APPLICATIONS FOR CALCULUS

IN SCIENCE AND ENGINEERING

by

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ABSTRACT

As a result of the recent curriculum reform process at the Colorado School of Mines, a set of problems has been developed to address the issues of inspiring student interest in calculus, of helping students to connect calculus to other subjects and of helping students to recognize mathematics written in different forms. The problems are taken from subject material used in courses offered by other departments on the CSM campus. The focus in the problems is on the mathematics involved in solving the problem. Included with the problems is a discussion which gives the reasons for developing the problems, the methodology used to develop the problems, how to use the problems, difficulties with notation in the problems and intended evaluation of their use.

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DEDICATION

To my nephews and nieces,

Gretchen

Ross

Kate

Patrick

Braden

Becca

Sierra

Chapter 1

INTRODUCTION

In the recent curriculum reform process at the Colorado School of Mines, comments were made by faculty members from several different departments that students were finishing their calculus courses unprepared or underprepared to do the mathematics in their courses. Also, there were comments that students have difficulty translating the calculus they know into new concepts where problems might appear to be different. This led to the idea of developing a set of calculus problems based on concepts from courses offered in other departments on campus. By showing students calculus used in this manner, it is hoped that students will see the necessity of learning and knowing calculus and that seeing problems presented in different forms will help them with the process of translating calculus into different areas.

Reasons for the development of this course are supported by student and faculty surveys. Support for this idea is also given through development of similar projects conducted at several universities through grants from the National Science Foundation [3] and through other publications. Within the past few years, calculus for engineers textbooks have begun to appear which use a myriad of calculus and engineering problems as have other publications devoted to both engineering and nonengineering applications for calculus.

Evaluation of some of these problems has been conducted by introducing them to Calculus II and Calculus III honors students during the 1996/1997 academic year. Honors Calculus II students are students who have advanced placement credit based on a score of 4 to 5 on the AB exam for calculus. However there is a need for further evaluation that will consist of determining how well the students are able to solve problems, whether their overall problem solving capability improves and whether they are able to carry over what they learned into other courses. It will take a coordinated study over several semesters to obtain results. This model will be tested on students who did not have advanced placement credit in the fall of 1997.

The problems in this collection are intended as a supplement to the regular calculus curriculum. Although they cover a wide range of calculus topics, there are several topics not covered at all. The problems have been developed using material taken from courses offered through the various departments on the CSM campus. It is not the intention for these problems to teach engineering or scientific concepts, but to concentrate on the mathematics being used to develop the concept. Within each problem, a concept is introduced and the necessary equations are derived for the students. From this a mathematics problem or series of problems is posed which students are expected to solve. The student only works through the mathematics - he or she is not expected to fully grasp the physical concept - only understand where the problem comes from and how to develop the mathematics.

This problem set was developed by collecting course material from professors in other departments. Many of the problems come from textbooks similar to the ones being used in courses. In some cases, problems were written directly by the professors, but in most instances they were developed by this writer. From the collection of material, an introduction to the concept was written and the problem was developed. All essential material is explained or derived. Students are only expected to work through the actual mathematics.

Chapter 2

WHY DEVELOP THESE APPLICATION PROBLEMS?

During the ongoing curriculum revision process at the Colorado School of Mines, many comments have been made that students were unable to use or apply what they have learned in calculus to other subjects. These comments resulted in the idea of taking concepts out of subject areas and concentrating on the mathematics involved. The result is to make calculus more relevant to these subjects and to help students see how calculus is applied. Interviews and written surveys with faculty members across campus have supported this idea and many faculty members have contributed material to this project. As a result, this thesis will present several problems involving different topics of calculus as it is applied to problems in Engineering, Chemical Engineering, Chemistry, Physics, Geology, Mining Engineering and Environmental Science among other areas. It is intended that these problems supplement the curriculum in ways that are not found in standard textbooks and that are specific to CSM.

Why is it desirable to have problems which apply calculus to a number of different fields? The goals for these problems are threefold. It is expected that these problems will provide:

- ways to inspire and interest students in calculus;
- ways to connect calculus to other subjects;
- ways to help students recognize mathematics written in different symbols.

These goals were developed from comments made by faculty members from both inside and outside the Mathematical and Computer Sciences (MCS) Department. Faculty members teaching calculus often feel that students are just trying to get credit in the course and that many students do not realize that they will encounter many problems using calculus in their chosen majors. Faculty members teaching in other disciplines often comment that students do not recognize the type of problem they are facing when the symbols have been changed. They say that this often leads to reteaching many calculus topics that students should know.

Specific comments which have motivated this project came through a faculty survey and through faculty interviews. One engineering professor states that "It is very important for the students to see that math is not just an abstract concept - but an extremely useful and necessary subject to be combined with science and engineering." That students are often, but not always, missing this point is shown by the faculty survey and through a math diagnostic exam given by the Physics Department to all Physics I students.

To get a more concise understanding for what needed to be done in constructing these problems, a faculty survey was developed and sent out to faculty members in each of the major science and engineering departments on campus. The survey consisted of three questions and a summary of the response is as follows. The first question asked for specific lists of mathematical topics of skills which pose difficulty for students. The response included topics such as complex variables, matrix manipulation, definite integrals, differentiation, simple first order differential equations, algebra, trigonometry, interpretation and construction of graphical relationships, application of symmetry information to simplify integrals in two or three dimensions, vector differences and differentials and understanding the chain rule. There was one reply stating that students generally exhibited the required mathematical knowledge. The second question asked if each respondee felt that students have forgotten the math, never learned the math or are confused about the change in vocabulary or symbols. Responses indicated that a change in vocabulary or symbols creates the most problems, followed by students not learning the math in the first place. Only one person felt that students forgetting what they have learned was a problem. The third question asks how much time is spent in reteaching math. Most people responded that little reteaching was done, but that students are expected to review on their own. However, the physics department states that a large amount of classroom time must be spent in teaching mathematical concepts. All of the people who have responded to this survey have contributed information from which problems were developed for this project.

At the beginning of the spring, 1997 semester, a math diagnostic exam was given to all Physics I students. The exam consisted of nine basic computational, algebra and calculus problems. Of 220 students who took the exam, 23% scored between 70% and 100% on the test, and 66% scored between 40% and 70% on the test. Possible conclusions from this are that students don't know their math, or that they are having a difficult time translating what they do know to problems that look different compared to what is found in their calculus textbook.

The problems presented in this thesis have been designed to address the issues that have been exposed by the survey, the diagnostic test and other comments. The problems consist of a background explanation and are written in the symbols and terminology used in the course from which they were taken. However, students need only solve the mathematics in the problem and be able to distinguish between the different uses of notation and symbols.

Further reasons for developing this problem set are based on recent calculus curriculum reform efforts where an emphasis is being placed on problem solving. Throughout the last ten years, many efforts have been made to improve the quality of instruction in calculus and to make calculus more relevant to other science and engineering subjects. Much of this came about when too many students were leaving science and engineering programs because they found the strict lecture technique dull and unwelcoming [10]. Textbooks and supplemental materials developed from this effort can be valuable references for additional problems beyond what has been produced in this thesis.

Beginning in 1986, a calculus reform effort was started by the Mathematical Association of America (MAA) and was supported by grants from the National Science Foundation (NSF). The reforms have moved away from the lecture and drill and practice teaching to using methods which have students working in small groups solving multi-step problems that come from a variety of disciplines, using graphing calculators and computers and writing lengthy explanations of their solutions [29]. Students are expected to move comfortably between symbolic, verbal, numerical and graphical representation of mathematical ideas [25]. Although criticisms have been made about calculus reform, such that, it is over simplified and too reliant on calculators, many of the critics will admit that the calculus reform movement has had a positive impact on the attitudes of mathematics faculty toward teaching [10].

One of the significant results of the reform movement was the publication of the "Harvard" Calculus series by Deborah Hughes Hallett, et al. These books are based on two principles [22] [15], "The rule of three: every topic should be presented geometrically, numerically and algebraically and The Way of Archimedes: Formal definitions and procedures evolve from the investigation of practical problems." The book generally starts a new concept with a practical problem and proceeds into a generalization of the concept. This design came into play when the Harvard faculty's response to the question of where students stumble mathematically in their courses was that students could not make use of what they had been taught [14]. It is the contention of the authors that this new method promotes thinking about problems in different ways by asking students to explain their reasoning.

Other textbooks and applications problems have recently been published and can be used as further resources for applications. The "Resources for Calculus" collection published by MAA [27] has a collection of problems and projects from a variety of engineering and scientific fields. "Calculus for Engineering and the Sciences" by Elgin H. Johnston and Jerold C. Mathews [18], applies many calculus concepts to engineering problems as does "Calculus for Engineers" by Robin Carr and Bill Goh [5]. New editions of standard texts also include new applications and creative use of technology.

In conjunction with the calculus reform effort, several universities are conducting programs to build application problems in a manner similar to this thesis. The NSF has multi-year grants funding these efforts [3]. Three key universities in this project are Rensselaer Polytechnic Institute, University of Pennsylvania and Dartmouth College. They are working together with consortia of universities. All of the universities involved in this project are not only constructing applications problems, but intend to integrate them into their curricula. Many of these applications will be available on the World Wide Web for general use.

Rensselaer Polytechnic University has created Project Links, which is a library of hypertext documents that will link calculus, differential equations, mechanics, linear systems and probability and statistics to different fields such as biomedical engineering, chemical engineering, electrical and computer systems engineering, mechanical engineering, physics, chemistry and biology [4]. They will be working in conjunction with the University of Delaware, Sienna College, Virginia Polytechnic Institute, Central State University, Hudson Valley Community College and the University of Maryland. The objectives for Project Links are to stimulate greater cooperation in educational development among faculty in mathematics and other disciplines, to encourage interactive teaching and learning strategies and to produce instructional materials for use in studio-type courses, to create a library of hypertext modules that link topics in mathematics usually studied by students of engineering or science and to continue pioneering efforts in the application of contemporary technology for educational purposes [4]. Information about Project Links and what they have completed so far can be found on the World Wide Web at http://www-links.math.rpi.edu.

The NSF project at the University of Pennsylvania has developed the Middle Atlantic Consortium for Mathematics and its Applications Throughout the Curriculum(MACMATC). The goals of the consortium are to integrate research and real-world applications into the basic mathematics curriculum and to achieve more effective integration of advanced mathematics and computing into the upper-level curricula of disciplines that use it [7]. The consortium consists of the University of Pennsylvania, Villanova, Polytechnic University and the Community College of Philadelphia. Each school is working on a series of application problems which include image analysis, focusing on methodologies that link geometry and computing, mathematics and finance, mathematical thermodynamics, mathematical connections to business, social science and everyday life, a problem of flight speed vs. travel time to introduce students to linear and nonlinear phenomena, elasticity in economics as an application for advanced calculus and Snell's law for light refraction in geometric optics [2]. As this is an ongoing project, evaluation is only just beginning. Information about MACMATC including a lengthy description can be found on the World Wide Web at http://www.math.upenn.edu/ ugrad/macmatc/gendesc.html.

Dartmouth College has formed the Center for Mathematics Education which is responsible for its Mathematics Across the Curriculum (MATC) project. The Center is responsible for collecting and housing materials and resources generated by MATC, as well as a collection of standards, curriculum and articles relevant to the integration of mathematics at various levels [24]. Information about the center can be found on the World Wide Web at http://www.dartmouth.edu/ matc/GRANT.REV.7.html.

Chapter 3

FORMULATION OF THE PROBLEMS

The problems in this thesis were chosen by considering how calculus is applied to other subjects and to foster student interest in calculus. Nearly all of the problems were taken from material used in courses taught by departments other than the MCS Department. Choices were made in collaboration with the professors who teach the courses. In choosing the problems, professors were asked to pick topics where students had difficulty with the mathematics involved. The problems have then been redesigned so that there is just an introduction to the topic from which the problem has been drawn and the main focus is on the mathematics contained within the problem.

There are also a few problems that are unrelated to a specific course, but were created to enhance certain subject areas. They still have a relationship to science or engineering, but look at a concept in a different manner. An example of this is the problem asking students to evaluate gradients and directional derivatives using contours on a topographic map.

The problems were collected through personal visits with professors and by corresponding with others through e-mail. Everyone who was approached for input into this project responded enthusiastically and many interesting concepts and ideas came out of the discussions. The amount of material available for developing problems is considerably more than was used in this thesis, so prospects for expanding the number and types of problems are excellent.

The problems were developed as material became available. Many of the problems have a short introductory section which tells where the problem comes from and gives some background material needed to do the problem. Since the problems are being applied to a subject, a description of the notation is included. Often, in order to work the problem, students must be able to identify the parameters by their meaning and make correct deductions toward the solution of the problem using those parameters. The procedure used for solving the problems concentrates on the mathematics needed. In some of the problems, students are led step by step through the solution, filling in the mathematical details. In some cases, two or three variations of the same problem are given and after being led through the first part students are expected to set up and work out the next part without as much guidance. Any equations that need to be derived from the written information are given with a brief explanation. In other problems, students will have to deduce the solution with little guidance by relying on their knowledge of mathematics. Although most problems lead to a specific solution, a few are open ended allowing for some interpretation. Many of the solutions involve using a mathematical procedure for developing formulas which are commonly used in other courses.

Students are not expected to grasp the full meaning of the subject until they get to the proper course, however, they should be able to develop a good understanding of the mathematics involved in solving these problems. It is not the intent of these problems to teach material in other subject areas, but to use that material to help students make use of their mathematical knowledge now and in the future.

Chapter 4

USING THE PROBLEMS

How these problems are used is up to the instructor, depending on what works best for him or her. They can be worked on individually by students, but are probably more easily solved by students working in groups. What needs to be done to work the problem is not always obvious and groups will stimulate discussion and generate ideas about the problems.

Chapter six contains two content tables for this problem set, one listing the problems by calculus topic and the other listing the problems according to the source. The problems have been organized according to the calculus topic they cover so that it is possible to choose from several problems covering one topic, depending on what draws the most interest. The problems are intended to supplement the material in the textbook. When students approach these problems, they should already have been introduced to the calculus concepts involved.

During the fall semester of 1996, some of these problems were given to students in the Calculus II Honors class. Students were given time in class to work on problems and were asked to work in groups. Many of the students found that the interaction in the group helped them formulate ideas and after some discussion, groups were able to come up with solutions. Students were asked to come up with a single paper explaining their group's solution and the paper was expected to be well written and neat. Since solutions vary from finding a formula or specific number to writing an estimate, how students prepare their solutions is important and should be done carefully. Although it sometimes takes a lot of effort to get students started in group work, once they begin to see the benefits, they find the group work to be helpful.

Although care was taken to insure the legitimacy of each problem, it is not possible to be an expert in all areas, so inaccuracies may exist. If a student thinks that there is an inaccuracy in a problem, this is an excellent opportunity for that student to do some extra exploration on that subject by making contact with professors in other departments.

Time to solve the problems varies greatly. Some problems can be used as examples during class or in short problem solving sessions; others will take two to three hours to finish. The longer problems can be shortened, but in some cases the point of the problem may be lost. An estimated time for completion has been listed with each problem.

Chapter 5

USE OF NOTATION

The use of different notations continually causes problems for students. Scientists and engineers use notation that matches the ideas they are trying to convey and to better define formulas so that they are easier to use. It often appears much different than what is seen in a calculus book. For example, in a calculus course, students may be asked to solve $\frac{dy}{dx} = -2y$ for y whereas in a reaction rate problem in a chemistry course the problem may look like $\frac{d[\mathrm{A}]}{\mathrm{dt}} = -k[\mathrm{A}][\mathrm{C}]$ where students solve for [A]. It should be easy for students to see that both problems can be solved using identical steps, and that their solutions will have an exponential form, depending on an initial condition. This problem occurs in yet another form in electric circuits as $\frac{dv}{dt} = -\frac{v}{CR}$. Again, the symbols are different, but the method to obtain the solution is identical. Often, students who can solve the first problem with ease will experience total confusion on the second and third problems.

An effort has been made to keep the notation true to the original problem to expose students to the different symbols. Although an attempt has been made to give meaning to the symbols used in these problems, not all of the notation is clearly explained since it would mean delving much deeper into the scientific subject. That is not the primary intent of this study. However, students should be able to distinguish between what are constants and what are variables and understand the meaning of different functions.

To help ease notation difficulties, many of the problems have a "Troublesome Notation" section describing anticipated areas where notation may be misinterpreted. These sections have been written for students and are areas instructors may want to make students aware of. Some of the more common notation difficulties are listed below.

- Symbols versus Numbers: In many of the problems in this thesis, we are deriving the mathematics behind an idea or concept. Hence, each symbol takes on a different meaning and it is important to understand the meaning since some may represent constants and others functions of variables. Most of the problems will be worked in terms of symbols, so students must have an understanding of what they are solving for.

- Vector Notation: Designation of vectors varies greatly. In some texts, vectors are denoted in boldface, $\mathbf{a}$, in others with an arrow, $\vec{a}$, and yet in others as just plain a. Most calculus books use the boldface designation, so when the others occur, confusion can result. To add to this, unit vectors can be written like any vector as shown or may be written as $\hat{a}$. Vector components may also take on several forms. If a_1, a_2 and a_3 are components of **a**, we may denote **a** as (a_1, a_2, a_3), or $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. If things are not crazy enough at this point, when denoting the magnitude of a vector, **a**, we may write just plain a or use the absolute value symbols, $|a|$.

- Subscripts: Often in engineering, subscripts are used on variables to give the variable a clearer meaning in the equation. For example, in $\frac{dM}{dt} = \rho\,(q_{in} - q_{out})$, q_{in} represents a flow quantity in and q_{out} represents a flow quantity out. Another problem may use q_x to indicate a flow quantity in the x-direction. Confusion may result when a subscript is used to indicate a partial derivative. For instance, q_x in another problem context may indicate the partial derivative of q with respect to x. Sometimes subscripts may just look confusing. For example, although $dF_A = d(F_{AO}(1 - X_A))$ appears to have a lot going on, the subscripts are being used only to help keep track of the meaning of the variables.

- Length and Distance Notation: Many different symbols can be used to indicate length or distance. Often in the description of the problem one symbol, say s, is used to indicate distance in a formula, yet in a specific problem, x is used for distance since the distance is on the x-axis or in the x-direction.

There are many other difficulties not listed here that will occur when these problems are presented to students. Students need to learn to be aware of differences in notation and to recognize patterns in types of problems written in different forms. It is also important that students learn to keep track of the meaning of the different symbols and variables in a problem. This will help avoid confusion.

Chapter 6

SUMMARY OF THE PROBLEMS

The problems in this thesis are summarized in tables here to give the user quick access to information about them. Table 1 is a listing of problems by calculus topic and table 2 is a listing of problems by subject. Some problems apply more than one area of calculus, so they may be listed twice. The problems have been organized by calculus topic as much as possible.

Table 6.1: List of Problems by Calculus Topic

Calculus Topic Subproblem	Problem No.
Cross products	
Moments on force systems (Engineering)	7.19
Curvature	
Temperature profile with heat generation (Chemical Engineering)	7.26

Calculus Topic Subproblem	Problem No.
Cylindrical coordinates	
Temperature profile with heat generation (Chemical Engineering)	7.26
Differentiation	
Motion of a collar on a swinging arm (Engineering)	7.1
Motion of a kicked football (Engineering, Physics)	7.4
Projectile motion (Engineering, Physics)	7.4
Directional derivatives	
Concentrations in a field (Chemical Engineering)	7.27
Gradients and directional derivatives from a topographic map (Geology)	7.27
First order differential equation with initial conditions	
Age of a rock formation (Geology)	7.13
Conservation of mass used to determine the flow through a system (Chemical Engineering)	7.22
Current in a resistor-inductor circuit (Engineering)	7.7
Dissapation of smoke in a room (Environmental Science)	7.6
Effect of friction on a fluid moving through a pipe (Chemical Engineering)	7.23

Calculus Topic Subproblem	Problem No.
First order differential equation with initial conditions	
Flow of a gas mixture through a tank (Chemical Engineering)	7.12
Growth rates for the human population on earth (Environmental Science)	7.11
Mass balance in tank flow (Chemical Engineering)	7.14
Motion of an object given its acceleration (Engineering, Physics)	7.8
Output of a plug flow reactor using reaction rates (Chemical Engineering)	7.15
Potential difference in a resistor-capacitor circuit (Engineering)	7.7
Reactions in a batch reaction system (Chemical Engineering)	7.10
Transfer of heat through different objects (Chemical Engineering)	7.9
Gradients	
Concentrations in a field (Chemical Engineering)	7.27
Gradients and directional derivatives from a topographic map (Geology)	7.27

Calculus Topic Subproblem	Problem No.
Indefinite Integration	
Estimating groundwater discharge and recharge from hydrographs (Geology)	7.5
Integration	
Electric potential at a point for different types of charged objects (Physics)	7.30
Estimating groundwater discharge and recharge from hydrographs (Geology)	7.5
Mass balance in tank flow (Chemical Engineering)	7.22
Output of a plug flow reactor using reaction rates (Chemical Engineering)	7.15
Potential difference due to continuous charge distributions (Physics)	7.20
Strength of a magnetic field at a point near a charged object (Physics)	7.31
Lagrange Multipliers	
Famine relief (Economics)	7.28

Calculus Topic Subproblem	Problem No.
Line Integrals	
Forces acting on particles on described paths (Physics)	7.29
Strength of a magnetic field at a point in a charged wire loop (Physics)	7.31
Work done by a spring (Physics)	7.29
Work done by friction (Physics)	7.29
Work required for a rock cutting machine to chip rock from the face of an excavation (Mining Engineering)	7.21
Mass Moment of Inertia	
Wear on steel spheres used in a ball mill (Mining Engineering)	7.24
Maxima	
Projectile motion (Engineering, Physics)	7.4
Moment at a point on a mechanism (Engineering)	7.3
Motion of a kicked football (Engineering, Physics)	7.4
Minima	
Movement of a secret agent (Physics)	7.2

Calculus Topic Subproblem	Problem No.
Multiple Integration	
Conservation of mass used to determine the maximum velocity of a fluid moving through a pipe (Chemical Engineering)	7.22
Effect of friction on a fluid moving through a pipe (Chemical Engineering)	7.23
Partial Differentiation	
Check solutions for the heat equation, the ideal gas law and the wave equation (Engineering, Physics)	7.25
Polar Coordinates	
Conservation of mass used to determine the maximum velocity of a fluid moving through a pipe (Chemical Engineering)	7.22
Simultaneous Equations	
Demands on a water storage tank (Environmental Science)	7.10
Spherical Coordinates	
Temperature profile with heat generation (Chemical Engineering)	7.26

Calculus Topic Subproblem	Problem No.
Surface Integrals	
Electric flux over a surface (Physics)	7.32
Magnetic flux over a surface (Physics)	7.32
Vectors	
Action of vehicles on a mine haul road (Mining Engineering)	7.17
Concentrations in a field (Chemical Engineering)	7.27
Electric flux over a surface (Physics)	7.32
Gradients and directional derivatives from a topographic map (Geology)	7.27
Forces acting on a moving object (Engineering)	7.16
Force vectors (Engineering)	7.19
Relative velocity of a moving object (Physics)	7.16
Rescue problem (Physics)	7.18
Strength of an electric field at a point (Physics)	7.30

Table 6.2: List of Problems by Source

Problem / Subproblem	Source / Calculus Topic	Problem No.
Reaction Rates	Chemistry, Environmental Science	
Reactions in a batch reaction system.	first order differential equation with initial conditions.	7.10
Amount of nitrate in drinking water sources.	simultaneous equations.	7.10
Demands on a water storage tank.	first order differential equation with initial conditions.	7.10
Flow of Fluids in Pipes and Tanks	Chemical Engineering	
Conservation of mass used to determine the maximum velocity through a pipe.	multiple integrals, polar coordinates.	7.22
Conservation of mass used to determine the flow through a system.	first order differential equation with initial conditions.	7.22

Problem Subproblem	Source Calculus Topic	Problem No.
Friction Factors in Flow through a Pipe	Chemical Engineering	
Effect of friction on a fluid moving through a pipe.	first order differential equation with initial conditions, multiple integration.	7.23
Gas Balance	Chemical Engineering	
Flow of a gas mixture through a tank.	first order differential equation with initial conditions.	7.12
Heat Transfer	Chemical Engineering	
Transfer of heat through different objects.	first order differential equation with initial conditions.	7.9
Mass Balance	Chemical Engineering	
Draining and filling of tanks.	first order differential equation, using integration techniques.	7.14
Reaction in a Plug Flow Reactor	Chemical Engineering	
Output of a plug flow reactor using reaction rates.	first order differential equation, using integration techniques.	7.15

Problem Subproblem	Source Calculus Topic	Problem No.
Temperature Profile		Chemical Engineering
Unsteady-state heat conduction.	gradients, curvatures, cylindrical and spherical coordinates.	7.26
Famine Relief		Economics
Distribution of food.	Lagrange multipliers.	7.28
Dynamics		Engineering
Tangential velocity and acceleration.	partial differentiation.	7.1
RC and RL Circuits		Engineering
Potential difference in a resistor-capacitor circuit.	first order differential equation with initial conditions.	7.7
Current in a resistor-inductor circuit.	first order differential equation with initial conditions.	7.7
Statics		Engineering
Moments on swinging booms.	finding a maximum.	7.3
Moments on force systems.	vectors, cross products.	7.19

Problem Subproblem	Source Calculus Topic	Problem No.
Heat and Wave equations	Engineering, Physics	
Checking solutions of the heat equation, wave equation and ideal gas law.	partial differentiation.	7.25
Objects in Motion	Engineering, Physics	
Motion in a recoil mechanism.	first order differential equation with initial conditions.	7.8
Motion of an object affected by gravity.	first order differential equation with initial conditions.	7.8
Acceleration of a particle.	first order differential equation with initial conditions.	7.8
Movement of a secret agent.	vectors, finding a minimum.	7.2
Projectile Motion	Engineering, Physics	
Parametric equations for position to find velocity and acceleration.	differentiation, finding a maximum.	7.4
Motion of a kicked football.	differentiation, graphing, finding a maximum.	7.4

Problem Subproblem	Source Calculus Topic	Problem No.
Logistic Growth of Human Population	Environmental Science	
Growth rates for the human population on earth.	first order differential equation with initial conditions.	7.11
Smoke in a Room	Environmental Science	
Dissapation of smoke in a room.	first order differential equation with initial conditions.	7.6
Geolgical Dating	Geology	
Age of a rock formation.	first order differential equation with initial conditions.	7.13
Hydrographs	Geology	
Using a hydrograph to estimate groundwater discharge and recharge.	integration, indefinite integration.	7.5
Mapping	Geology	
Gradients and directional derivatives from a topographic map.	vectors, gradients, directional derivatives.	7.27
Ball Mill Problem	Mining Engineering	
Evaluation of the wear of steel spheres in a ball mill.	mass moment of inertia, integration.	7.24

Problem Subproblem	Source Calculus Topic	Problem No.
Haul Roads		Mining Engineering
Action of a vehicle on a haul road out of a mine.	vectors.	7.17
Rock Excavation		Mining Engineering
Energy required for a rock cutting machine to chip rock from the face of an excavation.	line integral, dot product.	7.21
Electric Fields as Vector Fields		Physics
Strength of an electric field at a point.	vectors, setting up and evaluating an integral.	7.30
Electric Flux		Physics
Electric flux over a surface.	surface integrals, vectors.	7.32
Magnetic flux over a surface.	surface integral, dot product.	7.32
Magnetic Fields		Physics
Strength of a magnetic field at a point near a charged object.	cross product, setting up and evaluating an integral.	7.31
Strength of a magnetic field at a point in a charged wire loop.	line integral, cross product.	7.31

Problem / Subproblem	Source / Calculus Topic	Problem No.
Vector Problems		Physics
Sight of an airplane crash - Rescue problem.	vectors.	7.18
Relative velocity of a ferry.	vectors.	7.16
Motion on a pitched baseball.	vectors.	7.16
Wind acting as a force.	vectors.	7.16
Work and Potential Difference		Physics
Electric potential at a point for different types of charged objects.	finding the differential element, setting up integrals, line integrals.	7.29
Work Using Line Integrals		Physics
Work done by friction.	line integral, dot product.	7.20
Work done by a spring.	line integral, dot product.	7.20
Forces acting on particles on described paths.	line integral, dot product.	7.20

Chapter 7

PROBLEMS

The thirty-two problems in this chapter are the focus of the thesis. They should be self explanatory and a solution follows each problem.

7.1 Motion of a Collar on a Swinging Arm

Calculus Topic:	Differentiation
Department:	Engineering
Subject Area:	Dynamics
Time Needed:	20 minutes
Reference:	[1]

The mechanism shown in figure 7.1 is dynamic, that is when forces are applied, movement takes place. In this case the motion is angular, which means that position, acceleration and velocity depend on the angle the object moves through and the distance from the center of movement. Velocity and acceleration can be calculated in several different ways, for instance, with respect to angular or rectangular coordinates.

Troublesome Notation: Both $\vec{e}_r$ and $\vec{e}_\theta$ are defined as unit vectors in this problem, which is similar to the notation used for the vectors $\vec{v}$ and $\vec{a}$. In calculus unit vectors are indicated by the hat symbol, $\hat{\imath}$.

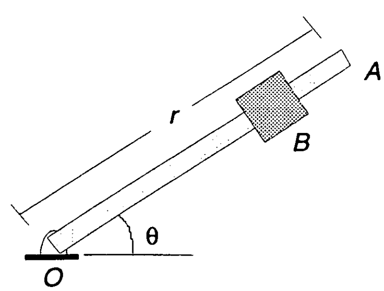

Figure 7.1: Collar on a swinging arm moving through the angle θ.

