GEOMETRIC PHASE OF ENTANGLEMENT FOR NON-ADIABATIC, TWO-PHOTON OPTICAL VORTEX EVOLUTION IN A DIELECTRIC TRAP

by

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ABSTRACT

Despite the attention it has commanded, the resources it has garnered, and the promising theoretical and technological milestones it has passed, quantum computing is in its infancy. A wide range of paradigms are currently being explored, motivated by the need to strike a balance between controllable interactions within the system and isolation from interactions with the environment. Within each of these, strategies that exploit geometric holonomies to store and process information seem especially promising, because they offer a measure of protection against environmental degradation. This tactic has yet to receive much attention for optical quantum computing however. In fact, there is a dearth of information associated with the geometric phase accumulation of entangled photon states—a linchpin for carrying out universal holonomic quantum logic.

A theoretical investigation has therefore been carried out that elucidates the relationship between photon entanglement and geometric phase. While intended to be helpful to quantum computing, the focus is on basic science. A particularly simple setting is chosen in which photons propagate along an axis while being laterally confined by a harmonic trap. Treating the propagation axis as time, linear combinations of two-dimensional modes are used to construct optical vortices that orbit about the trap center. Two-photon entangled states are created in terms of these, and the geometric phase is calculated over a single, shared orbital period. This setting makes it possible to scrutinize the relationship between the degree of entanglement and the geometric phase accumulated.

Attention is focused on the Geometric Phase of Entanglement (GPE), defined as the geometric phase above and beyond that accumulated by each single-photon state independently. General requirements were identified for which a GPE is supported. A set of especially tractable problems were then explored to show how the GPE is influenced by vortex tilt, orbit radius, and degree of entanglement.

The insights obtained comprise a foundation for subsequent experimental investigations. Towards that end, particular attention was also given to explain how the requisite two-photon states can be produced and to identify two physical settings for which the harmonic trap can be realized: a sequence of cylindrical lenses; and multi-mode dielectric fibers.
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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos \mu$</td>
<td>A vector orthogonal to one of the single-photon states that can be added to it to obtain the other single-photon state</td>
</tr>
<tr>
<td>$\phi_{SoM}$</td>
<td>Azimuthal angle on SoM</td>
</tr>
<tr>
<td>$w_0, w_p, w_s, w_i$</td>
<td>Beam waists</td>
</tr>
<tr>
<td>$c_{p_1, p_2}$</td>
<td>Coefficients of entangled state used for type-I SPDC</td>
</tr>
<tr>
<td>$a_{p,l}, b_{p,l}, c_{p,l}, d_{p,l}$</td>
<td>Coefficients of stationary modes for A, B, C, or D state</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Dielectric character</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Dimensionless parameter of dielectric character</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Eigenvalue</td>
</tr>
<tr>
<td>$\vec{E}$</td>
<td>Electric field</td>
</tr>
<tr>
<td>$</td>
<td>\Psi\rangle$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Entanglement magnitude</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Entanglement relative phase</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Final time</td>
</tr>
<tr>
<td>$\Phi_{Geom}$</td>
<td>Geometric phase</td>
</tr>
<tr>
<td>$\Phi_{GPE}$</td>
<td>Geometric phase of Entanglement</td>
</tr>
<tr>
<td>$\Phi_{GPE,\Psi}$</td>
<td>Geometric phase of Entanglement for state $\Psi$</td>
</tr>
<tr>
<td>$\Phi_{Geom,A}, \Phi_{Geom,B}, \Phi_{Geom,C}, \Phi_{Geom,D}$</td>
<td>Geometric phase of single photon state A, B, C, or D</td>
</tr>
<tr>
<td>$\Phi_{Geom,\Psi}$</td>
<td>Geometric phase of state $\Psi$</td>
</tr>
<tr>
<td>$p, l$</td>
<td>Indices of Laguirre-Gaussian modes</td>
</tr>
<tr>
<td>$x_0$</td>
<td>Initial vortex radius</td>
</tr>
<tr>
<td>$x_{0,A}, x_{0,B}, x_{0,C}, x_{0,D}$</td>
<td>Initial vortex radius of states A, B, C, or, D</td>
</tr>
<tr>
<td>$</td>
<td>p, l\rangle$</td>
</tr>
<tr>
<td>$x, y, \rho$</td>
<td>Length across beam, measured from beam center</td>
</tr>
<tr>
<td>$z$</td>
<td>Length along beam</td>
</tr>
</tbody>
</table>
Magnitude of inner product of A and B states \[ \sin^2 \mu \]
Normalizing constant \[ N \]
Phase of inner product of A and B states \[ \nu \]
Polar angle on SoM \[ \theta_{\text{SoM}} \]
Position on fiber bundle for state on SoM \[ \chi_{\text{SoM}} \]
Pump beam used for SPDC \[ \Upsilon(\rho, \phi) \]
Single-photon states \[ |A\rangle, |B\rangle, |C\rangle, |D\rangle \]
Speed of light in vacuum \[ c_0 \]
Stationary states \[ |\psi_{p,l}(z)\rangle \]
Vortex tilt angle \[ \xi \]
Vortex tilt angle of states A, B, C, or, D \[ \xi_A, \xi_B, \xi_C, \xi_D \]
Vortex tilt magnitude \[ \theta \]
Vortex tilt magnitude of states A, B, C, or, D \[ \theta_A, \theta_B, \theta_C, \theta_D \]
Wave angular frequency \[ \omega \]
Wavenumber \[ k \]
LIST OF ABBREVIATIONS

Geometric Phase .................................................. Geom
Geometric Phase of Entanglement .................................. GPE
Laguerre-Gaussian ..................................................... LG
Multi-Mode Fiber ...................................................... MMF
Orbital Angular Momentum .......................................... OAM
Sphere of Modes ...................................................... SoM
Spontaneous parametric downconversion ............................. SPDC
ACKNOWLEDGMENTS

I would like to thank and acknowledge my family, fellow graduate students, and my advisor Professor Mark Lusk for their support.
CHAPTER 1
INTRODUCTION

This thesis analyzes the geometric phase of entangled, and single-photon, optical vortices in a harmonic dielectric trap; both for use in linear optical quantum computing, and to advance the scientific communities’ understanding of geometric phase and entanglement. Quantum computing is a field newly branching out in many different directions. Trapped ion, superconducting qubits, neutral atom, and optical quantum computing to name only a few. All of these methods run into numerous issues, not the least of which is noise and interaction from the environment. Numerous methods have been proposed to circumvent this, and one promising avenue is the use of geometric holonomies. A holonomy is simply a description of how curvature on closed path changes a vector. Their geometric nature allows for the filtering out of noise associated with the dynamics of the system, and therefore error and decoherence. However, the use of geometric phase and holonomies has been neglected in the optical fields of quantum computing. One of the goals of this thesis is to rectify this, as well as contributing in general to enhance understanding of geometric phase in relation to entanglement.

Over the centuries, geometric phase has evolved from a source of mystery, to a subject of study, to a foundation for new technology. Whether the setting is the classical motion of a mechanical object across a curved surface, the propagation of light in the curved four-dimensional space of general relativity, or the progression of a quantum state that traces out a trajectory in a curved parameter space, the basic mathematics are the same. The essence of geometric phase can be cleanly abstracted within the context of differential geometry as a scalar measure of holonomy associated with cyclic, parallel transport[1]. In other words, geometric phase is simply a change in the phase of a vector due to the curvature of the space it moves through. A simple example of this is the parallel transport of a vector around a sphere, as shown in Figure 1.1. A path around the equator, a geodesic, returns the vector to its original orientation, while a lap around any other latitude line will result in the vector being oriented differently. The angular discrepancy is the geometric phase.

A tangible example of geometric phase accumulation is the precession of a Foucault pendulum that amounts to a free-swinging motion in the absence of any torque imposed by the supporting structure. If placed at the equator, the pendulum will not precess and the plane of oscillation will remain fixed in the reference frame of the earth. If placed at the North pole, the pendulum will appear to precess to any observer in the earth’s frame, slowly rotating at exactly the rate the earth rotates. For an observer on the earth, after one full rotation, the pendulum’s plane of oscillation will be back to where it began. An
Figure 1.1 Vector transport around three different latitude lines. A cycle around the equator, a geodesic, preserves vector orientation. Vector motion at high latitudes exhibits an orientation that is closer to being fixed with respect to the space, not the surface of the sphere. Lastly, vector transport at mid-latitude rotations clearly does not return the vector to its original orientation after one loop.

An inertial observer not rotating with the earth will observe the plane of oscillation as fixed. Any other position on the earth may leave the pendulum oscillating in a different plane than the one it started with after a full rotation of the earth. This is a holonomy in which the vector representing the pendulum’s oscillation plane is changed by transport on a closed loop. A full plot of the precession rate versus the latitude of the pendulum can be seen in Figure 1.2. This is an example of a nontrivial geometric phase of which the classical version is also called the Hannay angle [2]. The physics underlying pendulum precession is also responsible for the Coriolis effect [3, 4] that drives the rotation of hurricanes [5] and affects the direction of wind and sea currents. In this sense, geometric phase plays an integral role in our daily lives.

A plane wave of light propagating through vacuum will exhibit an oscillatory dynamic phase at a given position that is the product of its temporal frequency and time, $\omega t$. However, if the underlying medium causes the polarization of the light to change, then a geometric phase (the Pancharatnam phase) is also accumulated [6]. This can be observed by measuring the interference between the evolving beam and a reference beam. If the evolution of the polarization is cyclic, the geometric phase is equal to one-half the solid angle enclosed by the trajectory when plotted on a Poincaré sphere.
Figure 1.2 A plot of the precession per day and precession rate, versus the latitude of the pendulum on the earth.

There are other ways that geometric phase can be generated in optical settings. For instance, the transit of light through optical elements (such as cylindrical lenses) can be used to construct a closed trajectory on a Sphere of Modes (SoM) that is analogous to the Poincaré sphere of polarizations [7, 8]. Geometric phase generated this way is the result of exchange of angular momentum between the light and the optical system.

The use of optical elements to generate geometric phase is a specific way of engineering inhomogeneous and anisotropic dielectric media to produce such an effect, and this underlies the optical Magnus effect [9, 10]. A more familiar mechanical counterpart is exploited to make balls follow a curved path as they spin relative to the air. In that case, rotation causes a difference in pressures on different sides of the object, due to a difference in the way the fluid is deflected. The ball experiences an unbalanced force perpendicular to its direction of motion and subsequently drifts to one side.

This thesis is particularly concerned with how the Magnus effect generates geometric phase in optical vortex modes. An optical vortex is a point in a beam of light where the field phase becomes undefined, wrapping around the point in a vortex shaped spiral. The charge or spin of vortex modes is directly linked to the orbital angular momentum (OAM) of light and, as mentioned previously, that is the motive force for generating geometric phase. The generation of geometric phase by optical vortices passing through a cylindrical lens has been and continues to be investigated both theoretically and experimentally [8, 11, 12]. However, while the geometric phase generated by optical vortices moving through a lens has been investigated, the basic physics is quite different than beams that are guided through dielectric...
guides [13, 14]. This thesis newly investigates geometric phase accumulation for vortices in a harmonic
dielectric trap.

A harmonic dielectric trap is a setting in which the guide is characterized by a lateral dielectric profile
that is harmonic, but only perpendicular to the axis of propagation of the beam. In other word the
dielectric constant changes parabolically laterally away from the center of the beam. This produces
behavior similar to a particle trapped in a two-dimensional harmonic potential well. This trapping
redirects light and so plays a role analogous to that of lenses to generate geometric phase in previous
investigations. In particular, while optical vortices in free space move in straight lines, optical vortices
within a harmonic trap instead orbit the center somewhat like particles in a harmonic potential[15]. Such
behavior motivates one of the unanswered questions addressed in this thesis:

*Can an optical trap cause geometric phase to be accumulated, and how is this accumulation linked to
composition of underlying propagation modes?*

A second important element of this investigation is the spontaneous parametric downconversion
(SPDC), used to generate quantum-entangled pairs of photons[16]. This has enabled some of the most
fundamental experiments in quantum mechanics, including the first experimental validations of Bell’s
inequality[17]. SPDC continues to be central to much research on the frontiers of quantum information
such as quantum cryptography[18]. Type-I SPDC will be utilized in this thesis to generate pairs of
entangled vortex states. This leads to the second primary question of this thesis:

*Is there a geometric phase associated with the entanglement of two optical vortex states that are orbiting
within a trap?* Geometric phase accumulation for entangled systems has, until now, been somewhat
neglected by the linear optical quantum computing community, and this thesis hopes to contribute to
rectifying that through this question.

The answers to these questions serve to further some specific areas of the field of holonomic optical
quantum computing (Quantum computing using geometric phase generated on closed paths in Hilbert
space). There are many ways to encode information in a photon for the purposes of quantum computation.
The polarization of the photon, the mode composition of the photon(temporal-spatial modes), or even
which of multiple paths the photons takes[19, 20]. In addition to encoding information, there must also be
a way to apply quantum logic gates to the encoded information. Nonlinear optics provide one possible
toolbox for this, however they are very difficult to scale to large sizes. Possibly a more promising scale-able
source of gates is linear optics.

Two possible tools in the toolbox of linear optical quantum computing for the near future are the
Orbital Angular Momentum (OAM) Approach[21, 22] and the Multimode Fiber (MMF)
Approach[21, 23, 24]. The former uses light in free space acted upon by lenses, mirrors, and prisms while
the latter uses light within a harmonic dielectric fiber. This latter approach is exactly the setting investigated in this thesis in our first question.

Entanglement will be an important aspect of any quantum computer, no matter what form it takes. After all you cannot do much interesting quantum or classical computation with a single bit, and entanglement is a part of most two qudit or more gates. This is the most clear application of the results of our second question. The case-by-case analysis of this question is done specifically in the MMF setting in this thesis. However, the formulas and the properties of those formulas derived should also apply to the OAM setting.

Why is quantum computing using geometric phase, rather than total phase, of particular interest? One of the primary reason is that Holonomic quantum computing may provide robustness against noise. The advantages of adiabatic geometric phase, for use in noisy holonomic quantum computing, and non-adiabatic geometric phase for use in low coherence-time holonomic quantum computing have been written about extensively in recent years[25]. The properties of geometric phase may have advantages in overcoming the weaknesses of each of these two forms of quantum computing, noise in the former and decoherence in the latter.

The answers to our two questions, newly obtained in this thesis, will allow for the prediction or engineering of geometric phase and GPE for use in future holonomic linear optical quantum computing experiments using the OAM or MMF approaches. All of this lays a foundation for how the work in this thesis is motivated by, and contributes to, the field of optical quantum computing. However, this thesis is also generally motivated by pushing forward the basic science, and academic understanding of, geometric phase and entanglement. Entanglement has been one of the most discussed and argued about phenomena within quantum mechanics in the last century. From Einstein’s famous quote that "I am at all events convinced that He does not play dice.", to concepts of ”spooky action at a distance”, and more recent loophole free tests of Bell’s inequality[26] entanglement has captured the imagination of generations of physicists as an example of the difficult to understand behaviors of quantum mechanics. Work has begun in recent decades to investigate the ties between entanglement and geometric phase, and it has been found that the former does affect the latter[27]. However, this mine into deeper understanding of quantum mechanics has only begun to be explored. This thesis seeks to provide an additional avenue into this investigation through the analysis of GPE in the setting of linear optics.

1.1 Approach

This thesis seeks to answer, through derivation of formulas and analyses of the resulting plots, the questions of how geometric phase is generated by vortices in a dielectric trap and how entanglement affects
geometric phase. Figure 1.3 shows an infographic of the flow of ideas in this thesis. It can be seen how the subjects of geometric phase and optical vortices combine to allow investigation into the geometric phase of optical vortices in a dielectric trap. Derivation of general formulas for this geometric phase branch into investigations on how the parameters of specific states affect the geometric phase. Type-I SPDC is then used to add entanglement into the picture.

