused on either one type or the other, it is clearly phenomenologically relevant to extend the discussion to a combination of the two. There are a variety of product structures proposed in string theory and we have begun an analysis of some of the simpler ones.

In addition to gravitation, there are of course the remaining forces of the Standard Model. While these are already understood in the standard fiber bundle context, it is not yet clear how they fit into the composite bundles required by gravitation. I am currently investigating the addition of internal gauge symmetries to the composite bundle formalism. Starting with a base space M we can first fiber over M a composite bundle as done in chapter 3. If we wish to add additional forces, such as electromagnetism, then we need an additional principal bundle fibered over the intermediate base space E [14]. This is in contrast to the usual case where the internal symmetry fibers are fibered over spacetime rather then a bundle related to spacetime. This leads us to suspect additional connections to the anomaly mechanisms in string theory. For instance the condition enforced in the Type I and heterotic strings is that the first Pontryagin classes of the spacetime manifold and the principal gauge bundle be equivalent. If we fiber over the base space E additional principal bundles related to internal gauge degrees of freedom we will change the topology of the bundle space E. It may well be the case that the ensuing condition of global sections of E leads to the availability of a 3-form H on M whose curvature is given by $\mathrm{d}H = TrR^2 - TrF^2$ where R is the curvature of the spacetime bundle and F is the curvature of the additional internal gauge bundle. Finding this condition in a consistent fiber bundle formalism of gravitation which includes additional internal forces would be a very interesting result providing further confirmation of the relationship between the composite gauge theory of gravity and string theory.

It should be stressed that the formalism in which we work does not a priori include extended degrees of freedom. The theory is formulated in terms of composite fiber bundles which are naturally motivated by the inclusion of translations as gauge symmetries. For typical fiber bundles the pullback of bundle quantities to the base space results in field-type degrees of freedom. In the present case it is not entirely clear what type of degrees of freedom are ultimately described by the pullbacks of composite bundle quantities to the overall base. We at least have shown some indication that extended degrees of freedom seem relevant to the construction. This is in contrast to approaches like higher gauge theory wherein the assumption of extended degrees of freedom are a starting point [74]. The higher gauge program is billed as a gauge theory of extended degrees of freedom. However the additional layers of categorification in higher gauge theory make it difficult to rewrite the theory in a manner similar to standard gauge theories. An interesting prospect is to look for a standard bundle analog for higher Yang-Mills theories. Since these theories require 2-bundles (a categorification of a bundle in which we genuinely have two bundles to work with) we are led to believe that the only typical bundles capable of being interpreted as a 2-bundle would be composite bundles. The composites come naturally with two bundles. However at least one complication in identifying the two approaches is that the only higher gauge theories of gravitation proposed feature flat connections as a direct result of the formalism [16, 17]. The requirement of flat connections requires that the spin connection in the composite bundle formulation vanish, which seems too strong a requirement. We are currently investigating further the relation of composite bundles and higher gauge theory.

We would be remiss without an account of the experimental relevance of gauge theories of gravity. Currently all of the viable gauge theories of gravity include torsion. There is however no consensus as to the effects spacetime torsion has on particle dynamics, and hence its observation. This is partly reflected in that there are several possible ways of including torsion as a propagating degree of freedom [75–77]. Carroll and Field provide a review of some of the possible spacetime torsion theories in [78]. With the large set of possible theories and consequent interactions, any experimental evidence of deviations from Einstein's theory of general relativity would give us a much needed trailhead in the forest of gauge gravity. Luckily there has been an increasing interest in the observation of torsion. A partial review of the tests of spacetime torsion up until 2002 is given in [79]. With the launch of gravity probe B to measure frame dragging there has arisen a discussion of the possibility of using it as a test of spacetime torsion [80–82]. The more compelling conclusion of this exchange seems to be that gravity probe B cannot place bounds on spacetime torsion. A recently proposed test using laser systems is described in [83] by March et al. Garcia de Andrade investigated the use of a spin-polarized cylinder as a terrestrial test of torsion effects on structured matter. He considered explicit data obtained by Ritter et al. during a spin-polarized torsion balance experiment [84], finding that although the spacetime torsion induced is incredibly small it is sufficient to suggest an experiment in the spirit of Einstein de Haas to test for spacetime torsion [85].