This is a motion problem involving radial and tranverse components, i.e., it is dependent on the values of r and θ. In Figure 7.2, $\vec{e}_r$ is called the unit radial vector

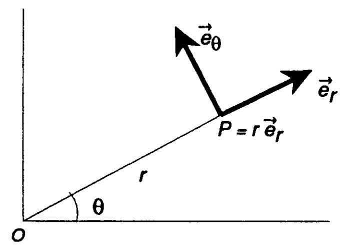

Figure 7.2: Tangential and radial vectors.

and $\vec{e}_\theta$ is the unit tangential vector. Hence the position of the particle P is $r\vec{e}_r$. The following relationships hold for the radial and tangential vectors:

$$\frac{d\vec{e}_r}{d\theta} = \vec{e}_\theta \qquad \text{and} \qquad \frac{d\vec{e}_\theta}{d\theta} = -\vec{e}_r \tag{7.1}$$

Using the chain rule, we can express the time derivatives of the unit vectors $\vec{e}_r$ and $\vec{e}_\theta$ as follows:

$$\frac{d\vec{e}_r}{dt} = \frac{d\vec{e}_r}{d\theta}\frac{d\theta}{dt} = \vec{e}_\theta\frac{d\theta}{dt} \qquad \text{and} \qquad \frac{d\vec{e}_\theta}{dt} = \frac{d\vec{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\vec{e}_r\frac{d\theta}{dt} \tag{7.2}$$

To obtain the velocity $\vec{v}$ of the particle P, we express the position vector $\vec{r}$ of P as the product of the scalar r and the unit vector $\vec{e}_r$ and differentiate with respect to t.

This yields:

$$\vec{v} = \frac{d}{dt}(r\vec{e}_r) = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta. \tag{7.3}$$

To find the acceleration of particle P we differentiate again with respect to t which yields:

$$\begin{aligned}\vec{a} = \frac{d\vec{v}}{dt} &= \frac{d^2r}{dt^2}\vec{e}_r + \frac{dr}{dt}\frac{d\vec{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta + r\frac{d^2\theta}{dt^2}\vec{e}_\theta + r\frac{d\theta}{dt}\frac{d\vec{e}_\theta}{dt} \\ &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\vec{e}_\theta \end{aligned} \tag{7.4}$$

In Figure 7.1, the rotation of the 0.5 m arm OA about O is defined by the relation $\theta = 0.13t^2$, where θ is expressed in radians and t in seconds. Collar B slides along the arm in such a way that its distance from O is $r = 0.5 - 0.12t^2$. In this case r is expressed in meters and t in seconds. After the arm OA has rotated through an angle of θ, we want to determine

1. the magnitude of the total velocity of the collar,
2. the magnitude of the total acceleration of the collar, and
3. the magnitude of the relative acceleration of the collar with respect to the arm.

After the arm has rotated through an angle of 30°, determine

4. the magnitude of the total velocity of the collar, and
5. the magnitude of the total acceleration of the collar.

Solutions to: Motion of a Collar on a Swinging Arm

1. We are given $\theta = 0.13t^2$ and $r = 0.5 - 0.12t^2$. We also know

$\vec{v} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$. Therefore

$$\vec{v} = -0.24t\vec{e}_r + \left(0.5 - 0.12t^2\right)(0.30t)\,\vec{e}_\theta = -0.24t\vec{e}_r + \left(0.15t - 0.036t^3\right)\vec{e}_\theta$$

$$v = \sqrt{(-0.24t)^2 + (0.15t - 0.036t^3)^2}$$

2. We know that $\vec{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\vec{e}_\theta$. Therefore

$$\vec{a} = \left(-0.24 - (0.26t)^2(0.5 - 0.12t^2)\right)\vec{e}_r + \left((0.5 - 0.12t^2)(0.26) + 2(-0.24t)(0.26t)\right)\vec{e}_\theta$$

$$= \left(-0.24 - 0.0338t^2 + 0.008112t^4\right)\vec{e}_r + \left(0.13 - 0.156t^2\right)\vec{e}_\theta$$

$$a = \sqrt{(-0.24 - 0.0338t^2 + 0.008112t^4)^2 + (0.13 - 0.156t^2)^2}$$

3. The acceleration of the collar with respect to the arm is

$\vec{a}_{B/OA} = \frac{d^2r}{dt^2}\vec{e}_r = -0.24\vec{e}_r$. The magnitude of the acceleration is

$$a_{B/OA} = 0.24\ \frac{\text{m}}{\text{s}^2}.$$

4. When $\theta = 30^\circ = 0.524$ rad, $0.524 = 0.13t^2 \quad \Rightarrow \quad t = 2.01$ s

$$v = \sqrt{(-0.24t)^2 + (0.15t - 0.036t^3)^2}$$

$$= \sqrt{(-0.24(2.01))^2 + (0.15(2.01) - 0.036(2.01)^3)^2} = 0.48 \ \frac{\text{m}}{\text{s}}$$

5. $$a = \sqrt{(-0.24 - 0.0338t^2 + 0.008112t^4)^2 + (0.13 - 0.156t^2)^2}$$

$$= \sqrt{(-0.24 - 0.0338(2.01)^2 + 0.008112(2.01)^4)^2 + (0.13 - 0.156(2.01)^2)^2}$$

$$= 0.560 \ \frac{\text{m}}{\text{s}^2}$$

7.2 Secret Agent Problem

Calculus Topic:	Minima
Department:	Physics
Subject Area:	Mechanics
Time Needed:	30 minutes
Reference:	[17]

In this problem it is necessary to develop an equation representing time in terms of distance traveled, then find a minimum.

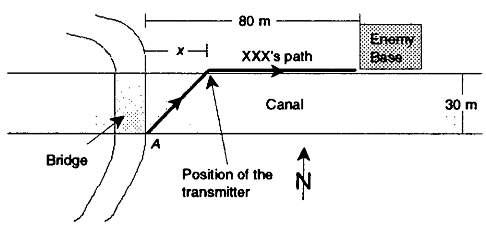

Figure 7.3: Secret Agent XXX's path.

Your mission, should you decide to accept it, is to guide Secret Agent XXX to her destination in a minimum amount of time. The secret agent is to travel under darkness from the south end of the bridge to the secret enemy base as shown in figure 7.3. She will have to swim across the canal and then creep along its bank to get there. You are to place an ultrasonic transmitter at the canal's edge; the secret agent will swim directly toward the transmitter and then creep from the transmitter to the warehouse. Determine the placement distance x from the north end of the bridge for the transmitter that will minimize the XXX's travel time. Your only information is that XXX's swimming speed is half of her creeping speed and that the width of the canal is 30 m and the distance from the north end of the bridge to secret base is 80 m. Develop an expression for the travel time in terms of x and the fixed parameters involved in the problem, then find the minimum.

Solutions to: Secret Agent Problem

The distance from A to the transmitter is

$$d = \sqrt{30^2 + x^2}$$

and the time required for the agent to swim across the canal is

$$t_s = \frac{d}{v_s} = \frac{\sqrt{30^2 + x^2}}{v_s}. \quad (v_s = \text{swimming velocity})$$

The distance the agent must creep is $80 - x$, so the creeping time is

$$t_c = \frac{80 - x}{v_c}. \quad (v_c = \text{creeping velocity})$$

The total travel time for the agent is

$$t = t_s + t_c = \frac{\sqrt{30^2 + x^2}}{v_s} + \frac{80 - x}{v_c}.$$

Since $2v_s = v_c$, $\qquad t = \frac{1}{v_c}\left(2\sqrt{30^2 + x^2} + 80 - x\right)$

To find the minimum t with respect to x, take the derivative and set

it equal to zero.

$$\frac{dt}{dx} = \frac{1}{v_c}\frac{d}{dx}\left(2\sqrt{900 + x^2} + 80 - x\right) = \frac{1}{v_c}\left(\frac{2\left(\frac{1}{2}\right)(2)\,x}{\sqrt{900 + x^2}} - 1\right)$$

$$= \frac{1}{v_c}\left(\frac{2x}{\sqrt{900 + x^2}} - 1\right) = 0$$

Solving for x yields

$$\frac{2x}{\sqrt{900 + x^2}} = 1 \qquad \Rightarrow \qquad 900 + x^2 = 4x^2 \qquad \Rightarrow \qquad 3x^2 = 900 \qquad \Rightarrow$$

$x = \pm 17.3$ m $\quad +17.3$ m puts the transmitter in the correct position.

7.3 Moments on Swinging Booms

Calculus Topic: Maxima

Department: Engineering

Subject Area: Statics

Time Needed: 1 hour

Reference: [12]

The mechanism shown in figure 7.4 is static, that is, the forces placed on it do not cause any movement in the system. This is the basis for all of the calculations in these two problems. Much of the preliminary work for these problems is shown to give an idea of where the equations come from. The second problem is an example of how mathematics can become complicated. It is suggested that graphing or numerical techniques with the aid of a symbolic manipulation program be used to solve this problem.

Troublesome Notation: The study of Statics is centered around the action of forces. The boldface and arrow notation is not used here, but the student is expected to keep track of both the magnitude and direction of the forces. Components of force vectors are indicated by subscripts, for example, F_x and F_y.

1. A beam supports three loads of given magnitude and a fourth load whose magnitude is a function of position as shown in figure 7.4. If $b = 1.5$ m and the loads are to be replaced with a single equivalent force, determine

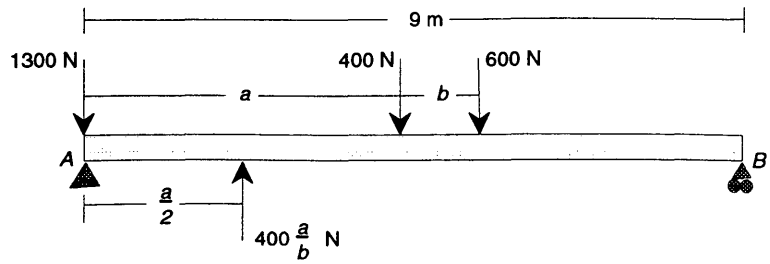

Figure 7.4: A static beam with three constant loads and one variable load.

(a) the value of a so that the distance form support A to the line of action of the equivalent force is maximum and

(b) the magnitude of the equivalent force and its point of application on the beam.

We want to reduce all loads with a single equivalent force which we will call R. Summing all of the forces in the vertical direction produces

$$-1300 + 400\frac{a}{b} - 400 - 600 = -R \tag{7.5}$$

Summing the moments about A produces

$$\frac{a}{2}\left(400\frac{a}{b}\right) - a\,(400) - (a+b)\,(600) = -LR \tag{7.6}$$

where L is the distance from A to R. Using (7.5) and (7.6) to solve for L produces

$$L = \frac{1000a + 600b - 200\left(\frac{a^2}{b}\right)}{2300 - 400\left(\frac{a}{b}\right)}$$

$$\text{or with } b = 1.5 \text{ m}, \quad L = \frac{-\frac{4}{3}a^2 + 10a + 9}{-\frac{8}{3}a + 23} \tag{7.7}$$

Now use (7.7) to find the value of distance a that maximizes length L.

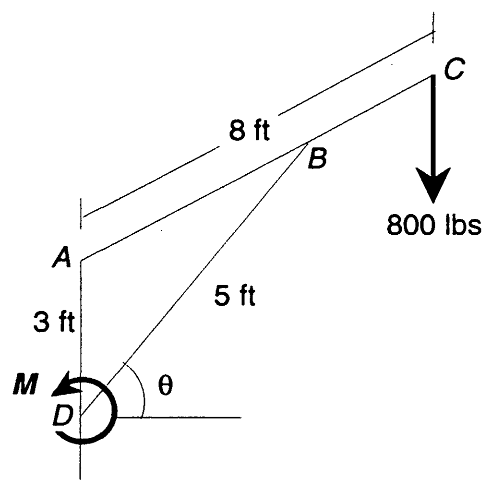

Figure 7.5: A boom with loads.

2. In the mechanism shown in figure 7.5, the position of boom AC is controlled by arm BD. For the loading shown, we want to be able to determine the couple M required to hold the system in equilibrium for different values of θ ranging from

the reaction force at A. As a part of the design process of the mechanism, determine

(a) the value of θ for which M is maximum and the corresponding value of M.

(b) the value of θ for which the reaction at A is maximum and the corresponding magnitude of this reaction.

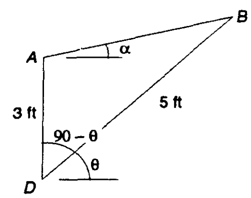

Figure 7.6: Trigonometry of the boom.

Using trigonometric properties on figure 7.6, we can find that

$$\alpha = \tan^{-1}\left(\frac{5\sin\theta - 3}{5\cos\theta}\right) = \tan^{-1}\left(\frac{\sin\theta - \frac{3}{5}}{\cos\theta}\right)$$

Using the free body diagram in figure 7.7 and summing the moments counterclockwise around point A gives the magnitude of the force at B, which is

$$B = \frac{8F}{5} \cdot \frac{\cos^2\alpha}{\cos\theta}.$$

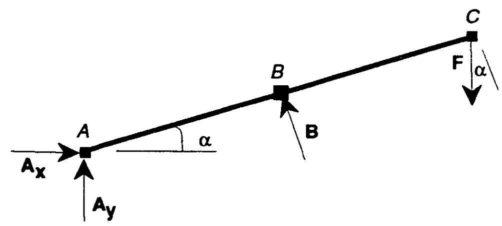

Figure 7.7: Free body diagram of bar AC.

Also from figure 7.7, we can find the magnitude of the force at A by summing the forces in the x- and y-directions on bar AC. Since the motion is static, the sum of the forces is 0. Summing the forces in the x-direction produces

$$A_x - B\sin\alpha = 0 \qquad \Longrightarrow \qquad A_x = B\sin\alpha = \frac{8F}{5} \cdot \frac{\cos^2\alpha\sin\alpha}{\cos\theta}.$$

Summing the forces in the y-direction produces

$$A_y + B\cos\alpha - F = 0 \qquad \Longrightarrow \qquad A_y = F\left(\frac{5\cos\theta - 8\cos^3\alpha}{5\cos\theta}\right).$$

Hence the magnitude of the force at A is

$$A(\theta) = A = \left(A_x^2 + A_y^2\right)^{\frac{1}{2}}$$

To find the couple M at D, consider the free body diagram in figure 7.8. By

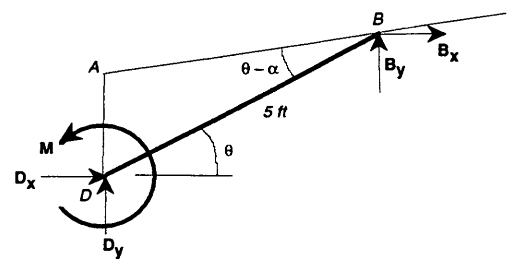

Figure 7.8: Free body diagram of bar BD.

summing the moments counterclockwise around point D, we find

$$M - 5\,(B\cos(\theta-\alpha)) = 0 \qquad \Longrightarrow \qquad M(\theta) = M = \frac{8F\cos^2\alpha\cos(\theta-\alpha)}{\cos\theta}$$

Now compute parts (a) and (b) and remember that α is a function of θ. The mathematics in this problem is difficult, so determining the maximum by the method used in calculus may not be feasable. What other ways can a maximum be determined? It might be better to use a symbolic manipulation program to help with this.

Solutions to: Moments on Swinging Booms

1. *a.* We are given that $b = 1.5$ m and that $L = \dfrac{-\frac{4}{3}a^2 + 10a + 9}{-\frac{8}{3}a + 23}$.

To maximize L over a, set $\dfrac{dL}{da} = 0$.

$$\frac{dL}{da} = \frac{2\,(1143 - 276a + 16a^2)}{(-69 + 8a)^2} = 0 \quad \Rightarrow \quad 1143 - 276a + 16a^2 = 0 \quad \Rightarrow$$

$a = \{6.91\text{m},\ 10.34\text{m}\}$. a must be 6.91 m since AB is only 9 m long.

b. $R = 1300 - 400\dfrac{a}{b} + 400 + 600 = 1300 - 400\dfrac{6.91}{1.5} + 400 + 600 = 458$ N

$$L = \frac{-\frac{4}{3}a^2 + 10a + 9}{-\frac{8}{3}a + 23} = \frac{-\frac{4}{3}(6.91)^2 + 10(6.91) + 9}{-\frac{8}{3}(6.91) + 23} = 3.16 \text{ m}$$

2. *a.* We have $M(\theta) = \dfrac{8F\cos^2\alpha\cos(\theta - \alpha)}{\cos\theta}$ where $\alpha = \tan^{-1}\left(\dfrac{\sin\theta - \frac{3}{5}}{\cos\theta}\right)$

and
$$\frac{dM}{d\theta} = \frac{200F\cos\left[\theta - \arctan\left[\sec\theta\left(-\left(\frac{3}{5}\right) + \sin\theta\right)\right]\right]\sin\theta}{34 - 30\sin\theta}$$
$$+\frac{50F\cos\left[\theta - \arctan\left[\sec\theta\left(-\left(\frac{3}{5}\right) + \sin\theta\right)\right]\right](45 - 15\cos 2\theta - 68\sin\theta)}{(-17 + 15\sin\theta)^2}$$
$$+\frac{150F\cos\theta\,(-3 + 5\sin\theta)\sin\left[\theta - \arctan\left[\sec\theta\left(-\left(\frac{3}{5}\right) + \sin\theta\right)\right]\right]}{(-17 + 15\sin\theta)^2}$$

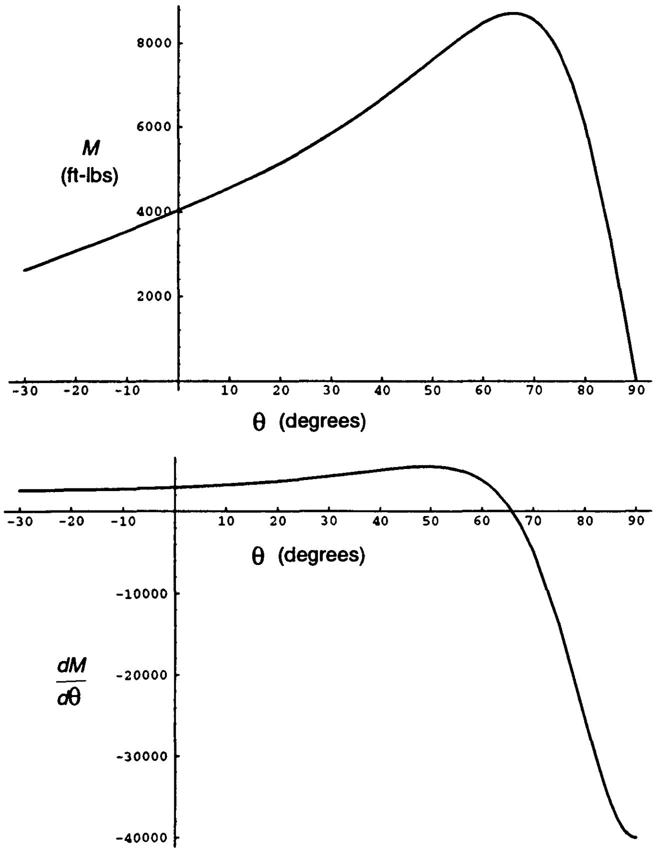

Figure 7.9: M vs. θ and $\dfrac{dM}{d\theta}$ vs. θ.

As can be seen, setting $\dfrac{dM}{d\theta} = 0$ and solving for θ would be difficult.

Figure 7.9 shows the graphs for M and $\dfrac{dM}{d\theta}$. From these, it is easy to see that a maximum for M occurs at approximately $\theta = 66^{\circ}$. Numerically checking the equation shows that there is a maximum of $M = 8684.32$ ft-lbs at $\theta = 65.9^{\circ}$.

b. We have $A(\theta) = F\sqrt{\left(\frac{8}{5}\frac{\cos^2\alpha\sin\alpha}{\cos\theta}\right)^2 + \left(\frac{5\cos\theta - 8\cos^3\alpha}{5\cos\theta}\right)^2}$

where $\alpha = \tan^{-1}\left(\frac{\sin\theta - \frac{3}{5}}{\cos\theta}\right)$ and

$$\frac{dA}{d\theta} = \frac{2000F\cos^3\theta\,(-3+5\sin\theta)}{(17-15\sin\theta)^3} + \frac{600F\cos\theta\,(-5+3\sin\theta))\,(-3+5\sin\theta)^3}{(-17+15\sin\theta)^4}$$

$$+2F\sec^2\theta\left(5\cos\theta - \frac{250\sqrt{2}}{(\sec^2\theta\,(17-15\sin\theta))^{3/2}}\right)$$

$$\left(\frac{-5\sin\theta + \frac{375\sec^3\theta\,(-45+15\cos 2\theta+68\sin\theta)}{\sqrt{2}\,(\sec^2\theta\,(17-15\sin\theta))^{5/2}}}{25}\right) + \frac{400\left(-3+5\sin^2\theta\right)\sin 2\theta}{(17-15\sin\theta)^3}$$

$$+\frac{\frac{1}{25}\left[2\sec^2\theta\left(5\cos\theta - \frac{250\sqrt{2}}{(\sec^2\theta\,(17-15\sin\theta))^{3/2}}\right)^2\tan\theta\right]}{\frac{2}{25}\sqrt{\sec^2\theta\left(5\cos\theta - \frac{250\sqrt{2}}{(\sec^2\theta\,(17-15\sin\theta))^{3/2}}\right)^2 + \frac{200\cos^2\theta\left(-3+5\sin^2\theta\right)}{\left(17-15\sin^3\theta\right)}}}$$

As can be seen, setting $\frac{dA}{d\theta} = 0$ and solving for θ would be difficult.

Figure 7.10 shows the graphs for A and $\frac{dA}{d\theta}$. From these, it is easy to see that a maximum for A occurs at approximately $\theta = 68^o$. Numerically checking the equation shows that there is a maximum of $A = 1436.06$ lbs at $\theta = 68.5^o$.

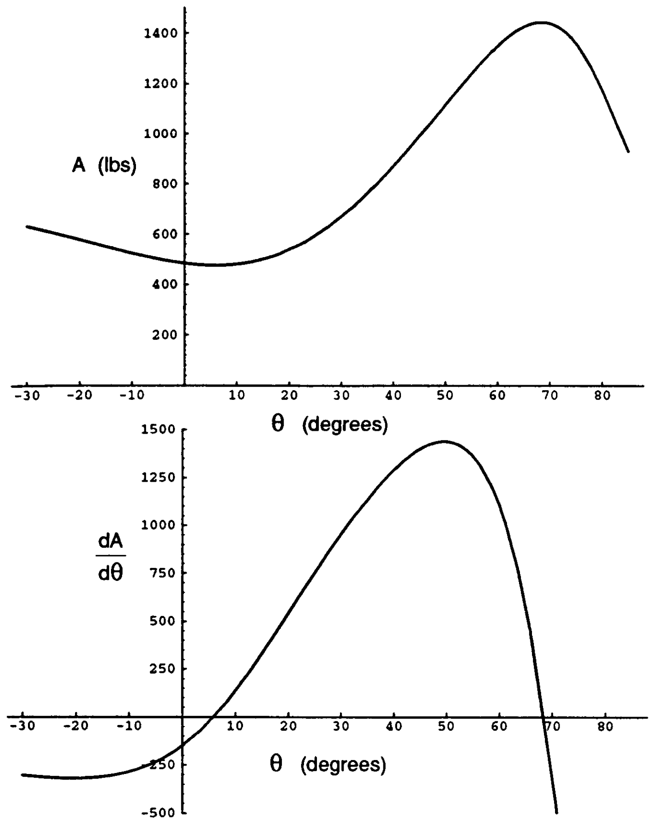

Figure 7.10: A vs. θ and $\frac{dA}{d\theta}$ vs. θ.

7.4 Projectile Motion

Calculus Topic:	Differentiation, Parametric Equations
Department:	Physics
Subject Area:	Mechanics
Time Needed:	40 minutes
Reference:	[11]

Projectile motion is the motion of an object in flight where gravity is the only force acting on it. Thus acceleration remains constant while velocity and position change. We look at the motion of a projectile in two dimensions, horizontal and vertical. Problem 1 illustrates how the motion equations can be used to determine the position and velocity of a projectile as shown if figure 7.11.

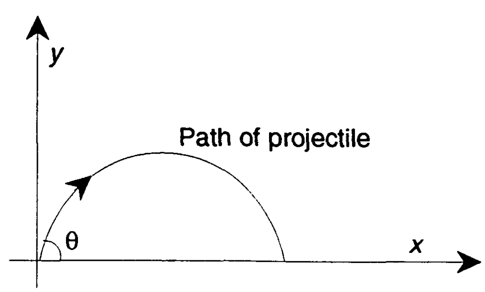

Figure 7.11: The motion of a projectile.

Troublesome notation: In these problems we use x to indicate the horizontal position, y to indicate the vertical position x' to indicate horizontal velocity and y' to indicate vertical velocity. In physics, subscripts may be used to express the same meaning. For example, s_x and s_y might indicate horizontal and vertical position and v_x and v_y might indicate horizontal and vertical velocity. Also note that v_o is used for initial velocity even though x' and y' indicate velocity at any time.

1. The motion of a projectile as shown in figure 7.11 can be described by the following parametric equations:

$$x(t) = (v_o \cos\theta)\, t + x_o$$

$$y(t) = -\frac{1}{2}gt^2 + (v_o \sin\theta)\, t + y_o$$

where v_o is the velocity with which the object is propelled, (x_o, y_o) is the launch point, θ is the angle from the horizontal and g is the acceleration due to gravity. Given $(x_o, y_o) = (0, 0)$,

(a) Find $x'(t)$ and $y'(t)$.

(b) When will the maximum height be reached by the object ($y' = 0$)?

(c) What is the maximum height the object reaches?

(d) Find the range of the object (When $y = 0$, find x).

(e) What θ will maximize the range of the object?

2. A football player is attempting to kick a field goal at a distance of 40 yards from the goal posts. The cross-bar of the goal posts is 3 yards above the ground level and he kicks the ball is such a way that the initial angle with the ground is 40°. For this problem, $g = 32\frac{\text{ft}}{\text{s}^2}$.

   (a) If he kicks it directly in a line with the center of the goal posts, find the minimum velocity he must impart to the football to assure that it passes over the crossbar.

   (b) Construct a graph showing the height of the ball at each ten yard marker. Also show the point where the peak of the trajectory occurs.

   (c) If the kicker imparts a velocity of 25 yards per second to the football and kicks it directly at the uprights 50 yards away, find the range of possible initial angles (to the nearest degree) with the ground for which the ball will pass above the crossbar.

Solutions to: Projectile Motion

1. *a.* $x'(t) = v_o \cos\theta$

   $y'(t) = -gt + v_o \sin\theta$

   b. Maximum height is reached when $t = \dfrac{v_o \sin\theta}{g}$

   c. Maximum height is $y(t) = \dfrac{v_o^2 \sin^2\theta}{2g} + y_o$

   d. The range of the object is $x(t) = \dfrac{v_o^2 \sin 2\theta}{2g} + x_o$

   e. $\dfrac{dx}{d\theta} = \dfrac{d}{d\theta}\left(\dfrac{v_o^2 \sin 2\theta}{2g} + x_o\right) = \dfrac{v_o^2 \cos 2\theta}{g} = 0$

   $\cos 2\theta = 0 \quad \Rightarrow \quad \theta = 45°$

   The angle that will maximize the range is $\theta = 45°$

2. *a.* Use $x(t) = (v_o \cos\theta)\, t + x_o$ and $y(t) = -\frac{1}{2}gt^2 + (v_o \sin\theta)\, t + y_o$

   Be sure to change yards to feet.

   Using $x(t)$: $120 = v_o \cos 40t + 0 \quad \Rightarrow \quad t = \dfrac{120}{v_o \cos 40}$

Using $y(t)$ and substituting for t:

$$9 = -\frac{1}{2}(32)\left(\frac{120}{v_o \cos 40}\right)^2 + v_o \sin 40 \left(\frac{120}{v_o \cos 40}\right)$$

$$9 = -\frac{392621}{v_o^2} + 100.69 \quad \Rightarrow \quad v_o = 65.4\frac{\text{ft}}{\text{s}}$$

b.

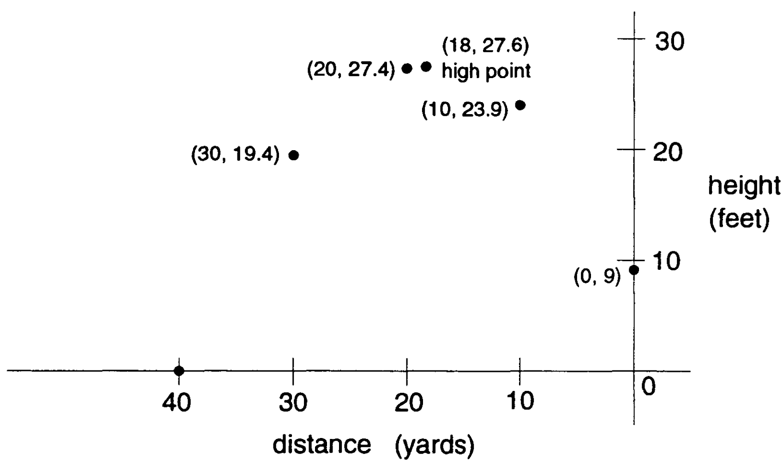

Figure 7.12: Graph of the motion of a football.

2. c. Use $x(t) = (v_o \cos\theta)\, t + x_o$ and $y(t) = -\frac{1}{2}gt^2 + (v_o \sin\theta)\, t + y_o$

Using $x(t)$: $150 = 75t\cos\theta + 0 \quad \Rightarrow \quad t = \dfrac{150}{75\cos\theta} = \dfrac{2}{cos\theta}$

Be sure to change yards to feet.

Using $y(t)$ and substituting for t:

$$9 = -\frac{1}{2}(32)\left(\frac{2}{\cos\theta}\right)^2 + 75\sin\theta\left(\frac{2}{\cos\theta}\right)$$

$$-64\sec^2\theta + 150\tan\theta = 9$$

$$-64\left(1 + \tan^2\theta\right) + 150\tan\theta = 9$$

$$-64\tan^2\theta + 150\tan\theta = 73$$

$$64\tan^2\theta - 150\tan\theta + 73 = 0$$

Using the quadratic formula: $\tan\theta = .686, 1.65 \quad \Rightarrow \quad 34.6^{\circ} < \theta < 58.8^{\circ}$

for the football to clear the crossbar.