An introduction to geometric phase is given in Chapter 2 to develop a general conceptual framework. The specific relations used in subsequent chapters are presented and their use explained. Previous work on geometric phase and entanglement is also presented.

In Chapter 3, the generation of optical vortices in dielectric traps is explained. Optical vortices and their behaviors are covered in general as well as specifically in a harmonic dielectric trap. The assumptions made about the optical setting are stated and, using these, the Laguerre-Gaussian modes that comprise optical vortices are derived.

In Chapter 4, the basic expression for geometric phase is applied for combinations of optically trapped vortex modes to quantify geometric phase. Towards this end, previous work on the geometric phase of untrapped vortices is covered to provide a foundation for the beginning of original work in this thesis—i.e. the generation of geometric phase using trapped optical vortices. A general expression for the geometric phase of any superposition of single-photon modes is found. Using this, it is shown that a single-photon state is required to have at least two modes that have nondegenerate eigenvalues for the geometric phase to be nontrivial. The specific cases of the geometric phase for single-vortex and double-vortex single-photon states are then covered. The manner in which geometric phase depends on the radius and tilt of these vortices is explored, as well as how the single-vortex and double-vortex states differ, especially at the origin.

Chapter 5 focuses on how entanglement affects geometric phase. It is shown how previous work to engineer specific states using Type-I SPDC can be used to generate entangled states of the forms sought[28]. Equations for the geometric phase of a class of entangled state are then derived. General conditions for these to be nontrivial are discussed. While some of the conditions are similar to the single-photon cases, there are many fascinating circumstances where an otherwise trivial phase can have a nontrivial contribution.

The rest of the chapter focuses on entangled states of a more specific form. The Geometric Phase of Entanglement (GPE) is then formulated to quantify the way in which entanglement affects geometric phase. It is shown that this is the result of the overlap of the single-photon states. This is further demonstrated with specific example cases.

Finally the thesis is concluded. In the conclusion the results are covered in summary with some additional conjecture. The limitations on the work in this thesis are discussed, though many of these could
be lifted with more time in the future. Additionally, what future work could be built upon this thesis is analyzed, some of which is already in the works.

Figure 1.3 A map of the subjects in this thesis. Green boxes show previous work of others, while yellow boxes are subjects newly explored in this thesis. The largest boxes are the overarching subjects of each chapter, approximately placed vertically in the order they appear in this thesis. The smaller boxes show the subjects that fall within each of these subjects. Arrows are drawn to show how the subjects relate to each other, and more specifically, how subjects flow into each other to create other subjects.
CHAPTER 2
AN INTRODUCTION TO GEOMETRIC PHASE

This chapter is a simple introduction to the concept of the Berry phase and geometric phase in general. It begins with an introduction to the concept of parallel transport and how it may bend vectors transported on a manifold. It is then shown how the Berry phase is an analog to this parallel transport for vectors in Hilbert spaces. The Bloch sphere is discussed as one manifold that these vectors may be transported on, but how the manifolds in this thesis are more complicated is also discussed. Finally the much more general non-cyclic non-adiabatic Aharonov-Anandan formulation used in this thesis is then introduced.

2.1 Parallel Transport

The geometric phase in the context of differential geometry is a result of the more general result of moving vectors through curved spaces using parallel transport. It is a general mathematical phenomenon, not specific to any one area of science. If a vector \( \mathbf{v} \) is transported along a curve \( C \) that is defined parametrically by \( \lambda \) and coordinates \( x^\alpha(\lambda) \), then this transport is defined by the parallel transport differential equation

\[
\frac{Dv^\alpha}{d\lambda} = \frac{dv^\alpha}{d\lambda} + \Gamma^\alpha_{bc}v^b dx^c = 0.
\]  

Here \( D \) is the covariant derivative, which is related to the non-covariant derivative by the addition of the second term that contains \( \Gamma^\alpha_{bc} \), the metric connection. This is a tensor generated from the metric of the space, which in turn is generated by how the space curves relative to a Cartesian space. In flat Cartesian space, this term vanishes, and the covariant derivative is identical to the regular derivative. One way to think about this conceptually is that any smooth manifold looks somewhat flat over very small increments. If a vector is moved—as if in Cartesian space—over a very small increment, then in the limit as that increment becomes zero the vector’s evolution through curved space is governed by the differential equation Eq. (2.1). In flat Cartesian space this movement would not change the vector; however, it is found that this does occur in spaces with curvature. This change in the vector due to transport on curvature can be clearly seen in Figure 1.1. After returning to the same spot through a cyclic evolution, some of the vectors are misaligned with their initial position. This misalignment is a geometric phase. It is due to the curvature of the space itself and it will change depending on the path, but not how fast the path is taken. This is what causes the Foucault pendulum to precess and generates the Berry phase from the curvature of manifolds in Hilbert space.
As an aside, a geodesic is a path that transports its own tangent vectors while keeping them parallel to the tangent. It is a generalization of the concept of a straight line to curved space. It is also the path that any object in free fall, without some force other than gravity acting on it, takes in general relativity. An example of a geodesic is a great circle on a sphere, such as the equator parallel transport shown in Figure 1.1. In general, such paths in parameter space do not generate geometric phase as a result of the curvature on a path but may still generate it topologically.

2.2 Berry Phase as Parallel Transport

The quantum dynamics of interest in this investigation deals with the evolution of complex vectors in abstract parameter spaces, rather than the familiar space we live in and experience first-hand, and so is harder to visualize. However, the concept is still the same. We may follow the same processes as those found in Vanderbilt’s *Berry Phases in Electronic Structure Theory: Electric Polarization, Orbital Magnetization and Topological Insulators* [1] to see how geometric phase developed due to movement in physical space and movement in parameter spaces are related. Let us examine a set of N vectors around a loop \(|u_0\) to \(|u_N\) such that \(|u_N\) is in the same location in Hilbert space as \(|u_0\). That is, \(|u_0\) is the same state as \(|u_N\) but may have a phase factor difference. If the state is evolved around this loop adiabatically such that there is no dynamic phase then it is found that the geometric phase, in this case specifically the Berry phase, is the phase accumulated around the loop

\[
\Phi = -\text{Im} \ln[\langle u_0 | u_1 \rangle \langle u_1 | u_2 \rangle \ldots \langle u_{N-1} | u_0 \rangle].
\] (2.2)

Note that the negative sign is merely a convention, as phases are often written in the form \(e^{-i\phi}\). Let us now consider a different set of vectors, a gauge transform of the previous set

\[
|\tilde{u}_j\rangle = e^{-i\beta_j} |u_j\rangle
\] (2.3)

such that \(\beta_j\) is real. Eq. (2.2) produces the same result. This shows that it is gauge invariant. This is a property that is necessary for any physically measurable quantity: as if it is not gauge invariant, it cannot have any single value. Let us now impose some additional restrictions on this transformation. We will require that \(\beta_0 = 0\) so that \(|\tilde{u}_0\rangle = |u_0\rangle\). We shall also require that each vector is in phase with the subsequent vector, their inner product being real and positive. That is

\[
0 = -\text{Im} \ln[\langle \tilde{u}_j | \tilde{u}_{j+1} \rangle].
\] (2.4)
This makes the gauge transform a parallel transport gauge. In the infinitesimal limit, when \( N \) is allowed to go to infinity, this can be seen as the same as the parallel transport differential equation (2.1). In the same way that the covariant derivative must be 0, the change in the vector must be zero in the infinitesimal limit. Even without the infinitesimal limit this restriction reduces equation (2.2) to

\[
\Phi = -\text{Im} \ln[(\tilde{u}_{N-1}|\tilde{u}_0)].
\] (2.5)

We can view this inner product as the same as the dot product between the starting and ending vectors in (Figure 1.1). In both cases, the disjoint between the starting and ending vectors is a geometric phase. In the end, this is the result of the same general mathematical principles that cause Foucault’s pendulum to process, or gravity to pull two objects together.

### 2.3 The Bloch Sphere and Larger Hilbert Spaces

In the Foucault pendulum and parallel transport examples considered so far, geometric phase results from parallel transport on the surface of a sphere in three-dimensional space. These spheres are two-dimensional submanifolds embedded in a three-dimensional space. These are good examples, as people are accustomed to thinking about three-dimensional spaces and lower; however, in this thesis and in physics in general, we must consider much higher dimensional spaces and submanifolds.

In quantum mechanics, a good example to consider next is the Bloch sphere, of which the Poincaré sphere is a more specific example when the Bloch sphere is used to represent polarizations. It exhibits some of the same behavior seen with parallel transport on a sphere in three-dimensional space. However, Parallel transport on the Bloch sphere is not quite the same. To begin with, it is embedded in a four-dimensional space. It is a two orthonormal-state system, a two-dimensional Hilbert space. However, each Hilbert space dimension is parameterized by a complex number that is equivalent to two real numbers. As such, each Hilbert space dimension is actually two-dimensional. The Bloch sphere is restricted to a two-dimensional submanifold of this space. The first restriction that leads to this is that the state must be normalized. It is also restricted to only considering the relative phase of the two states that define the Hilbert space, not the overall phase. In other words, the state’s position on the fiber of the fiber bundle(a space that locally a product space, such as the Bloch Sphere) is eliminated as a parameter. These choices restrict the parameters such that states are now only in a two-dimensional submanifold of the four-dimensional space. This submanifold is similar to a sphere; however, it is important to note that unlike in previous examples, the Bloch sphere is actually a double mapping of a sphere. To fully traverse azimuthally, we must actually wrap around it twice to fully circle the space, unlike a regular sphere, which can be fully traversed by being wrapped once. This means that one full rotation around a Bloch sphere will always produce a phase
difference of \( \pi \), even on a geodesic, but a rotation around a sphere in real space on a geodesic will not. This is an example of a topological phase, not a phase from curvature, as it is due to the topological properties of the space, not its curvature. These properties are observed in the Sphere of Modes (SoM), a Bloch sphere representing a superposition of two Laguerre-Gaussian modes, explored in section 4.3.1.

In this thesis, spaces and submanifolds of much higher dimension than the Bloch sphere will be explored. For instance, the first single-vortex single-photon state, defined by Eq. (4.1), is a superposition of three orthonormal states, placing it in a six-dimensional Hilbert space. It is then restricted to a submanifold by both normalization and setting certain parameters that determine the physical properties of the vortex. This thesis’ two-vortex single-photon state, Eq. (4.22), is made up of four orthonormal modes, meaning it is in a submanifold of an eight-dimensional space. It is restricted in the same way with the same number of parameters as the single-vortex state, so its submanifold has the same number of dimensions. Nevertheless, it is still a different shaped manifold and a different dimensional space, so its behavior can be expected to be very different. Any two-particle state made from these, such as an entangled state, will both add together the dimensions of each particle’s Hilbert space and its submanifold, and add additional dimensions for the entanglement parameters. Therefore, this thesis deals with much larger and more complicated spaces than just the Bloch sphere. A general characterization of these spaces is much more difficult and time-consuming than the spaces of the two-sphere or Bloch sphere. However, in this thesis, some specific evolutions in these spaces can be explored.

2.4 Berry Phase and Aharonov-Anandan Phase

The Berry Phase described above is specifically a phase in adiabatic cyclic evolution of quantum states that results from the geometry of the parameter space of the Hamiltonian[29]. However, others have generalized Berry’s work to non-adiabatic and non-cyclic evolutions, such as the formulation by Aharonov and Anandan[30]. This thesis uses this latter formulation, and while it generally examines cyclic evolutions, this work could also be extended to non-cyclic cases. The following formula from Aharonov and Anandan is the one used for this thesis:

\[
\Phi_{\text{Geom}}[\Gamma] = \text{Arg} \langle \Psi(0)|\Psi(\tau) \rangle + i \int_0^\tau dt \langle \Psi(t)|d_t \Psi(t) \rangle
\]  

(2.6)

where \( \Gamma \) is some path in Hilbert space. For the purposes of this thesis, the time component is always equivalent to the \( z \) coordinate. The reason for this will become evident in the discussion of optical vortices in section 3.3. As such, the formula for the geometric phase will be written as
$$\Phi_{\text{Geom}}[\Gamma] = \text{Arg} \left\langle \Psi(0) | \Psi(\tau) \right\rangle + i \int_0^\tau dz \, d\, \langle \Psi(z) | d \Psi(z) \rangle.$$  

(2.7)

This formula can be used to calculate the geometric phase of non-cyclic evolutions, though this is beyond the scope of this thesis. In the case that the above formula is applied to an evolution in which the final state is not the initial state, then the resulting geometric phase from the formula is found to equal the geometric phase if the state had returned to its initial position in Hilbert space via a geodesic. As mentioned previously, a geodesic should not generate any additional geometric phase due to the curvature of the space, and so this should be zero if there is no topological phase gained by this path.

2.5 Geometric Phase and Entanglement

The relation between geometric phase and entanglement in the specific case of entangled spin pairs has been investigated and found to be significant[27]. A rich entanglement dependence was found in the case of spins precessing azimuthally on the Poincaré sphere, and work on entanglement and geometric phase has shown further developments over the last few decades. All this shows that the relationship between geometric phase and entanglement is an area worth investigating.

This thesis will discuss the relation between geometric phase and entanglement in the context of a new relevant quantity, the geometric phase of entanglement (GPE). The exact definition of this quantity is shown in section 5.2.1.1, but in summary, it is the difference between the geometric phase generated by the overall two-photon entangled state of a given form and the geometric phase generated by the two single-photon states used to construct it.
CHAPTER 3
OPTICAL SETTING: HARMONIC DIELECTRIC TRAP MODES AND OPTICAL VORTICES

This chapter introduces the optical setting used throughout the investigation. As mentioned in Chapter 1, the work of this thesis is conducted in the context of a harmonic dielectric trap. Any harmonic dielectric profile perpendicular to beam propagation direction will qualify as a harmonic dielectric trap. Some examples of this would be multi-mode fibers, engineered dielectric rods, or a laser mediated dielectric fluid. The former is of particular interest to the MMF approach to linear optical quantum computing as harmonic dielectric multi-mode fibers may be utilized in this approach[21, 23, 24]. However, the derivations within this chapter are applicable to any harmonic dielectric setting. As will be shown, light in these settings propagates as Laguerre-Gaussian modes. These combine to form a Gaussian beam with vortices embedded in them. Throughout this chapter details of dielectric traps are provided, and the relation between trap strength and Laguerre-Gaussian modes is explained. The concept of optical vortices is then introduced, and some of their properties are demonstrated.

3.1 Derivation of Laguerre-Gaussian Modes in a Dielectric Trap

In previous sections, it was mentioned that a dielectric trap is necessary for the cyclic evolution of vortices. The exact nature of the trap and how this leads to cyclic evolution is derived below. This follows unpublished research notes[31], which in turn follow previously published work in this area[32–35]. Beginning with Maxwell’s equations in a charge free, non-magnetic, non-conducting, inhomogeneous but linear dielectric medium,

$$\nabla \cdot (\varepsilon \vec{E}) = 0$$  \hspace{1cm} (3.1)

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} + \frac{\varepsilon}{c_0^2} \partial_t \vec{E} = 0.$$  \hspace{1cm} (3.2)

The dielectric character of a harmonic trap, $\varepsilon$, may be expressed as a radial function of position, where $\Omega$ is a non-dimensional parameter,

$$\varepsilon = \varepsilon_p (1 - \Omega^2) \rho^2.$$  \hspace{1cm} (3.3)

Where $\rho$ is the distance from the center of the trap, and $\varepsilon_p$ is a constant. Examples of such a harmonic dielectric include fiber optics, engineered dielectric rods, or a laser-mediated dielectric fluid. The formula above does constrain the maximum radius of the medium to
\[ R_{\text{max}} = \frac{1}{\Omega} \sqrt{\frac{\epsilon_p - 1}{\epsilon_p}}. \]  \hspace{1cm} (3.4)

Separation of variables and linear polarization are then assumed:

\[ \vec{E}(\vec{r}, t) = \vec{E}(\vec{r}) e^{-i\omega t} \]  \hspace{1cm} (3.5)

\[ \vec{E}(\vec{r}) = \vec{e}\psi(\vec{r}) e^{ikz}, \]  \hspace{1cm} (3.6)

where \( \psi \) is a complex scalar function of position. The wavenumber is defined by

\[ k = \frac{\omega \sqrt{\epsilon_p}}{c_0}. \]  \hspace{1cm} (3.7)

Where \( \omega \) is the angular frequency and \( c_0 \) is the speed of light in vacuum. The second of these assumptions is required for the paraxial approximation setting used throughout this thesis.