In closing we are currently entering a new era of observational cosmology sparked by the positive identification of gravitational waves [86]. Further upgrades to the LIGO system, along with other gravitational wave observatories, will eventually provide us with new and invaluable data to test the validity of gravitational theories. With the ability to peer back into the history of our universe beyond the era of recombination, gravitational wave astronomy may well provide echoes of the activity of gravity at an age when the entire universe was small enough that the topology of its extra dimensions was relevant. In that situation, having the correct theory of gravity in hand to accommodate any and all spacetime topologies would be essential. This thesis has been an attempt to get that ball rolling.

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APPENDIX A - CALCULATING CURVATURE VIA FORMS

To demonstrate the use of our machinery consider the Schwarzschild metric [39],

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{rc^2}\right) c\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2. \tag{A.1}$$

Equation A.1 is the spherical solution to the vacuum Einstein equations, Equation 2.19. We can re-express the metric in terms of dual frames as,

$$\mathrm{d}s^2 = -\hat{\theta}^0 \otimes \hat{\theta}^0 + \hat{\theta}^1 \otimes \hat{\theta}^1 + \hat{\theta}^2 \otimes \hat{\theta}^2 + \hat{\theta}^3 \otimes \hat{\theta}^3, \tag{A.2}$$

where we have written the dual frames as,

$$\hat{\theta}^0 = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} c\mathrm{d}t, \qquad \hat{\theta}^1 = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} \mathrm{d}r$$
$$\hat{\theta}^2 = r\mathrm{d}\theta \qquad \hat{\theta}^3 = r\sin\theta\mathrm{d}\phi. \tag{A.3}$$

In standard general relativity the torsion vanishes and the covariant derivative satisfies the metricity condition $\nabla_\alpha g_{\mu\nu} = 0 \Rightarrow \omega_{\alpha\beta} = -\omega_{\beta\alpha}$ [36]. With these two conditions and the Cartan Structure equations we can calculate the components of the curvature 2-form. The metricity condition implies that the diagonal components of the connection vanish and the vanishing torsion equation gives,

$$\mathrm{d}\hat{\theta}^i + {\omega^i}_k \wedge \hat{\theta}^k = 0. \tag{A.4}$$

To solve this equation consider a specific index value, say $i = 0$, then after summing over k each of the coefficients of the basis 2-forms must vanish separately,

$$\left(1 - \frac{2GM}{rc^2}\right)^{-1/2} \left(-\frac{GM}{r^2c^2} c\mathrm{d}t + {\omega^0}_1\right) \wedge \mathrm{d}r = 0 \quad \Rightarrow {\omega^0}_1 = \frac{GM}{r^2c^2} c\mathrm{d}t \tag{A.5}$$
$${\omega^0}_2 \wedge r\mathrm{d}\theta = 0 \quad \Rightarrow {\omega^0}_2 = 0 \tag{A.6}$$
$${\omega^0}_3 \wedge r\sin\theta\mathrm{d}\phi = 0 \quad \Rightarrow {\omega^0}_3 = 0 \tag{A.7}$$

Following a similar procedure the remaining components of the connection not related by index manipulation can be calculated to be [36],

$$\omega^2{}_1 = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} \mathrm{d}\theta, \quad \omega^3{}_1 = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} \sin\theta \mathrm{d}\phi, \quad \omega^3{}_2 = \cos\theta \mathrm{d}\phi. \tag{A.8}$$

Using the connection coefficients Equation 2.35a can be used to calculate the non-vanishing components of the curvature 2-form R. Consider $i = 0, j = 1$ in Equation 2.35a,

$$\mathrm{d}\omega^1{}_0 + \omega^1{}_0 \wedge \omega^0{}_0 + \omega^1{}_1 \wedge \omega^1{}_0 + \omega^1{}_2 \wedge \omega^2{}_0 + \omega^1{}_3 \wedge \omega^3{}_0 = R^1{}_0 \tag{A.9}$$

Since $\omega^0{}_0 = \omega^1{}_1 = \omega^2{}_0 = \omega^3{}_0 = 0$ Equation A.9 reduces to,

$$\mathrm{d}\left(-\frac{GM}{r^2c^2} c\mathrm{d}t\right) = -\frac{2GM}{r^3c^2}\mathrm{d}r \wedge c\mathrm{d}t = -\frac{2GM}{r^3c^2}\hat{\theta}^1 \wedge \hat{\theta}^0 = R^1{}_0. \tag{A.10}$$