7.5 Groundwater Flow

Calculus Topic:	Integration, Indefinite Integration
Department:	Geology
Subject Area:	Groundwater Hydrology
Time Needed:	20 minutes
Reference:	[20]

An aquifer is a layer of porous rock which can store water and allow water to move through it. Figure 7.13 shows how groundwater in an aquifer can be recharged and discharged. A hydrograph, shown in figure 7.14, is a chart which shows cycles of

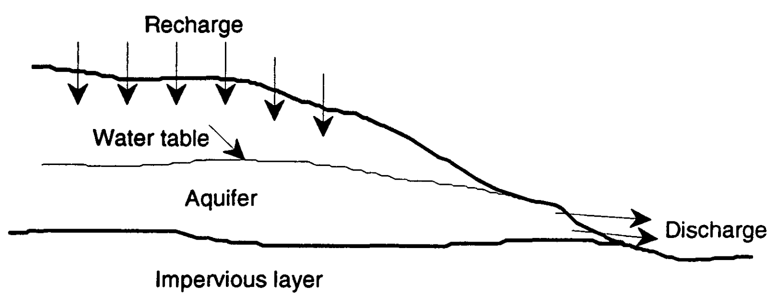

Figure 7.13: Recharge and discharge of groundwater.

discharge and recharge of groundwater in an aquifer. It measures the flow rate, Q, against time, t. Note that Q is on a log scale and t is on a linear scale. It is possible to determine the volume of water discharged by making an analysis of the curve. If t is the amount of time passed from a given date, then at $t = 0$ the discharge rate is $Q = Q_o$. For the purposes of this excercise, we will say that at some time $t = t_\ell$, the

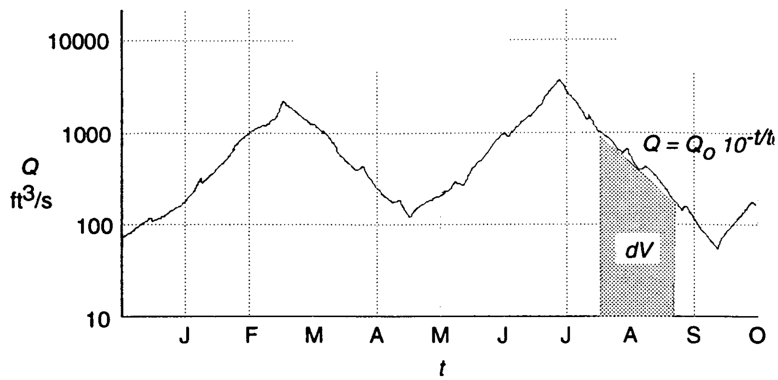

Figure 7.14: Example of an Hydrograph.

discharge $Q = 0.1 \cdot Q_o$. Hence at any time t,

$$Q = Q_o \cdot 10^{-t/t_\ell} \tag{7.8}$$

1. We want to find the volume V of water discharged during a time period, say from t_1 to t_2. Since Q is a rate of flow, we have $Q = \dfrac{dV}{dt}$. Find the volume of groundwater being discharged by setting up and evaluating the integral from t_1 to t_2.

2. Find the total amount of groundwater that could discharge, that is, the amount of groundwater that could discharge from $t = 0$ to $t = \infty$.

3. Calculate the total amount of groundwater that could discharge after the end of a recession if the recession ends at $t = t_1$.

Solutions to: Groundwater Flow

1. We are given $Q = \frac{dV}{dt}$ and that $Q = Q_o 10^{-t/t_\ell}$. Combining these produces

$$dV = Q_o 10^{-t/t_\ell} dt \quad \Rightarrow \quad \int_0^V dV = Q_o \int_{t_1}^{t_2} 10^{-t/t_\ell} dt \quad \Rightarrow$$

$$V\big|_0^V = -Q_o \frac{t_\ell}{10^{t/t_\ell} \ln 10}\bigg|_{t_1}^{t_2} \quad \Rightarrow \quad V = Q_o \left(\frac{t_\ell}{10^{t_1/t_\ell} \ln 10} - \frac{t_\ell}{10^{t_2/t_\ell} \ln 10} \right)$$

2. $$\int_0^V dV = Q_o \int_0^\infty 10^{-t/t_\ell} dt \quad \Rightarrow \quad V = \lim_{A \to \infty} Q_o \int_0^A 10^{-t/t_\ell} dt \quad \Rightarrow$$

$$V = \lim_{A \to \infty} Q_o \frac{-t_\ell}{10^{t/t_\ell} \ln 10}\bigg|_0^A \quad \Rightarrow \quad V = \lim_{A \to \infty} Q_o \left(\frac{-t_\ell}{10^{A/t_\ell} \ln 10} - \frac{-t_\ell}{10^{0/t_\ell} \ln 10} \right)$$

$$\Rightarrow \quad V = 0 - Q_o \frac{-t_\ell}{\ln 10} \quad \Rightarrow \quad V = Q_o \frac{t_\ell}{\ln 10}$$

3. $\int_0^V dV = Q_o \int_{t_1}^\infty 10^{-t/t_\ell} dt \quad \Rightarrow$ following the same solution as (2) $\Rightarrow$

$$V = \lim_{A \to \infty} Q_o \left(\frac{-t_\ell}{10^{A/t_\ell} \ln 10} - \frac{-t_\ell}{10^{t_1/t_\ell} \ln 10} \right) \quad \Rightarrow \quad V = 0 - Q_o \frac{-t_\ell}{10^{t_1/t_\ell} \ln 10}$$

$$\Rightarrow \quad V = Q_o \frac{t_\ell}{10^{t_1/t_\ell} \ln 10}$$

7.6 Smoke in a Room

Calculus Topic:	First Order Differential Equation
Department:	Environmental Science
Subject Area:	Air Pollution
Time Needed:	40 minutes
Reference:	[5]

We will use a mass balance equation to find the amount of formaldehyde released by smoking cigarettes in a living room. The solution of a steady state mass balance involves solving a simple equation where

$$\text{input rate} = \text{output rate} + \text{degrading rate.}$$

This can be written as

$$E_{in} = QC + kVC \tag{7.9}$$

where V is the volume of the room, E_{in} is the input rate of smoke, k is a reaction rate coefficient, Q is the rate air moves through the room and C is the concentration of formaldehyde being measured.

1. A living room with a volume of 50 m^3 has two people smoking in it, each smoking 3 cigarettes per hour. An individual emits approximately 1.40 mg of formaldehyde for each cigarette. Formaldehyde degrades with a reaction rate

coefficient of $\frac{0.40}{\text{hour}}$. The forced air furnace moves air through the room at a rate of 50 $\frac{\text{m}^3}{\text{hr}}$. Estimate the steady state concentration of formaldehyde, C, assuming complete mixing in the room. How does the value compare with the threshold for eye irritation of about 0.05 ppm $\left(\frac{0.05 \text{ ppm} \cdot 30 \frac{\text{g}}{\text{mol}}}{24.45} = 0.061 \frac{\text{mg}}{\text{m}^3}\right)$?

The next problem involves an unsteady state mass balance. Therefore we must consider solving the following equation where $\frac{dC}{dt}$ is the rate of change of concentration of formaldehyde in the room:

$$V\frac{dC}{dt} = E_{in} - QC - kVC. \tag{7.10}$$

2. Solve problem 1 assuming an unsteady state mass balance. First solve (7.10) for C. To find C, it might be helpful to use a change of variables, i. e., let $a = \frac{E}{V}$ and $b = \frac{Q}{V} + k$ then use the separation of variables method. Let $C = C_0$ at $t = 0$ be an initial condition for this problem. Since the mass balance will vary with time, find the concentration of formaldehyde, C, at $t = 1$ hr, $t = 2$ hrs, $t = 5$ hrs and $t = 10$ hrs with $C_o = 0$. Graph C vs. t. What is happening to the concentration of formaldehyde in the room as time goes by?

Solutions to: Smoke in a Room

1. We want to use $E_{in} = QC + kVC$ to solve for C.

We are given $E_{in} = \left(\frac{1.40 \text{ mg}}{\text{cigarette}}\right)\left(\frac{3 \text{ cigarettes}}{\text{person} \cdot \text{hour}}\right)(2 \text{ people}) = \frac{8.4 \text{ mg}}{\text{hr}}$,

$Q = 50 \ \frac{\text{m}^3}{\text{hr}}$, $\quad V = 50 \text{ m}^3 \quad$ and $\quad k = \frac{0.40}{\text{hr}}$

Solving for C, $\quad C = \frac{E_{in}}{Q + kV} = \frac{8.4}{50 + (0.40)(50)} = 0.12 \ \frac{\text{mg}}{\text{m}^3}$.

$0.12 \ \frac{\text{mg}}{\text{m}^3}$ is roughly twice the amount of the threshold for eye irritation.

Eyes will suffer irritation in this room.

2. First, we need to solve the differential equation,

$$V\frac{dC}{dt} = E_{in} - QC - kVC \quad \text{for } C.$$

If we use the suggested change of variables, $a = \frac{E}{V}$ and $b = \frac{Q}{V} + k$, then

$$\frac{dC}{dt} = a - bC \quad \Rightarrow \quad \frac{dC}{a - bC} = dt \quad \Rightarrow \quad \int \frac{dC}{a - bC} = \int dt \quad \Rightarrow$$

$$-\frac{1}{b} \ln (a - bC) = t + constant.$$

Since $C = C_o$ at $t = 0$, then $constant = -\frac{1}{b} \ln (a - bC_o)$.

Therefore $-\frac{1}{b}\ln(a - bC) = t - \frac{1}{b}\ln(a - bC_o)$.

Rearranging and solving for C: $-\frac{1}{b}\left(\ln(a - bC) - \ln(a - bC_o)\right) = t \quad \Rightarrow$

$$\ln\left(\frac{a - bC}{a - bC_o}\right) = -bt \quad \Rightarrow \quad \frac{a - bC}{a - bC_o} = e^{-bt} \quad \Rightarrow \quad a - bC = (a - bC_o)\,e^{-bt}$$

$$\Rightarrow \quad bC = a - (a - bC_o)\,e^{-bt} \quad \Rightarrow \quad C = \frac{a}{b} - \left(\frac{a}{b} - C_o\right)e^{-bt}$$

Therefore $C = \frac{E_{in}}{Q + kV} - \left(\frac{E_{in}}{Q + kV} - C_o\right)e^{\left(\frac{Q}{V}+k\right)t}$

$$C = \frac{8.4}{50 + (0.40)(50)} - \left(\frac{8.4}{50 + (0.40)(50)} - 0\right)e^{\left(\frac{50}{50}+0.40\right)(1)} = 0.090\ \frac{\text{mg}}{\text{m}^3}$$

$$C = \frac{8.4}{50 + (0.40)(50)} - \left(\frac{8.4}{50 + (0.40)(50)} - 0\right)e^{\left(\frac{50}{50}+0.40\right)(2)} = 0.112\ \frac{\text{mg}}{\text{m}^3}$$

$$C = \frac{8.4}{50 + (0.40)(50)} - \left(\frac{8.4}{50 + (0.40)(50)} - 0\right)e^{\left(\frac{50}{50}+0.40\right)(5)} = 0.120\ \frac{\text{mg}}{\text{m}^3}$$

$$C = \frac{8.4}{50 + (0.40)(50)} - \left(\frac{8.4}{50 + (0.40)(50)} - 0\right)e^{\left(\frac{50}{50}+0.40\right)(10)} = 0.120\ \frac{\text{mg}}{\text{m}^3}$$

As time goes by, the concentration of formaldehyde levels out.

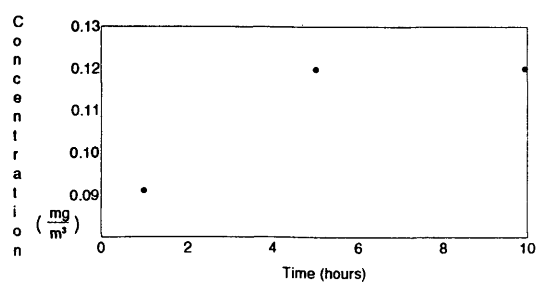

Figure 7.15: Unsteady state concentration of formaldehyde over time.

7.7 *RC* and *RL* Circuits

Calculus Topic:	First Order Differential Equations
Department:	Electrical Engineering
Subject Area:	Circuits
Time Needed:	1 hour
Reference:	[16]

RC Circuits

A *RC* circuit is a simple one-loop circuit containing only a capacitor and a resistor, as shown in figure 7.16. A capacitor is an electronic device that can store energy and release it over time. A resistor is an electronic device that resists the movement of energy through the circuit. In this circuit, there are no current or voltage sources, so any current or voltage is due entirely to the charge initially stored in the capacitor. If we consider what is happening in the circuit over a time t, then

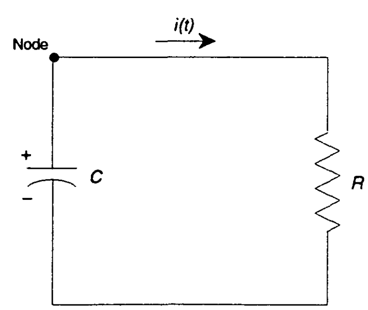

Figure 7.16: Example of a *RC* circuit.

we will say that at $t = 0$, the energy stored in the capacitor is V_o volts. Kirchoff's

Current Law (KCL) states that the sum of the currents entering and leaving a node equals zero. In a resistor, the current flowing through is directly proportional to the voltage across it, which gives the relationship

$$v = iR$$

where v is the potential difference (voltage) measured in volts (V), i is the current measured in amperes (A) and R is the resistance measured in ohms (Ω). In a capacitor,

$$i = C\frac{dv}{dt}$$

where C is the capacitance measured in farads and $\frac{dv}{dt}$ is the change in potential difference over time.

Troublesome Notation: Understanding the meaning of the variables in this problem is important. In motion problems, $\frac{dv}{dt}$ is the rate of change of velocity, or acceleration, but in electric circuit problems, $\frac{dv}{dt}$ is the rate of change in potential difference in a circuit. Also, the function notation in this problem can be confusing. Voltage, v, work, w, and power, p are all functions of time even though it is not always indicated.

1. Use this information to work the following exercises on RC circuits.

(a) Use the KCL to set up a first-order differential equation based on the current moving through the node shown in the circuit in figure 7.16.

(b) If the initial charge of the capacitor is V_o, find the voltage, $v(t)$ in the circuit at some time t.

All of the energy in the circuit is stored in the capacitor. At any time t the energy stored in the capacitor is $w_C(t) = \frac{1}{2}Cv^2(t)$ where $w_C(t)$ represents the energy of the capacitor and is often measured in joules (J). Energy is dissipated by the resistor. The rate at which energy is dissipated is called power, hence if power is denoted by p and measured in watts (W), $p = \dfrac{dw}{dt}$.

(c) If we know that the power dissipated by a resistor is

$$p_R(t) = \frac{v^2}{R} = \frac{v(t)^2}{R}$$

find the total energy $w_R(t)$ dissipated by the resistor from time 0 to some time t, where $w_R(0) = 0$ and $v(t)$ is the potential difference found in part (b).

(d) Given that the initial energy in the capacitor was $w_C(0) = \frac{1}{2}CV_o^2$, show that the energy lost by the capacitor is the same as the energy dissipated through the resistor.

RL Circuits

A *RL* circuit is a simple one-loop circuit containing only an inductor and a resistor, as shown in figure 7.17. An inductor is an electronic device which stores energy in the form of a magnetic field. As current running through an

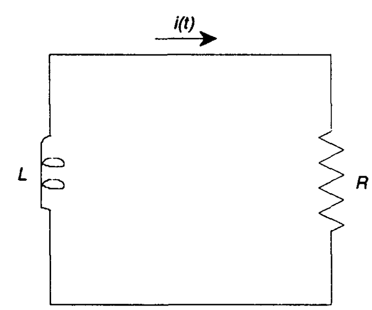

Figure 7.17: Example of a *RL* circuit.

inductor changes, a voltage is developed across the terminals of the inductor. Therefore, the relationship between inductance and potential difference is

$$v = L\frac{di}{dt}$$

where v is the potential difference (voltage), L is the inductance measured in henrys (H) and $\frac{di}{dt}$ is the change in current with respect to time. If we consider what is happening in the circuit over a time t, then we will say that at $t = 0$, the current through the inductor is I_o amperes. Kirchoff's Voltage Law (KVL) states that the sum of the voltage drops around any closed loop equals zero. In a resistor, the voltage across its terminals is directly proportional to the current flowing through it, which gives the relationship

$$v = iR$$

where v is the potential difference (voltage) measured in volts (V), i is the current measured in amperes (A) and R is the resistance measured in ohms (Ω).

2. Use this information to work the following exerciese on RC circuits.

   (a) Use the KVL to set up a first-order differential equation based on the voltage moving around the path shown in the circuit in figure 7.17.

   (b) If the current through the inductor at $t = 0$ is I_o, find the current, $i(t)$ in the circuit at some time t.

   (c) What happens if the initial time is not $t = 0$? Let $i(10) = I_o$ and find $i(t)$ given $R = 2\ \Omega$ and $L = 1$ H. (Hint: it is necessary to follow the procedures for parts (a) and (b).)

   (d) What are the similarities/differences between the equations for the RC and RL circuits?

Solutions to: RC and RL Circuits

1. *a.* $C\dfrac{dv}{dt} + \dfrac{v}{R} = 0$

   b. $C\dfrac{dv}{dt} + \dfrac{v}{R} = 0 \quad \Rightarrow \quad \dfrac{dv}{v} = -\dfrac{1}{CR}dt \quad \Rightarrow \quad \displaystyle\int_{V_o}^{v} \frac{dv}{v} = -\int_0^t \frac{1}{CR}dt$

   since $v(0) = V_o$. Therefore $v(t) = V_o e^{\frac{t}{RC}}$

   c. $p_R = \dfrac{dw_R}{dt} \quad \Rightarrow \quad dw_R = p_R dt = \dfrac{v^2}{R}dt = \dfrac{v_o^2 e^{\frac{-2t}{RC}}}{R} \quad \Rightarrow$

   $\displaystyle\int_0^{w_R} dw_R = \int_0^t \frac{v_o^2 e^{\frac{-2t}{RC}}}{R} \quad \Rightarrow \quad w_R = -\frac{CV_o^2}{2} e^{\frac{-2t}{RC}}\bigg|_0^t \quad \Rightarrow$

   $w_R = -\dfrac{CV_o^2}{2}\left(1 - e^{\frac{-2t}{RC}}\right)$

   d. We want to show that $w_C(0) - w_C(t) = w_R(t)$.

   $w_C(0) - w_C(t) = \dfrac{1}{2}CV_o^2 - \dfrac{CV_o^2}{2}e^{-\frac{2t}{RC}} = \dfrac{1}{2}CV_o^2\left(1 - e^{-\frac{2t}{RC}}\right) = w_R(t)$

2. *a.* $L\dfrac{di}{dt} + iR = 0$

   b. $L\dfrac{di}{dt} + iR = 0 \quad \Rightarrow \quad \dfrac{di}{i} = -\dfrac{R}{L}dt \quad \Rightarrow \quad \displaystyle\int_{I_o}^{i} \frac{di}{i} = -\int_0^t \frac{R}{L}dt$

   since $i(0) = I_o$. Therefore $i(t) = I_o e^{-\frac{Rt}{L}}$

c. From (a) and (b), $\dfrac{di}{i} = -\dfrac{R}{L}dt \quad \Rightarrow \quad \displaystyle\int_{I_o}^{i} \frac{di}{i} = -\int_{10}^{t} \frac{R}{L}dt$

since $i(10) = I_o$. Therefore $\ln i\big|_{I_o}^{i} = -\dfrac{R}{L}t\Big|_{10}^{t} \quad \Rightarrow$

$\ln \dfrac{i}{I_o} = -\dfrac{R}{L}(10 - t) \quad \Rightarrow \quad i(t) = I_o e^{-\frac{R}{L}(10-t)}$

For $R = 2\ \Omega$ and $L = 1$ H, $i = I_o e^{-2(10-t)}$.

d. In a *RC* circuit, the potential difference is changing with respect to time because of the action of the capacitor. In a *RL* circuit, the current is changing with respect to time because of the inductor. Both require a first order differential equation with initial conditions to solve.

7.8 Objects in Motion

Calculus Topic:	Integration, Maxima and Minima
Department:	Physics
Subject Area:	Kinematics
Time Needed:	1 hour
Reference:	[1]

The motion of an object moving in one direction can be described if we know its position at any time t. If the position of the object is given by $x(t)$, then its velocity, $v(t) = \frac{dx(t)}{dt}$ and its acceleration, $a(t) = \frac{d^2x(t)}{dt^2}$.

Troublesome notation: Although acceleration, $a(t)$, velocity, $v(t)$, and position, $x(t)$ are functions of time in each of these problems, they are often written in a shortened form as a, v and x. Also, position is represented by different variables, x and y. x generally refers to a horizontal position and y generally refers to a vertical position. It is not uncommon for physists to use s to indicate position.

1. Certain types of guns have breaking mechanisms to reduce recoil when the gun is fired. This mechanism consists of a piston attached to the barrel which moves in a fixed cylinder filled with oil. As the barrel recoils with an initial velocity v_o the piston moves and oil is forced through orifices in the piston, causing the piston and the barrel to decelerate at a rate directly proportional to their velocity, that is, $a = -kv$. Express:

(a) v in terms of t.

(b) x in terms of t.

(c) v in terms of x.

2. Considering the fact that the gravitational attraction of the earth is weaker as an object moves away, we will define the acceleration due to gravity at an altitude y above the surface of the earth to be

$$a = \frac{-9.81}{\left(1 + \frac{y}{6.37 \times 10^6}\right)^2}$$

where a is measured in $\frac{\text{m}}{\text{s}^2}$ and y in meters. Using this expression, compute the height reached by a bullet fired vertically upward from the surface of the earth with the following initial velocities: 200 $\frac{\text{m}}{\text{s}}$, 2000 $\frac{\text{m}}{\text{s}}$ and 11.18 $\frac{\text{km}}{\text{s}}$.

3. The acceleration of a particle is described as $a = k \sin\left(\frac{\pi t}{T}\right)$. Knowing that both the velocity and the position coordinate of the particle are zero when $t = 0$, determine

(a) the equations of motion, that is, find v in terms of t and x in terms of t.

(b) the maximum velocity.

(c) the position at $t = 2T$.

(d) the average velocity during the interval $t = 0$ to $t = 2T$.

Solutions to: Objects in Motion

1. *a.* $a = -kv \;\Rightarrow\; \dfrac{dv}{dt} = -kv \;\Rightarrow\; \dfrac{dv}{v} = -kdt$

Integrating both sides and using the initial condition that $v(0) = v_o$:

$$\int_{v_o}^{v} \frac{dv}{v} = -k\int_0^t dt \;\Rightarrow\; \ln v\big|_{v_o}^{v} = -kt\big|_0^t \;\Rightarrow\; \ln v - \ln v_o = -kt \;\Rightarrow$$

$$\ln\frac{v}{v_o} = -kt \;\Rightarrow\; \frac{v}{v_o} = e^{-kt} \qquad \text{Therefore} \quad v = v_o e^{-kt}.$$

b. $v = v_o e^{-kt} \;\Rightarrow\; \dfrac{dx}{dt} = v_o e^{-kt} \;\Rightarrow\; dx = v_o e^{-kt} dt$

Integrating both sides and using the initial condition that $x(0) = 0$:

$$\int_0^x dx = v_o\int_0^t e^{-kt}dt \;\Rightarrow\; x\big|_0^x = -\frac{v_o}{k}e^{-kt}\Big|_o^t$$

Therefore $x = v_o e^{-kt} - v_o e^0 = v_o\left(e^{-kt} - 1\right)$

c. $a = -kv = \dfrac{dv}{dt} \;\Rightarrow\; -kv = \dfrac{dv}{dx}\dfrac{dx}{dt} = v\dfrac{dv}{dx} \;\Rightarrow\; dv = -kdx$

Integrating both sides and using the initial condition that $v(0) = v_o$:

$$\int_{v_o}^{v} dv = -k\int_0^x dx \;\Rightarrow\; v\big|_{v_o}^{v} = -kx\big|_0^x \;\Rightarrow\; v - v_0 = -k(x - 0)$$

Therefore $v = v_o - kx$

2. $$a = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy} \quad \Rightarrow \quad ady = vdv \quad \Rightarrow \quad \frac{-9.81}{1+\dfrac{y}{6.37\times 10^6}}dy = vdv$$

Integrating both sides and using the initial condition that $v(0) = v_o$:

$$\int_0^y \frac{-9.81}{1+\dfrac{y}{6.37\times 10^6}}dy = \int_{v_o}^{v} vdv \quad \Rightarrow \quad \left.\frac{6.25\times 10^7}{1+\dfrac{y}{6.37\times 10^6}}\right|_0^y = \left.\frac{v^2}{2}\right|_{v_o}^{v} \quad \Rightarrow$$

$$\frac{6.25\times 10^7}{1+\dfrac{y}{6.37\times 10^6}} - 6.25\times 10^7 = \frac{v^2 - v_o^2}{2}$$

At the maximum height, $v = 0$. Solving for y produces

$$6.25\times 10^7 - \frac{6.25\times 10^7}{1+\dfrac{y}{6.37\times 10^6}} = \frac{v_o^2}{2} \quad \Rightarrow$$

$$1+\frac{y}{6.37\times 10^6} = \frac{2\,(6.25\times 10^7)}{-v_o^2 + 2\,(6.25\times 10^7)} \quad \Rightarrow$$

$$y = 6.37\times 10^6\left(\frac{1.25\times 10^8}{1.25\times 10^8 - v_o^2} - 1\right)$$

a. $$y = 6.37\times 10^6\left(\frac{1.25\times 10^8}{1.25\times 10^8 - 200^2} - 1\right) = 2040 \text{ m}$$

b. $$y = 6.37\times 10^6\left(\frac{1.25\times 10^8}{1.25\times 10^8 - 2000^2} - 1\right) = 210 \text{ km}$$

c. At 11.18 $\frac{\text{km}}{\text{s}}$ the bullet goes on infinitely.

3. *a.* $$a = \frac{dv}{dt} \quad \Rightarrow \quad \frac{dv}{dt} = k\sin\left(\frac{\pi t}{T}\right) \quad \Rightarrow \quad dv = k\sin\left(\frac{\pi t}{T}\right)dt$$

Integrating both sides and using the initial condition that $v(0) = 0$:

$$\int_0^v dv = \int_0^t k \sin\left(\frac{\pi t}{T}\right) dt \quad \Rightarrow \quad v|_0^v = -\frac{kT}{\pi}\cos\left(\frac{\pi t}{T}\right)\Big|_0^t \quad \Rightarrow$$

$$v = \frac{kT}{\pi}\left(1 - \cos\left(\frac{\pi t}{T}\right)\right)$$

$$v = \frac{dx}{dt} = \frac{kT}{\pi}\left(1 - \cos\left(\frac{\pi t}{T}\right)\right) \quad \Rightarrow \quad dx = \frac{kT}{\pi}\left(1 - \cos\left(\frac{\pi t}{T}\right)\right) dt$$

Integrating both sides and using the initial condition that $x(0) = 0$:

$$\int_0^x dx = \int_0^t \frac{kT}{\pi}\left(1 - \cos\left(\frac{\pi t}{T}\right)\right) dt \quad \Rightarrow$$

$$x|_0^x = \left[\frac{kT}{\pi}t - \frac{kT^2}{\pi^2}\sin\left(\frac{\pi t}{T}\right)\right]_0^t \quad \Rightarrow \quad x = \frac{kT}{\pi}t - \frac{kT^2}{\pi^2}\sin\left(\frac{\pi t}{T}\right)$$

b. To find the maximum velocity, set $\frac{dv}{dt} = 0$ and find t.

$$\frac{dv}{dt} = k\sin\left(\frac{\pi t}{T}\right) = 0 \quad \Rightarrow \quad t = \{0, T, 2T, \cdots\}$$

Maximums occur at $\{T, 3T, 5T, \cdots\}$. Hence the maximum velocity is

$$v_{max} = \frac{kT}{\pi}\left(1 - \cos\left(\frac{\pi T}{T}\right)\right) = \frac{2kT}{\pi}$$

c. At $t = 2T$, the position of the particle is

$$x = \frac{kT}{\pi}(2T) - \frac{kt^2}{\pi^2}\sin\left(\frac{\pi T}{T}\right) = \frac{2kT^2}{\pi}$$

d. Average velocity $= \dfrac{\text{change in position}}{\text{change in time}}$. From $t = 0$ to $t = 2T$,

$$\text{Average velocity} = \frac{\frac{2kT^2}{\pi}}{2T} = \frac{kT}{\pi}$$

7.9 Heat Transfer

Calculus Topic: First Order Differential Equations

Department: Chemical Engineering

Subject Area: Transport Processes

Time Needed: 40 minutes

Reference: [8]

The transfer of heat through a material can be described as follows:

$$\begin{pmatrix} \text{rate of} \\ \text{heat in} \end{pmatrix} \begin{pmatrix} \text{rate of} \\ \text{generation} \\ \text{of heat} \end{pmatrix} = \begin{pmatrix} \text{rate of} \\ \text{heat out} \end{pmatrix} \begin{pmatrix} \text{rate of} \\ \text{accumulation} \\ \text{of heat} \end{pmatrix} \tag{7.11}$$

If the heat transfer occurs only by conduction, then it can be described by Fourier's Law of Conduction which is:

$$\frac{q_x}{A} = -k\frac{dT}{dx} \tag{7.12}$$

where q_x is the heat-transfer rate in the x direction in watts (W), A is the cross-sectional area normal to the direction of flow of heat in m^2, T is the temperature in degrees K, x is the distance of flow in m and k is the thermal conductivity in $\frac{\text{W}}{\text{m}\cdot\text{K}}$.

Troublesome Notation: Don't confuse the subscript on q_x to mean a partial derivative with respect to x. Here it means that q is in the direction of x. and is part of the variable. The subscript notation can be very confusing in this problem, but it is only being used to help keep track of the meaning of the variables.

1. Consider the transfer of heat through a circular pipe whose cross-section is shown in figure 7.18. In figure 7.18, the pipe has an inside radius of r_1 where

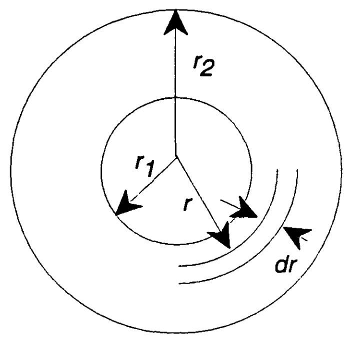

Figure 7.18: Cross-section of a circular pipe.

the temperature is T_1, an outside radius of r_2 where the temperature is T_2 and a length of L. Heat is flowing radially from the inside surface to the outside. The area normal to the direction of flow is $A = 2\pi r L$.