By invoking the paraxial approximation, that \( |\partial_{zz}\psi| << |k\partial_z\psi| \), the following equation of motion is obtained:

\[ i\partial_z\psi = \left( -\frac{1}{2k} \nabla_\perp^2 + \frac{1}{2} k\Omega^2 \rho^2 \right)\psi + \frac{1}{2k} \left( \frac{2\Omega^2(1 + \Omega^2 \rho^2)}{(\epsilon/\epsilon_p)^2} \right). \]  \hspace{1cm} (3.8)

The latter term may be disregarded when

\[ k >> \Omega, \]  \hspace{1cm} (3.9)

which is equivalent to

\[ R_{\text{max}} >> \lambda. \]  \hspace{1cm} (3.10)

This is an easy condition to meet, as many fiber cores are on the order of 50\( \mu \)m while the wavelengths used in them are on the order of 0.5\( \mu \)m [31]. The equation of motion is now

\[ i\partial_z\psi = \left( -\frac{1}{2k} \nabla_\perp^2 + \frac{1}{2} k\Omega^2 \rho^2 \right)\psi. \]  \hspace{1cm} (3.11)

### 3.2 Non-Dimensionalization of the Equation of Motion

Now that the paraxial equation of motion has been derived, it may be non-dimensionalized using
\[ z \to \frac{1}{\Omega} z, \rho \to \frac{1}{\sqrt{k\Omega}}\rho. \]  

(3.12)

This converts the paraxial equation into

\[ i\partial_z \psi = (-\frac{1}{2} \nabla^2_\perp + \frac{1}{2} \rho^2)\psi. \]  

(3.13)

The following form may be assumed for the solutions to the paraxial equation:

\[ \psi_{p,l}(\rho, \phi, z) = u_{p,l}(\rho, \phi)e^{-i\varepsilon_{p,l}z}. \]  

(3.14)

This gives the following eigenvalue problem:

\[ \varepsilon_{p,l} u_{p,l}(\rho, \phi) = (-\frac{1}{2} \nabla^2_\perp + \frac{1}{2} \rho^2)u_{p,l}(\rho, \phi). \]  

(3.15)

It has solutions of the form

\[ u_{p,l}(\rho, \phi) = e^{-\rho^2/2 - i\phi\rho |l|/\sqrt{\pi}} \sqrt{\frac{p!}{(p + |l|)!}} L_{p}^{l}(|\rho|^2) \]  

(3.16)

where \( L \) are the Laguerre polynomials. These are the orthonormal modes in the \( \{r, \phi\} \) basis—i.e.

\[ \langle \rho, \phi | p, l \rangle = u_{p,l}(\rho, \phi). \]  

(3.17)

Stationary states are obtained as the product of these modes with their stationary phase rotation:

\[ |\psi_{p,l}(z)\rangle = |p, l\rangle e^{-i\varepsilon_{p,l}z}. \]  

(3.18)

Here the eigenvalues are given by

\[ \varepsilon_{p,l} = 1 + |l| + 2p. \]  

(3.19)

### 3.3 Optical Vortices

Optical vortices are defined as a singular point, in a slice of a beam of light, where the field magnitude goes to zero and the field phase is undefined. Around this location, the phase undergoes at least one full cycle. This results in a shape similar to a corkscrew when plotted, and LG modes are a good example of such character. The number of times the phase goes through a full cycle of \( 2\pi \) defines the topological charge of the vortex, a charge of one for one corkscrew and a charge of two for two corkscrews. The direction of rotation defines the sign of the charge, which can be positive or negative. Examples of this can
be seen in Figure 3.1. There the field magnitude of a vortex at the center of a slice of a Gaussian beam can be seen. In the subsequent two figures, the phases of plus-one and minus-one vortices are plotted as functions of position. It is important that the center of the corkscrew, where the phase becomes undefined, is also the center of the vortex where the field magnitude must become zero. It must become zero, as otherwise the effects of a field with undefined phase could be measured, which does not make physical sense. In the last figure a charge plus-two vortex can be seen, with a pair of corkscrews. This is equivalent to two charge plus-one vortices placed at the same position. A vortex can also be described with zero charge, meaning that there is no point of discontinuity in the phase; but then, there is essentially no vortex. The spinning of the phase, that is the topological charge of the vortex, actually corresponds to the orbital angular momentum the light carries. This can be observed when the light is absorbed by a particle and transfers this angular momentum to it. This is different from the spin angular momentum of the light. As an analogy, one may think of the spin angular momentum of a particle as analogous to the angular momentum of a planet’s rotation, and its orbital angular momentum as the angular momentum of that planet’s orbit around its star.

There may be multiple vortices in a single photon field, and in fact vortex modes are a valid alternate basis for describing a field. Figure 3.2 (a) shows the field magnitude of a pair of vortices offset from the center of a Gaussian beam. Part (b) shows the phase of these two vortices, both plus-one vortices; note how the corkscrew shape is still present but must merge together between them. If one were to force these vortices together into the same point, these corkscrews would merge to become the double corkscrew shape seen in Figure 3.1 (d).

As light moves at a constant rate in the z direction, the beam direction, the time coordinate and z coordinate may be treated as the same. With this in mind, the field may be treated as a two-dimensional field evolving in the z-time direction. Vortices may also be “tilted,” distorting the vortex field as if the slice of field was rotated by some angle. This elongates the vortex into a more elliptical shape. An example of this can be seen in Figure 3.3. It can be seen in (a) that the depression of the vortex is elongated along the direction of the tilt, and in (b) that the phase corkscrew is steeper there and less steep everywhere else. Tilting the vortex, along with the charge and number of vortices, can change the path the vortex takes in real space along with the path the state takes in parameter space[15]. This of course may have effects on the geometric phase, as investigated in this thesis.

Though optical vortices can exist in light in many settings, we will be examining them specifically in a harmonic dielectric within this thesis. This setting has many advantages to the analysis of vortices. To begin with, this makes Laguerre-Gaussian modes the eigenbasis, allowing us to more easily find the geometric phase of states written in this basis. Laguerre-Gaussian modes are a natural choice for the vortex
basis, as it is easy to construct vortices out of a very small number of them. For instance, the \( p = 0, l = 1 \) mode on its own is a charge one vortex at the origin embedded in a Gaussian beam. It can then be moved from its origin by simply adding the \( p = 0, l = 0 \) mode. The vortices produced from these modes will be covered in more detail in Chapter 4. This dielectric trap setting also ensures that vortices will always take a cyclic path. An optical vortex in free space may shoot off in a straight line, never returning to its original state\[36\]. However, in a harmonic dielectric, it acts much like a particle does in a harmonic trap, orbiting the center of the path in a circle or ellipse. The tilt of the vortex actually determines the ellipticity of the orbit, with the orbit’s shape mimicking the shape of the vortex. The direction of the orbital ellipse also corresponds to the direction the vortex is tilted in, with the ellipse pointing along the direction of the tilt. This holds true for single-vortex states in the harmonic dielectric trap setting. Double-vortex states of the type we examine in this paper still take a cyclic path in this setting that is similarly governed by their tilt,
Figure 3.2 Shown in these figures are: (a) The magnitude of a pair of vortices placed at the sides of a Gaussian beam. (b) The phase of the two charge one vortices.

Figure 3.3 (a) Magnitude of a vortex tilted by $2\pi/5$ placed at the center of a Gaussian beam; (b) Phase of this tilted vortex. Note the steepness of the change in phase along the direction of the tilt.

but it is not exactly an ellipse; only very similar. Outside of this setting, and with single-photon states with multiple vortices of differing charges, the behavior of their trajectories may be much more complex.

Beams of laser light that contain vortices are generated by using holographic methods with spatial light modulators. Any initial vortex shape or number of vortices required can be created in this way.

Spontaneous parametric downconversion may be used to create entangled vortex states of the type seen in this thesis. This thesis is written in the setting of collinear, Type-I spontaneous parametric downconversion. This is covered in detail in section 5.1, where it is shown how[28, 37] the desired states can be engineered using a combination of vortices placed in the pump beam and by extracting signal and idler beams of differing waists.
CHAPTER 4
SINGLE-PHOTON VORTEX STATES AND THEIR GEOMETRIC PHASE IN A HARMONIC TRAP

As a first step towards elucidating the relationship between geometric phase and entanglement, the geometric phase of single-photon states must first be examined. Previous work on this front will be reviewed, such as the geometric phase of vortices through lenses. Setting the stage for analyses of their phase, the development of a general single-photon framework that can be used to examine the evolution of arbitrarily tilted vortices within a harmonic trap is shown. This facilitates our new analysis of the general properties of geometric phase in such a trapped setting.

Specific cases are then considered. Both single-vortex and double-vortex (but still single-photon) states are investigated as functions of initial position, path, and vortex tilt. In each case, geometric phase is calculated. These specific cases offer insights that, taken together, allow important conclusions to be drawn about the general nature of single-photon vortices that are harmonically confined.

4.1 Previous Work

As previously mentioned, work has already been conducted on the geometric phase of optical vortices moving through lenses [8, 11, 12]. There it was shown that lenses may be used to cause a vortex state to develop geometric phase. For a vortex state composed of only two modes, a Sphere of Modes (SoM) may be constructed representing the magnitude and relative phase of those modes. This SoM is analogous to the Bloch sphere discussed in section 2.3, and shares the properties discussed there. The lenses will then allow the evolution of the vortex state in circular arcs on that SoM. These paths develop a geometric phase equal to one-half of the enclosed solid angle, as has been previously shown for other evolutions on the SoM. It will be shown in section 4.3.1 that when there are only two Laguerre-Gaussian modes in a harmonic dielectric trap then the state may again be represented as a point on a SoM of modes. In this case, the evolution of the state in the harmonic trap is again a circular arc, and again, it is shown that the geometric phase is one-half of the solid angle enclosed by the path. However, when more than two modes are required to represent the state, it may no longer be represented on a SoM and the geometric phase no longer has this simple relation.

It is also important to note that the dynamics of the vortices in this thesis, the ways in which their parameters change their trajectory, have already been investigated[15]. It is by building on this work that we continue the investigation into vortices in the fresh context of the geometric phase of vortices in a harmonic trap.
4.2 Geometric Phase of an Arbitrary Linear Combination of LG Modes

First, consider the most general case that can be constructed from the stationary states of a harmonically trapped paraxial beam. This begins with an explanation of the construction of this state which is subsequently used to find a general formula for its geometric phase.

4.2.1 Evolving States as Linear Combinations of Eigenmodes

General single-photon states can be constructed as a normalized sum of the orthonormal Laguerre-Gaussian eigenmodes as follows:

\[
|A(z)\rangle = \sum_{p,l} a_{p,l} |\psi_{p,l}(z)\rangle.
\] (4.1)

Here the eigenmodes are

\[
|\psi_{p,l}(z)\rangle = |p, l\rangle e^{-i\varepsilon_{p,l} z},
\] (4.2)

and the eigenenergies, which generate stationary phase character, are

\[
\varepsilon_{p,l} = 1 + |l| + 2p.
\] (4.3)

The derivative of these stationary states is simply

\[
|\partial_z A(z)\rangle = -i \sum_{p,l} \varepsilon_{p,l} a_{p,l} |\psi_{p,l}(z)\rangle.
\] (4.4)

Since the eigenmodes are orthonormal, the inner product between two such linear combinations of evolving states is

\[
\langle A(z_1)|B(z_2)\rangle = \sum_{m,n} a_{m,n}^* b_{m,n} e^{i\varepsilon_{m,n}(z_1 - z_2)}.
\] (4.5)

For the same value of axial position (time), this reduces to

\[
\langle A(z)|B(z)\rangle = \sum_{m,n} a_{m,n}^* b_{m,n}.
\] (4.6)

Because it will be useful in calculating geometric phase, the inner product of one evolving state with the derivative of another such state is

\[
\langle A(z)|\partial_z B(z)\rangle = -i \sum_{m,n} \varepsilon_{m,n} a_{m,n}^* b_{m,n}.
\] (4.7)
The value of this form quickly becomes clear when we attempt to calculate the second term of the geometric phase, Eq. (2.7). As these inner products have no z-dependence, the integral becomes easily solvable once the inner product has been evaluated.

4.2.2 Geometric Phase of Orbiting Trapped Vortices

In the expression given for geometric phase, Eq. (2.7), the term Arg $\langle \Psi(0)|\Psi(\tau) \rangle$ is the total phase difference of the state after evolution[30]. Since the eigenenergies $\varepsilon_{m,n}$ are always integers in our non-dimensional setting as per Eq. (4.3), all cyclic processes have a period that is an integer fraction of $2\pi$. The total phase is therefore always zero.

$$\text{Arg } \langle A(0)|A(\tau) \rangle = 0. \quad (4.8)$$

The term, $i \int_0^\tau dz \langle \Psi(z)|d_z\Psi(z) \rangle$, in the geometric phase expression of Eq. (2.7) is the negative of the dynamic phase. Throughout this thesis, reference will be made to both the dynamic and total phase, with their difference being the geometric phase. Our Laguerre-Gaussian mode evolution formulation allows us to evaluate the geometric phase for all single-photon states in this setting as simply

$$i \int_0^\tau dz \langle A(z)|d_zA(z) \rangle = \tau \sum_{m,n} \varepsilon_{m,n} a_{m,n}^* a_{m,n}. \quad (4.9)$$

Our overall geometric phase of this state for a cyclic evolution of $\tau = 2\pi$ is then

$$\Phi_{\text{Geom},A} = 2\pi \varepsilon_{A}. \quad (4.10)$$

It is important to note that in the special case for which state $|A\rangle$ is composed of a combination of degenerate states (i.e., states with the same eigenvalue) the geometric phase is zero. This is because all eigenvalues are integers, and the geometric phase expression reduces to

$$\Phi_{\text{Geom},A} = 2\pi \varepsilon_{A}. \quad (4.11)$$

As is clear from Eq. (4.3), many of the stationary states share the same eigenvalue. For a single-photon state to have a nontrivial geometric phase after one full cycle, it must be a superposition of two or more stationary states with non-degenerate eigenvalues.

4.3 Single Charge One Vortex Geometric Phase

A single-vortex state of this form may be defined by just three modes with carefully constructed coefficients. Being made up of three orthonormal basis states places the state in SU(3)[38]. These
coefficients are defined by the variables $x_0$, $\theta$, and $\xi$, defining the initial radius, ellipticity, and angle of the vortex respectively. We will also refer to these variables as the starting radius, polar angle, and azimuthal angle respectively. The latter two are referred to as such because they determine the degree and radial direction of the tilt of our vortices, as shown in section 3.3.

\[ |A(z)\rangle = a_{0,0} |\psi_{0,0}(z)\rangle + a_{0,1} |\psi_{0,1}(z)\rangle + a_{0,-1} |\psi_{0,-1}(z)\rangle \]  

(4.12)

\[ a_{0,0} = -\frac{\sqrt{2}x_0(\cos \xi - i \cos \theta \sin \xi)}{\sqrt{\frac{1}{2}(1 + x_0^2)(3 + \cos(2\theta)) + x_0^2 \cos(2\xi) \sin^2 \theta}} \]  

(4.13)

\[ a_{0,1} = \frac{\sqrt{2}e^{i\xi} \sin^2(\frac{\theta}{2})}{\sqrt{\frac{1}{2}(1 + x_0^2)(3 + \cos(2\theta)) + x_0^2 \cos(2\xi) \sin^2 \theta}} \]  

(4.14)

\[ a_{0,-1} = \frac{e^{-i\xi}(1 + \cos \theta)}{\sqrt{2}\sqrt{\frac{1}{2}(1 + x_0^2)(3 + \cos(2\theta)) + x_0^2 \cos(2\xi) \sin^2 \theta}} \]  

(4.15)

Such vortices will trace out an elliptical path, as shown in Figure 4.1. Only in the special case of a rectilinear vortex (i.e. one with no tilt) is the orbit is circular. The field magnitude and phase are shown, along with the trajectory the vortex takes. In Figure 4.4 a similar vortex is seen, but now with tilt. Note the changes not only in the shape of the vortex but also in its trajectory.