In total the non-vanishing components of the curvature 2-form are [36],

$$\begin{aligned} R^2{}_1 &= \frac{GM}{r^3c^2}\hat{\theta}^1 \wedge \hat{\theta}^2 & R^2{}_3 &= \frac{2GM}{r^3c^2}\hat{\theta}^2 \wedge \hat{\theta}^3 & R^2{}_0 &= \frac{GM}{r^3c^2}\hat{\theta}^0 \wedge \hat{\theta}^2 \\ R^1{}_3 &= \frac{GM}{r^3c^2}\hat{\theta}^1 \wedge \hat{\theta}^3 & R^0{}_3 &= \frac{2GM}{r^3c^2}\hat{\theta}^0 \wedge \hat{\theta}^3 & R^1{}_0 &= -\frac{2GM}{r^3c^2}\hat{\theta}^1 \wedge \hat{\theta}^0. \end{aligned} \tag{A.11}$$

The curvature 2-form is related to the Riemann tensor in the Lorentz basis by $R^i{}_j = R^i{}_{jk\ell}\hat{\theta}^k \wedge \hat{\theta}^\ell$. Once we have calculated the curvature 2-form we can calculate the Riemann tensor and in turn rotate back to the coordinate basis,

$$R^\alpha{}_{\lambda\mu\nu} = e^\alpha{}_i e^j{}_\lambda e^k{}_\mu e^\ell{}_\nu R^i{}_{jk\ell}. \tag{A.12}$$

From the six non-vanishing curvature 2-form components the symmetries of the Riemann tensor generate the remaining 18 components. In addition, raising and lower indices in the coordinate basis involves summation over a metric of often complicated functions. While in the non-coordinate basis indices are raised and lowered using the Minkowski metric. We can see the benefit of using a non-coordinate basis, the expressions are simpler to calculate and there are fewer of them.

APPENDIX B - BUILDING A G/H BUNDLE

When discussing the composite bundles in chapter 3 I worked with elements of G/H as if they were just group elements. I also did not explain how we can identify the elements of the group G/H as translations. In this appendix I will go a little further into the details of the bundle $P(M, G/H)$.

If a group G has a subgroup $H \subset G$ we denote the left coset as $gH = \{gh|h \in H\}$ and the right coset $Hg = \{hg|h \in H\}$ [87]. Furthermore we call a subgroup H such that the left and right cosets are equal, $gH = Hg$, a normal subgroup [87]. The above statement often causes confusion as noted by [87]. The equation $gH = Hg$ is not saying that the subgroup H commutes with every element g of G, rather it is what I call "sloppy" multiplication. For every $g \in G$ there exists $h, h' \in H$ with $gh = h'g$. As an example relevant to the composite bundles consider $ISO(1, 2)$, the Poincaré group with two spatial dimensions and one timelike dimension. To see that the translations are the normal subgroup of the Poincaré group first fix $h = T_{x^1}$, a translation along the x^1-axis. Next choose an element of G, e.g. $g = R_{x^1x^2}(\theta)$, a rotation in the two spatial dimensions as shown in Figure B.1. Then we can find a $T' \in \mathbb{R}^{1,2}$

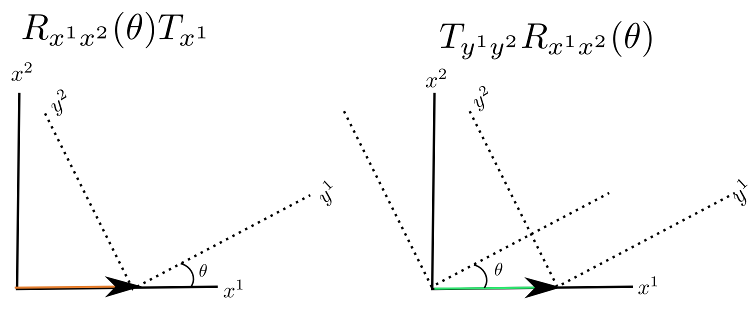

Figure B.1: An example of normal subgroups is shown. On the left we first translate by T_{x^1} then preform a rotation by θ. On the right we instead first rotate by an angle θ then we translate in both of the new axis, notice $T_{x^1} \neq T_{y^1y^2}$.

such that $R_{x^1x^2}T_{x^1} = T'R_{x^1x^2}$. Pictorially it is easy to see that the new translation is not simply a translation along the x^1-axis.