(a) By letting $r = x$, evaluate Fourier's Law of Conduction for this pipe, that is, solve for q.

(b) Now find the heat conduction through a hollow sphere, assuming the sphere has an inside radius of r_1 where the temperature is T_1, an outside radius of r_2 where the temperature is T_2 and that heat is flowing radially from the inside surface to the outside. What is A for a sphere?

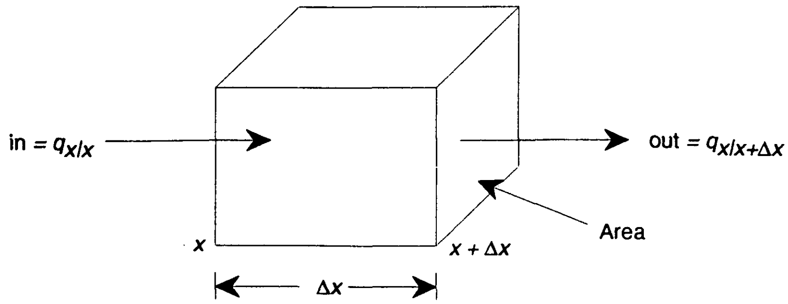

Figure 7.19: Balance for heat transfer in a control volume.

A different way of looking at the idea expressed in (7.11) is shown in figure 7.19. In this case, the heat transfer equation becomes

$$q_{x|x} + \dot{q}\,(\Delta x \cdot A) = q_{x|x+\Delta x} + 0 \tag{7.13}$$

where $q_{x|x}$ is the rate of heat transfer in the x-direction going into the control volume, $q_{x|x+\Delta x}$ is the rate of heat transfer in the x-direction coming out of the control volume $\dot{q}$ is the rate of heat generated per unit volume and A is the cross-sectional area of the control volume. In order to evaluate the continuous movement of heat through the block, we need to rearrange the terms, divide by Δx and let $\Delta x \to 0$. Then,

$$\frac{-dq_x}{dx} + \dot{q} \cdot A = 0 \tag{7.14}$$

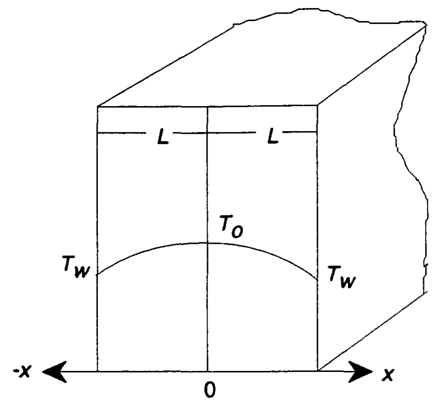

Figure 7.20: Wall with internal heat generation.

2. In the plane wall shown in figure 7.20, heat is being generated inside the wall and is conducted only in the x-direction. The temperature T_w at $x = L$ and $x = -L$ is held constant.

   (a) Substitute equation (7.12) into equation (7.14) for q_x and solve for $\frac{d^2T}{dx^2}$.

   (b) Solve for the temperature profile T by integrating twice.

   (c) The boundary conditions are at $x = \pm L$, $T = T_w$ and at $x = 0$. $T = T_0$. Solve for the integration constants using these conditions to get a equation for T.

Solutions to: Heat Transfer

1. *a.* We are given that $\frac{q_r}{A} = -k\frac{dT}{dr}$ where $A = 2\pi r L$.

Therefore $\frac{q_r}{2\pi r L} = -k\frac{dT}{dr} \quad \Rightarrow \quad \frac{q_r}{2\pi r L}dr = -kdT \quad \Rightarrow$

$$\frac{q_r}{2\pi L}\int_{r_1}^{r_2}\frac{dr}{r} = -k\int_{T_1}^{T_2} dT \quad \Rightarrow \quad \frac{q_r}{2\pi L}\ln r\Big|_{r_1}^{r_2} = -kT|_{T_1}^{T_2} \quad \Rightarrow$$

$$\frac{q_r}{2\pi L}\ln\frac{r_1}{r_2} = k(T_1 - T_2) \quad \Rightarrow \quad q_r = \frac{2\pi L k(T_1 - T_2)}{\ln\frac{r_1}{r_2}}.$$

b. Start with $\frac{q_r}{A} = -k\frac{dT}{dr}$. For a hollow sphere, $A = 4\pi r^2$.

Therefore $\frac{q_r}{4\pi r^2} = -k\frac{dT}{dr} \quad \Rightarrow \quad \frac{q_r}{4\pi r^2}dr = -kdT \quad \Rightarrow$

$$\frac{q_r}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = -k\int_{T_1}^{T_2} dT \quad \Rightarrow \quad \frac{q_r}{4\pi}\left[-\frac{1}{r}\right]_{r_1}^{r_2} = -kT|_{T_1}^{T_2} \quad \Rightarrow$$

$$\frac{q_r}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = k(T_1 - T_2) \quad \Rightarrow \quad q_r = \frac{4\pi k(T_1 - T_2)}{\frac{1}{r_1} - \frac{1}{r_2}}.$$

2. *a.* The two equations are $\frac{q_x}{A} = -k\frac{dT}{dx}$ and $\frac{-dq_x}{dx} + \dot{q}A = 0$

Therefore $-\frac{d}{dx}\left(-kA\frac{dT}{dx}\right) + \dot{q}A = 0 \quad \Rightarrow \quad \frac{d^2T}{dx^2} = -\frac{\dot{q}}{k}$

b. $$\int d\left(\frac{dT}{dx}\right) = -\frac{\dot{q}}{k}\int dx \quad \Rightarrow \quad \frac{dT}{dx} = -\frac{\dot{q}}{k}x + c_1$$

$$\int dT = \int \left(-\frac{\dot{q}}{k}x + c_1\right) dx \quad \Rightarrow \quad T = -\frac{\dot{q}}{2k}x^2 + c_1 x + c_2$$

c. $x = 0, T = T_o \quad \Rightarrow \quad T_o = c_2.$

Hence $T = -\dfrac{\dot{q}}{2k}x^2 + c_1 x + T_o$. We also have $T = T_w$ at $x = \pm L, \quad \Rightarrow$

$$T_w = -\frac{\dot{q}}{2k}L^2 \pm c_1 L + T_o \quad \Rightarrow \quad T_w - T_o + \frac{\dot{q}}{2k}L^2 = \pm c_1 L \quad \Rightarrow \quad c_1 = 0.$$

Therefore, the final form for the equation is $T = -\dfrac{\dot{q}}{2k}x^2 + T_o$.

7.10 Reaction Rate Problems

Calculus Topic:	First Order Differential Equations
Department:	Chemistry, Environmental Science
Subject Area:	Environmental Monitoring
Time Needed:	2 hours
Reference:	[8]

Since working with reaction rates involves some knowledge of chemistry, the initial set up for each of these problems is done. There is some vocabulary here which may be confusing, but the meaning of the words does not play a part in the problems you have to solve. It is left to the reader to find the meaning of any unknown words.

Troublesome Notation: The variables in these problems are written differently that most math students are used to seeing, which may make the problems seem difficult. For instance in problem (1), you will be given $\frac{d[A]}{dt}$ and be asked to find [A]. [A] represents the amount of chemical A in the reaction and acts just like solving for x in $\frac{dx}{dt}$. You use the same methods for finding a derivative, or in this case, for solving a differential equation.

1. In a batch reaction system, a chemical reaction is partially autocatalytic when the product C affects the reaction rate as shown below:

$$2\text{A} \rightarrow \text{C} + \text{other products} \tag{7.15}$$

$$r_A = -k[\mathrm{A}][\mathrm{C}] \tag{7.16}$$

In this problem, r_A represents the reaction rate, k is a constant and [A] and [C] are amounts of reactant and product. Since the reaction is partially autocatalytic, we need to consider the derivatives of the reactants and the products. Since this occurs in a batch reaction system, we have

$$r_A = \frac{d[\mathrm{A}]}{dt}. \tag{7.17}$$

(a) First we want to develop an expression for r_A in terms of [A] and $[\mathrm{C_o}]$. The stoichiometry of (7.15) produces the following relationship:

$$[\mathrm{A_o}] - [\mathrm{A}] = 2\left([\mathrm{C}] - [\mathrm{C_o}]\right) \tag{7.18}$$

where $[\mathrm{A_o}]$ and $[\mathrm{C_o}]$ represent initial amounts and [A] and [C] represent amounts at some time t during the reaction. We want r_A in terms of [A] and $[\mathrm{C_o}]$, so using (4), we will make the following substitution.

$$\begin{aligned} [\mathrm{C}] &= [\mathrm{C_o}] + \Delta[\mathrm{C}] \\ &= [\mathrm{C_o}] + ([\mathrm{C}] - [\mathrm{C_o}]) \\ &= [\mathrm{C_o}] + \frac{1}{2}\left([\mathrm{A_o}] - [\mathrm{A}]\right) \end{aligned} \tag{7.19}$$

From (7.16) and (7.17) we find that

$$\frac{d[\mathrm{A}]}{dt} = -k[\mathrm{A}][\mathrm{C}]. \tag{7.20}$$

Substituting for [C] using (7.19) produces

$$\frac{d[\mathrm{A}]}{dt} = -k[\mathrm{A}]\left([\mathrm{C_o}] + \frac{1}{2}([\mathrm{A_o}] - [\mathrm{A}])\right). \tag{7.21}$$

Solve for [A] using where we initially have $[\mathrm{A_o}]$ at $t_o = 0$ and we end with some amount [A] at some time t.

(b) Plot [A] and [C] versus t for $[\mathrm{A_o}] = 10\ \frac{\mathrm{mol}}{\mathrm{m^3}}$, $[\mathrm{C_o}] = 1\ \frac{\mathrm{mol}}{\mathrm{m^3}}$, and $k = 1\ \frac{\mathrm{m^3}}{\mathrm{mol\cdot d}}$ in a batch reaction system.

2. The reaction-rate data given below were obtained using a batch reaction system for the reaction

$$\mathrm{A} \rightarrow \text{products.}$$

We want to determine an appropriate rate expression and the rate coefficient. To solve this problem, we seek an expression for r_A where k is constant for the given data. Since this is a batch reaction system where A → products., then

$$r_A = \frac{d[\mathrm{A}]}{dt} \tag{7.22}$$

Time min	A_f $\frac{g}{m^3}$
0	30.00
0.5	12.00
1	7.50
2	4.29
4	2.31
8	1.20
16	0.61
32	0.31

Table 7.1: Reaction Data

Consider the order of [A]. For the n^{th} order of [A], $r_A = \frac{d[A]}{dt} = -k[A]^n$. For example, for the zeroth order of [A], $r_A = \frac{d[A]}{dt} = -k[A]^0$. Start at the zeroth order and find an expression for k given that the amount of reactant at $t = 0$ is A_o and the amount of reactant at some time t is A_f. Now use the data in Table 7.1 to check if k is constant. If not, increase the order of [A] by one and check k again. Continue increasing the order of [A] until you find a constant k.

3. A town requiring 1.0 $\frac{m^3}{s}$ of drinking water has two sources, a local well with 60 $\frac{g}{m^3}$ nitrate and a distant reservoir with 10 $\frac{g}{m^3}$ nitrate. What flow rates of well and reservoir water are needed to meet the 45 $\frac{g}{m^3}$ nitrate drinking water standard and minimize the use of more expensive reservoir water?

Set up two equations. The first is a mass balance equation.

$$\begin{pmatrix} \text{total mass} \\ \text{of water} \end{pmatrix} = \begin{pmatrix} \text{sum of the masses of water} \\ \text{from the different sources} \end{pmatrix}$$

The second is a materials balance using the amount of nitrate in the water.

$$\begin{pmatrix} \text{amount of nitrate} \\ \text{in a source of water} \end{pmatrix} = \begin{pmatrix} \text{mass of} \\ \text{water} \end{pmatrix} \cdot \begin{pmatrix} \text{concentration} \\ \text{of nitrate} \end{pmatrix}$$

$$\begin{pmatrix} \text{total amount of} \\ \text{nitrate in the water} \end{pmatrix} = \begin{pmatrix} \text{sum of the amounts of nitrate in} \\ \text{the water from different sources} \end{pmatrix}$$

In this case, mass and amount of nitrate are given in flow rates. The density of water is constant, so the mass balance equation can be written in terms of $\frac{\text{m}^3}{\text{s}}$ with the same result as if mass were written in terms of grams. The density of nitrate is also constant, so the amount of nitrate can be written in terms of $\frac{\text{g}}{\text{m}^3}$.

4. A water storage tank receives a constant feed rate of 0.2 $\frac{\text{m}^3}{\text{s}}$, and the demand varies according to the relationship $0.2\left(1 - \dfrac{\cos \pi t}{43200}\right) \frac{\text{m}^3}{\text{s}}$. The tank is cylindrical with a cross-sectional area of 1000 m^2. If the depth at $t = 0$ is 5 m, plot the water depth as a function of time. To solve this problem, we need to solve the

following mass balance equation.

$$\frac{d(\rho V)}{dt} = \rho Q_{in} - \rho Q_{out} \tag{7.23}$$

where V is the volume of the water in the cylinder, ρ is the density of the water (in this case a constant) and Q is the mass of the water moving in and out of the tank. Since ρ is a constant, (7.23) reduces to

$$\frac{dV}{dt} = Q_{in} - Q_{out} \tag{7.24}$$

Let $V = Ah$, where A is the cross-sectional area of the cylinder. Hence A remains constant and h varies according to how much water is in the tank. Use this information and (7.24) to solve for h. Assume that time ranges from $t = 0$ to $t = t$. Then construct the plot.

5. Given a reaction A $\rightarrow$ B with $r_A = -kC_A$, determine the time $\frac{t}{\tau}$ to achieve 95 percent of the steady-state for a step increase in influent concentration from 0 to C_{A_i} for a single CFSTR. Assume the value of $\tau k = 1.0$.

To achieve 95 percent of the steady-state for a step increase in influent concentration from 0 to C_{A_i} we want $C_A/C_{A_i} = 0.95$. To help find this we will use a mass balance equation which, in general, shows

$$\text{Accumulation} = \text{Mass in} - \text{Mass out} + \text{Mass generated}$$

Using information from our problem, the mass balance equation becomes

$$V\frac{dC_A}{dt} = QC_{A_i} - QC_A + r_A V \tag{7.25}$$

where r_A is the reaction rate of chemical A, V is the volume, Q is the flow rate, C_{A_i} is the initial concentration of chemical A and C_A is the concentration of chemical A at some time t. Let $V = \tau Q$, where τ is a detention time constant. From this, (7.25) becomes

$$\begin{aligned} \frac{dC_A}{dt} &= \frac{1}{\tau}C_{A_i} - \frac{1}{\tau}C_A - kC_A, \quad \text{or} \\ \tau\frac{dC_A}{dt} &= C_{A_i} - C_A - \tau k C_A \qquad (7.26) \end{aligned}$$

Use (7.26) to find the ratio C_A/C_{A_i} and consider what happens as $t \to \infty$. Then find the ratio $\frac{t}{\tau}$.

Solutions to: Reaction Rate Problems

1. *a.* $\frac{d[\mathrm{A}]}{dt} = -k[\mathrm{A}]\left([\mathrm{C_o}] + \frac{1}{2}([\mathrm{A_o}] - [\mathrm{A}])\right) \quad \Rightarrow$

$$\frac{d[\mathrm{A}]}{[\mathrm{A}]\left([\mathrm{C_o}] + \frac{1}{2}([\mathrm{A_o}] - [\mathrm{A}])\right)} = -kdt \quad \Rightarrow$$

$$\int_{[\mathrm{A_o}]}^{[\mathrm{A}]} \frac{d[\mathrm{A}]}{[\mathrm{A}](2[\mathrm{C_o}] + [\mathrm{A_o}] - [\mathrm{A}])} = -\frac{1}{2}\int_0^t kdt$$

$$\int_{[\mathrm{A_o}]}^{[\mathrm{A}]} \frac{d[\mathrm{A}]}{[\mathrm{A}](2[\mathrm{C_o}] + [\mathrm{A_o}] - [\mathrm{A}])} = \frac{1}{2[\mathrm{C_o}] + [\mathrm{A_o}]}\left[\ln \frac{[\mathrm{A_o}]}{(2[\mathrm{C_o}] + [\mathrm{A_o}]) - [\mathrm{A}]}\right]_{[\mathrm{A_o}]}^{[\mathrm{A}]}$$

$$= \frac{1}{2[\mathrm{C_o}] + [\mathrm{A_o}]}\left[\ln \frac{[\mathrm{A}]}{(2[\mathrm{C_o}] + [\mathrm{A_o}]) - [\mathrm{A}]} - \ln \frac{[\mathrm{A_o}]}{(2[\mathrm{C_o}] + [\mathrm{A_o}]) - [\mathrm{A_o}]}\right]$$

$$= \frac{1}{2[\mathrm{C_o}] + [\mathrm{A_o}]}\left[\ln \frac{2[\mathrm{A}][\mathrm{C_o}]}{[\mathrm{A_o}](2[\mathrm{C_o}] + [\mathrm{A_o}] - [\mathrm{A}])}\right]$$

Now, $\frac{1}{2[\mathrm{C_o}] + [\mathrm{A_o}]}\left[\ln \frac{2[\mathrm{A}][\mathrm{C_o}]}{[\mathrm{A_o}](2[\mathrm{C_o}] + [\mathrm{A_o}] - [\mathrm{A}])}\right] = -\frac{1}{2}k\int_0^t dt = -\frac{1}{2}kt \quad \Rightarrow$

$$-\frac{1}{2}kt(2[\mathrm{C_o}] + [\mathrm{A_o}]) = \ln \frac{2[\mathrm{A}][\mathrm{C_o}]}{[\mathrm{A_o}](2[\mathrm{C_o}] + [\mathrm{A_o}] - [\mathrm{A}])} \quad \Rightarrow$$

$$\frac{2[\mathrm{A}][\mathrm{C_o}]}{[\mathrm{A_o}](2[\mathrm{C_o}] + [\mathrm{A_o}] - [\mathrm{A}])} = e^{-\frac{1}{2}kt(2[\mathrm{C_o}]+[\mathrm{A_o}])} \quad \Rightarrow$$

$$[\mathrm{A}] = \frac{(2[\mathrm{A_o}][\mathrm{C_o}] + [\mathrm{A_o}]^2)\,e^{-\frac{1}{2}kt(2[\mathrm{C_o}]+[\mathrm{A_o}])}}{2[\mathrm{C_o}] + [\mathrm{A_o}]e^{-\frac{1}{2}kt(2[\mathrm{C_o}]+[\mathrm{A_o}])}}$$

b. $[\mathrm{A}] = \frac{(2(10)(1) + (10)^2)\,e^{-\frac{1}{2}1t(2(1)+(10))}}{2(1) + 10e^{-\frac{1}{2}1t(2(1)+(10))}} = \frac{60e^{-6t}}{1 + 5e^{-6t}}$

$$[\mathrm{C}] = [\mathrm{C_o}] + \frac{1}{2}([\mathrm{A_o}] - [\mathrm{A}]) = 1 + \frac{1}{2}\left(10 - \frac{60e^{-6t}}{1+5e^{-6t}}\right) = 6 + \frac{30e^{-6t}}{1+5e^{-6t}}$$

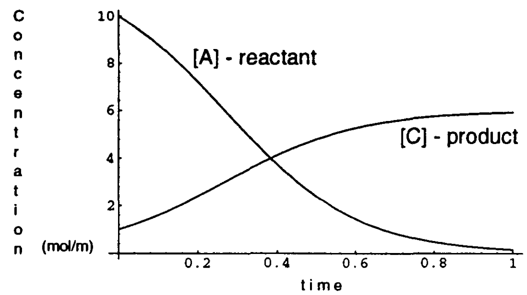

Figure 7.21: Plot for a batch reaction system.

2. Since this is a batch reaction, $r_A = \dfrac{d[\mathrm{A}]}{dt}$

0$^{\text{th}}$ order: $r_A = \dfrac{d[\mathrm{A}]}{dt} = -k[\mathrm{A}]^0 \quad \Rightarrow \quad \displaystyle\int_{[\mathrm{A_o}]}^{[\mathrm{A_f}]} d[\mathrm{A}] = \int_0^t -k dt \quad \Rightarrow$

$$k = \frac{[\mathrm{A_o}] - [\mathrm{A_f}]}{t} \quad \frac{\mathrm{g}}{\mathrm{m^3 \cdot min}}$$

1$^{\text{st}}$ order: $r_A = \dfrac{d[\mathrm{A}]}{dt} = -k[\mathrm{A}]^1 \quad \Rightarrow \quad \displaystyle\int_{[\mathrm{A_o}]}^{[\mathrm{A_f}]} \frac{d[\mathrm{A}]}{[\mathrm{A}]} = \int_0^t -k dt \quad \Rightarrow$

$$k = -\frac{1}{t} \ln \frac{[\mathrm{A_f}]}{[\mathrm{A_o}]} \quad \mathrm{min}^{-1}$$

2$^{\text{nd}}$ order: $r_A = \dfrac{d[\mathrm{A}]}{dt} = -k[\mathrm{A}]^2 \quad \Rightarrow \quad \displaystyle\int_{[\mathrm{A_o}]}^{[\mathrm{A_f}]} \frac{d[\mathrm{A}]}{[\mathrm{A}]^2} = \int_0^t -k dt \quad \Rightarrow$

$$k = -\frac{1}{t}\left(\frac{1}{[\mathrm{A_f}]} - \frac{1}{[\mathrm{A_o}]}\right) \quad \frac{\mathrm{m^3}}{\mathrm{g \cdot min}}$$

At $t = 0$, $[\mathrm{A}] = [\mathrm{A_o}] = 30$ g.

Time	$[\mathrm{A_f}]$	$k_0 = \frac{[\mathrm{A_o}]-[\mathrm{A_f}]}{t}$	$k_1 = \frac{1}{t}\ln\frac{[\mathrm{A_f}]}{[\mathrm{A_o}]}$	$k_2 = \frac{1}{t}\left(\frac{1}{[\mathrm{A_f}]} - \frac{1}{[\mathrm{A_o}]}\right)$
0	30.00			
0.5	12.00	36	-1.8	0.1
1	7.50	22.5	-1.4	0.1
2	4.29	12.86	-0.97	0.1
4	2.31	6.92	-0.64	0.1
8	1.20	3.60	-0.40	0.1
16	0.61	1.84	-0.24	0.1
32	0.31	0.92	-0.14	0.1

Table 7.2: Reaction calculations for zeroth, first and second order equations.

Therefore, since k_2 is constant, it is the rate coefficient. The rate expression is

$$k_2 = \frac{1}{t}\left(\frac{1}{[\mathrm{A_f}]} - \frac{1}{[\mathrm{A_o}]}\right).$$

3. Let Q be the flow rate in $\frac{\mathrm{m}^3}{\mathrm{s}}$ and C be the concentration of nitrate in $\frac{\mathrm{g}}{\mathrm{m}^3}$.

The two equations become

Mass balance $\quad Q_{total} = Q_{reservoir} + Q_{well}$

Materials balance $\quad Q_{total}C_{(NO_3)total} = Q_{reservoir}C_{(NO_3)reservoir} + Q_{well}C_{(NO_3)well}$

We know that $Q_{total} = 1.0\ \frac{\mathrm{m}^3}{\mathrm{s}}$, $C_{(NO_3)total} = 45\ \frac{\mathrm{g}}{\mathrm{m}^3}$, $C_{(NO_3)reservoir} = 10\ \frac{\mathrm{g}}{\mathrm{m}^3}$

and $C_{(NO_3)well} = 60\ \dfrac{\text{g}}{\text{m}^3}$. Therefore, $Q_{reservoir} = Q_{well} - 1$ and

$$45 = (Q_{well} - 1)(10) + Q_{well}(60) \quad \Rightarrow \quad Q_{well} = 0.7\ \frac{\text{m}^3}{\text{s}} \text{ and } Q_{reservoir} = 0.3\ \frac{\text{m}^3}{\text{s}}.$$

4. $$\frac{d(\rho V)}{dt} = \rho Q_{in} - \rho Q_{out} \quad \Rightarrow \quad \frac{dV}{dt} = Q_{in} - Q_{out} \text{ since } \rho \text{ is constant.}$$

$V = Ah$, where $A = 1000\ \text{m}^2$, $Q_{in} = 0.2\ \frac{\text{m}^3}{\text{s}}$ and $Q_{out} = 0.2\left(1 - \frac{\cos \pi t}{43200}\right)$.

Therefore, $$1000\frac{dh}{dt} = 0.2 - 0.2\left(1 - \frac{\cos \pi t}{43200}\right). \quad \Rightarrow$$

$$dh = \frac{1}{1000}\left(0.2 - 0.2\left(1 - \frac{\cos \pi t}{43200}\right)\right) \quad \Rightarrow \quad dh = \frac{0.2}{1000}\left(\cos\frac{\pi t}{43200}\right) \quad \Rightarrow$$

$$\int_5^h dh = \int_0^t \frac{0.2}{1000}\left(\cos\frac{\pi t}{43200}\right) \quad \Rightarrow \quad h\big|_5^h = \frac{0.2}{1000}\frac{43200}{\pi}\left(\sin\frac{\pi t}{43200}\right)\Big|_0^t \Rightarrow$$

$$h = 5 + \frac{8640}{1000\pi}\sin\frac{\pi t}{43200}.$$ The plot is as follows.

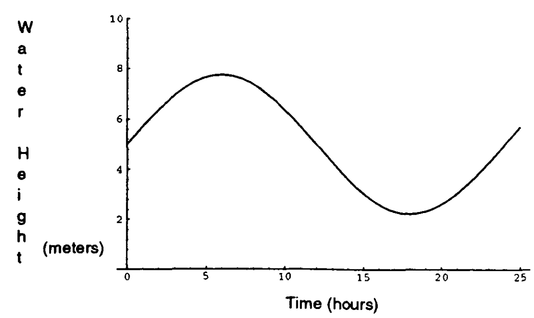

Figure 7.22: Change in water depth in a tank over time.

5. We are given $\tau \frac{dC_A}{dt} = C_{A_i} - C_A - \tau k C_A$. Solving for C_A gives

$$\int_0^{C_A} \frac{dC_A}{C_{A_i} - C_A(1+\tau k)} = \int_0^t \frac{dt}{\tau} \quad \Rightarrow$$

$$-\frac{1}{1+\tau k} \ln\left(C_{A_i} - C_A(1+\tau k)\right)\Big|_0^{C_A} = \frac{t}{\tau}\Big|_0^t \quad \Rightarrow$$

$$-\frac{1}{1+\tau k}\left[\ln\left(C_{A_i} - C_A(1+\tau k)\right) - \ln C_{A_i}\right] = \frac{t}{\tau} \quad \Rightarrow$$

$$-\frac{1}{1+\tau k} \ln \frac{C_{A_i} - C_A(1+\tau k)}{C_{A_i}} = \frac{t}{\tau} \quad \text{We want the form } \frac{C_A}{C_{A_i}}, \text{ so}$$

$$-\frac{1}{1+\tau k} \ln\left(1 - \frac{C_A}{C_{A_i}}(1+\tau k)\right) = \frac{t}{\tau} \quad \Rightarrow \frac{C_A}{C_{A_i}}(1+\tau k) = e^{-(1+\tau k)\frac{t}{\tau}} + 1$$

$$\frac{C_A}{C_{A_i}} = \frac{e^{-(1+\tau k)\frac{t}{\tau}} + 1}{1+\tau k} \qquad \text{As } t \to \infty \quad \frac{C_A}{C_{A_i}} \to \frac{1}{1+\tau k}.$$

Therefore since we want $\frac{C_A}{C_{A_i}} = 0.95$, $\frac{C_A}{C_{A_i}} = 0.95\frac{1}{1+\tau k}$

Substituting back and using $\tau k = 1.0$, $\frac{t}{\tau} = -\frac{1}{2}\ln\left[1 - 0.95\left(\frac{1}{2}\right)(2)\right] = 1.5$

7.11 Logistic Growth

Calculus Topic: First Order Differential Equation

Department: Environmental Science

Subject Area: Population Growth

Time Needed: 40 minutes

Reference: [6]

To solve this problem we need to use the logistic growth model

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{7.27}$$

where K is the carrying capacity of the environment (maximum population that the environment can support), r is the logistic growth factor and N is the population at time t. rN represents the exponential growth in this problem and $\left(1 - \frac{N}{K}\right)$ represents the environmental resistance to growth. We will apply (7.27) to the following information in steps.

Suppose human population growth follows a logistic curve until it stabilizes at 7.3 billion people. Starting with its 1985 population of 4.84 billion people and the growth rate, R_o, at that time of 1.7 % per year, we want to find the predicted population for 1994?