Over one orbit cycle, these states generate a geometric phase of

\[ \Phi_{Geom,A} = 2\pi \sum_{n=-1}^{1} \varepsilon_{0,n} a_{0,n} a_{0,n} = 2\pi + \frac{2\pi}{1 + x_0^2} \frac{2\cos(2\xi) \sin^2 \theta}{3 + \cos(2\theta)}. \]  

(4.16)

This geometric phase is next examined for several special cases.

4.3.1 Case Examination: A Single Rectilinear Vortex

Consider the simplest case of a charge one rectilinear vortex (i.e. a vortex with no tilt). An example of the field magnitude and phase of such a vortex is shown in Figure 4.1. The general single-vortex state is reduced to that of a rectilinear vortex by setting $\theta$ to 0 or $\pi$, depending on the sign of the charge desired. In either case, the geometric phase simplifies to

\[ \Phi_{Geom,A} = 2\pi + \frac{2\pi}{1 + x_0^2}. \]  

(4.17)

A plot of this phase is given in Figure 4.2. When interpreting these plots, it is important to keep in mind that any phase is only meaningful modulo $2\pi$. Any value at which the phase is an integer multiple of
Figure 4.1 The field magnitude (a) and field phase (b) of a rectilinear vortex, along with the trajectory and direction shown as a black line and arrow. $x_0$ has been set to one. $x$ and $y$ have been non-dimensionalized with $1/\sqrt{k\Omega}$ where $k$ is the wavenumber and $\Omega$ is a non-dimensional parameter of the dielectric character as in Eq. (3.12).

$2\pi$ is equivalent to saying that the geometric phase is zero. For instance, the geometric phase is trivial (zero) when the orbit radius of the vortex is zero. As the orbit radius is increased, the geometric phase becomes finite, but then is gradually returned to zero as the orbit radius goes to infinity. While such trends are common in the dynamics considered in this thesis, it is not universal. For instance, the double-vortex state examined next can have a nontrivial geometric phase at zero initial radius, while the single-vortex state examined cannot. One way of seeing this is that single-vortex states are composed of only modes with degenerate eigenvalues for zero and infinite orbit radii, while the double-vortex state with zero initial radius is still composed of modes with non-degenerate eigenvalues.

In a simple single-vortex rectilinear case, the geometric phase can also be interpreted in terms of a solid angle subtended by the evolving state on a SoM. This is because the rectilinear vortex is the normalized superposition of only two stationary states. This amounts to three real-valued degrees of freedom that can be mapped onto a SoM. To see this, choose $\theta = 0$. Then the evolving state simplifies to

$$|A(z)\rangle = e^{-i(z-\xi+i\pi)} \left( \frac{x_0}{\sqrt{1+x_0^2}} |0,0\rangle + \frac{1}{\sqrt{1+x_0^2}} |0,-1\rangle e^{-i(\pi)} \right). \quad (4.18)$$

This can be mapped onto a SoM by setting
Figure 4.2 The geometric phase of a single rectilinear vortex as a function of orbit radius. The geometric phase is trivial at both zero and as it trends towards infinity. It can be seen that this phase goes through a single cycle of $2\pi$. The horizontal axis, initial vortex radius, has been non-dimensionalized with $1/\sqrt{k\Omega}$ where $k$ is the wavenumber and $\Omega$ is a non-dimensional parameter of the dielectric character as in Eq. (3.12)

$$\theta_{\text{SoM}} = 2 \arccos \left( \frac{x_0}{\sqrt{1 + x_0^2}} \right) = 2 \arcsin \left( \frac{1}{\sqrt{1 + x_0^2}} \right)$$

$$\phi_{\text{SoM}} = z - \pi$$

$$\chi_{\text{SoM}} = -z - \pi + \xi. \quad (4.19)$$

Here $\theta_{\text{SoM}}$ and $\phi_{\text{SoM}}$ are our polar and azimuthal angles on the SoM and $\chi_{\text{SoM}}$ is our position on the fiber of the fiber bundle. This allows us to show the state in a standard form for states on the SoM:

$$|A\rangle = e^{i\chi_{\text{SoM}} \left( \cos \left( \frac{\theta_{\text{SoM}}}{2} \right) |0,0\rangle + \sin \left( \frac{\theta_{\text{SoM}}}{2} \right) |0,-1\rangle e^{-i\phi_{\text{SoM}}}. \right )} \quad (4.20)$$

Any superposition of two orthonormal states can be mapped onto a Bloch sphere, in this case a sphere of modes, by redefining their coefficients in terms of their relative magnitude, relative phase, and overall phase. This is all we have done in this instance.

In terms of these alternate coordinates, the geometric phase, Eq. (4.17), is now

$$\Phi_{G,A} = 2\pi + 2\pi \sin^2 \left( \frac{\theta_{\text{SoM}}}{2} \right). \quad (4.21)$$

This is $2\pi$ plus one-half the solid angle enclosed by the path on the SoM. In Figure 4.3 an image of a SoM, along with various paths and the area they enclose is shown. The geometric phase is equal to one-half of the solid angle enclosed on the SoM for a large class of evolutions in the special unitary group SU(2). This group can be used to model many quantum-mechanical phenomena, such as the evolution of spins[39].
Figure 4.3 The area enclosed by three different paths on the SoM composed of the $p = l = 0$ and $p = 0, l = -1$ modes. The solid angle enclosed is $4\pi$ multiplied by the fraction of the sphere area enclosed. Therefore the geometric phase is $2\pi$ multiplied by the fraction of the area enclosed.

Even more generally, the Berry flux, a quantity that can be used to calculate the Berry phase based on the area enclosed by the path, is equal to $2\pi C$ on any closed two-dimensional manifold, where $C$ is the Chern number[1].

Most of the states of interest in this thesis cannot be assigned to the SoM because they are superpositions of more than two stationary states. It is always possible, of course, to characterize even the most complex entangled geometric phases in Chapter 5 as tracing out trajectories in a higher-dimensional space, where a higher-dimensional Berry flux counterpart to solid angles may also lend itself to a geometric interpretation.

### 4.3.2 Case Examination: A Single Tilted Vortex

Building on the analysis of rectilinear vortices, the geometric phase of tilted vortices can now be considered. Illustrative examples of such vortices are shown in Figure 4.4 for the case of nonzero polar angle, $\theta$. The effect of having both nonzero polar angle, $\theta$, and nonzero azimuthal angle, $\xi$, is shown in section Figure 4.5. In these figures the relationship between these parameters, vortex tilt, and vortex trajectory that was discussed in 3.3 can be seen. This behavior, and the dynamics of vortices in general have already been shown in detail by others[15].

As a starting point, consider what happens to the geometric phase as a function of radius as the vortex is tilted from the vertical, $\theta \neq 0$ or $\pi$, while keeping an azimuthal orientation of $\xi = 0$. The result is shown in Figure 4.6, where it is clear that the concavity of the geometric phase profile increases with greater tilt. However, the effect here is not particularly large.
Figure 4.4 Field magnitude (a) and field phase (b) of a vortex tilted by $\theta = \pi/4$ along the vertical, along with its trajectory and direction shown as a black line and arrow. The other parameters have been set to $x_0 = 1$ and $\xi = z = 0$. The trajectory is elongated into an ellipse just as the vortex is. $x$ and $y$ have been
non-dimensionalized with $1/\sqrt{k\Omega}$ where $k$ is the wavenumber and $\Omega$ is a non-dimensional parameter of the
dielectric character as in Eq. (3.12).

A much larger effect can be seen if $\xi$ is also introduced. Figure 4.7 shows the geometric phase for a
vortex at a given starting radius plotted against $\xi$ for multiple values of $\theta$. The effect of $\xi$ on geometric
phase increases with tilt and is especially pronounced as the polar tilt approaches $\pi/2$. It should be kept in
mind that, at the point of this extreme polar tilt, the Laguerre-Gaussian vortex state has collapsed to just
one of its two Hermite-Gaussian constituents. The effect of $\xi$ is maximized or minimized when it is a
multiple of $\pi/2$. It becomes trivial (zero) when $\theta = \pi/2$ and $\xi$ is an odd multiple of $\pi/2$. In Figure 4.8 it
can be observed that the vortex has simply become a line that cuts the Gaussian in half. It is now simply
the TEM$_{1,0}$ Hermite-Gaussian mode. The magnitude of the field also does not evolve in this circumstance,
though if $\xi$ is changed such that it no longer cuts through the center, then the field magnitude does evolve
somewhat, and the geometric phase may become nontrivial. It can also be observed in Figure 4.9 that,
while the geometric phase is always trivial at infinite radius, at non-infinite but very large radius the right
choice of $\theta$ and $\xi$ can always, though in a smaller and smaller region, make the geometric phase
significantly nontrivial. This, however, is not true as the vortex approaches the origin; then, the geometric
phase always approaches a trivial value no matter the choice for $\theta$ and $\xi$. This helps to demonstrate that
the size and shape of the vortex trajectory is related to the geometric phase.
Figure 4.5 Field magnitude (a) and field phase (b) of a vortex tilted by \( \theta = \pi/4 \) along an angle corresponding to \( \xi = \pi/4 \), along with its trajectory and direction shown as a black line and arrow. The other parameters have been set to \( x_0 = z = 0 \). The trajectory direction also changes with the tilt direction. 

**4.4 The Geometric Phase of a Pair of Charge one Vortices**

Although we have been considering the geometric phase of a single orbiting vortex for each photon, it is also helpful to differentiate between photon and vortex by briefly examining the single-photon geometric phase of a state in which there are two orbiting vortices. This setting will be re-visited as part of the entangled state in appendix A.2.

It is possible to construct a state with two orbiting vortices by adding together four LG modes:

\[
|D(z)\rangle = d_{0,0} |\psi_{0,0}(z)\rangle + d_{0,2} |\psi_{0,2}(z)\rangle + d_{0,-2} |\psi_{0,-2}(z)\rangle + d_{1,0} |\psi_{1,0}(z)\rangle .
\] (4.22)

Here

\[
d_{0,0} = \frac{\sin^2 \theta + 2x_0^2 (i \cos \xi + \cos \theta \sin^2 \xi)}{\sqrt{8 \cos^8 \left(\frac{\pi}{2}\right) + 8 \sin^8 \left(\frac{\pi}{2}\right) + \sin^4 \theta + (\sin^2 \theta - 2x_0^2 (\cos \xi + i \cos \theta \sin \xi)^2)(\sin^2 \theta + 2x_0^2 (i \cos \xi + \cos \theta \sin \xi)^2)}},
\] (4.23)

\[
d_{0,2} = \frac{2 \sqrt{2} e^{2i \xi} \sin^2 \left(\frac{\pi}{2}\right)}{\sqrt{8 \cos^8 \left(\frac{\pi}{2}\right) + 8 \sin^8 \left(\frac{\pi}{2}\right) + \sin^4 \theta + (\sin^2 \theta - 2x_0^2 (\cos \xi + i \cos \theta \sin \xi)^2)(\sin^2 \theta + 2x_0^2 (i \cos \xi + \cos \theta \sin \xi)^2)}},
\] (4.24)
Figure 4.6 The geometric phase of a single tilted vortex vs its radius, at various tilts, and with $\xi = 0$. Similar behavior to the rectilinear case can be observed, but now the concavity of the slope increases with tilt. The horizontal axis, initial vortex radius, has been non-dimensionalized with $1/\sqrt{k\Omega}$ where $k$ is the wavenumber and $\Omega$ is a non-dimensional parameter of the dielectric character as in Eq. (3.12).

\[ d_{0,-2} = \frac{2\sqrt{2}e^{-2i\xi}\cos^2(\frac{\theta}{2})}{\sqrt{8\cos^8(\frac{\theta}{2}) + 8\sin^8(\frac{\theta}{2}) + \sin^4\theta + (\sin^2\theta - 2x_0^2(\cos\xi + i\cos\theta\sin\xi)^2)(\sin^2\theta + 2x_0^2(i\cos\xi + \cos\theta\sin\xi)^2)}}. \]

(4.25)

and

\[ d_{1,0} = \frac{-\sin^2\theta}{\sqrt{8\cos^8(\frac{\theta}{2}) + 8\sin^8(\frac{\theta}{2}) + \sin^4\theta + (\sin^2\theta - 2x_0^2(\cos\xi + i\cos\theta\sin\xi)^2)(\sin^2\theta + 2x_0^2(i\cos\xi + \cos\theta\sin\xi)^2)}}. \]

(4.26)

The vortices constructed are diametrically opposed and have the same charge and tilt. Just as in the single-vortex case, the parameters $x_0$, $\theta$, and $\xi$ determine the starting distance from the origin, the tilt of the vortices and trajectories, and the direction of that tilt respectively. In this construction both vortices share the same tilt, charge, and path but are on opposite sides of the trap center. The path these vortices follow is altered in the same way by the tilt as the single-vortex case. The path is still close to an ellipse shape; however, it is not exactly an ellipse. An example of the magnitude and phase of the field of this state for rectilinear vortices has already been shown in section 3.3; however, a more complicated example can be seen in Figure 4.10. This is an example of a two-vortex state with some tilt. Note how the two vortices mirror each other.

After one cycle, this state generates a geometric phase of
The geometric phase of a single tilted vortex versus $\xi$ for various values of $\theta$. $x_0$ has also been set to 1. The azimuthal influence is maximized for $\xi = \theta = \pi/2$, where it forces the geometric phase to become trivial.

$$\Phi_{Geom,D} = 2\pi \left( \sum_{m=0, n=-2}^{1, 2} \varepsilon_{m,n} d_{m,n}^* d_{m,n} = 2\pi (d_{0,0}^* d_{0,0} + 3d_{0,2}^* d_{0,2} + 3d_{0,-2}^* d_{0,-2} + 3d_{1,0}^* d_{1,0}). \right) $$

(4.27)

As for a single vortex, there are only three parameters in this single-photon state. Their influence is examined by considering several special cases.

### 4.4.1 Case Examination: Vortex Instability

Figure 4.11 gives a plot of the geometric phase as a function of half the initial separation of the vortices. At zero tilt, the behavior is similar to the single-photon state, with trivial phases at both zero and infinite orbit radii, although here the rate of change is greater and it goes through multiple cycles of $2\pi$ as the orbit size is increased.

The two-vortex dynamics become qualitatively distinct when tilt is introduced. In particular, it is now possible for the system to accumulate a nontrivial geometric phase when both vortices begin the cycle sitting at the center of the trap. This can be seen on the far left of the plot, where the radius is zero, in lines that correspond with a tilt near $\theta = \pi/2$. At this point there is no difference between two charge one vortices and a single charge two vortex. A reason for this nontrivial geometric phase at the origin is that the $p = 0, l = 0$ mode coefficient does not go to zero at the origin, unlike in the single-vortex state.

The reason for this becomes clear when Figure 4.12 is examined. There is a tilted single-vortex state at the origin at time zero in panel (a), and the state simply exhibits a phase rotation with time, as shown in...
Figure 4.8 Field magnitude of a single-vortex state with $\xi = \theta = \pi/2$, $x_0 = 1$, and $z = 0$. This is a Hermite-Gaussian mode, and no geometric phase is accumulated. $x$ and $y$ have been non-dimensionalized with $1/\sqrt{k}\Omega$ where $k$ is the wavenumber and $\Omega$ is a non-dimensional parameter of the dielectric character as in Eq. (3.12)

panel (b). Contrast this with the evolution of a tilted double-vortex state at the origin at time zero, shown in panel (c). Its evolution reveals, in panel (d), that it is unstable and immediately decomposes into two single-charge vortices. Interestingly, these periodically come back together into a single charge two vortex at the origin. The instability of higher-order vortex has been well-studied in the literature but, to the best of our knowledge, such investigations are always associated with free-space propagation and not trapped states[12].