With normal subgroups defined we can define the quotient group of G by a normal subgroup H by $G/H = \{gH | g \in G\}$ [87]. Each element $g \in G$ defines a class $[g] \in G/H$ with the unit element defined by the unit of G as $[e] = \{eh | h \in H\}$. If we have two classes with representatives g, g' we can compose them as, $[g] * [g'] = [gg']$ for $*$ the product of G/H [36]. Then for $gh \in [g]$ and $g'h' \in [g']$, since we have H normal in G, there exists $h'' \in H$ such that $g'h'' = hg'$ and we have,

$$[g] * [g'] \to ghg'h' = gg'h''h' \in [gg']. \tag{B.1}$$

It is easy to see that unit element keeps us within the same class,

$$[g] * [e] \to gheh' = ghh' \in [g]. \tag{B.2}$$

The composite bundle required a bundle $P(M, G/H)$. With the details of normal subgroups fleshed out, the setup of the bundle $P(M, G/H, G)$ follows closely to chapter 2. What we should notice is that in this case the group H is the Lorentz group. The easiest place to see the importance of the Lorentz group as the closed subgroup for the decomposition of $P(M, G)$ is in building the right action. As in typical bundles $P(M, G/H, G) \cong M \times G/H$, so locally we have for some $u \in P$ the trivialization $u = (x, [g])$. The structure group of the bundle is G and so we can define a right action R_g on a point $u \in P$ as $R_g u = ug$. As is typical, the right action moves us through the fiber space and for any elements $u, v \in P$ within the same fiber space, u and v are related by a right action $v = ub$ for $b \in G$. In our trivialization we have $(x, [g']) = (x, [g])b$ and so the group elements are related as $[g'] = [g]b$. Expanding this relationship by choosing $gh \in [g]$ we find,

$$[g]b \to ghb = bg''h' = g'h' \in [g'], \text{ for } g' = bg''. \tag{B.3}$$

For the right action to produce another element in the fiber space we need $gh \in [g]$ to act as a translation. In this case then we can use the normal subgroup relation to rewrite $ghb = bg''h'$ as I have done in Equation B.3 and identify $bg'' = g'$ since $b, g'' \in G$. Constructing the right action of G on the bundle we have seen the formal equivalence of G/H for $G = ISO(p, q)$ and $H = SO(p, q)$ with the translations $R^{p,q}$. We can see this by manipulation of the equations

I have displayed in the beginning of this appendix, however it is nice to see it be determined as a necessity of the right action of G moving us to another class in G/H. With this formal equivalence we can treat elements of G/H as simply translations as I have done in chapter 3 and chapter 4.

APPENDIX C - FINITELY GENERATED HOMOLOGY GROUPS

In this appendix we will briefly discuss the construction of compactly supported cohomology and the proof that $\dim H_c^r < \infty$. Since the manifold we will be creating our composite bundle over is non-compact we will instead work with the open sets which cover M. To begin we first define a relative chain group. Suppose we have some topological space X and A a subspace of X. Then we have a subset of the chain group $C_n(A) \subset C_n(X)$. The boundary operator $\partial : C_n \to C_{n-1}$ acting on $C_n(A)$ is the restriction of ∂ to A and so gives back C_{n-1}. The relative chain group is then given by the quotient $C_n(X, A) = C_n(X)/C_n(A)$. The relative homology group is defined as before as a quotient of the kernel and image of the boundary map ∂. The dual system is the relative cohomology $H^r(X, A; G)$ (for a finite abelian group G) and is defined as in Equation 4.3 using relative groups instead [64]. An element in $H^r(X, A; G)$ is a class which vanishes on a set $A \subset X$. We can immediately see the usefulness of this cohomology when the set A is replaced by its complement $X - A$. The cohomology $H^r(X, X - A; G)$ has elements which vanish outside of the set A, exactly the condition we need to define cohomology with compact support.

The final piece we need to build the cohomology with compact support for the composite gauge theory of gravity is a set of compact subsets of the base manifold M. Since we put a Minkowski fiber metric on the tangent space we can know the space we are working with is paracompact and is normal[7]. With these conditions, each locally finite cover $\mathcal{U}$ has a refinement $\mathcal{U}' = \{U' | \text{for } m \in \mathcal{U}', \cup\{U' \in \mathcal{U}' | m \in \mathcal{U}' \subset U\}\}$ [62]. A space which meets this criterion is called strongly paracompact [62]. Furthermore if each open cover has a countable subcover then the space is called finally compact and the countable subcover is called a shrinking [62]. With this information we can then find a sequence of compact sets,

$$k_1 \subset \mathring{K}_2 \subset K_2 \subset \cdots \subset \mathring{K}_j \subset K_j, \tag{C.1}$$

[7]Every metric space is normal [40].