1. Let t^* be the time at which $N = \frac{K}{2}$. Show that

$$N = \frac{K}{1 + e^{-r(t-t^*)}}.$$

2. If N_o is the population at $t = 0$, show that

$$t^* = \frac{1}{r} \ln\left(\frac{K}{N_o} - 1\right).$$

3. If R_o is the growth rate at $t = 0$, show that

$$r = \frac{R_o}{1 - \frac{N_o}{K}}.$$

4. Now find the population for 1994 from the information given in the opening problem.

Solutions to: Logistic Growth

1. $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$

$$\frac{dN}{rN\left(1 - \frac{N}{K}\right)} = dt \quad \Rightarrow \quad \frac{1}{r}\int \frac{dN}{N\left(1 - \frac{N}{K}\right)} = \int dt \quad \Rightarrow$$

$$\frac{1}{r}\ln\left(\frac{N}{1 - \frac{N}{K}}\right) = t + c, \text{ c is a constant} \quad \Rightarrow \quad \ln\left(\frac{N}{1 - \frac{N}{K}}\right) = rt + c \quad \Rightarrow$$

$$\frac{N}{1 - \frac{N}{K}} = e^{rt+c} = ce^{rt} \quad \Rightarrow \quad \frac{KN}{K - N} = ce^{rt} \quad \Rightarrow$$

$$KN + Nce^{rt} = Kce^{rt} \quad \Rightarrow \quad N\left(K + ce^{rt}\right) = Kce^{rt} \quad \Rightarrow$$

$$N = \frac{Kce^{rt}}{K + ce^{rt}} = \frac{K}{1 + Kce^{-rt}}.$$

Using the initial condition, $N(t^*) = \frac{K}{2}$, $\frac{K}{2} = \frac{K}{1 + Kce^{-rt^*}} \quad \Rightarrow$

$$\frac{1}{2} = \frac{1}{1 + Kce^{-rt^*}} \quad \Rightarrow \quad 1 + Kce^{-rt^*} = 2 \quad \Rightarrow \quad ce^{-rt^*} = \frac{1}{K} \quad \Rightarrow$$

$c = \frac{e^{rt^*}}{K}$ Substituting for c, $N = \frac{K}{1 + K\left(\frac{e^{rt^*}}{K}\right)\left(e^{-rt}\right)} \quad \Rightarrow$

$$N = \frac{K}{1 + e^{-r(t - t^*)}}$$

2. If $N(0) = N_o$, then $N_o = \frac{K}{1 + e^{rt^*}} \quad \Rightarrow$

$$\frac{K}{N_o} = 1 + e^{rt^*} \quad \Rightarrow \quad \frac{K}{N_o} - 1 = e^{rt^*} \quad \Rightarrow \quad rt^* = \ln\left(\frac{K}{N_o} - 1\right) \quad \Rightarrow$$

$$t^* = \frac{1}{r}\ln\left(\frac{K}{N_o} - 1\right)$$

3. $$\frac{dN}{dt} = R_o N_o = rN_o\left(1 - \frac{N_o}{K}\right) \quad \Rightarrow \quad R_o = r\left(1 - \frac{N_o}{K}\right) \quad \Rightarrow \quad r = \frac{R_o}{1 - \frac{N_o}{K}}$$

4. We are given that $K = 7.3$ billion people, $N_o = 4.84$ billion people

and $R_o = \frac{1.7\%}{\text{year}}$.

$$r = \frac{R_o}{1 - \frac{N_o}{K}} = \frac{0.017}{1 - \frac{4.84}{7.3}} = 0.0504 \text{ /year}$$

$$t^* = \frac{1}{r}\ln\left(\frac{K}{N_o} - 1\right) = \frac{1}{0.0504}\ln\left(\frac{7.3}{4.84} - 1\right) = -13.4 \text{ years}$$

Note: The negative sign on t^* means that the population reached half the carrying capacity 13.4 years before 1985.

$$N = N = \frac{K}{1 + e^{-r(t-t^*)}} = \frac{7,300,000,000}{1 + e^{-0.0504((1994-1985)-(-13.4))}} = 5.2 \text{ billion people.}$$

7.12 Flow of a Gas Mixture through a Tank

Calculus Topic:	Differential Equations
Department:	Chemical Engineering
Subject Area:	Process Simulation and Analysis
Time Needed:	40 minutes
Reference:	[27]

When a gas is placed in a tank, it expands to fill the whole volume of the tank. Because of this pressure must be considered as the measurable representation in this problem. The ideal gas law

$$PV = nRT \tag{7.28}$$

can be used to relate pressure to the other conditions in the tank. In (7.28), P is the pressure in the tank, V is the total volume of the tank, n is the total number of mols of gas, R is a gas constant and T is the absolute temperature.

Consider the tank system in Figure 7.23 which has a two-component ideal gas mixture at a constant temperature. The gas mixture is being added to the tank at a flow rate (mols per amount of time) of F_i with a_i being the percent of the composition of component A. The flow rate of the gas out of the tank is F_o with a being the percent of the composition of A leaving the tank since a is the percent of the composition of A in the tank. In the tank at any time, there are N mols of gas mixture of which N_A mols are of composition of A, at pressure P and a is the percent of the compositon

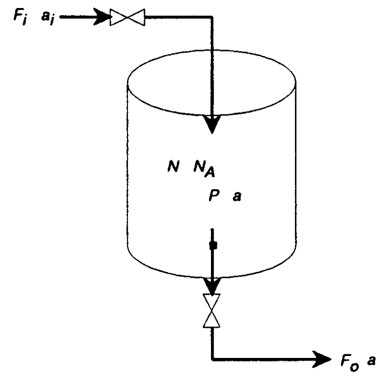

Figure 7.23: Ideal gas mixture at a constant volume and temperature.

of A in the tank. From (7.28) we can find that

$$N = \frac{PV}{RT} \qquad \text{so that for A} \qquad N_A = aN = \frac{aPV}{RT}.$$

Considering an overall mole balance for the system,

$$\frac{dN}{dt} = \frac{d\left(\frac{PV}{RT}\right)}{dt} = F_i - F_o \tag{7.29}$$

The mole balance for component A is

$$\frac{dN_A}{dt} = \frac{d\left(\frac{aPV}{RT}\right)}{dt} = a_iF_i - aF_o \tag{7.30}$$

Troublesome Notation: The differential notation is also confusing when substitution takes place. For example, $\frac{dN}{dt} = \frac{d\frac{PV}{RT}}{dt}$ is a simple substitution of $\frac{PV}{RT}$

for N. $\frac{d\frac{PV}{RT}}{dt}$ simplifies to $\frac{V}{RT}\frac{dP}{dt}$ since V, R and T are constants.

1. Let $F_i = 0$ so that the tank is being emptied. Let the gas leave the tank at a rate proportional to the difference between the tank pressure and the pressure of the atmosphere, P_{atm}, a constant. Then the overall mole balance equation is

$$-k\left(P - P_{atm}\right) = \frac{d\left(\frac{PV}{RT}\right)}{dt}$$

   where the ideal gas law is used to write N is terms of P. Considering that V, T and R are constant and that when $t = 0$, the pressure is P_o, find the pressure as a function of t at any time.

2. Now consider a non ideal gas which behaves according to the non ideal gas law

$$PV = ZnRT \qquad \text{where} \qquad Z = 1 + \frac{BP}{RT} \tag{7.31}$$

   where B represents the degree of departure from non-ideality at low pressures and is constant for constant temperature processes. As in problem (1), Let $F_i = 0$ so that the tank is being emptied, but this time assume that the tank is emptying at a constant rate, that is, F_o is constant. If the pressure in the tank is P_o at $t = 0$, set up the differential equation and solve for P.

Solutions to: Flow of a Gas Mixture through a Tank

1. We are given $\frac{d\left(\frac{PV}{RT}\right)}{dt} = -k\left(P - P_{atm}\right) \quad \Rightarrow \quad \frac{dP}{P - P_{atm}} = -\frac{kRT}{V}dt \quad \Rightarrow$

$$\int_{P_o}^{P} \frac{dP}{P - P_{atm}} = -\frac{kRT}{V}\int_0^t dt \quad \Rightarrow \quad \ln\left(P - P_{atm}\right)\Big|_{P_o}^{P} = -\frac{kRT}{V}t\Big|_0^t \quad \Rightarrow$$

$$\ln\left(\frac{P - P_{atm}}{P_o - P_{atm}}\right) = -\frac{kRTt}{V} \quad \Rightarrow \quad P = P_{atm} + \left(P_o - P_{atm}\right)e^{-\frac{kRTt}{V}}$$

2. We are given that $PV = ZnRT$ where $Z = 1 + \frac{BP}{RT}$. In our system,

$$n = N, \text{ so } N = \frac{PV}{ZRT} = \frac{PV}{\left(1 + \frac{BP}{RT}\right)RT} = \frac{PV}{RT + BP}$$

The mass balance for this system is $\frac{dN}{dt} = F_o$ where F_o is a constant.

$$\frac{dN}{dt} = \frac{d}{dt}\left(\frac{PV}{RT + BP}\right) = \frac{V\frac{dP}{dt}\left(RT + BP\right) - PVB\frac{dP}{dt}}{\left(RT + BP\right)^2} = \frac{dP}{dt}\frac{VRT}{\left(RT + BP\right)^2}$$

Hence $\frac{dP}{dt}\frac{VRT}{\left(RT + BP\right)^2} = F_o \quad \Rightarrow \quad \frac{dP}{\left(RT + BP\right)^2} = \frac{F_o}{VRT}dt \quad \Rightarrow$

$$\int_{P_o}^{P} \frac{dP}{\left(RT + BP\right)^2} = \frac{F_o}{VRT}\int_0^t dt \quad \Rightarrow \quad -\frac{1}{RT + BP}\Big|_{P_o}^{P} = \frac{BF_o}{VRT}t\Big|_0^t$$

$$\frac{1}{RT + BP} - \frac{1}{RT + BP_o} = -\frac{BF_o t}{VRT} \quad \Rightarrow$$

$$P = \frac{\left(VRT\right)\left(RT + BP_o\right)}{-B\left(BF_o t\right)\left(RT + BP_o\right) + VRT} - \frac{RT}{B}$$

7.13 Geological Dating

Calculus Topic:	Differential Equations
Department:	Geology
Subject Area:	Historical Geology
Time Needed:	40 minutes
Reference:	[24]

The age of rock formations can be determined by measuring the amounts of certain radioactive elements in the rocks. Chemical elements can be classified into isotopes, which are defined by the number of protons and neutrons in the element. Most isotopes are stable, however some are radioactive; that is, they emit particles and decay to either a different isotope of the same element or to a different element all together. For example, carbon-14 decays to a nitrogen isotope and rubidium-87 decays to a strontium isotope. Each unstable isotope decays at a different rate. The rate of decay is measured in terms of its half-life which is the amount of time is takes for one-half of the atoms in any sample of that element to decay. A property of half-life is that once one-half of the initial amount of unstable isotope has decayed, only one-half of the amount of isotope that is left will decay over the next half-life.

1. Uranium-234 decays to a thorium isotope and has a half-life of 248,000 years. At the time a certain rock formation developed, 1000 grams of U-234 formed in a sample of the rock. Using the definition of half-life, approximate the amount

of U-234 left in that sample of rock after one million years.

2. From the method you used to find the solution to problem (1), find the amount of an isotope left after T years if its half life is H and the original amount of isotope is N. What type of relationship develops between N and T?

To determine the age of a rock, a geologist needs to know the amount of unstable isotope left in the rock, the amount of the stable element produced and the rate of decay of the unstable isotope. The rate of decay of the isotope is directly proportional to the amount of isotope remaining, that is,

$$\frac{dN}{dt} = -kN$$

where N is the amount of unstable isotope at any time t and k is a decay constant.

3. If N_o is the amount of unstable isotope at $t = 0$, solve for $N(t)$ taking into account the initial condition. Sketch the solution curve.

4. Derive the equation for converting between the decay constant k and the half-life T. That is, derive $kT = \ln 2$.

5. Use the results from problems (3) and (4) to solve problem (1).

Solutions to: Geological Dating

1. Approximately four half-lives have occured in one million years,

End of first half-life: 500 g is left

End of first half-life: 250 g is left

End of first half-life: 125 g is left

End of first half-life: 62 g is left

or, $n = 1000\frac{1}{2}^{1000000/248000} = 61.1$grams remain.

2. The amount remaining will be $N\left(\frac{1}{2}\right)^{T/H}$

3. We are given $\frac{dN}{dt} = -kN \quad \Rightarrow \quad \frac{dN}{N} = -kdt \quad \Rightarrow$

$$\int_{N_o}^{N} \frac{dN}{N} = -k\int_0^t dt \quad \Rightarrow \quad \ln N|_{N_o}^{N} = -kt|_0^t \quad \Rightarrow \quad \ln\frac{N}{N_o} = -kt \quad \Rightarrow$$

$$N = N_o e^{-kt}$$

4. $\frac{1}{2} = (1)e^{-kT} \quad \Rightarrow \quad \ln\frac{1}{2} = -kT \quad \Rightarrow \quad -\ln 2 = -kT \quad \Rightarrow \quad \ln 2 = kT$

5. $N = N_o e^{-kt} \quad \Rightarrow \quad N = (1000)e^{(\ln 2/248000)(1000000)} = 61.1$ grams.

7.14 Mass Balance in Flow through a Tank

Calculus Topic:	Differential Equations
Department:	Chemical Engineering
Subject Area:	Simulation and Analysis
Time Needed:	90 minutes
Reference:	[27]

A mass balance equation for a tank is being filled and drained at the same time is formed according to the following conservation principal:

$$\begin{pmatrix} \text{rate of change} \\ \text{of mass} \end{pmatrix} = \begin{pmatrix} \text{input rates} \\ \text{of flow} \end{pmatrix} - \begin{pmatrix} \text{output rates} \\ \text{of flow} \end{pmatrix} \tag{7.32}$$

Since the mass M of the fluid is the same as the density ρ times the volume, we can write (7.32) as

$$\frac{dM}{dt} = \rho q_i - \rho q_o \tag{7.33}$$

where t is time, q_i is the volumetric rate of flow (amount of volume per amount of time) into the tank and q_o is the volumetric rate of flow out of the tank.

Troublesome Notation: Often in science and engineering, subscripts are used on variables to give them a clearer meaning in an equation. For example, in this problem, q_i denotes the rate of flow in. Subscripts can also indicate partial derivatives or vector components, which is not the case here. The differential notation confusing when substitution takes place. For example, $\frac{dM}{dt} = \frac{d(\rho Ah)}{dt}$ is a simple substitution of ρAh for M. $\frac{d(\rho Ah)}{dt}$ simplifies to $\rho A\frac{dh}{dt}$ since ρ and A are constants.

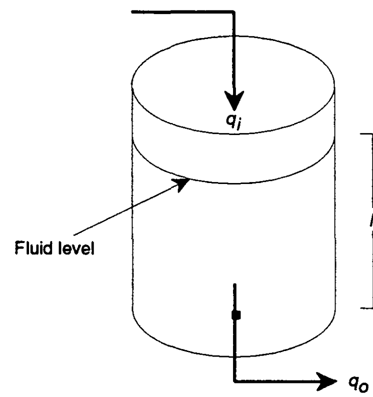

Figure 7.24: Cylindrical tank being both filled and drained.

1. In figure 7.24, a cylindrical tank is being filled and drained at the same time so the conservation principal in (7.32) applies. The amount of the liquid through the cylindrical tank depends on the height h since the cross-sectional area A remains constant. The mass of the fluid is also dependent on the height and is calculated as $M = \rho Ah$. Using (7.33), the conservation of mass problem becomes

$$\frac{d(\rho Ah)}{dt} = \rho A\frac{dh}{dt} = \rho(q_i - q_o)$$

or

$$\frac{dh}{dt} = \frac{(q_i - q_o)}{A} \tag{7.34}$$

(a) If both q_i and q_o are linear functions of time, i. e., $q_o = k_1 t$ and $q_i = k_2 t$, and the height of the liquid at $t = 0$ is h_o, find the height of the liquid in the cylindrical tank as a function of time.

(b) If q_i is a function of time and q_o is a function of height so that $q_i = kt$ and $q_o = ch$ where c and k are constants, then (7.34) becomes

$$\rho A \frac{dh}{dt} = \rho (ch - kt) . \tag{7.35}$$

Using a technique for solving differential equations called the integrating factor, (7.35) can be solved by rewriting it as

$$d\left(e^{\frac{ct}{A}} h\right) = \frac{kte^{\frac{ct}{A}}}{A} dt.$$

Assuming that the height of the liquid at $t = 0$ is h_o, find h as a function of time.

2. Now we will use conservation of mass to determine the flow rate through the cone shaped tank shown in Figure 7.25.

(a) The volume of a cone is $V = \frac{1}{3}\pi\frac{w^2}{4}h$. In Figure 7.25 there are two similar cones, one which makes up the tank and the other formed by the water level. Since these cones are similar, we can write the relation $\frac{w}{h} = \frac{W}{H}$. Set

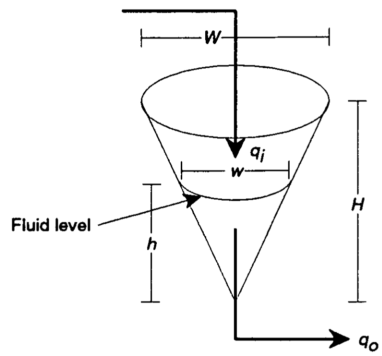

Figure 7.25: Cone shaped tank being both filled and drained.

up an equation which gives the mass of the liquid in the tank at any time t with M in terms of ρ, W, H and h.

(b) Set up the differential equation according to (7.32).

(c) If both q_i and q_o are linear functions of time, i. e., $q_o = k_1 t$ and $q_i = k_2 t$, and the height of the liquid at $t = 0$ is h_o, find the height of the liquid in the cylindrical tank as a function of time.

(d) If $q_i = 0$ and $q_o = ch^2$ where c is a constant, and the height of the liquid at $t = 0$ is h_o, find the height of the liquid in the tank at any time t until the tank is completely drained.

Solutions to: Mass Balance in Flow through a Tank

1. *a.* We are given $\dfrac{dh}{dt} = \dfrac{(q_i - q_o)}{A}$. Substituting $q_o = k_1 t$ and $q_i = k_2 t \Rightarrow$

$$\frac{dh}{dt} = \frac{(k_2 t - k_1 t)}{A} \quad \Rightarrow \quad dh = \frac{(k_2 t - k_1 t)}{A} dt \quad \Rightarrow$$

$$\int_{h_o}^{h} dh = \int_0^t \frac{(k_2 t - k_1 t)}{A} dt \quad \Rightarrow \quad h|_{h_o}^{h} = \left. \frac{(k_2 t^2 - k_1 t^2)}{2A} \right|_0^t \quad \Rightarrow$$

$$h = \frac{(k_2 t^2 - k_1 t^2)}{2A} + h_o$$

b. We are given $\dfrac{dh}{dt} = \dfrac{(q_i - q_o)}{A}$. Substituting $q_o = ch$ and $q_i = kt \Rightarrow$

$\dfrac{dh}{dt} = \dfrac{(kt - ch)}{A} \quad \Rightarrow \quad \dfrac{dh}{dt} + \dfrac{ch}{A} = \dfrac{kt}{A}$. The integrating factor, $\mu(t) = e^{\frac{ct}{A}}$

produces $d\left(e^{\frac{ct}{A}} h\right) = \dfrac{kte^{\frac{ct}{A}}}{A} dt \quad \Rightarrow \quad \int d\left(e^{\frac{ct}{A}} h\right) = \dfrac{k}{A} \int te^{\frac{ct}{A}} dt \quad \Rightarrow$

$$e^{\frac{ct}{A}} h = \frac{k}{A}\left[\frac{A}{c} te^{\frac{ct}{A}} - \int e^{\frac{ct}{A}} dt\right] \quad \Rightarrow \quad e^{\frac{ct}{A}} h = \frac{k}{A}\left[\frac{A}{c} te^{\frac{ct}{A}} - \frac{A}{c} e^{\frac{ct}{A}}\right] + c_1 \quad \Rightarrow$$

$h = \dfrac{k}{c}(t-1) + c_1 e^{-\frac{ct}{A}}$. Since $h = h_o$ at $t = 0$, $c_1 = h_o + \dfrac{k}{c}$.

Therefore $h = \dfrac{k}{c}(t-1) + \left(h_o + \dfrac{k}{c}\right) e^{-\frac{ct}{A}}$.

2. *a.* We are given $V = \dfrac{1}{3}\pi \dfrac{w^2}{4} h$, $M = \rho V$, $\dfrac{w}{h} = \dfrac{W}{H}$ and $\dfrac{dM}{dt} = \rho(q_i - q_o)$.

$$M = \rho V = \frac{1}{3}\rho\pi \frac{w^2}{4} h = \frac{1}{3}\rho\pi \frac{1}{4}\left(\frac{W}{H} h\right)^2 h = \frac{\rho\pi W^2 h^3}{12H^2}$$

b. The differential equation is $\dfrac{dM}{dt} = \dfrac{\pi \rho W^2 h^2}{12H^2}\dfrac{dh}{dt} = \rho\left(q_i - q_o\right) \quad \Rightarrow$

$$\frac{\pi W^2 h^2}{12H^2}\frac{dh}{dt} = \left(q_i - q_o\right)$$

c. Since $q_o = k_1 t$ and $q_i = k_2 t$, $\dfrac{\pi W^2 h^2}{12H^2}\dfrac{dh}{dt} = \left(k_2 t - k_1 t\right) \quad \Rightarrow$

$$\frac{\pi W^2 h^2}{12H^2}dh = \left(k_2 t - k_1 t\right)dt \quad \Rightarrow \quad \frac{\pi W^2}{12H^2}\int_{h_o}^{h} h^2 dh = \left(k_2 - k_1\right)\int_0^t t dt \quad \Rightarrow$$

$$\frac{\pi W^2}{36H^2}h^3\Big|_{h_o}^{h} = \frac{1}{2}\left(k_2 - k_1\right)t^2\Big|_0^t \quad \Rightarrow \quad h^3 = \frac{18H^2}{\pi W^2}\left(k_2 - k_1\right)t^2 - h_o^3 \quad \Rightarrow$$

$$h = \left(\frac{18H^2}{\pi W^2}\left(k_2 - k_1\right)t^2 - h_o^3\right)^{1/3}$$

d. Since $q_o = ch^2$ and $q_i = 0$, $\dfrac{\pi W^2 h^2}{12H^2}\dfrac{dh}{dt} = -ch^2 \quad \Rightarrow \quad \dfrac{\pi W^2}{12H^2}\dfrac{dh}{dt} = -c \quad \Rightarrow$

$$\frac{\pi W^2}{12H^2}dh = -cdt \quad \Rightarrow \quad \frac{\pi W^2}{12H^2}\int_{h_o}^{h} dh = -c\int_0^t dt \quad \Rightarrow$$

$$\frac{\pi W^2}{12H^2}h\Big|_{h_o}^{h} = -ct|_0^t \quad \Rightarrow \quad h = -ct + \frac{\pi W^2}{12H^2}h_o.$$

7.15 Reactions in a Plug Flow Reactor

Calculus Topic:	First Order Differential Equation
Department:	Chemical Engineering
Subject Area:	Kinetics
Time Needed:	40 minutes
Reference:	[18]

Different types of reactors such as batch reactors and plug flow reactors produce different results from reactions between fluids. Each reaction can be described by a material balance equation as follows.

$$\begin{pmatrix} \text{rate of} \\ \text{reactant} \\ \text{flow into} \\ \text{element} \\ \text{of volume} \end{pmatrix} = \begin{pmatrix} \text{rate of} \\ \text{reactant} \\ \text{flow out} \\ \text{of element} \\ \text{of volume} \end{pmatrix} + \begin{pmatrix} \text{rate of reactant} \\ \text{loss due to} \\ \text{chemical reaction} \\ \text{within the element} \\ \text{of volume} \end{pmatrix} + \begin{pmatrix} \text{rate of} \\ \text{accumulation} \\ \text{of reactant} \\ \text{in element} \\ \text{of volume} \end{pmatrix} \tag{7.36}$$

In a plug flow reactor, the composition of the fluid varies from point to point along a flow path. This means that a material balance equation for a reaction component must be made for a differential element of volume dV. We set up the material balance equation according to (7.36).

$$F_A = (F_A + dF_A) + (-r_A)\, dV + 0 \qquad (7.37)$$

where F_A is the input of reactant A in moles per time, $F_A + dF_A$ is the output of A in moles per time, $(-r_A)dV$ is the disappearance of A in moles per time and r_A is rate of reaction of A. If we let F_{AO} be the molar feed rate of component A to the reactor, then the input of A in moles per time becomes $F_A = F_{AO}(1 - X_A)$ and $dF_A = d(F_{AO}(1 - X_A)) = -F_{AO}dX_A$ where X_A is the fraction of reactant that is converted into product. Substituting this into (7.37) gives

$$F_{AO}dX_A = (-r_A)\, dV \qquad (7.38)$$

Troublesome Notation: The subscripts in this problem can be very confusing, but in each case they are being used to give a specific meaning to the variable and to distinguish between similar quantities. The subscript A refers to element A and the subscript O refers to an original (initial) amount.

1. Using separation of variables, write a differential equation placing volumes on one side and rates on the other side. Set up the limits of integration considering that the reaction starts at zero volume and ends at some volume and the amount of reactant converted to product starts at zero and ends at some final fraction,

X_{Af}. Decide which terms are constant and which are variables.

2. Let τ be the time required to process one reactor volume of feed measured at specified conditions. Then $\tau = \dfrac{C_{AO}V}{F_{AO}}$. where C_{AO} is the initial concentration of A in moles per liter. Using the result from problem (1), find τ. (Hint: It will be possible to evaluate one side of the equation, but you do not have enough information to evaluate the other side.)

3. A homogeneous gas reaction A $\longrightarrow$ 3R has a reported rate at 215° C of $-r_A = kC_A^{1/2}\ \frac{\text{moles}}{\text{liter}\cdot\text{sec}}$. Find the space-time ($\tau$) needed for 80% (i. e., $X_A = 0.8$) conversion of a 50% A - 50% inert feed to a plug flow reactor operating at 215° C and 5 atm. $C_{AO} = 0.0625\ \frac{\text{moles}}{\text{liter}}$.

   For this type of reaction,

$$C_A = C_{AO}\left(\frac{1 - X_A}{1 + \epsilon_A X_A}\right)$$

   and $k = 10^{-2}\ \frac{\text{moles}^{1/2}}{\text{liter}^{1/2}\cdot\text{s}}$. Since the reaction is 50% A - 50% inert feed, $\epsilon_A = 1$. Use this substitution to solve the problem.

Solutions to: Reactions in a Plug Flow Reactor

1. We are given that $F_{AO} dX_A = (-r_A)\, dV$. The differential equation takes the form $\frac{dV}{F_{AO}} = \frac{dX_A}{-r_A}$. With integration limits, $\int_0^V \frac{dV}{F_{AO}} = -\int_0^{X_{A_f}} \frac{dX_A}{r_A}$.

F_{AO} is the only constant in the equation.

2. Since $\tau = \frac{C_{AO}V}{F_{AO}}$, then $V = \frac{\tau F_{AO}}{C_{AO}}$. Evaluating the left side of the integral produces $\int_0^V \frac{dV}{F_{AO}} = \frac{V}{F_{AO}} = \frac{\tau}{C_{AO}}$. Hence we now have $\frac{\tau}{C_{AO}} = \int_0^{X_{A_f}} \frac{dX_A}{-r_A}$. We need more information about r_A to evaluate the right side. Therefore $\tau = C_{AO} \int_0^{X_{A_f}} \frac{dX_A}{-r_A}$

3. We are given that $-r_A = kC_A^{1/2}$ where $C_A = C_{AO}\left(\frac{1 - X_A}{1 + \epsilon_A X_A}\right)$ and that $\epsilon_A = 1$. First we need to find $\int_0^{X_{A_f}} \frac{dX_A}{-r_A}$.

$$\tau = C_{AO} \int_0^{X_{A_f}} \frac{dX_A}{-r_A} = \frac{C_{AO}}{k\sqrt{C_{AO}}} \int_0^{X_{A_f}} \left(\frac{1 - X_A}{1 + X_A}\right)^{-1/2} dX_A$$

$$= \frac{\sqrt{C_{AO}}}{k} \int_0^{X_{A_f}} \left(\frac{1 + X_A}{1 - X_A}\right)^{1/2} dX_A = \frac{\sqrt{C_{AO}}}{k} \int_0^{X_{A_f}} \frac{1 + X_A}{(1 - X_A)^{1/2}} dX_A$$

$$= \frac{\sqrt{C_{AO}}}{k} \left(\arcsin X_A - \sqrt{1 - X_A^2}\right)\Big|_0^{X_{A_f}}$$

$$= \frac{\sqrt{C_{AO}}}{k}\left(\left(\arcsin X_{A_f} - \sqrt{1 - {X_{A_f}}^2}\right) - \left(\arcsin 0 - \sqrt{1-0}\right)\right)$$

$$= \frac{\sqrt{0.0625}}{10^{-2}}\left(\arcsin 0.8 - \sqrt{1 - 0.8^2} + 1\right)$$

$$= 8.30 \text{ seconds}$$

7.16 Vector Problems

Calculus Topic:	Vectors
Department:	Physics
Subject Area:	Mechanics
Time Needed:	40 minutes

Motion problems can be solved using vectors. The key to solving these problems is to make a sketch of the vectors before attempting to do the mathematics. Since each problem is different, no general formulas can be given here.

Troublesome Notation: In these problems, two different methods of denoting vectors are used. One is $\vec{F}$ with its magnitude denoted as F and the other is $\mathbf{a}$ with its magnitude denoted as a. In each problem, the x-, y- and z-components are given in terms of the unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$. In most calculus texts, a boldface $\mathbf{a}$ is used for vector notation, $|a|$ would designate its magnitude and (a_1, a_2, a_3) would designate its components.

1. A ferry carrying cars and passengers across a river is traveling east with a velocity of 15 miles per hour relative to the water. The current is going directly south at this point in the river. What is the velocity of the ferry relative to the river bed?

2. A pitcher throws a curve ball with an acceleration vector $\mathbf{a} = 24\hat{\jmath} - 32\hat{k}$, an initial velocity vector $\mathbf{v_o} = 120\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and an initial position vector of

$\mathbf{r_o} = 5\hat{k}$. The origin of the coordinate system is at the pitcher's mound. the $x-$axis is from the mound to home plate. The $z-$axis is vertical. It is 60 feet from the mound to home plate. Determine the y and z components of the position vector when $x = 60$ feet.

3. A wind with a force $\vec{F}(t) = 2t\hat{\imath} - 5t^2\hat{\jmath}$ N is acting on a ball with a mass of 10 kg. If the ball is moving on a horizontal surface with no friction, find the position of the ball at $t = 6$ s if it is at rest at $t = 0$. It may be useful to recall Newton's Second Law of Motion which states that force = the product of mass and acceleration and can be written in vector form $\vec{F} = m\vec{a}$. [19]

Solutions to: Vector Problems

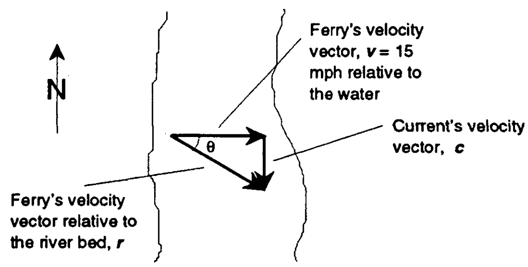

Figure 7.26: Vector diagram of the boat crossing the river.