One takeaway is that geometric phase is accumulated by the re-direction of vortices by an inhomogeneous dielectric, the vortex analog of the bending of light that generates geometric phase for plane waves propagating through inhomogeneous, anisotropic media. A vortex sitting at the center of the trap does not have an opportunity to be influenced by the inhomogeneous dielectric profile. This is analogous to a ball sitting at the bottom of a bowl, versus a ball with some momentum circling or oscillating. However, we do not strictly require a vortex to generate geometric phase, as we have seen that a vortex fully tilted on its side (in which case it is no longer a single point of undefined phase and no longer really a vortex) may still generate a geometric phase. A more general guess we could make then is that it is required that the magnitude of the field evolves in some way to generate geometric phase. The evolution of the field in these cases can be seen as the optical Magnus effect exerting the equivalent of a force on the light, pushing it and making it move[10, 40]. If this evolution entails a transfer of momentum from the light to the medium or vice versa, then this may well in fact be a real force, and more evidence for the tie between momentum transfer and geometric phase.
Figure 4.9 The geometric phase of a single-vortex state for two regimes: (a) large initial distance from origin, \( x_0 = 15 \); and (b) small initial distance from origin, \( x_0 = 0.05 \). For large initial distance, there is only a narrow region in which geometric phase is nontrivial, but it always spans the full range of possible nontrivial values. At very small distances from the origin, though, the geometric phase is very close to trivial no matter the choice of tilt variables.

### 4.4.2 Case Examination: Tilt Degree and Direction

We will spend little time on the general effects of tilt degree and direction on the double-vortex case. We can see several different plots at different radii in Figure 4.13. At extremely large radius a similar form as in our single-vortex case can be seen, and at very small radius we see the case we have already examined above. Note that it has no \( \xi \) dependence despite having \( \theta \) dependence. Between these two extremes, a variety of different behaviors that slowly mediate between them can be observed.
Figure 4.10 Field magnitude and field phase of a tilted, double-vortex state at specific time slice $z = \pi/3$. The parameters in this case are $x_0 = 1$ and $\theta = \xi = \pi/4$ radians. $x$ and $y$ have been non-dimensionalized with $1/\sqrt{k\Omega}$ where $k$ is the wavenumber and $\Omega$ is a non-dimensional parameter of the dielectric character as in Eq. (3.12).

Figure 4.11 The geometric phase of our double-vortex state vs the starting radius $x_0$ for various values of theta. $\xi$ is set to zero. Unlike the single-vortex case it is now possible to have a nontrivial geometric phase at the origin by tilting the vortex. The horizontal axis, initial vortex radius, has been non-dimensionalized with $1/\sqrt{k\Omega}$ where $k$ is the wavenumber and $\Omega$ is a non-dimensional parameter of the dielectric character as in Eq. (3.12).
Figure 4.12 The field magnitude of a tilted single-vortex state (a-b) versus a double-vortex state (c-d) both with parameters $x_0 = 0$ and $\theta = \xi = \pi/3$ radians. Both are shown at two different times, $z = 0$ (a,c) and $z = \pi/4$ (b,d). Note how a tilted double-vortex state split and evolves, but a single-vortex state remains constant. $x$ and $y$ have been non-dimensionalized with $1/\sqrt{k\Omega}$ where $k$ is the wavenumber and $\Omega$ is a non-dimensional parameter of the dielectric character as in Eq. (3.12).
Figure 4.13 The geometric phase of an evolving double-vortex state versus tilt angles for various radii. At large radii, it is somewhat similar to the single-vortex case; however, at smaller radii it is very different. The plots above demonstrate the wide array of different dependencies the geometric phase has between these extremes.
Armed with the foundational material presented in previous chapters, the primary focus of this thesis can now undertaken: an elucidation of the relationship between geometric phase and the degree of entanglement of two-photon vortex states. A methodology for producing such states is reviewed, and a general class of entangled vortex states is then constructed. This allows some relatively abstract observations to be made about the relationship between geometric phase and entanglement. Attention is then turned to states of a more specialized form, where we are able to cleanly identify a component of the geometric phase that is intrinsic to the entanglement itself. We dub this the \textit{Geometric Phase of Entanglement} (GPE), and it proves very useful in explaining how entanglement influences the accumulation of geometric phase.

We then go on to show how a GPE emerges as a projection of the constituent single-photon states; if these two states are orthogonal, then the GPE is zero, and the same is true if they are parallel. Useful relationships are then identified between the eigenvalues of the constituent states and the GPE. This is followed by the examination of several key cases that foster physical intuition for how entanglement, tilt, and vortex orbit size conspire to produce a GPE.

### 5.1 Creating Entangled States With Type-I Spontaneous Parametric Downconversion

Perhaps the most logical place to begin an investigation of the geometric phase of entangled states is to explain how such states can be produced. Towards this end, suppose that we seek to generate a two-photon state of the following form:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{N}}(\cos(\frac{\alpha}{2})|A(0)\rangle_1|B(0)\rangle_2 + \sin(\frac{\alpha}{2})|B(0)\rangle_1|A(0)\rangle_2),$$

where $N$ is a normalizing constant, $\alpha$ defines the entanglement magnitude, and $\beta$ is the relative phase of the two terms.

Following the work of Torres et al. [28], Type-I Spontaneous Parametric Downconversion (SPDC) produces entangled OAM states that involve many two-photon states:

$$|\Psi(0)\rangle = \sum_{p_1, l_s, p_2, l_i} C_{p_1, p_2}^{l_s, l_i} |p_1, l_s\rangle_1 |p_2, l_i\rangle_2,$$

where $N$ is a normalizing constant, $\alpha$ defines the entanglement magnitude, and $\beta$ is the relative phase of the two terms.
where \( p \) and \( l \) identify Laguerre-Gaussian modes. The coefficients, \( C_{p_1, p_2}^{l_1, l_2} \), represent the probability of measuring each two-photon state, and these can be calculated as projections of the pump beam, \( \Upsilon(\rho, \phi) \), onto each two-photon pair of such modes:

\[
C_{p_1, p_2}^{l_1, l_2} \sim \int_0^\infty \int_0^{2\pi} \rho d\rho d\phi \Upsilon(\rho, \phi) u(\rho, \phi)_{p_1, l_1}^* u(\rho, \phi)_{p_2, l_2}. \tag{5.3}
\]

It is highly desirable to maximize the probability of measuring the two-photon state of interest, such as that given in Eq. (5.1). One way this can be accomplished is to filter out all unwanted modes using a specific type of multi-mode fiber. There is an alternate approach, though, in which the probabilities of the desired modes can be engineered.

Torres [28] showed how to optimize two-photon modes by strategically placing vortices within a Gaussian pump beam so that it has the following form:

\[
\Upsilon(r, \phi) = A_0 \prod_{j=1}^{M} (r e^{i\phi} - r_j e^{i\phi_j}) e^{-r^2/w_0^2}. \tag{5.4}
\]

Here \( r_j \) and \( \phi_j \) are the coordinates of the jth vortex and \( w_0 \) is the beam waist. The \( C_{p_1, p_2}^{l_1, l_2} \) coefficients can then be expressed as functions of vortex positions in the pump beam. In settings for which a 1:1 relationship exists, it can be inverted to place the vortices so as to optimize the population of the 2-photon states that are desired. This is the essence of the method Torres lays out, and the example below serves to demonstrate its efficacy.

### 5.1.1 Example of an Optimized 2-Photon State

Suppose we want to produce a pair of maximally entangled rectilinear charge-one vortices at two different radii. As discussed in Section 4.3, such a state can be represented as

\[
|\mathbf{A}(z)\rangle = -\frac{x_{0,A}}{\sqrt{1 + x_{0,A}^2}} |\psi_{0,0}(z)\rangle + \frac{1}{\sqrt{1 + x_{0,A}^2}} |\psi_{0,1}(z)\rangle, \tag{5.5}
\]

\[
|\mathbf{B}(z)\rangle = -\frac{x_{0,B}}{\sqrt{1 + x_{0,B}^2}} |\psi_{0,0}(z)\rangle + \frac{1}{\sqrt{1 + x_{0,B}^2}} |\psi_{0,1}(z)\rangle, \tag{5.6}
\]

where \( x_{0,A} \) and \( x_{0,B} \) are the orbit radii of the the vortices in the following entangled state:

\[
|\Psi(z)\rangle = \frac{1}{\sqrt{2N}} (|\mathbf{A}(z)\rangle_1 |\mathbf{B}(z)\rangle_2 + |\mathbf{B}(z)\rangle_1 |\mathbf{A}(z)\rangle_2) \tag{5.7}
\]

Here the normalization factor is given by
\[ N = 1 + \langle A | B \rangle \langle B | A \rangle. \]  

As shown for collinear Type-1 SPDC by Torres [28], the weighting coefficients of the two-photon states can then be expressed in terms of pump vortex locations as:

\[ C_{l_s,l_i}^{l_s,l_i} \sim (-1)^{M-l_s-l_i} \frac{2^{[(l_s+|l_s|)/2]}}{3^{(l_s+|l_s|+|l_i|)/2}} \frac{A_0 a_0^{l_s+l_i}}{\Gamma((l_s+|l_s|+|l_i|)/2+1)} \frac{\Gamma((l_s+|l_s|)/2+1)}{B_{M-l_s+l_i}} \]  

where

\[ B_m = \sum_{j_1} \sum_{j_2} \ldots \sum_{j_m} \prod_{l=1}^{m} \rho_{j_l} e^{i \phi_{j_l}}. \]  

These relations imply that

\[ C_{0,0}^{0,0} = \frac{2}{\sqrt{2N}} \frac{x_{0,A}}{\sqrt{1+x_{0,A}^2}} \frac{x_{0,B}}{\sqrt{1+x_{0,B}^2}}, \]  

\[ C_{1,1}^{1,1} = \frac{2}{\sqrt{2N}} \frac{1}{\sqrt{1+x_{0,A}^2}} \frac{1}{\sqrt{1+x_{0,B}^2}} \]  

\[ C_{0,1}^{0,1} = C_{1,0}^{1,0} = -\frac{1}{\sqrt{2N}} \frac{1}{\sqrt{1+x_{0,A}^2}} \frac{1}{\sqrt{1+x_{0,B}^2}} (x_{0,A} + x_{0,B}). \]  

The pump beam has vortices with the following polar coordinates:

\[ r_1 = r_2 = \frac{2}{3} \sqrt{x_{0,A} x_{0,B}}, \]  

\[ \phi_1 = -\phi_2 = \frac{1}{2} \sec^{-1} \left( \frac{4x_{0,A} x_{0,B}}{(x_{0,A} - x_{0,B})^2} \right). \]  

The solution relies on an invertibility that is valid only for certain ratios of vortex radii. For instance, if \( x_{0,A} \) is larger than \( x_{0,B} \) it is required that \( x_{0,B} \geq (3 - 2\sqrt{2}) x_{0,A} \) for the solutions to be real. With this in mind, consider an example when we set \( x_{0,A} = 1 \) and \( x_{0,B} = 0.5 \). In this case it is found that the desired state makes up 21\% of all generated states. The distribution of generated states can be seen in Figure 5.1. The pump beam magnitude and phase that would create such a distribution is shown in Figure 5.2. It is significant to note that, without such vortex optimization, the population of three of the desired output modes would be zero: \( C_{0,1}^{0,1} = C_{1,0}^{1,0} = C_{1,1}^{1,1} = 0 \). This implies that engineering vortices into the pump beam
is actually required, in this case, to produce the entangled state; it is not simply an issue of improving the likelihood of it being measured.

Figure 5.1 A plot of the magnitude of each Laguerre-Gaussian mode versus the $l$ index of each photon. Our desired modes are highlighted in red.

5.1.2 Tunable Entanglement

The previous example was billed as an example of optimizing the presence of a two-photon state that is maximally entangled. In fact, though, all two-photon states produced by using the Torres approach [28] are maximally entangled. In the previous example, this can be seen in the equal weighting of the coefficients listed in Eq. (5.13) as well as the equal heights of the $|01\rangle$ and $|10\rangle$ states in Figure 5.1. In fact, the more general setting of Eq. (5.9) gives the weighting coefficients in a form such that $C_{ls,li}^{0,0} = C_{ls,li}^{0,0}$ for all choices of signal and idler OAM. But it is precisely the weighting of these two states that in which the entanglement is manifested as is clear from Eq. (5.1). Therefore the approach of Torres is not sufficiently general for our purposes.

In a beautiful extension of the original Torres approach, Yao et al. [37] recognized that there is no need to constrain the signal and idler beam so that they have the same waist size as the pump. Under the assumption of Type-I collinear SPDC, a more general set of weighting coefficients, $C_{ls,li}^{ps,pi}$, were derived that now depend on the non-dimensional waist ratios, $\gamma_s := w_s/w_p$ and $\gamma_i := w_i/w_p$.
Figure 5.2 Plots of the pump intensity (left) and pump phase (right) that would maximize an entangled set of rectilinear vortices with $x_{0,A} = 1$ and $x_{0,B} = 0.5$. $x$ and $y$ have been non-dimensionalized with $1 / \sqrt{k \Omega}$, where $k$ is the wavenumber and $\Omega$ is a non-dimensional parameter of the dielectric character as in Eq. (3.12).

\[
C^{l_p, l_s, l_i}_{p_s, p_i} = \delta_{l_p, l_s + l_i} \sqrt{\frac{2}{\pi w_0^2}} \frac{2 \sigma_i + 1}{(1 + \gamma_s^2 + \gamma_i^2)^{\sigma_i + 1}} \sum_{k=0}^{p_p} \sum_{i=0}^{p_s} \sum_{j=0}^{p_i} \frac{(-2)^{k+i+j} \gamma_s^{2i} \gamma_i^{2j}}{(1 + \gamma_s^2 + \gamma_i^2)^{k+i+j}} \frac{(\sigma_i + k + i + j)!}{(p_p - k)!((l_p + k)!k!(p_s - i)!((l_s + i)!i!(p_i - j)!((l_i + j)!j)!}(5.16)
\]

with $\sigma_l := (|l_p| + |l_s| + |l_i|)/2$. A quick inspection of this result indicates that $C^{l_p, l_s, l_i}_{p_s, p_i}$ and $C^{l_i, l_s, p_i}_{p_i, p_s}$ can be distinguished by choosing signal and idler waists that are different.

The Yao method can be used to optimize the population of desired states by engineering vortices into the pump beam, just as mapped out by Torres. However, the example given below has a more modest goal of simply showing how to produce the following state:

\[
|\psi[0]\rangle = \cos(\alpha/2) |01\rangle + \sin(\alpha/2) |10\rangle (5.17)
\]

This state can be produced with pump beam for which $\{p_p = 0, l_p = 1\}$, and there are only two non-zero coefficients that, when normalized, are simply

\[
C^{0,1}_{0,0,\text{norm}} = \frac{\gamma_i}{\sqrt{\gamma_i^2 + \gamma_s^2}}, \quad C^{1,0}_{0,0,\text{norm}} = \frac{\gamma_s}{\sqrt{\gamma_i^2 + \gamma_s^2}}. (5.18)
\]

A comparison of Eqs. (5.17) and (5.18) then reveals that any degree of entanglement, $\alpha$, can be achieved with the following beam waist ratios:

\[
\gamma_s = \cos(\alpha/2), \quad \gamma_i = \sin(\alpha/2). (5.19)
\]
This demonstrates that the Yao [37] methodology can be used to tune entanglement. Her approach can be combined with the vortex engineering method of Torres [28] to construct the states of interest in the current investigation and to optimize their populations.