such that $\cup_{j\in\mathbb{N}}\mathring{K}_j = M$ where $\mathring{K}$ denotes the interior of the set K [62]. If we now do as above and create a cochain group $C^n(M, M-K_j, \mathbb{Z})$, then the compactly supported cochain group is $C_c^n = \cup_{j\in\mathbb{Z}} C^n(M, M-K_j, \mathbb{Z})$ [64]. Additionally we notice that the coboundary operator moves us within the relative cochain group since if $f \in C_c^n(M;\mathbb{Z})$ vanishes on $M - K_j$ then so does δf and so we can write the cohomology with compact support just as in Equation 4.3,

$$H_c^r(M;\mathbb{Z}) = \frac{Z_c^r(M;\mathbb{Z})}{B_c^r(M;\mathbb{Z})}, \tag{C.2}$$

only now we work with the compactly supported cocycle and coboundary groups. This allows us to build a cohomology theory for the composite bundles even though we do not require the base space to be compact. Even with the compactly supported cohomology we still require the same conditions to hold in order to build a composite theory of gravitation, vanishing Stiefel-Whitney classes, vanishing Euler class and vanishing first Pontryagin class. And we are left showing that $H^4(M;\mathbb{Z})$ can be decomposed into only a free part. To do so we would like to use the same lemma from [64], although there will be some differences. The reason Hatcher's lemma requires compact connected spacetimes is so that the homology groups are finitely generated. Thus to use the same reasoning we must have that the compactly supported homology groups we have created are finitely generated. If we can show this we are free to use his lemma.

We have that provided that a space M has a finite good cover then $\dim H_c^r(M;\mathbb{R}) < \infty$ [88]. To see this we will follow the exposition of Bott and Tu and begin with the Mayer-Vertoris sequence for two open sets U, V such that $U \cup V = M$ [64],

$$\cdots \xrightarrow{\delta_{r-1}} H_c^r(M;\mathbb{R}) \xrightarrow{\alpha_r} H_c^r(U;\mathbb{R}) \oplus H_c^r(V;\mathbb{R}) \xrightarrow{\beta_r} H_c^r(U\cap V;\mathbb{R}) \xrightarrow{\delta_r} H_c^{r+1}(M;\mathbb{R}) \cdots, \tag{C.3}$$

where α_r, β_r and δ_r are linear maps induced by the inclusion maps,

$$i_1 : U \to M \qquad i_2 : U \to M \tag{C.4a}$$
$$j_1 : U\cap V \to U \qquad j_2 : U \cap V \to V. \tag{C.4b}$$

We will use Equation C.1 since the open sets $\mathring{K}_j$ form a finite good cover. A sequence such as Equation 4.1 is called exact if $\mathrm{im}(d_k) = \ker(d_{k+1})$, the Mayer-Vertoris sequence is a called a long exact sequence. We can now use induction to show the result, suppose Equation C.1 contained only one set. Then this set would have to be M itself so M is contractible to a point. It is well known that the homology and cohomology groups of a point-set are all finite, in fact they are all zero except the for the first group [64, 68]. Then assume that for a cover which has $k-1$ sets the cohomology groups $H^r_c(M,\mathbb{R})$ are all finite. Take M to have a finite good cover with k sets $\{W_1, \cdots, W_k\}$. If we let $U = \cup_{j=1}^{k-1} W_j$ and $V = W_k$ then $U \cup V$ admits a finite good cover given by $\{W_1 \cap W_k, \cdots W_{k-1} \cap W_k\}$. Since U, V and $U \cap V$ admit covers with $k-1$ sets their cohomologies are finite dimensional by our assumption. Finally using Equation C.3 we have,

$$\cdots \to H_c^{r-1}(U \cap V; \mathbb{R}) \xrightarrow{\delta_{r-1}} H_c^r(M; \mathbb{R}) \xrightarrow{\alpha_r} H_c^r(U; \mathbb{R}) \oplus H_c^r(V; \mathbb{R}) \to \cdots, \tag{C.5}$$

and since we know that $\dim(H^r_c(U;\mathbb{R}) \oplus H^r_c(V;\mathbb{R})) < \infty$ by assumption we have,

$$\dim(\mathrm{im}(\alpha_r)) \leq \dim(H^r(U; \mathbb{R}) \oplus H^r(V; \mathbb{R})) < \infty. \tag{C.6}$$

The Mayer-Vertoris sequence is exact so $\ker(\alpha_r) = \mathrm{im}(\delta_{r-1})$ and δ_{r-1} is a linear map starting from a finite dimensional space so $\dim(\mathrm{im}(\delta_{r-1})) < \infty$ meaning $\dim(\ker(\alpha_r < \infty))$. Finally since the dimension of the kernel and image of α_r are finite we have $\dim H^r_c(M;\mathbb{R}) < \infty$.