1. From figure 1, $|\mathbf{r}| = \sqrt{c^2 + v^2} = \sqrt{c^2 + 15^2}$ and $\theta = \tan^{-1}\frac{c}{r} = \tan^{-1}\frac{c}{15}$

Therefore $\mathbf{r} = \sqrt{c^2 + 15^2} \quad \tan^{-1}\frac{c}{15}$ degrees south of east.

2. $\mathbf{a} = 24\hat{\jmath} - 32\hat{k}, \quad \mathbf{v_o} = 120\hat{\imath} - 3\hat{\jmath} + 4\hat{k}, \quad \mathbf{r_o} = 5\hat{k}$

$$\mathbf{v} = \int \mathbf{a}dt = c_1\hat{\imath} - (24t + c_2)\,\hat{\jmath} + (-32t + c_3)\,\hat{k}$$

At $t = 0$, $v = \mathbf{v_o} \quad \Rightarrow \quad c_1 = 120, \quad c_2 = -3, \quad c_3 = 4.$

Therefore $\mathbf{v} = 120\hat{\imath} - (24t - 3)\,\hat{\jmath} + (-32t + 4)\,\hat{k}$

$$\mathbf{r} = \int \mathbf{v}dt = \int \left(120\hat{\imath} - (24t - 3)\,\hat{\jmath} + (-32t + 4)\,\hat{k}\right) dt$$

$$= (120t + c_4)\,\hat{\imath} - \left(12t^2 - 3t + c_5\right)\hat{\jmath} + \left(-16t^2 + 4t + c_6\right)\hat{k}$$

At $t = 0,\, r = \mathbf{r_o} \quad \Rightarrow \quad c_4 = 0, \quad c_5 = 0, \quad c_6 = 5.$

Therefore $\mathbf{r} = 120t\hat{\imath} - \left(12t^2 - 3t\right)\hat{\jmath} + \left(-16t^2 + 4t + 5\right)\hat{k}$

When $x = 60,\; 120t = 60 \quad \Rightarrow \quad t = \dfrac{1}{2}.$

The y-component is $12\left(\dfrac{1}{2}\right)^2 - 3\left(\dfrac{1}{2}\right) = \dfrac{3}{2}.$

The z-component is $-16\left(\dfrac{1}{2}\right)^2 + 4\left(\dfrac{1}{2}\right) + 5 = 3.$

3. We are given that $\vec{F}(t) = 2t\hat{\imath} - 5t^2\hat{\jmath}$ N and that mass, $m = 10$ kg.

Since $\vec{F} = m\vec{a}$, then $\vec{a} = \dfrac{\vec{F}}{m} = \dfrac{2t\hat{\imath} - 5t^2\hat{\jmath}}{10} = \dfrac{1}{5}t\hat{\imath} - \dfrac{1}{2}t^2\hat{\jmath}.$

$$\vec{a} = \frac{d\vec{v}}{dt} \quad \Rightarrow \quad d\vec{v} = \left(\frac{1}{5}t\hat{\imath} - \frac{1}{2}t^2\hat{\jmath}\right)dt \quad \Rightarrow \quad \int d\vec{v} = \int \left(\frac{1}{5}t\hat{\imath} - \frac{1}{2}t^2\hat{\jmath}\right)dt$$

$$\Rightarrow \quad \vec{v} = \left(\frac{1}{10}t^2 + c_1\right)\hat{\imath} - \left(\frac{1}{6}t^3 + c_2\right)\hat{\jmath} \quad \text{Since } \vec{v}(0) = \vec{0},\; c_1 = c_2 = 0.$$

Therefore $\vec{v} = \dfrac{1}{10}t^2\hat{\imath} - \dfrac{1}{6}t^3\hat{\jmath}.$

$$\vec{v} = \frac{d\vec{s}}{dt} \quad \Rightarrow \quad d\vec{s} = \left(\frac{1}{10}t^2\hat{\imath} - \frac{1}{6}t^3\hat{\jmath}\right)dt \quad \Rightarrow \quad \int d\vec{s} = \int \left(\frac{1}{10}t^2\hat{\imath} - \frac{1}{6}t^3\hat{\jmath}\right)dt$$

$$\Rightarrow \quad \vec{2} = \left(\frac{1}{30}t^3 + c_3\right)\hat{\imath} - \left(\frac{1}{24}t^4 + c_4\right)\hat{\jmath} \quad \text{Since } \vec{s}(0) = \vec{0},\; c_3 = c_4 = 0.$$

Therefore $\vec{s} = \dfrac{1}{30}t^3\hat{\imath} - \dfrac{1}{24}t^4\hat{\jmath}.$

At $t = 6,\; \vec{s}(6) = \dfrac{1}{30}(6)^3\hat{\imath} - \dfrac{1}{24}(6)^4\hat{\jmath} = \dfrac{36}{5}\hat{\imath} - 54\hat{\jmath}.$

7.17 Haul Road Design

Calculus Topic:	Vectors
Department:	Mining Engineering
Subject Area:	Mine Design
Time Needed:	20 minutes
Reference:	[10]

A haul road up the side of the pit of the Turnagain Copper Mine consists of three straight sections that lead from the pit bottom to the crusher on top of the highwall. The road sections can be described by the following vectors, which are written in units of miles.

Section 1	Section 2	Section 3
$1.31\hat{\imath} + 0.78\hat{\jmath} + 0.016\hat{k}$	$0.93\hat{\imath} + 1.50\hat{\jmath} - 0.233\hat{k}$	$0.142\hat{\imath} + 2.49\hat{\jmath} - 0.046\hat{k}$

1. To climb out of the pit, do haul trucks travel from section 1 to section 3, or vice versa?

2. What is the elevation difference between the bottom of the pit and the crusher station?

3. If the x-axis is aligned with east and the y-axis with north, what bearing is the pit bottom in relation to the crusher station?

4. If the units were kilometers (1.00 km = 0.62137 mi), how would the bearing in problem (3) change?

Solutions to: Haul Road Design

1. The trucks go from section three to section one. In this direction, they rise $0.046 + 0.233 - 0.016$ miles.

2. $0.046 + 0.233 - 0.016 = 0.263$ miles $= 1389$ feet

3. The pit is $1.31 + 0.93 + 0.142 = 2.382$ miles east of the crusher and $0.78 + 1.50 + 2.49 = 4.770$ miles north of the crusher. Hence, the pit is $\tan^{-1} \dfrac{4.77}{2.382} = 63.5°$ north of east or $26.5°$ east of north of the crusher.

4. The bearing would stay the same. Units have no bearing on the bearing.

7.18 Rescue Problem

Calculus Topic:	Vectors
Department:	Physics
Subject Area:	Mechanics
Time Needed:	2 hours

This problem uses properties of three dimensional vectors for finding a downed aircraft. Using consistent vector notation is important to the solution of the problem.

A ranger stationed in a remote section of the Rocky Mountains spots a small Cessna aircraft with smoke pouring from the engine. In order to give the search and rescue team as accurate information as possible, he immediately contacts a second ranger who can also see the plane. At the same time they both take bearings on the plane in order to calculate its exact position. The first ranger is located at $(0, 0, 6800)$ and sights the plane at $42°$ south of east and $11.5°$ above the horizon. The second ranger is located at $(17321, 10000, 7600)$ and sights the plane at $58°$ south of west and $3.4°$ above the horizon. Three minutes later, they both take a second bearing on the plane. The first ranger sights the plane at $41°$ east of north and $2.3°$ above the horizon. The second ranger sights the plane at $83°$ west of north and $1.6°$ below the horizon. All coordinates are in feet.

1. Draw a reasonably accurate sketch of the problem. Although you have three dimensions in the problem, it may be easier to work with a two dimensional

sketch with the elevations listed next to the points.

2. Calculate the coordinates of the plane at the two sightings.

3. Find both the displacement vector and the velocity vector (in feet per second) for the plane between the two points.

4. If the plane is heading toward a meadow averaging 6300 feet in elevation and assuming that the plane maintains its current direction and velocity, calculate the coordinates where it will crash and find the amount of time elapsed between the second sighting and the crash.

5. A rescue helicopter leaves an airfield at the same time the second sighting is taken by the rangers. It flies on a velocity vector of $-170\hat{\imath} + 248\hat{\jmath} + 7\hat{k}$. The coordinates of the airfield are $(150000, -200000, 4000)$. When will it reach the crash sight?

Solutions to: Rescue Problem

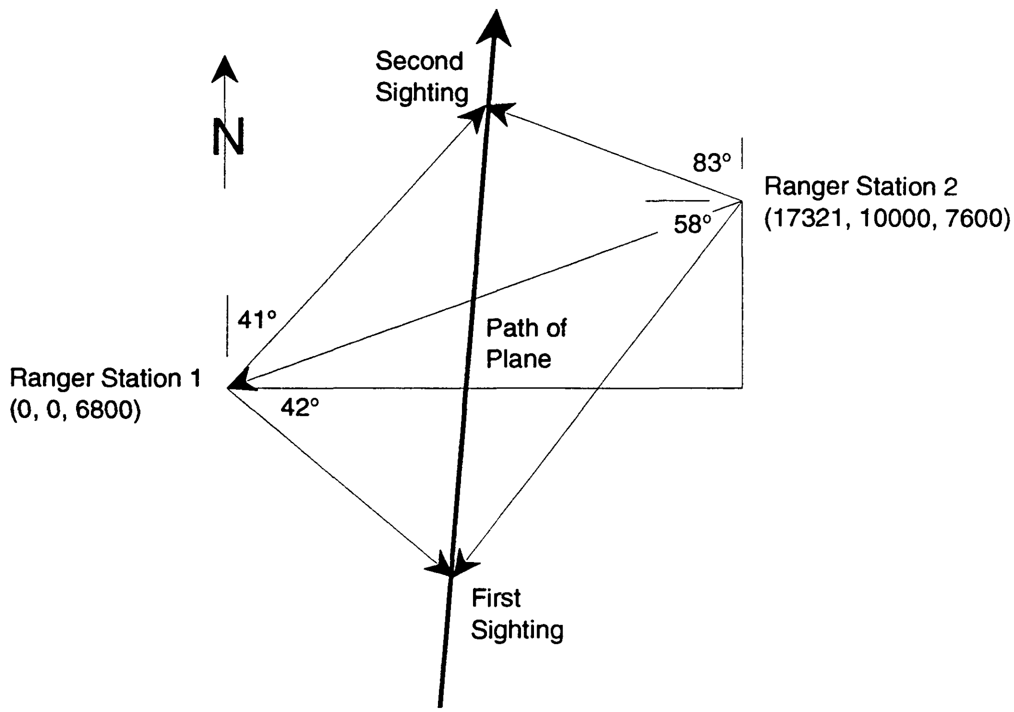

Figure 7.27: Sketch of the path of the plane.

2. Vector from Ranger Station 1 to the first sighting: $7085\hat{\imath} - 6379\hat{\jmath} + 1940\hat{k}$

   Vector from Ranger Station 1 to the second sighting: $9525\hat{\imath} + 10957\hat{\jmath} + 583\hat{k}$

   Vector from Ranger Station 2 to the first sighting: $-10236\hat{\imath} - 16379\hat{\jmath} + 1140\hat{k}$

   Vector from Ranger Station 2 to the second sighting: $-7796\hat{\imath} + 957\hat{\jmath} - 271\hat{k}$

3. Displacement vector of the plane from the first sighting to the second sighting: $2440\hat{\imath} + 17336\hat{\jmath} - 1357\hat{k}$

   The displacement unit vector: $0.139\hat{\imath} + 0.987\hat{\jmath} - 0.077\hat{k}$

   The velocity vector of the plane from the first sighting to the second sighting: $13.6\hat{\imath} + 96.3\hat{\jmath} - 7.5\hat{k}$

4. The vector form the second sighting to the crash: $1955\hat{\imath} + 13882\hat{\jmath} - 1083\hat{k}$

   The coordinates of the estimated crash point: $(11480, 24839, 6300)$

   The crash will occur in 2 minutes 24 seconds after the second sighting.

5. The helicopter flies at 300 $\frac{\text{ft}}{\text{s}}$ for a distance of 264094 ft. It will take 14 minutes 40 seconds for it to reach the sight.

7.19 Moments on Force Systems

Calculus Topic:	Vectors, Cross Products
Department:	Engineering
Subject Area:	Statics
Time Needed:	1 hour
Reference:	[12]

The mechanism shown in figure 7.28 is static, that is, the forces placed on it do not cause any movement in the system. This is the basis for all of the calculations in the following problems. Much of the preliminary work for these problems is shown to give an idea of where the equations come from.

Troublesome Notation: The study of Statics is centered around the action of forces, which are vectors. The arrow notation, $\vec{T}_{BD}$, is used here, and $|\vec{T}_{BD}|$ denotes the magnitude of $\vec{T}_{BD}$. It is important for the student to keep track of both the magnitude and direction of the forces. Components of force vectors are indicated by subscripts, for example, F_x and F_y.

1. Given the forces in the diagram in figure 7.28 and the coordinates of the following points as $A(0,0,0)$, $B(6,0,0)$ $C(10,0,0)$ $D(0,7,6)$ and $E(0,7,-6)$, find

   (a) $|\vec{T}_{BD}|$ and $|\vec{T}_{BE}|$ (the magnitude of the force vectors) and

   (b) $\vec{A}$ where $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$.

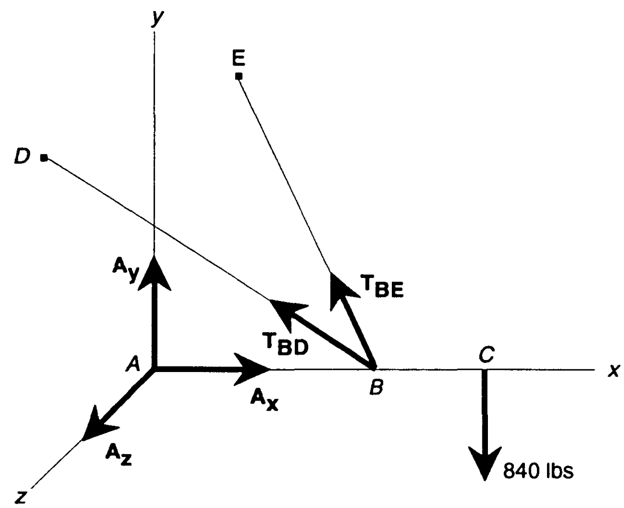

Figure 7.28: System for Problem 1.

Since this is a static system, the sum of the moments acting on A is 0. Each moment acting on A can be found by

$$\vec{M}_A = \vec{v} \times \vec{T}$$

where $\vec{T}$ is a force vector acting at a point and $\vec{v}$ is the displacement vector from A to $\vec{T}$. For example, $\vec{T}_{BD} = |\vec{T}_{BD}| \frac{-6\hat{\imath} + 7\hat{\jmath} + 6\hat{k}}{\sqrt{(-6)^2 + 7^2 + 6^2}} = \frac{|\vec{T}_{BD}|}{11} \left(-6\hat{\imath} + 7\hat{\jmath} + 6\hat{k}\right)$ and $\vec{v}_{BD} = 6\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = 6\hat{\imath}$. There are four forces working in this system ($\vec{A}$, $\vec{T}_{BD}$, $\vec{T}_{BE}$ and $\vec{C}$) and you need to find the moments for all four by evaluating the cross-product. Then use

$$\sum \vec{M}_A = 0$$

to set up an equation in which you can solve for $|\vec{T}_{BD}|$ and $|\vec{T}_{BE}|$.

To find A_x, A_y and A_z sum the forces in the x-, y- and z-directions and set the sum equal to zero. Consider only the x-components of the forces when summing the forces in the x-direction, the y-components of the forces when summing the forces in the y-direction and the z-components of the forces when summing the forces in the z-direction.

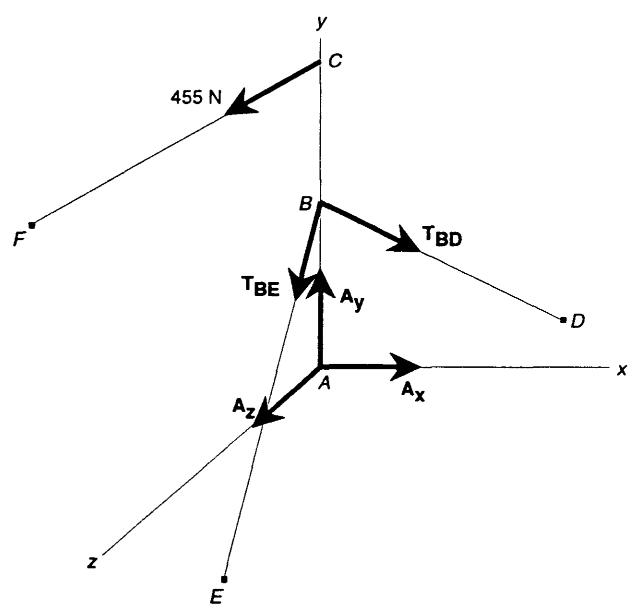

Figure 7.29: System for Problem 2.

2. Given the forces in the diagram in figure 7.29 and the coordinates of the following points as $A(0,0,0)$, $B(0,3,0)$ $C(0,6,0)$ $D(1.5,0,-3)$ $E(1.5,0,3)$ and $F(-3,0,2)$, find

(a) $|\vec{T}_{BD}|$ and $|\vec{T}_{BE}|$ (the magnitude of the force vectors) and

(b) $\vec{A}$ where $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$.

Since this is a static system, the sum of the moments acting on A is 0. Each moment acting on A can be found by

$$\vec{M}_A = \vec{v} \times \vec{T}$$

where $\vec{T}$ is a force vector acting at a point and $\vec{v}$ is the displacement from A to $\vec{T}$. For example, $\vec{T}_{BD} = |\vec{T}_{BD}|\frac{1.5\hat{\imath} - 3\hat{\jmath} - 3\hat{k}}{\sqrt{1.5^2 + (-3)^2 + (-3)^2}} = \frac{|\vec{T}_{BD}|}{4.5}\left(1.5\hat{\imath} - 3\hat{\jmath} - 3\hat{k}\right)$ and $\vec{v}_{BD} = 0\hat{\imath} + 3\hat{\jmath} + 0\hat{k} = 3\hat{\jmath}$. There are four forces working in this system and you need to find the moments for all four by finding the cross-product. Then use

$$\sum \vec{M}_A = 0$$

to set up an equation in which you can solve for $|\vec{T}_{BD}|$ and $|\vec{T}_{BE}|$.

To find A_x, A_y and A_z sum the forces in the x-, y- and z-directions and set the sum equal to zero. Consider only the x-components of the forces when summing the forces in the x-direction, the y-components of the forces when summing the forces in the y-direction and the z-components of the forces when summing the forces in the z-direction.

Solutions to: Moments on Force Systems

1. a. $\vec{T}_{BD} = |\vec{T}_{BD}|\dfrac{-6\hat{\imath}+7\hat{\jmath}+6\hat{k}}{\sqrt{(-6)^2+7^2+6^2}} = \dfrac{|\vec{T}_{BD}|}{11}\left(-6\hat{\imath}+7\hat{\jmath}+6\hat{k}\right)$

$$\vec{T}_{BE} = |\vec{T}_{BD}|\frac{-6\hat{\imath}+7\hat{\jmath}-6\hat{k}}{\sqrt{(-6)^2+7^2+(-6)^2}} = \frac{|\vec{T}_{BD}|}{11}\left(-6\hat{\imath}+7\hat{\jmath}-6\hat{k}\right)$$

$$\sum \vec{M}_A = \vec{A}\times\vec{0} + \vec{T}_{BE}\times(6,0,0) + \vec{T}_{BD}\times(6,0,0) + \vec{C}\times(10,0,0)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 0 & 0 \\ -6 & 7 & -6 \end{vmatrix}\frac{|\vec{T}_{BE}|}{11} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 0 & 0 \\ -6 & 7 & 6 \end{vmatrix}\frac{|\vec{T}_{BD}|}{11} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 10 & 0 & 0 \\ 0 & -840 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad \left(\frac{36}{11}|\vec{T}_{BE}| - \frac{36}{11}|\vec{T}_{BD}|\right)\hat{\jmath} + \left(\frac{42}{11}|\vec{T}_{BE}| - \frac{42}{11}|\vec{T}_{BD}| - 8400\right)\hat{k} = 0$$

Therefore, $\dfrac{36}{11}|\vec{T}_{BE}| - \dfrac{36}{11}|\vec{T}_{BD}| = 0$ and $\dfrac{42}{11}|\vec{T}_{BE}| - \dfrac{42}{11}|\vec{T}_{BD}| - 8400 = 0.$

Solving this system gives $|\vec{T}_{BE}| = |\vec{T}_{BD}| = 1100$ lbs

b. $\sum F_x \quad \Rightarrow \quad A_x - \dfrac{6}{11}|\vec{T}_{BD}| - \dfrac{6}{11}|\vec{T}_{BE}| = 0 \quad \Rightarrow$

$$A_x = \frac{6}{11}(1100) + \frac{6}{11}(1100) = 1200\text{lbs}$$

$\sum F_y \quad \Rightarrow \quad A_y + \dfrac{7}{11}|\vec{T}_{BD}| + \dfrac{7}{11}|\vec{T}_{BE}| - 840 = 0 \quad \Rightarrow$

$$A_y = \frac{7}{11}(1100) + \frac{7}{11}(1100) - 840 = -560\text{lbs}$$

$$\sum F_z \quad \Rightarrow \quad A_z - \frac{6}{11}|\vec{T}_{BD}| + \frac{6}{11}|\vec{T}_{BE}| = 0 \quad \Rightarrow$$

$$A_z = -\frac{6}{11}(1100) + \frac{6}{11}(1100) = 0$$

Therefore $\vec{A} = 1200\hat{\imath} - 560\hat{\jmath}$ lbs.

2. *a.* $$\vec{T}_{BD} = |\vec{T}_{BD}|\frac{1.5\hat{\imath} - 3\hat{\jmath} - 3\hat{k}}{\sqrt{1.5^2 + (-3)^2 + (-3)^2}} = \frac{|\vec{T}_{BD}|}{4.5}\left(1.5\hat{\imath} - 3\hat{\jmath} - 3\hat{k}\right)$$

$$\vec{T}_{BE} = |\vec{T}_{BE}|\frac{1.5\hat{\imath} - 3\hat{\jmath} + 3\hat{k}}{\sqrt{1.5^2 + (-3)^2 + 3^2}} = \frac{|\vec{T}_{BE}|}{4.5}\left(1.5\hat{\imath} - 3\hat{\jmath} + 3\hat{k}\right)$$

$$\vec{C} = 455\frac{-3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}}{\sqrt{(-3)^2 + (-6)^2 + 2^2}} = -195\hat{\imath} - 390\hat{\jmath} + 130\hat{k}$$

$$\sum \vec{M}_A = \vec{A} \times \vec{0} + \vec{T}_{BE} \times (0,3,0) + \vec{T}_{BD} \times (0,3,0) + \vec{C} \times (0,6,0)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & 0 \\ 1.5 & -3 & -3 \end{vmatrix} \frac{|\vec{T}_{BE}|}{4.5} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & 0 \\ 1.5 & -3 & 3 \end{vmatrix} \frac{|\vec{T}_{BD}|}{4.5} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 6 & 0 \\ -195 & -390 & 130 \end{vmatrix} = 0$$

$$\Rightarrow \quad \left(\frac{-9}{4.5}|\vec{T}_{BD}| + \frac{9}{4.5}|\vec{T}_{BE}| + 780\right)\hat{\imath} + \left(-|\vec{T}_{BD}| - |\vec{T}_{BE}| + 1170\right)\hat{k} = 0$$

Therefore, $-2|\vec{T}_{BD}| + 2|\vec{T}_{BD}| + 780 = 0$ and $-|\vec{T}_{BE}| - |\vec{T}_{BD}| + 1170 = 0.$

Solving this system gives $|\vec{T}_{BD}| = 780$ N and $|\vec{T}_{BE}| = 390$ N

b. $$\sum F_x \quad \Rightarrow \quad A_x + \frac{1.5}{4.5}|\vec{T}_{BD}| + \frac{1.5}{4.5}|\vec{T}_{BE}| - 195 = 0 \quad \Rightarrow$$

$$A_x = -\frac{1}{3}(780) - \frac{1}{3}(390) + 195 = -195\text{N}$$

$$\sum F_y \quad \Rightarrow \quad A_y - \frac{3}{4.5}|\vec{T}_{BD}| - \frac{3}{4.5}|\vec{T}_{BE}| - 390 = 0 \quad \Rightarrow$$

$$A_y = \frac{2}{3}(780) + \frac{2}{3}(390) + 390 = 1170\text{N}$$

$$\sum F_z \quad \Rightarrow \quad A_z - \frac{3}{4.5}|\vec{T}_{BD}| + \frac{3}{4.5}|\vec{T}_{BE}| + 130 = 0 \quad \Rightarrow$$

$$A_z = \frac{2}{3}(780) - \frac{2}{3}(390) - 130 = 130\text{N}$$

Therefore $\vec{A} = -195\hat{\imath} + 1170\hat{\jmath} + 130\hat{k}$ N.

7.20 Potential Difference Due to Continuous Charge Distributions

Calculus Topic:	Integration
Department:	Physics
Subject Area:	Electric Fields
Time Needed:	90 minutes
Reference:	[17] [23]

When a small charge called a test charge, q_0, is placed in an electric vector field $\vec{E}$, the electric force on the test charge, $\vec{F} = q_o\vec{E}$ is the vector sum of the individual forces exerted on q_o. Work is the amount of energy it takes for a force to displace an object, so the work done by the force $q_o\vec{E}$ for an infinitesimal displacement $d\vec{s}$ is given by

$$dW = \vec{F} \cdot d\vec{s} = q_o\vec{E} \cdot d\vec{s}$$

Change in potential energy, ΔU, is defined as the negative of work done by a conservative force so that

$$\Delta U = \int_a^b -q_o\vec{E} \cdot d\vec{s}.$$

Potential difference is directly related to potential energy by a constant equal to the reciprocal of the test charge. It is defined as the amount of work per unit charge that an external force must apply to displace the charge, say from a to b. Hence

$$\Delta V = \frac{\Delta U}{q_o} = -\int_a^b \vec{E} \cdot d\vec{s}. \tag{7.39}$$

For this problem, we are concerned with what happens when we introduce a continuous charge distribution as opposed to a single test charge. For a continuous charge distribution we need to consider a charge element dq and see what effect that this charge distribution has on the potential difference when the distance from the charged object is kept constant. Using (7.39), and given that $-\vec{E} \cdot d\vec{s} = k\frac{dq}{r}$, the potential dV due to the charge element dq at some point P a distance r from dq is given by

$$dV = k\frac{dq}{r} \quad \Rightarrow \quad V = k\int_a^b \frac{dq}{r}$$

Troublesome notation: In this problem, vectors are denoted by $\vec{E}$ with the magnitude of $\vec{E}$ denoted as E and the components of $\vec{E}$ as E_x, E_y and E_z. In most calculus texts, a boldface $\mathbf{a}$ is used for vector notation, $|a|$ would designate its magnitude and (a_1, a_2, a_3) would designate its components. Both s and r are used here to describe a distance or displacement. In this case, $\vec{s}$ is a vector describing a displacement along a path that an object is being moved and r describes the distance from an object to a point. $\vec{E}$ represents a field of vectors, but can be represented by a single vector at a known point. Also in this problem, $\vec{E} \cdot d\vec{s}$ denotes a dot product between two vectors. There are two ways to evaluate a dot product. In calculus, the dot product is written as $\mathbf{a} \cdot \mathbf{b}$ and is either $a_1b_1 + a_2b_2 + a_3b_3$ for three component vectors or $|\mathbf{a}||\mathbf{b}|\cos\theta$ where θ is the angle between the vectors. This problem uses the second form.

In the following problems you will be finding the potential difference due to different types of charged objects.

1. For a uniformly charged ring of radius a and total charge Q

   (a) Find the electric potential at a point P located on the x-axis a distance x from the center of the ring. As shown in figure 7.30, the plane of the ring is chosen perpendicular to the x-axis. Your answer should be written in terms of x and a.

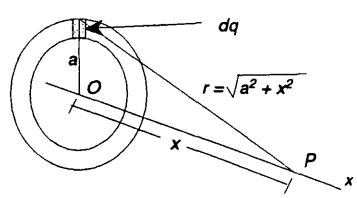

Figure 7.30: A uniformly charged ring of radius a.

   (b) What is the maximum electric potential along the x-axis?

   (c) Find the amount of work required to move a test charge from point P to the ring. ($W = -Vq_o$)

(d) Given that the strength of an electric field at a point on the x-axis is $\vec{E} = -\frac{dV}{dx}$, find $\vec{E}$ at point P. You may want to compare this to Problem 3 in the Electric Fields (7.30) section.

2. Find the electric potential along the axis of a uniformly charged disk of radius a and charge per unit area σ.

   (a) Sketch the problem showing the disk, the x-axis, P, the radius and the charge element. The charge element will be a ring of area $dA = 2\pi r dr$. Show dA and dr on your sketch.

   (b) Since the disk has a charge per unit area σ, dq can be written in terms of dA. Using this information, set up and evaluate the integral to find the potential difference.

3. A rod of length ℓ located along the $x-$axis has a uniform charge per unit length and a total charge Q. Find the electric potential at a point P along the $y-$axis at a distance d from the origin as shown in figure 7.31. Let $\lambda = \frac{Q}{\ell}$ be the unit charge so that $dq = \lambda dx$.

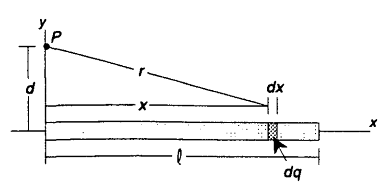

Figure 7.31: A uniformly charged rod of length ℓ.