5.2 General Description of Entangled Laguerre-Gaussian Mode Geometric Phase

Now that it has been established how entangled states can be generated, their geometric phase may be investigated. First, consider entangled, two-photon systems of the following general form:

\[ |\psi(z)\rangle = \frac{1}{\sqrt{N}} (e^{-i\beta/2} \cos(\frac{\alpha}{2}) |A(z)\rangle_1 |B(z)\rangle_2 + e^{i\beta/2} \sin(\frac{\alpha}{2}) |C(z)\rangle_1 |D(z)\rangle_2) \]  

(5.20)

where the normalizing constant is

\[ N = 1 + \frac{1}{2} \sin(\alpha)(e^{i\beta} \langle A|C \rangle \langle B|D \rangle + e^{-i\beta} \langle C|A \rangle \langle D|B \rangle) \]  

(5.21)

In terms of the Laguerre-Gaussian coefficients previously derived, Eq. (4.1), this normalization constant is

\[ N = 1 + \frac{1}{2} \sin(\alpha) e^{i\beta} \sum_{m,n} a^*_m a_m + e^{-i\beta} \sum_{m,n} c^*_m c_m \sum_{j,k} b^*_j b_j + e^{-i\beta} \sum_{m,n} d^*_m d_m \sum_{j,k} a^*_j a_j) \]  

(5.22)

As discussed within the single-photon setting, when the vortex cycle period, \( \tau \), is a multiple of \( 2\pi \), the initial state is the same as the final state (i.e., the total phase is equal to zero). This implies that the geometric phase is equal to the negative of the dynamic phase. This makes sense, as the total phase is zero the geometric phase and dynamic phase must add up to zero. The same is true here in the two-photon setting. The geometric phase after one cycle is then

\[ \Phi_{\text{Geom},\psi} = \frac{2\pi i}{N} (\cos(\frac{\alpha}{2})^2 (\langle A|d_z A \rangle + \langle B|d_z B \rangle) + \sin(\frac{\alpha}{2})^2 (\langle C|d_z C \rangle + \langle D|d_z D \rangle) \]  

\[ + \frac{1}{2} \sin(\alpha)(e^{i\beta} (\langle A|d_z C \rangle \langle B|D \rangle + \langle A|C \rangle \langle B|d_z D \rangle) + e^{-i\beta} (\langle C|d_z A \rangle \langle D|B \rangle + \langle C|A \rangle \langle D|d_z B \rangle) )) \]  

(5.23)

In terms of Laguerre-Gaussian mode coefficients, this is

\[ \Phi_{\text{Geom},\psi} = \frac{2\pi}{N} \cos(\frac{\alpha}{2})^2 (\sum_{m,n} \varepsilon_m (a^*_m a_m + b^*_m b_m) \]  

\[ + \sin(\frac{\alpha}{2})^2 (\sum_{m,n} \varepsilon_m (c^*_m c_m + d^*_m d_m) \]  

\[ + \frac{1}{2} \sin(\alpha)(e^{i\beta} \sum_{m,n} \varepsilon_m (a^*_m c_m \sum_{j,k} b^*_j b_j + b^*_m a_m \sum_{j,k} c^*_j c_j) \]  

\[ + e^{-i\beta} \sum_{m,n} \varepsilon_m (c^*_m a_m \sum_{j,k} d^*_j d_j + d^*_m b_m \sum_{j,k} e^*_j e_j)) \) \]  

(5.24)
It is clear from our equation that geometric phase accumulation is the result of a fractional weighting of Laguerre-Gaussian mode eigenvalues with coefficients that now depend on entanglement variables $\alpha$ and $\beta$.

5.2.1 The AB+BA Case

Now we turn to the more specialized form of entangled states that is the focus of this thesis:

$$|\Psi(z)\rangle = \frac{1}{\sqrt{N}} (e^{-i\beta/2} \cos(\frac{\alpha}{2}) |A(z)\rangle_1 |B(z)\rangle_2 + e^{i\beta/2} \sin(\frac{\alpha}{2}) |B(z)\rangle_1 |A(z)\rangle_2).$$  \hspace{1cm} (5.25)

Here the normalizing constant is

$$N = 1 + \sin(\alpha) \cos(\beta) \langle A|B \rangle \langle B|A \rangle,$$  \hspace{1cm} (5.26)

or, in terms of the LG coefficients,

$$N = 1 + \sin(\alpha) \cos(\beta) \sum_{m,n} a_{m,n}^* b_{m,n} \sum_{j,k} b_{j,k}^* a_{j,k}^*.$$

(5.27)

The geometric phase is now of a simpler, more digestible form:

$$\Phi_{Geom,\Psi} = \frac{2\pi i}{N} (\langle A|d_z A \rangle + \langle B|d_z B \rangle)$$

$$+ \sin(\alpha) \cos(\beta) (\langle A|d_z B \rangle \langle B|A \rangle + \langle A|B \rangle \langle B|d_z A \rangle)).$$  \hspace{1cm} (5.28)

In terms of the LG coefficients, it is

$$\Phi_{Geom,\Psi} = \frac{2\pi}{N} (\sum_{m,n} \epsilon_{m,n} (a_{m,n}^* a_{m,n} + b_{m,n}^* b_{m,n})$$

$$+ \sin(\alpha) \cos(\beta) \sum_{m,n} \epsilon_{m,n} (a_{m,n}^* b_{m,n} \sum_{j,k} b_{j,k}^* a_{j,k} + b_{m,n}^* a_{m,n} \sum_{j,k} a_{j,k}^* b_{j,k})).$$  \hspace{1cm} (5.29)

It can be seen that entanglement appears as a factor, $\sin(\alpha) \cos(\beta) \langle A|B \rangle \langle B|A \rangle$, in both the normalization and the cross terms (The terms containing inner products between $|A\rangle$ and $|B\rangle$). This skews the contributions to emphasize the relevance of cross terms in the total geometric phase.

5.2.1.1 The Geometric Phase of Entanglement

We define the GPE as the contribution to geometric phase that is above and beyond that accumulated by the single-photon states when considered separately:
\[ \Phi_{GPE, \Psi} = \Phi_{Geom, \Psi} - \Phi_{Geom, A} - \Phi_{Geom, B} \]
\[ = \frac{2\pi i}{N} (\langle A | d_z A \rangle + \langle B | d_z B \rangle - N \langle A | d_z A \rangle - N \langle B | d_z B \rangle + \sin(\alpha) \cos(\beta)(\langle A | d_z B \rangle \langle B | A \rangle + \langle A | B \rangle \langle B | d_z A \rangle)). \] 

(5.30)

This can be further simplified:

\[ \Phi_{GPE, \Psi} = \left( \frac{1}{N} - 1 \right) \left( \Phi_{Geom, A} + \Phi_{Geom, B} - 2\pi i \left( \frac{\langle A | d_z B \rangle}{\langle A | B \rangle} + \frac{\langle B | d_z A \rangle}{\langle B | A \rangle} \right) \right). \] 

(5.31)

The GPE is the product of two factors: (1) a factor involving the normalizing constant as seen on the left, which goes to zero whenever the states \( |A \rangle \) and \( |B \rangle \) are orthogonal; (2) the right-side factor composed of the sum of the geometric phases of \( |A \rangle \) and \( |B \rangle \), and another new term \( -2\pi i \left( \frac{\langle A | d_z B \rangle}{\langle A | B \rangle} + \frac{\langle B | d_z A \rangle}{\langle B | A \rangle} \right) \). This term on the right-side appears to be almost like the geometric phase of \( |A \rangle \) or \( |B \rangle \); however, instead of the time derivative in the inner product of \( |A \rangle \) or \( |B \rangle \), it is the inner product between \( |A \rangle \) and \( |B \rangle \) with the time derivative between them. This term is then not directly the overlap between \( |A \rangle \) and \( |B \rangle \), but is related. When \( |A \rangle \) and \( |B \rangle \) share only a single eigenvalue, \( \varepsilon \), then this right-side term simplifies and the GPE becomes

\[ \Phi_{GPE, \Psi} = \left( \frac{1}{N} - 1 \right) \left( \Phi_{Geom, A} + \Phi_{Geom, B} - 4\pi \varepsilon \right). \] 

(5.32)

This is not equivalent to requiring that \( |A \rangle \) and \( |B \rangle \) each only have a single eigenvalue, which would result in a GPE of zero; rather, it is a restriction that all of the nonzero modes shared between them both have only a single eigenvalue. It is important to note that, while the term \(-4\pi \varepsilon\) appears trivial, it is often not as it is multiplied by \(1/N - 1\).

The GPE is a valuable quantity for elucidating exactly how entanglement influences the accumulation of geometric phase, as born out in the following analysis of several special cases.

### 5.2.1.2 Orthogonality

If states \( |A \rangle \) and \( |B \rangle \) are orthogonal, the normalizing constant of Eq. (5.2.1) reduces to unity. The dynamic term then loses its cross terms and simplifies to

\[ \Phi_{Geom, dyn, \Psi} = i \int_0^t dt \langle A(z) | d_z A(z) \rangle + i \int_0^t dt \langle B(z) | d_z B(z) \rangle = \Phi_{Geom, dyn, A} + \Phi_{Geom, dyn, B}. \] 

(5.33)

This is simply a sum of the dynamic term of the geometric phases of the \( |A \rangle \) and \( |B \rangle \) states with no weighting. It is worth noting that this clean result is obtained with the setting for which the total phase is
zero. The GPE of (5.31) goes to zero whenever the states \( |A\rangle \) and \( |B\rangle \) are orthogonal. More generally, it is true that the entanglement dependence of at least the dynamic term of the geometric phase vanishes for any state of the form Eq. (5.25) when \( |A\rangle \) and \( |B\rangle \) are orthogonal and it is assumed that the entanglement variables \( \alpha \) and \( \beta \) are time independent. It applies not only to Laguerre-Gaussian modes, but any state that can be written as Eq. (5.25). This is evident from mathematical examination of the Aharanov-Anandan relation, Eq. (2.7), applied to states of this form. As the total phase term is always zero (As shown in section 4.2.2.) for the cases considered in this thesis this condition applies to the total geometric phase rather than just the dynamic phase portion.

We have shown that the geometric phase of entanglement goes to zero when the states are fully orthogonal, and it increases with state overlap. If the states are fully overlapped, though, the geometric phase of entanglement goes to zero again as the two-photon state is then separable. The focus of this thesis lies between these two extremes.

This relationship between lack of state orthogonality and the GPE can be understood more completely by decomposing state \( |B\rangle \) into components that are either orthogonal, \( |O\rangle \), or parallel to state \( |A\rangle \):

\[
|B\rangle = \sin \mu e^{i\nu} |A\rangle + \cos \mu |O\rangle .
\]  

(5.34)

The normalizing constant is then found to be

\[
N = 1 + \sin(\alpha) \cos(\beta) \sin^2 \mu .
\]

(5.35)

Additionally, this construction allows the inner products that appear in the GPE formula to be simplified to

\[
\Phi_{GPE,\psi} = \left( \frac{1}{N} - 1 \right) (\Phi_{Geom, A} - \Phi_{Geom, B} - 2\pi i \cot \mu (\langle A | d_z O \rangle e^{-i\nu} + \langle O | d_z A \rangle e^{i\nu})).
\]

(5.36)

It can now be clearly seen that the GPE is the weighted difference between the geometric phases of \( |B\rangle \) and \( |A\rangle \) along with an additional term \(-2\pi i \cot \mu (\langle A | d_z O \rangle e^{-i\nu} + \langle O | d_z A \rangle e^{i\nu})\). When this additional term vanishes then the GPE is just \( \frac{1}{N} - 1 \) multiplied by the difference in the single-photon states’ geometric phases. The term will vanish when \( \langle A | d_z O \rangle = 0 \). This can happen, for instance, when the states \( |A\rangle \) and \( |O\rangle \) share only a single Eigenvalue. To illustrate this further define \( |O\rangle \) as

\[
|O\rangle = \sum_{p,l} q_{p,l} |\psi_{p,l}(z)\rangle
\]

(5.37)

such that
\[ \langle A | O \rangle = \sum_{p,l} a_{p,l}^* a_{p,l} = 0 \]  \hspace{1cm} (5.38)

while, of course, still satisfying Eq. (5.34). If \( |A\rangle \) and \( |O\rangle \) share only a single eigenvalue then

\[ \langle A | d_z O \rangle = -i \varepsilon \sum_{p,l} a_{p,l}^* a_{p,l} = 0 \]  \hspace{1cm} (5.39)

and so the GPE reduces to

\[ \Phi_{GPE,\psi} = \left( \frac{1}{N} - 1 \right) (\Phi_{Geom,B} - \Phi_{Geom,A}). \]  \hspace{1cm} (5.40)

For single-vortex states of the form Eq. (4.12) this occurs whenever \( |A\rangle \) is placed at the origin, or we set \( x_{0,A} = \infty \) making the state a Gaussian bump. In those cases \( |A\rangle \) has only a single eigenvalue, and so must share only a single eigenvalue with \( |O\rangle \). In section 5.3.2 the state \( |A\rangle \) is reduced to a single-mode gaussian bump, and so this is the case.

To obtain the formulas above the \( |B\rangle \) state was decomposed, but there is no particular reason we cannot choose to instead decompose the \( |A\rangle \) state in the same way instead. If we define

\[ |A\rangle = \sin \mu e^{i \nu} |B\rangle + \cos \mu |O\rangle. \]  \hspace{1cm} (5.41)

such that

\[ |O\rangle = \sum_{p,l} a_{p,l} \psi_{p,l}(z) \]  \hspace{1cm} (5.42)

and

\[ \langle B | O \rangle = \sum_{p,l} b_{p,l}^* a_{p,l} = 0, \]  \hspace{1cm} (5.43)

we then obtain the same formula for GPE but with \( |B\rangle \) and \( |A\rangle \) reversed:

\[ \Phi_{GPE,\psi} = \left( \frac{1}{N} - 1 \right) (\Phi_{Geom,B} - \Phi_{Geom,A} - 2\pi i \cot \mu (\langle B | d_z O \rangle e^{-i \nu} + \langle O | d_z B \rangle e^{i \nu})). \]  \hspace{1cm} (5.44)

This has the same properties as Eq. (5.44), and reduces to

\[ \Phi_{GPE,\psi} = \left( \frac{1}{N} - 1 \right) (\Phi_{Geom,A} - \Phi_{Geom,B}) \]  \hspace{1cm} (5.45)

Whenever \( |B\rangle \) and \( |O\rangle \) share only a single eigenvalue. In sections 5.3.1 and 5.3.3 state \( |B\rangle \) is fixed to the origin, and so the GPE will be governed by Eq. (5.45). It is important to note that the GPE does not
always reduce to (5.40) or (5.45), however there are many situations in which it does.

In summary, the GPE is generated by two competing factors, which in turn are proportional to how different or similar states \(|A\rangle\) and \(|B\rangle\) are. When states \(|A\rangle\) and \(|B\rangle\) are identical the right-side factor of Eq. (5.31) goes to zero. When states \(|A\rangle\) and \(|B\rangle\) are completely orthogonal, the normalizing factor is one and the left-side factor goes to zero. This results in the many pseudo-bell-shaped curves that will be seen in the case studies taken up next. This relation between orthogonality and geometric phase of entanglement further suggests that there may be a relation between the geometric phase of entanglement and mutual information in entangled states.

5.2.2 Eigenvalues and Nontrivial Geometric Phases

As noted in Eq. (4.2.2), the geometric phase becomes trivial in the single-photon case of degenerate states. This holds, by extension, for analogous degeneracies in the two-photon states of Eq. (5.25):

\[
\Phi_{\text{Geom}, \Psi} = 4 \pi * \varepsilon. \tag{5.46}
\]

Keep in mind that the eigenvalues are integer-valued in the dimensionless setting of this thesis. Additionally, if the \(|A\rangle\) and \(|B\rangle\) states can each be described by degenerate states, but not necessarily the same states, then the geometric phase reduces to

\[
\Phi_{\text{Geom}, \Psi} = 2 \pi (\varepsilon_A + \varepsilon_B). \tag{5.47}
\]

Once again, the geometric phase is trivial. This may also be seen as an additional case in which orthogonality causes the geometric phase of entanglement to vanish, since the two states have no modes in common.