Solutions to: Potential Difference Due to Continuous Charge

1. a. $$V = k\int \frac{dq}{r} = k\int_0^{2\pi a} \frac{dq}{\sqrt{x^2+a^2}} = \frac{k}{\sqrt{x^2+a^2}}\int_0^{2\pi a} dq$$
$$= \frac{k}{\sqrt{x^2+a^2}} 2\pi a q = \frac{kQ}{\sqrt{x^2+a^2}}$$

b. $$\frac{dV}{dx} = \frac{kQx}{(x^2+a^2)^{3/2}} = 0 \quad \Rightarrow \quad x = 0$$

c. $$W = -Vq_o = \frac{-kQq_o}{\sqrt{x^2+a^2}}$$

d. $$\vec{E} = -\frac{dV}{dx}\hat{i} = -kQ\frac{d}{dx}\frac{1}{\sqrt{x^2+a^2}}\hat{i}$$
$$= -kQ\left(\frac{-1}{2}\right)\left(x^2+a^2\right)^{-3/2}(2x)\,\hat{i} = \frac{kQx}{(x^2+a^2)^{3/2}}\hat{i}$$

2. a.

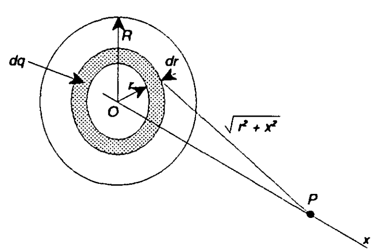

Figure 7.32: A uniformly charged disk.

$$b.\quad V = k\int \frac{dq}{r} = k\int_0^{2\pi a} \frac{dq}{\sqrt{x^2+r^2}} = k\int_0^{a} \frac{\sigma 2\pi r dr}{\sqrt{x^2+r^2}}$$

$$= \pi k\sigma \int_0^{a} \frac{2r dr}{\sqrt{x^2+r^2}} = 2\pi k\sigma \left(\sqrt{x^2+a^2} - x\right)$$

3. $$V = k\int \frac{dq}{r} = k\int_0^{\ell} \frac{dq}{\sqrt{x^2+d^2}} = k\lambda \int_0^{\ell} \frac{dx}{\sqrt{x^2+d^2}}$$

$$= k\lambda \ln\left(\frac{\ell + \sqrt{l^2+d^2}}{d}\right) = \frac{kQ}{\ell} \ln\left(\frac{\ell + \sqrt{l^2+d^2}}{d}\right)$$

7.21 Rock Excavator Bit Testing

Calculus Topic:	Line Integrals
Department:	Mining Engineering
Subject Area:	Cutting Machine Design
Time Needed:	30 minutes
Reference:	[10]

Rock excavation machines use hard metal cutters mounted on rotating cutterheads to chip rock from the working face of an excavation. Many different cutter designs have been used over the years. One of the ways that machine designers determine what cutters are appropriate for a particular excavating job is to perform a *punch-penetration test* in a sample of the rock to be dug. A single cutter is fixed to a hydraulic loading device and forced down into the rock until chips form on the rock surface. The load level and penetration depth are recorded several times per second and plotted. The purpose of this test is to determine how much energy that particular cutter needs to create a chip of that particular rock. Energy is represented on the data plot by the area under the load-penetration curve. When a chip forms the load falls quickly, creating a sawtooth in the curve. For a curve recorded by the CSM Excavation Engineering and Earth Mechanics Institute in a recent project, determine the energy required to form the first major rock chip.

The following equation is used to determine the energy required to form a rock chip:

$$U_{1-2} = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r}$$

where U_{1-2} is the work done by the loading device to make the cutter penetrate the rock, $\mathbf{F}$ is the applied force vector and $\mathbf{r}$ is the displacement vector which has magnitude s. Subscript 1 refers to the initial condition (the cutter is sitting with no load on the rock surface) and subscript 2 refers to the final condition (the cutter is at the penetration point of interest).

Troublesome Notation: In this problem, $\mathbf{F}\cdot\mathbf{r}$ denotes a dot product between two vectors. There are two ways to evaluate a dot product. In calculus, the dot product is written as $\mathbf{a}\cdot\mathbf{b}$ and is either $a_1b_1 + a_2b_2 + a_3b_3$ for three component vectors or $|\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the vectors. This problem uses the second form.

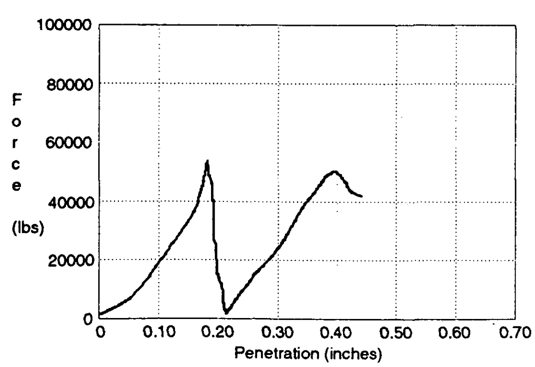

Figure 7.33: Results from a punch-penetration test.

Figure 7.33 shows the results of a particular test. Calculate the amount of work done to make both chips if the force was applied at 0, 20, and 40 degree angles to the displacement of the bit. We will assume a linear relationship here.

Solutions to: Rock Excavator Bit Testing

We are given the equation $U_{1-2} = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r}$ which can also be written as

$U_{1-2} = \int_{s_1}^{s_2} F \cos\theta ds$, where s is the magnitude of the displacement.

From the graph in figure 7.33, the first rock chip occurs at 0.18 inches with a force of 53000 lbs., and the second chip occurs at 0.19 in (0.40 in - 0.21 in) with a force of 48000 lbs.

The energy for the first chip penetrating at an angle of 0° is

$$U_{1-2} = \int_{s_1}^{s_2} F \cos\theta ds = \int_{0}^{0.18} F \cos\theta ds$$

$$= Fs \cos\theta|_0^{0.18} = (53000)(\cos 0)(0.18 - 0) = 9540 \text{ in-lbs}$$

The energy for the first chip penetrating at an angle of 20° is

$$U_{1-2} = \int_{s_1}^{s_2} F \cos\theta ds = \int_{0}^{0.18} F \cos\theta ds$$

$$= Fs \cos\theta|_0^{0.18} = (53000)(\cos 20)(0.18 - 0) = 8965 \text{ in-lbs}$$

The energy for the first chip penetrating at an angle of 40° is

$$U_{1-2} = \int_{s_1}^{s_2} F\cos\theta ds = \int_{0}^{0.18} F\cos\theta ds$$

$$= Fs\cos\theta|_0^{0.18} = (53000)(\cos 40)(0.18 - 0) = 7308 \text{ in-lbs}$$

The energy for the second chip penetrating at an angle of 0° is

$$U_{1-2} = \int_{s_1}^{s_2} F\cos\theta ds = \int_{0}^{0.18} F\cos\theta ds$$

$$= Fs\cos\theta|_0^{0.18} = (48000)(\cos 0)(0.40 - 0.21) = 9120 \text{ in-lbs}$$

The energy for the second chip penetrating at an angle of 20° is

$$U_{1-2} = \int_{s_1}^{s_2} F\cos\theta ds = \int_{0}^{0.18} F\cos\theta ds$$

$$= Fs\cos\theta|_0^{0.18} = (48000)(\cos 20)(0.40 - 0.21) = 8570 \text{ in-lbs}$$

The energy for the second chip penetrating at an angle of 40° is

$$U_{1-2} = \int_{s_1}^{s_2} F\cos\theta ds = \int_{0}^{0.18} F\cos\theta ds$$

$$= Fs\cos\theta|_0^{0.18} = (48000)(\cos 40)(0.40 - 0.21) = 6986 \text{ in-lbs}$$

7.22 Flow of Fluids in Pipes and Tanks

Calculus Topic:	Multiple Integration
Department:	Chemical Engineering
Subject Area:	Fluids
Time Needed:	1 hour
Reference:	[15]

To set up a conservation of mass equation, we will make use of the general conservation equation for mass, momentum or energy which is

$$\left.\frac{dN}{dt}\right|_S = \left.\frac{\partial N}{\partial t}\right|_{CV} + \int\int_{CS} n\rho V_n dA \tag{7.40}$$

where N is a flow quantity (mass, momemtum or energy), n is the flow quantity per unit, S is the system, CV is the control volume, CS is the control surface, V_n is the velocity distribution in direction n, ρ is the density of the medium and A is the cross-sectional area. The three parts of this equation can be written in the following manner:

$$\begin{pmatrix} \text{instantaneous} \\ \text{time rate of change} \\ \text{in } N \text{ for a system} \\ \text{of particles} \end{pmatrix} = \begin{pmatrix} \text{instantaneous time} \\ \text{rate of accumulation} \\ \text{of } N \text{ within the} \\ \text{control volume} \end{pmatrix} + \begin{pmatrix} \text{amount of } N \\ \text{leaving the control} \\ \text{volume minus the} \\ \text{amount of } N \text{ entering} \end{pmatrix}$$

From the general conservation equation, we can find the continuity equation, which is a statement of the conservation of mass. The flow quantity N becomes m (mass) and $n = m/m = 1$. Now, the general conservation equation becomes

$$\left.\frac{dm}{dt}\right|_S = \left.\frac{\partial m}{\partial t}\right|_{CV} + \int\int_{CS} \rho V_n dA.$$

This is called the control volume approach to flow through a pipe. For a system of particles, mass is constant, so we can further simply the equation to

$$\left.\frac{\partial m}{\partial t}\right|_{CV} + \int\int_{CS} \rho V_n dA = 0.$$

For a system with steady flow,

$$\int\int_{CS} \rho V_n dA = 0.$$

For incompressible fluids, ρ is a constant, so

$$\int\int_{CS} V_n dA = 0,$$

which means that the amount leaving is equal to the amount entering. When a liquid flows through a pipe, it has a velocity profile as shown in figure 7.34. An average

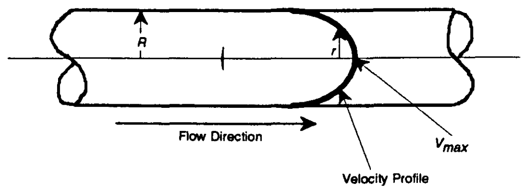

Figure 7.34: Velocity profile in a pipe.

velocity, V can be calculated for a profile as follows.

$$VA = \int\int_{CS} V_n dA \qquad \Longrightarrow \qquad V = \frac{\int\int_{CS} V_n dA}{A} \tag{7.41}$$

Water is moving through the pipe in a bullet shape which may approximate the shape of a parabola. The use of polar coordinates helps to describe the movement in the above equation.

Troublesome Notation: The symbol $\int\int_{CS}$ is a general method of indicating integration over a surface which is two-dimensional. In these problems, it is defined according to the shape of the duct and the type of flow. When setting up the problem, it will be necessary to define the limits for each part of the integral. Also note that V_n is velocity in the n-direction. In the problems, velocity may be in the z-direction, so we would define it as V_z. In these problems, V stands for velocity instead of v found in motion problems of physics. Don't confuse V with volume.

The following problems are associated with the flow of fluid through a pipe.

1. Find the average velocity over the cross-sectional area of a circular duct as in figure 7.34. The velocity distribution is given as

$$V_z = V_{max}\left(1 - \frac{r^2}{R^2}\right)$$

where R is the radius of the pipe.

2. Find the average velocity over the cross-sectional area of a circular duct when the velocity distribution is

$$V_z = V_{max}\left(1 - \frac{r}{R}\right)^{\frac{1}{5}}.$$

3. A pipeline with a 24 cm inside diameter is carrying liquid with an average velocity of 0.5 $\frac{\mathrm{m}}{\mathrm{s}}$. A reducer is placed in the line and the outlet diameter is 8 cm as shown in figure 7.35. What is the velocity at the end of the reducer? Given the velocity distribution of problem 1, what is the maximum velocity for each section of pipe?

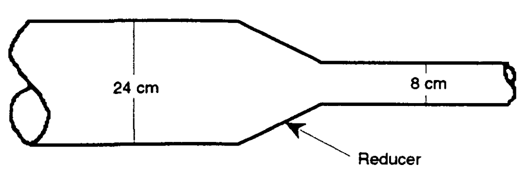

Figure 7.35: Pipe with a reducer.

4. A 4 foot diameter tank containing acetone is sketched in figure 7.36. The acetone is drained from the bottom of the tank by a pump so that the velocity of flow in the outlet pipe is constant at 3 $\frac{\text{ft}}{\text{s}}$. If the outlet pipe has an inside diameter of 1 in, determine the time required to drain the tank from a depth of 3 feet to a depth of 6 inches. Acetone has a density of 1.527 $\frac{\text{slugs}}{\text{ft}^3}$. The control volume is

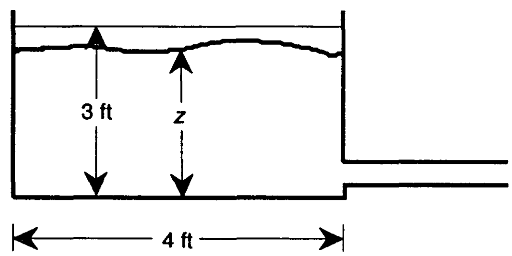

Figure 7.36: Cylindrical acetone tank.

chosen to be the volume of liquid in the tank at any time. The flow is unsteady and the continuity equation applies as follows:

$$\left.\frac{\partial N}{\partial t}\right|_{CV} + \int\int_{CS} n\rho V_n dA = 0.$$

Set up the equation and find t. Note that the integration is over several constants.

Solutions to: Flow of Fluids in Pipes and Tanks

1. The average velocity is $V = \dfrac{\int\int_{CS} V_z dA}{A}$ where $V_z = V_{max}\left(1 - \dfrac{r^2}{R^2}\right)$.

Since the pipe is circular, $A = \pi R^2$. Therefore, using polar coordinates,

$$V = \frac{1}{A}\int_0^{2\pi}\int_0^R V_{max}\left(1 - \frac{r^2}{R^2}\right) r dr d\theta = \frac{V_{max}}{\pi R^2}\int_0^{2\pi}\left[\left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right)\right]_0^R d\theta$$

$$= \frac{V_{max}}{\pi R^2}\int_0^{2\pi}\left(\frac{R^2}{2} - \frac{R^4}{4R^2}\right) d\theta = \frac{V_{max}}{\pi}\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{4}\right) d\theta$$

$$= \frac{1}{4}\frac{V_{max}}{\pi}\theta\bigg|_0^{2\pi} = \frac{1}{2}V_{max}$$

2. The average velocity is $V = \dfrac{\int\int_{CS} V_z dA}{A}$ where $V_z = V_{max}\left(1 - \dfrac{r}{R}\right)^{\frac{1}{5}}$.

Since the pipe is circular, $A = \pi R^2$. Therefore, using polar coordinates,

$$V = \frac{1}{A}\int_0^{2\pi}\int_0^R V_{max}\left(1 - \frac{r}{R}\right)^{\frac{1}{5}} r dr d\theta$$

$$= \frac{V_{max}}{\pi R^2}\int_0^{2\pi}\left[\left(\frac{-r + R}{R}\right)^{1/5}\left(\frac{5r^2}{11} - \frac{5rR}{66} - \frac{25R^2}{66}\right)\right]_0^R d\theta$$

$$= \frac{V_{max}}{\pi R^2}\int_0^{2\pi}\left(\frac{-R + R}{R}\right)^{1/5}\left(\frac{5R^2}{11} - \frac{5R^2}{66} - \frac{25R^2}{66}\right) d\theta$$

$$= \frac{25}{66}\frac{V_{max}}{\pi}\theta\bigg|_0^{2\pi} = \frac{25}{33}V_{max}$$

3. We know that the average velocity $V = \dfrac{\int\int_{CS} V_n dA}{A}$. From this we get

$VA = \int\int_{CS} V_n dA$. For the 24 cm end of the pipe, $V = 0.5\ \frac{\text{m}}{\text{s}}$ and

$A = 144\pi\ \text{cm}^2$. Therefore, $\int\int_{CS} V_n dA = (0.5)(144\pi) = 226.2$ For the

8 cm end of the pipe, we know that $\int\int_{CS} V_n dA = 226.2$ and $A = 16\pi\ \text{cm}^2$.

Therefore, $V = \frac{\int\int_{CS} V_n dA}{A} = \frac{226.2}{16\pi} = 4.5\ \frac{\text{m}}{\text{s}}$

4. We are given that $\left.\frac{\partial N}{\partial t}\right|_{CV} + \int\int_{CS} \rho V_n dA = 0.$

The volume of the tank is $\mathcal{V} = \pi\left(2^2\right) z = 12.57z\ \text{ft}^3$. The mass of

the acetone in the tank is $m = \rho \cdot \mathcal{V} = (1.523)(12.57z) = 19.2z$ slugs.

$\frac{\partial m}{\partial t} = 19.2\frac{\partial z}{\partial t} = 19.2\frac{dz}{dt}$ since the depth varies only with time.

$$\int\int_{CS} \rho V_n dA = \int\int_{out} \rho V_n dA - \int\int_{in} \rho V_n dA = (1.523)(\pi\left(\frac{1}{24}\right)^2(3) = 0.025.$$

Substituting, $\left.\frac{\partial N}{\partial t}\right|_{CV} + \int\int_{CS} \rho V_n dA = 19.2\frac{dz}{dt} + 0.025 = 0.$

Solving the differential equation produces $\int_3^{0.5} dz = -0.00130\int_0^t dt \quad \Rightarrow$

$0.5 - 3 = -0.00130(t - 0) \quad \Rightarrow \quad t = 1920\ \text{s}$

7.23 Friction Factor in Flow through a Pipe

Calculus Topic: First Order Differential Equations

Department: Chemical Engineering

Subject Area: Fluid Mechanics

Time Needed: 2 hours

Liquids flow through pipes in two different ways, laminar and turbulent. In laminar flow, the liquid moves in smooth layers through the pipe, and in turbulent flow liquid moves erratically in the pipe. Friction caused by contact of a liquid with the wall of a pipe acts on a fluid moving through a pipe. We will look at laminar flow in a pipe, making use of a momentum equation which will be derived from the general conservation equation,

$$\left.\frac{dN}{dt}\right|_S = \left.\frac{\partial N}{\partial t}\right|_{CV} + \int\int_{CS} n\rho V_n dA \tag{7.42}$$

where N is a flow quantity (mass, momemtum or energy), n is the flow quantity per unit, S is the system, CV is the control volume, CS is the control surface, V_n is the velocity distribution in direction n, ρ is the density of the medium and A is the cross-sectional area. The three parts of this equation may be better understood if written in the following manner:

$$\begin{pmatrix} \text{instantaneous} \\ \text{time rate of change} \\ \text{in } N \text{ for a system} \\ \text{of particles} \end{pmatrix} = \begin{pmatrix} \text{instantaneous time} \\ \text{rate of accumulation} \\ \text{of } N \text{ within the} \\ \text{control volume} \end{pmatrix} + \begin{pmatrix} \text{amount of } N \\ \text{leaving the control} \\ \text{volume minus the} \\ \text{amount of } N \text{ entering} \end{pmatrix}$$

If we consider linear momentum as we would find in laminar flow, we can find the result of forces acting on a control volume in a pipe as follows:

$$\sum F_i = \frac{d}{dt}(mV)_i \tag{7.43}$$

where $\sum F_i$ is the sum of the forces in the i direction. From (7.43) the conservation equation becomes

$$\sum \mathbf{F} = \frac{d}{dt}(m\mathbf{V})\bigg|_S = \frac{\partial}{\partial t}(m\mathbf{V})\bigg|_{CV} + \int\int_{CS} V(\mathbf{V} \cdot d\mathbf{A}). \tag{7.44}$$

Since we will be considering steady one-dimensional flow only, (7.44) becomes

$$\sum F_i = \int\int_{CS} V_i \rho V_n dA \tag{7.45}$$

Troublesome Notation: In this problem, vectors are denoted by **F** with the magnitude of **F** denoted as F and the components of **F** as F_x, F_y and F_z. In most calculus texts, a boldface **a** is used for vector notation, $|a|$ would designate its magnitude and (a_1, a_2, a_3) would designate its components. Also, $\mathbf{V} \cdot d\mathbf{A}$ denotes a dot product between two vectors. There are two ways to evaluate a dot product. In calculus, the dot product is written as $\mathbf{a} \cdot \mathbf{b}$ and is either $a_1b_1 + a_2b_2 + a_3b_3$ for three component vectors or $|\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the vectors. This problem uses the second form.

1. In figure 7.37, we will select an element representing the flow of a liquid through a circular pipe and then analyze the forces acting on that pipe. pA is a pressure force acting over the cross-sectional area and τdA_p is a shear force caused by the walls of the pipe. Since the diameter of the pipe is constant, the right-hand

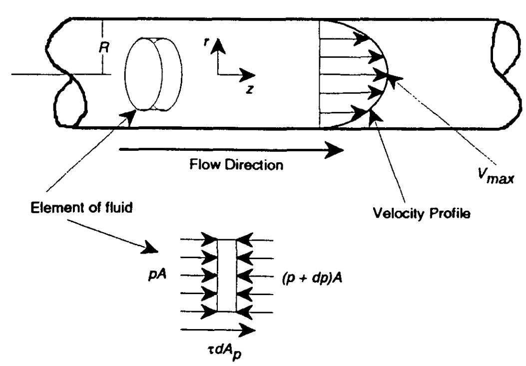

Figure 7.37: Forces acting on an element of fluid in a circular pipe.

side of (7.45) is zero. Summing the forces on the element gives

$$pA + \tau dA_p - (p + dp)\,A = 0 \qquad \Rightarrow \qquad \tau dA_p - Adp = 0 \tag{7.46}$$

Since the pipe is circular, $A = 2\pi r$ and the area over which the shear stress acts is $dA_p = 2\pi r dz$. This changes (7.46) to

$$\tau\,(2\pi r dz) - \pi r^2 dp = 0 \qquad \Rightarrow \qquad \frac{dp}{dz} = \frac{2\tau}{r} \tag{7.47}$$

$\dfrac{dp}{dz}$ is a pressure drop per length of pipe, which is caused by the friction in the pipe. We will assume that we have a Newtonian fluid which means that

$$\tau = \mu \frac{dV_z}{dr} \tag{7.48}$$

where μ is a constant depending on the type of fluid.

Now, for the circular pipe in figure 7.37,

(a) Substitute (7.48) for τ into (7.47) and solve for $\dfrac{dV_z}{dr}$. Then solve the differential equation for V_z.

(b) At the boundary, $r = R$, the velocity $V_z = 0$. Use this to solve for the constant of integration. Now what is V_z?

(c) Now find the average velocity using

$$V = \frac{\int\int_{CS} V_n dA}{A}$$

2. Consider the flow of a fluid through a rectangular duct as shown in figure 7.38.

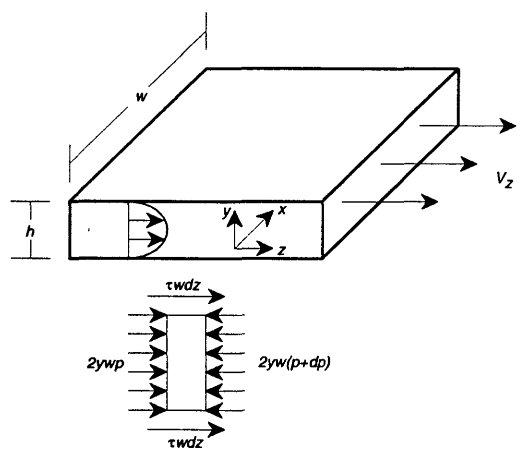

Figure 7.38: Forces acting on an element of fluid in a rectangular duct.

(a) As with the circular pipe in (7.46), find the sum of the forces acting on the element. Again, the right-hand side of (7.45) is zero since the size of the duct remains constant. Using the steps you used to find V_z in problem (1), find V_z for the rectangular duct. This time $\tau = \mu \frac{dV_z}{dy}$ and you will be integrating with respect to y.

(b) To evaluate the integration constant, use the fact that $y = \pm\frac{h}{2}$ when $V_z = 0$.

(c) Find the average velocity for the profile throught the rectangular duct.

3. Consider the flow of a fluid through an annulus as shown in figure 7.39. Pay particular attention to the shape of the flow element in figure 7.39 and to how the forces act on it.

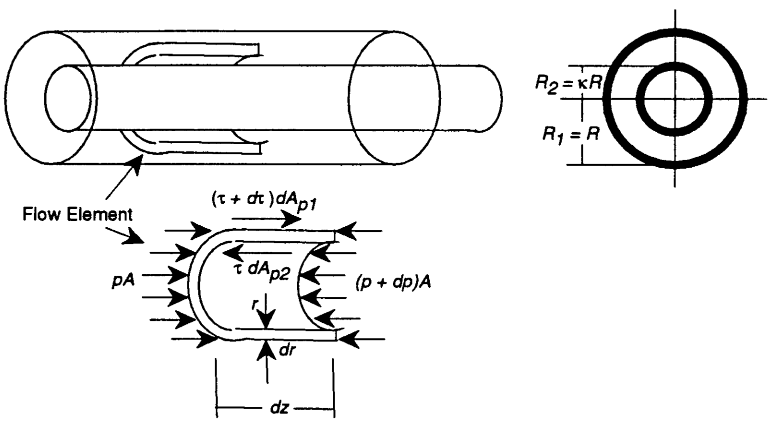

Figure 7.39: Forces acting on an element of fluid in an annulus.

(a) As with the circular pipe in (7.46), find the sum of the forces acting on the element. Again, the right-hand side of (7.45) is zero since the size of the duct remains constant.

(b) Find dA_{p_1} and dA_{p_2} and substitute into part (a).

(c) The term $dr\ d\tau$ is very small and can be taken to be zero. Assuming that $dr\ d\tau$ is zero, write the equation from part (b) in terms of $\frac{dp}{dz}$.

(d) For a Newtonian fluid, $\tau = \mu \frac{dV_z}{dz}$. Substitute this into the equation from part (c) and integrate with respect to r.

(e) There are two boundary conditions for this problem. When $V_z = 0$, $r = R$ and when $V_z = 0$, $r = \kappa R$, $(0 < \kappa < 1)$. Use these conditions to find the two integration constants, then give an equation in terms of V_z.

(f) Find the average velocity for a profile in an annulus.

Solutions to: Friction Factor in Flow through a Pipe

1. We are given $\tau = \mu \dfrac{dV_z}{dr}$ and $\dfrac{dp}{dz} = \dfrac{2\tau}{r}$.

a. $$\frac{dp}{dz} = \frac{2\tau}{r} = \frac{2}{r}\mu\frac{dV_z}{dr} \quad \Rightarrow \quad \frac{dV_z}{dr} = \frac{dp}{dz}\frac{r}{2\mu}$$

$$\int dV_z = \int \frac{dp}{dz}\frac{r}{2\mu}dr \quad \Rightarrow \quad V_z = \frac{dp}{dz}\frac{r^2}{4\mu} + c_1$$

b. Since $V_z = 0$ when $r = R$, then $0 = \dfrac{dp}{dz}\dfrac{R^2}{4\mu} + c_1 \quad \Rightarrow \quad c_1 = -\dfrac{dp}{dz}\dfrac{R^2}{4\mu}$.

$$\Rightarrow \quad V_z = \frac{dp}{dz}\frac{1}{4\mu}\left(r^2 - R^2\right) = -\frac{dp}{dz}\frac{R^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

c. See section 7.22 $$V = \frac{\int\int V_z dA}{A} = \frac{\dfrac{-R^2}{4\mu}\dfrac{dp}{dz}\displaystyle\int_0^{2\pi}\int_0^R \left(1 - \frac{r^2}{R^2}\right) r\,dr\,d\theta}{\pi R^2}$$

$$= \frac{\dfrac{-R^2}{4\mu}\dfrac{dp}{dz}\displaystyle\int_0^{2\pi}\left[\left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right)\right]_0^R d\theta}{\pi R^2} = \frac{\dfrac{-R^2}{4\mu}\dfrac{dp}{dz}\displaystyle\int_0^{2\pi}\left(\frac{1}{4}\right)d\theta}{\pi}$$

$$= \frac{\dfrac{-R^2}{16\mu}\dfrac{dp}{dz}\theta\Big|_0^{2\pi}}{\pi} = \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right)$$

2. *a.* $$\sum F_z = 0 = 2wyp + \tau w dz + \tau w dz - 2wy\left(p + dp\right) \quad \Rightarrow$$

$$2\tau w dz = 2wy dp \quad \Rightarrow \quad \frac{dp}{dz} = \frac{\tau}{y}$$

Since we are given that $\tau = \dfrac{dV_z}{dy}$, $\dfrac{y}{\mu}\dfrac{dp}{dz} = \dfrac{dV_z}{dy}$.

$$\int dV_z = \int \frac{y}{\mu}\frac{dp}{dz}dy \quad \Rightarrow \quad V_z = \frac{y^2}{2\mu}\frac{dp}{dz} + c_2$$

b. Since $V_z = 0$ when $y = \pm\frac{h}{2}$, then $0 = \frac{h^2}{8\mu}\frac{dp}{dz} + c_2 \quad \Rightarrow \quad c_2 = \left(-\frac{dp}{dz}\right)\frac{h^2}{8\mu}$

$$\Rightarrow \quad V_z = \frac{h^2}{2\mu}\left(-\frac{dp}{dz}\right)\left(\frac{1}{4} - \frac{y^2}{h^2}\right)$$

c. See section 7.22 $V = \dfrac{\int\int V_z dA}{A} = \dfrac{\int_0^x \int_{-h/2}^{h/2} \frac{h^2}{2\mu}\left(-\frac{dp}{dz}\right)\left(\frac{1}{4} - \frac{y^2}{h^2}\right)dydx}{xh}$

$$= \frac{\int_0^x \frac{h^2}{2\mu}\left(-\frac{dp}{dz}\right)\left(\frac{1}{4}y - \frac{y^3}{6h^2}\right)\Big|_{-h/2}^{h/2} dx}{xh} = \frac{\int_0^x \frac{h^2}{2\mu}\left(-\frac{dp}{dz}\right)\left(\frac{1}{4}h - \frac{2h^3}{48h^2}\right)dx}{xh}$$

$$= \frac{\int_0^x \frac{h^2}{12\mu}\left(-\frac{dp}{dz}\right)dx}{x} = \frac{\frac{h^2}{12\mu}\left(-\frac{dp}{dz}\right)x}{x} = \frac{h^2}{12\mu}\left(-\frac{dp}{dz}\right)$$

3. *a.* $\sum F_z = 0 = pA + (\tau + d\tau)\,dA_{p_1} - \tau dA_{p_2} - (p + dp)\,A$

b. $dA_{p_1} = 2\pi(r + dr)dz \qquad dA_{p_2} = 2\pi r dz \quad \Rightarrow$

$$-2\pi r dr dp + (\tau + d\tau)(2\pi)(r + dr)dz - \tau(2\pi r)\,dz = 0,$$

where the shell area, $A = 2\pi r dr$.

c. Since $dr d\tau \to 0$, set $dr d\tau = 0$. Then $2\pi r dr dp = 2\pi\tau dr dz + 2\pi r d\tau dz$

$$\Rightarrow \quad r dr \frac{dp}{dz} = \tau dr + r d\tau.$$

d. Since we are given that $\tau = \mu\frac{dV_z}{dz}$, $r dr \frac{dp}{dz} = \mu\frac{dV_z}{dr}dr + r d\left(\mu\frac{dV_z}{dr}\right)$

Dividing both sides by dr produces $\mu\frac{dV_z}{dr} + r\frac{dp}{dz} = r\frac{d}{dr}\left(\mu\frac{dV_z}{dr}\right)$

Using the product rule where $\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) = \frac{dV_z}{dr} + \frac{d}{dr}\left(r\frac{dV_z}{dr}\right)$, we get

$\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) = \frac{r}{\mu}\frac{dp}{dz}$. Integrating, $\int d\left(r\frac{dV_z}{dr}\right) = \int \frac{r}{\mu}\frac{dp}{dz}dr \quad \Rightarrow$

$r\frac{dV_z}{dr} = \frac{r^2}{2\mu}\frac{dp}{dz} + c_3$. Dividing by r and integrating again produces

$\int dV_z = \int \left(\frac{r}{2\mu}\frac{dp}{dz} + \frac{1}{r}c_3\right) dr \quad \Rightarrow \quad V_z = \frac{r^2}{4\mu}\frac{dp}{dz} + c_3 \ln r + c_4$

e. The boundary conditions are when $V_z = 0$, $r = R$ and

when$V_z = 0$, $r = \kappa R$, where $0 < \kappa < 1$.