To accumulate a nontrivial geometric phase, either \(|A\rangle\) or \(|B\rangle\) must be comprised of modes with at least two non-degenerate eigenvalues. This is also a requirement for the GPE to be nonzero, as it can be seen that Eq. 5.2.2 and Eq. 5.47 are merely sums of the single-photon geometric phases. However, this need not be the case for entangled states of the more general form in Eq. (5.20). This is covered in appendix A.1

5.3 Single-Vortex Entanglement

A detailed analysis is now undertaken of the GPE associated with states of the form

\[
|\Psi(z)\rangle = \frac{1}{\sqrt{N}} (e^{-i\beta/2} \cos(\frac{\alpha}{2}) |A(z)\rangle_1 |B(z)\rangle_2 + e^{i\beta/2} \sin(\frac{\alpha}{2}) |B(z)\rangle_1 |A(z)\rangle_2). \tag{5.48}
\]

Here, \(|A\rangle\) and \(|B\rangle\) are single-vortex states of the form Eq. (4.12) but with differing parameters: \(x_{0,A}, \theta_A, \xi_A\) and \(x_{0,B}, \theta_B, \xi_B\) respectively. The normalizing factor is
The geometric phase follows from Eq. (5.29):

$$N = 1 + \sin(\alpha) \cos(\beta) \sum_{n=-1}^{1} a_{0,n}^* b_{0,n} \sum_{k=-1}^{1} b_{0,k}^* a_{0,k}. \quad (5.49)$$

The geometric phase follows from Eq. (5.29):

$$\Phi_{\text{Geom, } \Psi} = \frac{2\pi}{N} \left( \sum_{n=-1}^{1} \varepsilon_{0,n}(a_{0,n}^* a_{0,n} + b_{0,n}^* b_{0,n}) \right) + \sin(\alpha) \cos(\beta) \sum_{n=-1}^{1} \varepsilon_{0,n}(a_{0,n}^* b_{0,n} \sum_{k=-1}^{1} b_{0,k}^* a_{0,k} + b_{0,n}^* a_{0,n} \sum_{k=-1}^{1} a_{0,k}^* b_{0,k}) \quad (5.50)$$

This is still much too complicated to understand by direct inspection, motivating the consideration of several special cases.

### 5.3.1 GPE of Rectilinear Vortices: One Vortex at the Origin

As for the single-photon states of Chapter 4, entangled rectilinear vortices offer a starting point for exploring the GPE. First, consider the very specific case in which one rectilinear vortex is at the trap center while the other circles about it at a fixed distance. For further simplicity, set the entanglement phase factor, $\beta$, to zero. The geometric phase is now only a function of the single-vortex orbit radius and the magnitude of the entanglement parameter, $\alpha$. Figure 5.3 shows the total geometric phase in panel (a) and the geometric phase of entanglement in panel (b). Panel (a) may appear to show the same behavior we would expect from the analogous single-photon state. If we subtract off the individual geometric phases of the single-photon $|A\rangle$ and $|B\rangle$ states, though, the resulting GPE exhibits a distinctly different dependence on radius. As we have set the vortex in state $|B\rangle$ to the origin the GPE takes on the form Eq. (5.45). The rise and fall of the GPE can be explained as follows. At zero radius there is, of course, a single eigenvalue, resulting in a trivial geometric phase. As the $|A\rangle$ and $|B\rangle$ states are identical so are their geometric phases, causing the right-side factor of Eq. (5.45) and therefore the GPE to be zero. As the radius increases, the coefficients of the states with other eigenvalues increase and so a nontrivial geometric phase and the GPE may be generated. Then, as the radius of the vortex from the trap center goes to infinity, the two states become orthogonal. This causes the geometric phase of entanglement to slowly return to zero as the left hand factor in Eq. (5.45) becomes zero. The overall geometric phase also becomes trivial, as individually each state now has a single eigenvalue. Between these two extremes, a peak can be seen where the effect of these two competing factors finds a balance. This is a clear example of what was discussed in section 5.2.1.2. It is important to note that GPE and total geometric phase do not always correlate exactly. It will be shown in section 5.3.3 that there are cases in which the geometric phase is nontrivial, in this case $\pi$, but
the GPE is zero.

Figure 5.3 The geometric phase (a) and the geometric phase of entanglement (b) plotted against the radius of the vortex for various values of the entanglement \( \alpha \). Other parameters are set to \( x_0, B = 0 \) and \( \xi_A = \xi_B = \theta_A = \theta_B = \beta = 0 \) radians. The horizontal axis, initial vortex radius, has been non-dimensionalized with \( 1/\sqrt{k \Omega} \) where \( k \) is the wavenumber and \( \Omega \) is a non-dimensional parameter of the dielectric character as in Eq. (3.12).

5.3.2 GPE for an Orbiting Vortex Entangled with a Gaussian Mode

Setting the radius of one vortex to infinity amounts to reducing one state to a Gaussian mode—i.e. the state has only a single Laguerre-Gaussian mode, \( p = 0 \) and \( l = 0 \), with mode coefficient

\[
a_{0,0} = -\frac{2(\cos \xi - i \cos \theta \sin \xi)}{\sqrt{3 + \cos(2\theta) + 2 \cos(2\xi) \sin^2 \theta}}. \tag{5.51}
\]

Though tilt variables \( \theta_A \) and \( \xi_A \) still appear in its coefficient, they do not appear anywhere in the geometric phase and do not affect it at all. In fact, they do not even affect the shape of the field. The magnitude of this term is unity, and the tilt parameters serve only to add an overall phase to the entire field.

In Figure 5.4, the dependence of the GPE on radius and entanglement magnitude is shown. The fact that state \( |A\rangle \) is composed of only a single mode now means that our GPE takes on the form of Eq. (5.40). Although the overall geometric phase is still positive, its GPE is now negative. This sign change of the GPE can be attributed to the switch from the form Eq. (5.45) to form Eq. (5.40) as well as a switch in which of the single photon \( |A\rangle \) or \( |B\rangle \) states has the largest geometric phase. This GPE function appears very similar to the negative of the geometric phase of entanglement from section 5.3.1, and it exhibits its change as the radius changes for the same reasons. However, there is one important difference between this state and a state at the origin: the tilt parameters of this state with vortex placed at infinity cannot affect the geometric phase of entanglement. This can be seen if one examines the function for geometric phase.
There is no dependence on the tilt variables of the state we have set to infinite radius. This should not be surprising, as the tilt parameters now only alter a constant overall phase on the field. However, a more general insight can be gained by examining how the stationary states contribute to the geometric phase. This mode is a simple Gaussian, so its parameters cannot possibly change the weighing of its modes. This means that though it is not orthogonal with the other vortex, its parameters do not change the magnitude of their overlap, their non-orthogonality. This serves as contrast to the next state we will examine. In the next section we will see an example where the tilt of a vortex placed at the origin, with a trivial geometric phase, determines the geometric phase of entanglement. This shows that while single-photon states composed of a vortex at the origin or a Gaussian both have trivial geometric phases individually, the former can affect the total geometric phase of the entangled two-photon state and GPE while the latter cannot.

Figure 5.4 The geometric phase (a) and the geometric phase of entanglement (b) plotted against the radius of the vortex for various values of the entanglement $\alpha$ for a vortex entangled with a Gaussian mode. The other parameters are set to $x_{0,A} = \infty$ and $\xi_A = \xi_B = \theta_A = \theta_B = \beta = 0$ radians. The horizontal axis, initial vortex radius, has been non-dimensionalized with $1/\sqrt{k\Omega}$ where $k$ is the wavenumber and $\Omega$ is a non-dimensional parameter of the dielectric character as in Eq. (3.12).

5.3.3 GPE for Tilted Vortices with One Fixed at the Origin

The rectilinear case summarized in Figure 5.3 is now generalized to allow the vortices to be tilted. For the sake of simplicity, however, the azimuthal angles are set to zero: $\xi_A = \xi_B = \beta = 0$, $x_{0,A} = 1$. That is, in this section one vortex, $|B\rangle$, is placed at the origin while the other vortex, $|A\rangle$, is placed at an initial radius of 1. The tilt magnitudes of both vortices, $\theta$, are then varied independently. In this situation, as has been shown, the vortex at the origin’s individual geometric phase will be trivial. However, it can still affect the overall geometric phase. State $|B\rangle$ is confined to the origin implying that $b_{0,0} = 0$. This leaves $|B\rangle$ with only the 0, 1 and 0, $-1$ states. As these states share an eigenvalue $|B\rangle$ has only a single eigenvalue and so the conditions for the GPE to simplify to Eq. 5.45 are again met. This also causes the $|A\rangle$ and $|B\rangle$ states
to be orthogonal when \( a_{0,1} b_{0,1} \) and \( a_{0,-1} b_{0,-1} \) are both zero. As previously discussed, the GPE vanishes if the states are orthogonal. This means the 0, 1 and 0, −1 states, which determine the tilt of the vortices, are entirely responsible for any overlap between the states and so too the normalizing constant that appears in Eq. (5.45). Therefore the difference or similarity of the tilt of the two vortices has an interesting effect on the GPE. In this case the geometric phase of entanglement simplifies to

\[
\Phi_{\text{GPE}, \Psi} = \frac{32 \pi (1 + \cos \theta_A \cos \theta_B)^2}{(7 + \cos(2\theta_A))(4(1 + \cos \theta_A \cos \theta_B)^2 + (7 + \cos(2\theta_A))(3 + \cos(2\theta_B)) \csc \alpha)}
\]  

(5.52)

In Figure 5.5 both the geometric phase (a) and the geometric phase of entanglement (b) plotted versus \( \theta_B \) can be observed. Of interest here is that the geometric phase of state \( |B\rangle \) is always trivial, but its tilt still affects the geometric phase of entanglement, and through that, the overall geometric phase of the entangled state. The entanglement has a significant effect, except when the vortices are set to opposite polar angles. This is because when one is set to \( \pi \) and the other is set to 0, the \( a_{0,1} b_{0,1} \) and \( a_{0,-1} b_{0,-1} \) terms vanish as mentioned. It is important to note here that even when the geometric phase of entanglement is zero, this does not mean the overall geometric phase is trivial. We can see this when we compare the geometric phase plots in this situation with the geometric phase of entanglement plots. When the geometric phase of entanglement goes to zero in this case a nontrivial factor of \( \pi \) can be observed in the overall geometric phase. This is because as the states \( |A\rangle \) and \( |B\rangle \) becomes orthogonal the geometric phase reduces to just the sum of the geometric phases of the \( |A\rangle \) and \( |B\rangle \) states individually, but while \( |B\rangle \) is trivial, \( |A\rangle \) is not. In Figure 5.6 we can again see the geometric phase of entanglement in the same
situation, but when $\theta_A = \pi$. This can be seen to be merely the mirror of our previous case. However, when $\theta_A = \pi/2$, as in Figure 5.7, there is no longer an angle $\theta_B$ can be set to that makes the geometric phase of entanglement vanish. This helps demonstrate that the relationship between the parameters and the geometric phase of entanglement is not always simple. We must examine how the parameters affect the coefficients of the stationary states to understand how the geometric phase of entanglement will function in general.

Figure 5.6 The geometric phase (a) and geometric phase of entanglement (b) of our single-vortex entangled state vs $\alpha$ and $\theta_B$ when $x_{0,A} = 1$ and $x_{0,B} = \beta = 0 = \xi_A = \xi_B = 0$ radians, but now set $\theta_A = \pi$ radians. This is simply the mirror of our previous figure.

Figure 5.7 The GPE of our single-vortex entangled state vs $\alpha$ and $\theta_B$ when $x_{0,A} = 1$ and $x_{0,B} = \beta = \xi_A = \xi_B = 0$ radians, but now with $\theta_A = \pi/2$ radians. There is no longer a zero GPE no matter the tilt.

The interaction between the two tilts can be even more clearly seen when we plot the geometric phase of entanglement against the tilt of both vortices at maximum entanglement, as in Figure 5.8. It can be
seen how the geometric phase of entanglement maximizes for similar angles, when the states are less orthogonal, and minimizes when the states are more orthogonal. This is a good example of the general rule of orthogonality discussed earlier. We should be careful, however, to not associate this rule of orthogonality strictly with vortex tilt. As seen in appendix A.2, when placed at the origin the single-vortex state’s tilt has absolutely no effect on the geometric phase of entanglement when entangled with the double-vortex state.

Figure 5.8 The geometric phase of entanglement of the single-vortex entangled state vs both vortex tilts when the parameters are set such that $x_{0,A} = 1$, $\alpha = \pi/2$ radians, and $x_{0,B} = \beta = \xi_A = \xi_B = 0$. It can be seen how the geometric phase of entanglement maximizes for similar angles, when the states are less orthogonal.

5.3.4 Case Examination: Entanglement Variables

It is also valuable to see how the entanglement magnitude and relative phase affect the geometric phase in the absence of any vortex tilt. To this end, the GPE is examined for three settings as function of the entanglement variables $\alpha$ and $\beta$: (1) one rectilinear vortex circling and the other at the origin; (2) both rectilinear vortices circling in the same way, and; (3) rectilinear vortices circling in opposite directions.

The first case is shown in the first plot of Figure 5.9. In this case state $|A\rangle$ has a nontrivial phase of $\pi$ and state $|B\rangle$ has an entirely trivial phase. The effects of both the magnitude and relative phase of entanglement can be seen here, with these variables each shifting the overall geometric phase off the $\pi$ value of state $|A\rangle$.

The second case is not shown in the figure here, as in this case state $|A\rangle$ is equal to state $|B\rangle$. Therefore entanglement has no effect, as the two terms in our entangled state are equal. It is found that the geometric phase of entanglement in this setup is zero. This state is only important to show that our geometric phase still makes sense in this case; that is, when the entanglement variables cannot have any
effect on the state, then they cannot have any effect on the geometric phase.

The last case is of particular interest and is shown in the second plot of Figure 5.9. In this case the vortices of both states are circling at the same radius, but in opposite directions. Individually they both generate a nontrivial geometric phase of $\pi$. When there is no entanglement the phases of these states merely add together to make a trivial $2\pi$ however, as can be clearly seen from the plot, nonzero entanglement makes this no longer trivial. This is a good example of how entanglement changes geometric phase to make it no longer a simple addition of the phases of the individual states.

![Figure 5.9](image)

Figure 5.9 The geometric phase plotted against the entanglement variables $\alpha$ and $\beta$. (a) Shows the geometric phase when one vortex is at the origin and the other is circling at a constant radius. Specifically, the parameters of the $|A\rangle$ and $|B\rangle$ states are set to $x_{0,A} = 1, \xi_A = \theta_A = x_{0,B} = \xi_B = \theta_B = 0$. (b) shows the geometric phase when both vortices are circling at the same constant radius, but different directions. Specifically, the parameters of the $|A\rangle$ and $|B\rangle$ states are set to $x_{0,A} = x_{0,B} = 1, \theta_B = \pi, \xi_A = \theta_A = \xi_B = 0$. Observe how the dependence on the entanglement variables is essentially the same, just with a sign flipped. The shapes seen here reflect the form the entanglement variables appear in in the first factor of Eq. (5.31).

### 5.4 Chapter Summary

A number of special settings have been considered to develop insight as to what GPE really is and how it is influenced by key vortex parameters. A summary of the key take-aways is now provided to pull these disparate results together.

For entangled states of the form (5.25) the difference between the sum of geometric phases of the $|A\rangle$ and $|B\rangle$ states and the geometric phase of the overall entangled state, the geometric phase of entanglement, after a full cycle is dependent on the overlap between stationary states of the $|A\rangle$ and $|B\rangle$ states. Specifically, it is zero whenever the $|A\rangle$ and $|B\rangle$ states are orthogonal, and in some circumstances when...
they have the same geometric phase. This sometimes corresponds to specific physical parameters, such as radius and tilt, or the vortex states described by $|A\rangle$ and $|B\rangle$, but not for all forms of vortex states. Additionally, for the geometric phase of the entangled state to be nontrivial, it is required that at least one of the single-photon states $|A\rangle$ or $|B\rangle$ have a nontrivial geometric phase. In this way, the geometric phase of the entangled state is still dependent on there being more than one eigenvalue in the $|A\rangle$ or $|B\rangle$ states.

These general rules apply not only to our vortices in this setting, but to any states when the conditions discussed section 5.2.1.2 are met. For vortices in our setting specifically, we have found how some specific physical parameters such as tilt or radius affect the geometric phase of entanglement for some specific vortex states, such as entangled single charge one vortices. However, it has also been shown that these relationships between the parameters and the geometric phase of entanglement do not hold in general for entanglement between states of different forms, such as between single-vortex and double-vortex states shown in appendix A.2.

Though beyond the scope of this thesis, the GPE can also be considered using a state operator formalism[41]. This allows it to be interpreted from the perspective of information theory, and a functional relationship can be made between the GPE and the reduced state entropies.
6.1 Results

This thesis has sought to elucidate the relationship between few-body optical vortex dynamics and the associated accumulation of geometric phase. Recent investigations have been carried out on the geometric phase of optical vortices transiting lenses [8, 11, 12], but analogous work has yet to be carried out in the much different setting [13, 14] of dielectrically trapped optical modes. This has motivated the first of two primary thesis questions:

*Can an optical trap cause geometric phase to be accumulated, and how is this accumulation linked to composition of underlying propagation modes?*

We have theoretically demonstrated that this is possible, but the relationship between geometric phase and vortex dynamics turns out to be quite nuanced. As explained in Section 4.2.2, there must be at least two modes with non-degenerate eigenvalues present for there to be a nontrivial geometric phase after a full cycle. Parameters that characterize the vortex dynamics, such as tilt angles and orbital radius, influence geometric phase by changing the weight of relevant system eigenvalues.

For a single charge-one vortex placed at the origin, the eigenvalues are always fully degenerate and the geometric phase is trivial. This is not the case, though, for double-vortex states provided both vortices are placed at the origin but are tilted. This is because the double-vortex state does not stay at the origin, while the single-vortex state does. For all single-photon cases examined, the tilt had no effect when the vortices remained at the origin. This demonstrates that it is the way the tilt of the vortex affects the trajectory of the vortex that, in turn, affects the geometric phase. However, this is not necessarily true for two-photon entangled states.

While the consideration of single-photon states has revealed new insights, it is an elucidation of the relationship between entanglement and geometric phase that is the primary focus of this thesis. A second thesis question was therefore posed:

*Is there a geometric phase associated with the entanglement of two optical vortex states that are orbiting within a trap?*

The answer is generally affirmative, but our investigation has shown that there are a variety of conditionals that should be attached. The geometric phase of entangled vortex states in an optical trap is, in general, not simply the sum of the geometric phase of the single-photon states. It cannot even always be...
equated to a weighted sum of their geometric phases. This motivated the consideration of a more restricted type of entangled state composed of $|AB\rangle$ and $|BA\rangle$ constituents.

We have thoroughly investigated the effect of entanglement for states of this form (Eq. (5.25)). It proved convenient to identify a *Geometric Phase of Entanglement (GPE)* as the difference between the geometric phase of such states and the sum of geometric phases associated with each photon separately. The GPE is dependent on the entanglement variables, $\alpha$ and $\beta$, and the overlap between constituent states. When the constituents are fully orthogonal, the GPE is always zero.

The tilt of a single-vortex state at the origin can still affect the geometric phase of the overall entangled state even though its single-photon geometric phase is zero. This is because vortex tilt influences orthogonality. This is not the case when the single-vortex state is entangled with a double-vortex state. These more complicated scenarios are considered in Appendix A.2.

The GPE was subsequently considered in even more restricted settings to provide additional insights. For instance, in one physically realizable setting, the GPE is proportional to the weighted geometric phases of its single-photon states along with a biasing cross-term. The cross-term amounts to a projection of one single-photon state in the direction of the time-derivative of another single-photon state. It therefore bears a resemblance to dynamic phase. This cross-term is worthy of additional research, and it may be that considering its classical counterpart may lead to physical interpretation of its role.

The cross-term can be recast under a set of even more restrictive assumptions, indirectly showing how orthogonality and eigenvalue degeneracy affect the GPE. In such settings, the GPE is proportional to the difference of the two single-photon geometric phases. Within this especially simple setting, a range of particular cases have been examined in detail that shed light on how to experimentally influence the GPE. It is worth noting that, in all cases examined, the geometric phase and the GPE are highly dependent on the overlap between states and the degeneracy of their eigenvalues.

### 6.2 Limitations

Any theoretical investigation necessarily includes a significant list of constraints, restrictions, and disclaimers. Some of these have already been discussed, such as a restriction to the setting of a harmonic dielectric trap. More general dielectric profiles could be explored, but care would need to be take to identify conditions such that the paraxial approximation is still valid. The dielectric profile could even be a function of axial position. Another restriction is that only cyclic evolutions are examined. This does not need to be the case, and later work could extend this to non-cyclic evolution. Most of the examination of entanglement, though not all, is restricted to entangled states of a particular form taken up in Chapter 5. A function for the geometric phase of a more general formulation is also identified, but specific cases of this...
form are not examined, and the work could be extended in this direction in the future.

Additionally, only two forms of vortex states are considered in detail for each photon, the single and double-vortex states. The world of possible vortex states is much wider than this and could be explored further in the context of this thesis. Furthermore all forms of single-photon vortex states in this thesis have had at most two different eigenvalues. This need not be the case and more cases with additional levels of complexity could be investigated in the same way. This could also facilitate investigation into how vortex braiding might play a role in the accumulation of geometric phase, as other states may have trajectories which better suit braiding.

6.3 Future Work

The work in the thesis could be expanded upon in several different directions.

The use of dielectrically trapped optical vortices is being investigated as a means of carrying out quantum logic [42]. The work of this thesis may find application there within the setting of two-qubit logic gates, where entanglement is called for. The insights for single-photon states obtained in this investigation are directly applicable to vortices which propagate through multi-mode fibers (MMF)[21, 23, 24]. Perhaps an even more straight-forward initial implementation, though, is within the setting of free-space beam propagation through a sequence of cylindrical lenses [8, 43–45]. While the specific setting of the cases analysed to find the behaviors of GPE was the MMF setting, and intended to be applied first there, the general formulas and properties of the GPE we have derived should apply equally well to the OAM approach [8, 21, 22, 43–45]. In these contexts the analyses in this thesis can be directly applied to future experimental efforts in linear optical quantum computing.

The method adopted for the generation of tunable entangled states using Type-I SPDC [37]readily lends itself to a variety of foundational experiments in which the degree of entanglement can be changed simply by setting up the measurement device so that it can select from a range of beam waists for the signal and idler photons. This would allow the GPE to be changed, in a quantitatively predictable way, without changing the device used to produce two-photon states.

More generally, many cases of single-vortex and double-vortex dynamics have been examined to quantify the relationship between orbit radius, tilt, and geometric phase. However, the space of possible vortex states is much larger and could be profitably explored for additional insights. Indeed, even with the single-vortex and double-vortex states examined, differences could be seen in how the geometric phase develops. We may find that other vortex states have entirely different properties that relate their physical parameters to the geometric phase.
The foundation laid by this thesis can also be extended to examine other types of traps. While harmonic traps have been particularly useful for this analysis, hard traps, the dielectric equivalent of an infinite cylindrical well in quantum mechanics, can also be produced experimentally. In fact, this characterizes the more standard type of dielectric fiber. The braiding of optical vortices is an ongoing area of research [46–48], and insights obtained in the current investigation may find application there. Finally, in an ongoing investigation [44], the GPE is considered within a state operator (density matrix) setting to explain how it can be viewed in terms of information gained by making a measurement—i.e. as an entropy of entanglement.
REFERENCES


This appendix contains additional thesis work that did not fit into the rest of the thesis but still has value to contribute to the subject.

## A.1 Additional exploration of trivial vs nontrivial geometric phase in entangled states

As mentioned in 5.2.2, the rules that govern the relationship between the eigenvalues of the states and the geometric phase of the entangled state for states of the form Eq. (5.25) do not necessarily apply for more general forms of entangled states. For instance, states of the form of Eq. (5.20) now have four contributions that must be considered instead of just two. If states $|A\rangle$, $|B\rangle$, $|C\rangle$, and $|D\rangle$ are individually degenerate for a different eigenvalue, then the geometric phase of each will be zero, but the overall entangled state will have a geometric phase of

$$\Phi_{Geom} = 2\pi \cos^2\left(\frac{\alpha}{2}\right)(\varepsilon_A + \varepsilon_B) + 2\pi \sin^2\left(\frac{\alpha}{2}\right)(\varepsilon_C + \varepsilon_D).$$

(A.1)

This may or may not be trivial depending on the choice of $\alpha$. In the case that $\varepsilon_A = \varepsilon_D$ and $\varepsilon_B = \varepsilon_C$, this reduces to (5.47), as one would expect. However, if $|A\rangle$ and $|C\rangle$ have a common degenerate eigenmode, and the same is true for $|B\rangle$ and $|D\rangle$, it is instead found that

$$\Phi_{Geom} = \frac{2\pi}{N}(\varepsilon_A + \varepsilon_B + \frac{1}{2} \sin(\alpha)(e^{i\beta}(\varepsilon_A \sum_{m,n} a^*_{m,n} c_{m,n} \sum_{j,k} b^*_{j,k} d_{j,k}) + e^{-i\beta}(\varepsilon_A \sum_{m,n} c^*_{m,n} a_{m,n} \sum_{j,k} d^*_{j,k} b_{j,k}) + \varepsilon_B \sum_{m,n} d^*_{m,n} b_{m,n} \sum_{j,k} c^*_{j,k} a_{j,k})).$$

(A.2)

Clearly, this is vastly more complicated. It may sometimes be trivial and sometimes nontrivial depending on the states, and not just the eigenvalues. For entangled states of the general form (5.20) it may be possible to have degree of entanglement still affect the geometric phase, even when all states are orthogonal. Additionally, it should be possible for all of the single-photon states $|A\rangle$, $|B\rangle$, $|C\rangle$, and $|D\rangle$ to have trivial geometric phases, and have the entangled state still have a nontrivial geometric phase. These are more examples of how the geometric phase can be highly influenced by entanglement. Even when states with individually trivial geometric phases are entangled they may then have an overall nontrivial geometric phase. This rich area of investigation was not tapped in the present investigation, where the focus is on states of the form of Eq. (5.25).
A.2 Single-Vortex with Double-Vortex Entanglement

The case of single-vortex with double-vortex entanglement was not covered in Chapter 5, however it was still investigated and is worth discussing.

It will be seen that many of the rules that have been found to apply to the states in that chapter will no longer apply. Whereas up till now we have examined states of the single-vortex form laid out in section 4.3 entangled with other states of the same form but with different parameters, we will now examine states of that form entangled with states of the double-vortex form laid out in section 4.4. The double-vortex state has an entirely different relation between its mode coefficients and its parameters. The overlap of the entangled states, and so the overlap of the modes, is one of the determining factors of GPE, and so it will be seen that some phenomena (such as the relative tilt of the vortices affecting the GPE) no longer apply.

Let the entangled state be written as

\[ |\psi(z)\rangle = \frac{1}{\sqrt{N}}(e^{-i\beta/2} \cos(\alpha/2)|C(z)\rangle_1 |D(z)\rangle_2 + e^{i\beta/2} \sin(\alpha/2)|D(z)\rangle_1 |C(z)\rangle_2) \]  

where \(|C\rangle\) is a single-vortex state of the form (4.12) with parameters \(x_{0,C}, \theta_C, \xi_C\) and \(|D\rangle\) is a two-vortex state of the form (4.22) with parameters \(x_{0,D}, \theta_D, \xi_D\). The normalizing constant becomes

\[ N = 1 + \cos \beta \sin \alpha c_{0,0}^* d_{0,0}^* d_{0,0}. \]  

Our geometric phase of entanglement is now

\[ \Phi_{GPE,\psi} = 2\pi \left( \frac{1}{1 + \cos \beta \sin \alpha c_{0,0}^* d_{0,0}^* d_{0,0}} - 1 \right) \left( \sum_{m=0, n=-2}^{1,2} \varepsilon_{m,n} d_{m,n}^* d_{m,n} - \sum_{n=-1}^{1} \varepsilon_{m,n} c_{m,n}^* c_{m,n} \right) \]  

It is important to note that the only overlapping mode is now the 0,0 mode. This means the entanglement dependence is entirely dependent on the \(c_{0,0}\) and \(d_{0,0}\) coefficients. If either of these is 0, then there is no dependence on entanglement. This can happen either whenever the single vortex state is placed at the origin, or when the double-vortex state has its parameters set such that

\[ \sin^2 \theta + 2x_{0,D}^2 (i \cos \xi_D + \cos \theta_D \sin^2 \xi_D) = 0. \]  

These cause either \(c_{0,0}\) or \(d_{0,0}\) to go to zero respectively, eliminating the geometric phase of entanglement. Either of these settings will also cause the individual respective state’s geometric phase, though not the entangled state’s geometric phase, to become trivial, as it will only have one eigenvalue. It should be possible, however, for us to have a geometric phase of entanglement of zero, and still have both individual single-photon states’ geometric phase and the overall entangled state geometric phase be nonzero. It would simply be required that both of the single-photon states be orthogonal, but for each to have multiple eigenvalues in their stationary states.
A.2.1 Case Examination: Single-Vortex State at the Origin

In section 5.3.3 it was shown how a single-vortex state at the origin can still affect the GPE through its overlap with another single-vortex state with nontrivial geometric phase. To contrast this, this section examines what happens when that single-vortex state at the origin is instead entangled with a double-vortex state. It will now be investigated how the overall geometric phase of the entangled state behaves when the single-vortex state is set at the origin. The coefficient $c_{0,0}$ is now zero. This is significant, as this is the only mode the single-vortex and double-vortex states share. Consequently, they will now be orthogonal. This in turn means that all dependence of the geometric phase on entanglement vanishes, and the geometric phase of entanglement is zero. This does not however mean that the overall geometric phase of the entangled state is trivial; as will be seen, it often is not.

The geometric phase of the single-vortex state on its own now becomes trivial. The overall entangled state geometric phase becomes only a function of the double-vortex state parameters. In fact, one can observe that the overall geometric phase of the entangled state is now equal to the double-vortex state with an additional trivial phase factor:

$$\Phi_{\text{Geom},\Psi} - \Phi_{\text{Geom},D} = 4\pi$$
$$\Phi_{\text{GPE},\Psi} - \Phi_{\text{Geom},D} = 4\pi$$ (A.6)

This is clearly not the same behavior seen before with entangled states involving only single-vortex states. This difference comes from, as stated, the fact that the single-vortex and double-vortex states share only a single mode. When the $p = l = 0$ state coefficient of the single-vortex state is zero as in this case and the case seen before, then an inner product between this state and the double-vortex state will always be zero, but another single-vortex state may have other nonzero mode coefficients for it to interact with, allowing for nonzero values.
B.1 Foucault pendulum precession vs latitude

For Figure 1.2 Graphs of precession period and precession per sidereal day vs latitude. The sign changes as a Foucault pendulum rotates anticlockwise in the Southern Hemisphere and clockwise in the Northern Hemisphere. The example shows that one in Paris precesses 271° each sidereal day, taking 31.8 hours per rotation. https://commons.wikimedia.org/wiki/File:Foucault_pendulum_precession_vs_latitude.svg. Licensed under Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0) https://creativecommons.org/licenses/by-nc-sa/4.0/