Hence, we have$\frac{R^2}{4\mu}\frac{dp}{dz} + c_3 \ln R + c_4 = 0$ and $\frac{\kappa^2 R^2}{4\mu}\frac{dp}{dz} + c_3 \ln(\kappa R) + c_4 = 0$.

Solving this system produces $c_3 = \frac{1}{\ln\kappa}\frac{R^2}{4\mu}\frac{dp}{dz}\left(1 - \kappa^2\right)$ and

$$c_4 = -\frac{R^2}{4\mu}\frac{dp}{dz}\left(1 + \frac{1-\kappa^2}{\ln\kappa}\ln R\right).$$

Therefore $V_z = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[1 - \frac{r^2}{R^2} - \frac{1-\kappa^2}{\ln\kappa}\ln\left(\frac{r}{R}\right)\right]$

f. See section 7.22

$$V = \frac{\int\int V_z dA}{A} = \frac{\int_0^{2\pi}\int_{\kappa R}^{R} \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[1 - \frac{r^2}{R^2} - \frac{1-\kappa^2}{\ln\kappa}\ln\left(\frac{r}{R}\right)\right] r dr d\theta}{\pi\left(R^2 - \kappa^2 R^2\right)}$$

$$= \frac{\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\int_0^{2\pi}\left[\int_{\kappa R}^{R} r dr - \int_{\kappa R}^{R}\frac{r^3}{R^2}dr - \int_{\kappa R}^{R} r\frac{1-\kappa^2}{\ln\kappa}\ln\frac{r}{R}dr\right] d\theta}{\pi R^2\left(1-\kappa^2\right)}$$

$$= \frac{\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\int_0^{2\pi}\left[\frac{R^2-\kappa^2R^2}{2} - \frac{R^4-\kappa^4R^4}{4R^2} - \frac{R^2\left(\kappa^2-1\right)+\kappa^2R^2\left(\kappa^2-1\right)(1-2\ln\kappa)}{4\ln\kappa}\right]d\theta}{\pi R^2\left(1-\kappa^2\right)}$$

$$= \frac{\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\int_0^{2\pi}\left[\frac{R^2}{4}\left(1-2\kappa^2+\kappa^4\right) - \frac{\left(\kappa^2-1\right)R^2+\kappa^2R^2}{4\ln\kappa} - \frac{\left(\kappa^2-1\right)\kappa^2R^2}{2}\right]d\theta}{\pi R^2\left(1-\kappa^2\right)}$$

$$= \frac{\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\int_0^{2\pi}\left[\frac{R^2}{4}\left(1-\kappa^2\right)^2 + \frac{R^2}{4\ln\kappa}\left(1-\kappa^2\right)^2 + \frac{\left(1-\kappa^2\right)\kappa^2R^2}{2}\right]d\theta}{\pi R^2\left(1-\kappa^2\right)}$$

$$= \frac{\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\int_0^{2\pi}R^2\left(1-\kappa^2\right)\left[\frac{\left(1-\kappa^2\right)}{4} + \frac{\left(1-\kappa^2\right)}{4\ln\kappa} + \frac{\kappa^2}{2}\right]d\theta}{\pi R^2\left(1-\kappa^2\right)}$$

$$= \frac{1}{\pi}\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\int_0^{2\pi}\left[\frac{(1-\kappa^2)\ln k + (1-\kappa^2) + 2\kappa^2\ln k}{4\ln\kappa}\right]d\theta$$

$$= \frac{1}{\pi}\frac{R^2}{16\mu}\left(-\frac{dp}{dz}\right)\int_0^{2\pi}\left[1+\kappa^2+\frac{1-\kappa^2}{\ln\kappa}\right]d\theta$$

$$= \frac{1}{\pi}\frac{R^2}{16\mu}\left(-\frac{dp}{dz}\right)\left[1+\kappa^2+\frac{1-\kappa^2}{\ln\kappa}\right]\theta\Big|_0^{2\pi}$$

$$= \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right)\left[1+\kappa^2+\frac{1-\kappa^2}{\ln\kappa}\right]$$

7.24 Ball Mill Problem

Calculus Topic: Mass Moment of Inertia

Department: Mining Engineering

Subject Area: Ore Processing

Time Needed: 30 minutes

Reference: [10]

A ball mill is used to break large chunks of ore into smaller pieces. It is a large barrel into which is put a quantity of ore and a quantity of steel spheres. As the mill turns about its axis, the ore and the spheres tumble around together. The spheres break the ore chunks by impact. The mill must be rotated fast enough that the material tumbles well, but not so fast that centripetal force keeps the material plastered up against the wall. The engineers who design ball mills need to know, among other things, how the mass moment of inertia of the steel spheres changes as they wear down (their radii decrease). The mass moment of inertia is calculated by

$$I = \int_m r^2 dm = \int_V r^2 \rho dV = \rho \int_V r^2 dV$$

where I is the mass moment of inertia, r is the perpendicular distance from the axis of rotation to the element dm, m is mass, V is the volume of the object and ρ is the density of the object (assumed constant in this case).

Troublesome Notation: The symbol $\int_V$ is a general method of indicating integration over a volume which is three-dimensional. When setting up the problem, it will be necessary to evaluate what may be a triple integral along with the limits for each part.

Plot how the mass moment of inertia of a steel sphere changes as its radius decreases with wear. The initial radius is 3.00 in, the final radius is 0.75 in. and the steel density is $7.88\frac{\text{g}}{\text{cm}^3}$. Consider 0.25 in increments.

Solutions to: Ball Mill Problem

We want to find $I = \rho \int_V r^2 dV$ for a sphere where r is the distance from the center of the sphere to its surface. The volume for a sphere is $V = \frac{4}{3}\pi r^3$.

Hence $I = \rho \int_0^r r^2 \left(\frac{4}{3}\pi r^2 dr\right) = \frac{4}{3}\pi\rho \int_0^r r^4 dr = \frac{4}{15}\pi\rho r^5$

Convert ρ to compatible units: $7.88 \left(\frac{\text{g}}{\text{cm}^3}\right)\left(\frac{2.54\text{cm}}{\text{in}}\right)^3\left(\frac{1\text{lb}}{454\text{g}}\right) = 0.284\ \frac{\text{lbs}}{\text{in}^3}$

For $r = 3.00$, $I = \frac{4}{15}\pi(0.284)(3.00)^5 = 57.9\ \text{lb}\cdot\text{in}^2$

For $r = 2.75$, $I = \frac{4}{15}\pi(0.284)(2.75)^5 = 37.5\ \text{lb}\cdot\text{in}^2$

For $r = 2.50$, $I = \frac{4}{15}\pi(0.284)(2.50)^5 = 23.3\ \text{lb}\cdot\text{in}^2$

For $r = 2.25$, $I = \frac{4}{15}\pi(0.284)(2.25)^5 = 13.7\ \text{lb}\cdot\text{in}^2$

For $r = 2.00$, $I = \frac{4}{15}\pi(0.284)(2.00)^5 = 7.63\ \text{lb}\cdot\text{in}^2$

For $r = 1.75$, $I = \frac{4}{15}\pi(0.284)(1.75)^5 = 3.91\ \text{lb}\cdot\text{in}^2$

For $r = 1.50$, $I = \frac{4}{15}\pi(0.284)(1.50)^5 = 1.81\ \text{lb}\cdot\text{in}^2$

For $r = 1.25$, $I = \frac{4}{15}\pi(0.284)(1.25)^5 = 0.727\ \text{lb}\cdot\text{in}^2$

For $r = 1.00$, $I = \dfrac{4}{15}\pi(0.284)(1.00)^5 = 0.238\ \mathrm{lb} \cdot \mathrm{in}^2$

For $r = 0.75$, $I = \dfrac{4}{15}\pi(0.284)(0.75)^5 = 0.0565\ \mathrm{lb} \cdot \mathrm{in}^2$

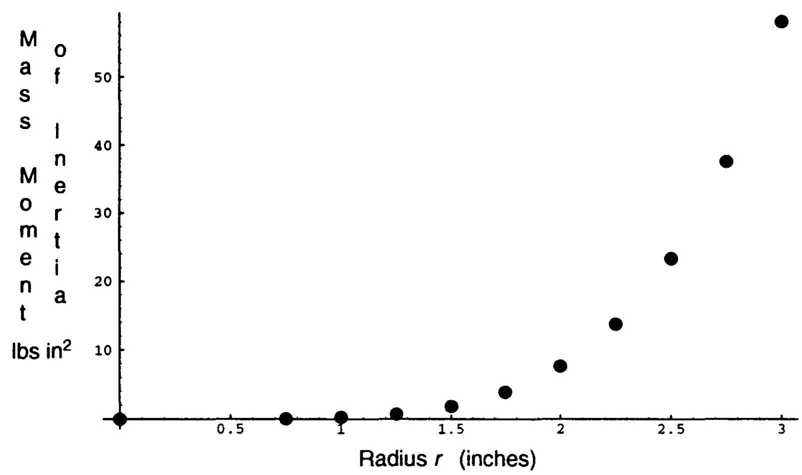

Figure 7.40: Change in the mass moment of inertia as the size of the steel spheres shrinks.

7.25 Heat and Wave Equations

Calculus Topic:	Partial Differentiation
Department:	Physics, Engineering
Subject Area:	Continuity
Time Needed:	40 minutes
Reference:	[25]

The heat, wave and ideal gas law equations are partial differential equations which can be easily verified by taking partial derivatives with respect to the variables shown in the equation. The following problems demonstrate this idea.

1. It is shown in physics that the temperature $u(x,t)$ at time t and at point x of a long insulated rod that lies along the $x-$axis satisfies the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

where k is a constant determined by the material from which the rod is made of. Show that $u(x,t) = e^{-n^2kt}\sin(nx)$ satisfies the one-dimensional heat equation for any constant n.

2. The Ideal gas law $pV = nRT$ (n is the number of moles of the gas, R is a constant) determines each of the three variables p, V and T (pressure, volume

and temperature) as function of the other two. Show that

$$\frac{\partial p}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p} = -1.$$

3. A string is stretched along the $x-$axis, fixed at each end and then set in vibration. In physics it is shown that the displacement $w(x,t)$ of the point of the string at location x at time t satisfies the one-dimensional wave equation. Another example occurs if you stand on an ocean shore and take a snapshot of the waves. The picture shows a regular pattern of peaks and valleys in an instant of time. If we stand in the water, we can feel the rise and fall of the water as the waves go by, that is, we see periodic vertical motion in time. This equation is

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

where w is the wave height, x is the distance variable, t is the time variable and c is the velocity with which the waves are propagated. The number c varies with the meduim and type of wave. Show that the following functions satisfy the one-dimensional wave equation.

   (a) $w = \sin(x + ct)$

   (b) $w = 5\cos(3x + 3ct) + e^{x+ct}$

Solutions to: Heat and Wave Equations

1. $u(x,t) = e^{-n^2kt}\sin(nx)$

$$\frac{\partial u}{\partial t} = -n^2ke^{-n^2kt}\sin(nx)$$

$$\frac{\partial u}{\partial x} = ne^{-n^2kt}\cos(nx) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = -n^2e^{-n^2kt}\sin(nx)$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \Rightarrow \qquad -n^2ke^{-n^2kt}\sin(nx) = k\left(-n^2e^{-n^2kt}\sin(nx)\right)$$

2. $pV = nRT$

$$p = \frac{nRT}{V} \qquad\qquad \frac{\partial p}{\partial V} = -\frac{nRT}{V^2}$$

$$V = \frac{nRT}{p} \qquad\qquad \frac{\partial V}{\partial T} = \frac{nR}{p}$$

$$T = \frac{pV}{nR} \qquad\qquad \frac{\partial T}{\partial p} = \frac{V}{nR}$$

$$\frac{\partial p}{\partial V}\cdot\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial p} = -\frac{nRT}{V^2}\cdot\frac{nR}{p}\cdot\frac{V}{nR} = -\frac{nRT}{pV} = -1$$

3. $\dfrac{\partial^2 w}{\partial t^2} = c^2\dfrac{\partial^2 w}{\partial x^2}$

a. $w = \sin(x + ct)$

$$\frac{\partial w}{\partial t} = c\cos(x + ct)$$

$$\frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x + ct)$$

$$\frac{\partial w}{\partial x} = \cos(x + ct)$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x + ct)$$

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \qquad \Rightarrow \qquad -c^2 \sin(x + ct) = c^2 \left(-\sin(x + ct)\right)$$

b. $w = 5\cos(3x + 3ct) + e^{x+ct}$

$$\frac{\partial w}{\partial t} = -15c \sin(3x + 3ct) + ce^{x+ct}$$

$$\frac{\partial^2 w}{\partial t^2} = -45c^2 \cos(3x + 3ct) + c^2 e^{x+ct}$$

$$\frac{\partial w}{\partial x} = -15 \sin(3x + 3ct) + e^{x+ct}$$

$$\frac{\partial^2 w}{\partial x^2} = -45 \cos(3x + 3ct) + e^{x+ct}$$

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \qquad \Rightarrow$$

$$-45c^2 \cos(3x + 3ct) + c^2 e^{x+ct} = c^2 \left(-45 \cos(3x + 3ct) + e^{x+ct}\right)$$

7.26 Temperature Profile with Heat Generation

Calculus Topic:	Differential Equation, Cylindrical and Spherical Coordinates
Department:	Chemical Engineering
Subject Area:	Transport Processes
Time Needed:	1 hour
Reference:	[9]

Temperature profiles with heat generation can be modeled through a partial differetial equation. The following equation represents an energy change and is a form of Fourier's second law of unsteady-state heat conduction. Here it is written in rectangular coordinates:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\nabla^2 T = \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{7.49}$$

where $\frac{k}{\rho c_p}$ is a thermal diffusivity term.

1. Rewrite equation (7.49) in cylindrical and spherical coordinates.

2. A solid cylinder in which heat generation is occurring uniformly as $\dot{q}$ Watts per cubic meter is insulated at the ends. The temperature of the surface of the cylinder is held constant at T_W degrees Kelvin. The radius of the cylinder is $r = R$ meters. Heat flows only in the radial direction. Derive the equation

for the temperature profile at steady state if the solid has a constant thermal conductivity as follows.

(a) Use the cylindrical form of Fourier's second law of unsteady-state heat conduction and add the term $\frac{\dot{q}}{\rho c_p}$ to the right-hand side. For steady state, we have

$$\frac{\partial T}{\partial t} = 0.$$

Since the heat conduction is only in the radial direction, then

$$\frac{\partial^2 T}{\partial z^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 T}{\partial \theta^2} = 0$$

What is the resulting equation?

(b) Show that the equation from part (a) can be written in the form:

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\dot{q}r}{k}$$

(c) Integrate the equation from part (b) twice and solve for T.

(d) The boundary conditions are when $r = 0$, $\frac{dT}{dr} = 0$ (by symmetry) and when $r = R$, $T = T_w$. Find the integration constants and solve the equation for the temperature profile T.

Solutions to: Temperature Profile with Heat Generation

1. Rectangular coordinates: $\dfrac{\partial T}{\partial t} = \dfrac{k}{\rho c_p}\nabla^2 T = \dfrac{k}{\rho c_p}\left(\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2}\right)$

   Cylindrical coordinates: $\dfrac{\partial T}{\partial t} = \dfrac{k}{\rho c_p}\left(\dfrac{\partial^2 T}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial T}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 T}{\partial \theta^2} + \dfrac{\partial^2 T}{\partial z^2}\right)$

   Spherical coordinates:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right]$$

2. *a.* We start with $\dfrac{\partial T}{\partial t} = \dfrac{k}{\rho c_p}\left(\dfrac{\partial^2 T}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial T}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 T}{\partial \theta^2} + \dfrac{\partial^2 T}{\partial z^2}\right) + \dfrac{\dot{q}}{\rho c_p} = 0$

   Since $\dfrac{\partial^2 T}{\partial z^2} = 0$ and $\dfrac{\partial^2 T}{\partial \theta^2} = 0$, we have $\dfrac{d^2 T}{dr^2} + \dfrac{1}{r}\dfrac{dT}{dr} = -\dfrac{\dot{q}}{k}$.

   Since the change is only in the r direction, the PDE becomes an ODE.

   b. Using the product rule, $\dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right) = r\dfrac{d}{dr}\left(\dfrac{dT}{dr}\right) + \dfrac{dT}{dr} = r\dfrac{d^2T}{dr^2} + \dfrac{dT}{dr}$.

   Multiplying the result from (a) by r produces $r\dfrac{d^2T}{dr^2} + \dfrac{dT}{dr} = -\dfrac{\dot{q}r}{k}$.

   Therefore $\dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right) = -\dfrac{\dot{q}r}{k}$.

   c. Separating variables, $\displaystyle\int d\left(r\frac{dT}{dr}\right) = -\frac{\dot{q}}{k}\int r\,dr \quad \Rightarrow \quad r\frac{dT}{dr} = -\frac{\dot{q}}{2k}r^2 + c_1$.

   Dividing both sides by r and separating variables again,

$$\int dT = \int \left(-\frac{\dot{q}}{2k} r + \frac{c_1}{r} \right) \quad \Rightarrow \quad T = -\frac{\dot{q}}{4k} r^2 + c_1 \ln r + c_2.$$

d. $\dfrac{dT}{dr} = 0$ when $r = 0 \quad \Rightarrow \quad c_1 = 0.$

$T = T_w$ when $r = R$, R is the radius of the cylinder, $\Rightarrow$

$c_2 = T_w + \dfrac{\dot{q}R^2}{4k}$. Therefore, our final equation is $t = \dfrac{\dot{q}\,(R^2 - r^2)}{4k} + T_w.$

7.27 Gradients

Calculus Topic:	Gradients and Directional Derivatives
Department:	Physics
Subject Area:	Physics, Chemical Engineering
Time Needed:	1 hour
Reference:	[14]

The directional derivative of a function, f, describes the rate of change of f in a certain direction. A gradient vector, ∇f for a function f points in the direction in which f increases most rapidly. The magnitude of ∇f is equal to the rate of increase of f with respect to distance. Directional derivatives and gradients can be applied to mapping problems, heat flow problems and solution mixing problems among others. The following problems are examples of some of these ideas.

1. Figure 7.41 is a topographic map showing the elevation $f(x, y)$ of the point (x, y) in feet above sea level. Find solutions to each of the following problems and give a written justification for your solution.

   (a) Estimate the slope of the terrain as you walk from B to C.

   (b) Estimate the direction you would go from C if you wish to descend by the quickest route. Give your answer in unit vector form.

   (c) Estimate the directional derivative at C (to the next contour line) in the direction of $-2\hat{\imath} + \hat{\jmath}$.

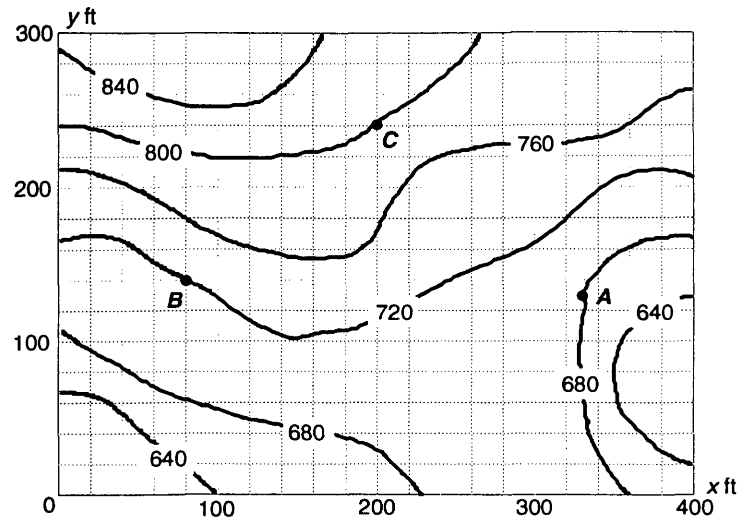

Figure 7.41: A topographic map with a contour interval of 40 ft.

(d) Estimate the direction from B that has a directional derivative of -0.406.

(e) Estimate the value of the gradient at A.

2. The concentration of salt in a body of salt water can be described by the function $f(x, y, z) = x^2 + y^2 z^4 + x^3 z^3$ at the point (x, y, z) (in feet). Your are standing in the water at the point $(-1, 1, -1)$.

(a) In which direction should you move if you want the concentration to increase the fastest?

(b) Suppose you start to move in the direction you found in part (a) at a speed of 2 $\frac{\text{m}}{\text{s}}$. How fast is the concentration changing? Explain your answer.

3. The temperature field at any point on a metal plate is given by

$$T(x, y) = \frac{1000}{x^2 + y^2 + 5}$$

(a) Where on the plate is it the hottest? What is the temperature at that point?

(b) At the point (3, 2), find the direction of the greatest increase in temperature. What is the magnitude of that greatest increase?

(c) At the point (3, 2), find the direction of the greatest decrease in temperature.

(d) Does the vector you found in part (b) point towards your answer from part (a)? If not, why not?

(e) Find a direction at the point (3, 2) where the temperature does not increase or decrease.

(f) What shape are the level curves of T?

Solutions to: Gradients

1. *a.* The slope from B to C is approximately $\dfrac{80}{\sqrt{100^2+120^2}} = 0.512.$

   b. The quickest descent is in the direction $\hat{\imath} - \hat{\jmath}$.

   c. The approximate directional derivative at C in the direction $-2\hat{\imath} + \hat{\jmath}$

   is $\dfrac{40}{\sqrt{60^2+30^2}} = 0.596.$

   d. The direction with a directional derivative of -0.406 is approximately

   $40\hat{\imath} - 90\hat{\jmath}$.

   e. The approximate gradient at A is $.943\left(-\hat{\imath} + \hat{\jmath}\right)$

2. $f(x,y,z) = x^2 + y^2z^4 + x^3z^3$

   a. $\nabla f = \dfrac{\partial f}{\partial x}\hat{\imath} + \dfrac{\partial f}{\partial y}\hat{\jmath} + \dfrac{\partial f}{\partial z}\hat{k} = \left(2x + 3x^2z^3\right)\hat{\imath} + 2yz^4\hat{\jmath} + \left(4y^2z^3 + 3x^3z^2\right)\hat{k}$

   From the point $(-1,1,-1)$ you would want to move in the direction

   $\left(2(-1) + 3(-1)^2(-1)^3\right)\hat{\imath} + 2(1)(-1)^4\hat{\jmath} + \left(4(1)^2(-1)^3 + 3(-1)^3(-1)^2\right)\hat{k}$

   $= -5\hat{\imath} + 2\hat{\jmath} - 7\hat{k}.$

b. $|\nabla f|$ gives us the rate of change in the direction with respect to distance.

$$= \sqrt{(-5)^2 + 2^2 + (-7)^2} = \sqrt{78}.$$

To find the rate of change with respect to time, rate of change $\left(\frac{\text{concentration}}{\text{time}}\right)$

$$= \text{rate of change } \left(\frac{\text{concentration}}{\text{distance}}\right) \cdot \text{rate of change } \left(\frac{\text{distance}}{\text{time}}\right)$$

$$= \sqrt{78} \cdot 2 = 2\sqrt{78} \text{ units of salt per second.}$$

3. $$T(x, y) = \frac{1000}{x^2 + y^2 + 5}$$

a. The hottest point on the plate is where $x = 0$ and $y = 0$. The temperature at that point is 200.

b. The gradient of T, $\nabla T = \frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath}$, at $(3, 2)$ will give the direction of the greatest increase in temperature.

$$\frac{\partial T}{\partial x} = \frac{(-1)(1000)(2x)}{(x^2 + y^2 + 5)^2} = -\frac{2000x}{(x^2 + y^2 + 5)^2}$$

$$\frac{\partial T}{\partial y} = \frac{(-1)(1000)(2y)}{(x^2 + y^2 + 5)^2} = -\frac{2000y}{(x^2 + y^2 + 5)^2}$$

At $(3, 2)$, $\nabla T = -\frac{2000(3)}{(3^2 + 2^2 + 5)^2}\hat{\imath} + -\frac{2000(2)}{(3^2 + 2^2 + 5)^2}\hat{\jmath}$

$$= -\frac{6000}{\sqrt{38}}\hat{\imath} - \frac{4000}{\sqrt{38}}\hat{\jmath} = \frac{2000}{\sqrt{38}}\left(-3\hat{\imath} - 2\hat{\jmath}\right)$$

c. The direction of the greatest decrease will be in the opposite direction

as the gradient, or $\frac{2000}{\sqrt{38}}\left(3\hat{\imath}+2\hat{\jmath}\right)$

d. It does point toward the origin, as expected.

e. The direction of no increase or decrease occurs perpendicular to

the gradient. The two choices are $\frac{2000}{\sqrt{38}}\left(3\hat{\imath}-2\hat{\jmath}\right)$ or $\frac{2000}{\sqrt{38}}\left(-3\hat{\imath}+2\hat{\jmath}\right)$.

f. The shape of the level curves are circles, centered at the origin.

7.28 Famine Relief

Calculus Topic:	Lagrange Multipliers
Department:	Economics
Subject Area:	Fund Allocation
Time Needed:	30 minutes
Reference:	[15]

An international organization must decide how to spend $27,000 they have alloted for famine relief in a remote area. They expect to divide the money between buying rice at $3/sack and beans at $9/sack. The number P of people who would be fed if they buy r sacks of rice and b sacks of beans is given by

$$P = r + 3b + \frac{r^3 b^3}{3 \times 10^{18}}.$$

What is the maximum number of people that can be fed, and how should the organization allocate the money?

Solutions to: Famine Relief

If we use Lagrange multipliers, let $G = 27000 - 3r - 9b = 0$.

$$\nabla P = \left(1 + \frac{3r^2b^3}{3 \times 10^{18}}\right)\hat{\imath} + \left(3 + \frac{3r^3b^2}{3 \times 10^{18}}\right)\hat{\jmath} = \left(1 + \frac{r^2b^3}{10^{18}}\right)\hat{\imath} + \left(3 + \frac{r^3b^2}{10^{18}}\right)\hat{\jmath}$$

$$\nabla G = -3\hat{\imath} - 9\hat{\jmath}$$

$$\nabla P = \lambda \nabla G \quad \Rightarrow \quad \left(1 + \frac{r^2b^3}{10^{18}}\right) = -3\lambda \text{ and } \left(3 + \frac{r^3b^2}{10^{18}}\right) = -9\lambda$$

Hence, $3 + \dfrac{3r^2b^3}{10^{18}} = 3 + \dfrac{r^3b^2}{10^{18}} \quad \Rightarrow \quad 3b = r \text{ or } r = 0 \text{ or } b = 0.$

From $G = 0$, we have: when $r = 0$, $b = 3000$, when $b = 0$, $r = 9000$ and

when $r = 3b$, $r = 3000$, $b = 1000$. Subsituting into P, $P(9000, 0) = 9000$,

$P(0, 3000) = 3(3000) = 9000$ and

$$P(4500, 1500) = 4500 + 3(1500) + \frac{4500^3 \cdot 1500^3}{3 \times 10^{18}} = 9102$$

Therefore, 9102 people can be fed with 1500 sacks of beans and 4500 sacks of rice.

7.29 Work done by Variable Force

Calculus Topic:	Line Integrals
Department:	Physics
Subject Area:	Work and Energy
Time Needed:	40 minutes
Reference:	[17]

A line integral can be used to determine the amount of work or energy it takes to move an object along a path. The amount of work W can depend on both a variable force $\vec{F}$ and a variable displacement $\vec{\ell}$. Consider the direction of the force and the path followed in figure 7.42. The path can be divided into a number of small

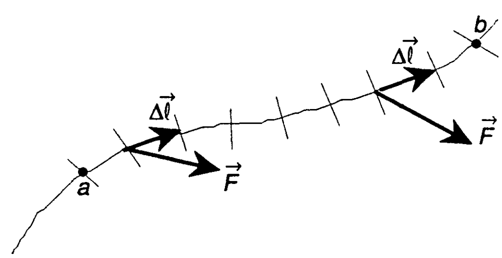

Figure 7.42: The work done by a variable force on a small line segment of the path is $\vec{F} \cdot \Delta\vec{\ell}$.

line segments approximated by a small displacement $\Delta\vec{\ell}$. If each displacement is so small that over that segment the force vector can be considered unchanging and the path be considered straight, then the work done by the force over that segment is

$$\Delta W = \vec{F} \cdot \Delta\vec{\ell}.$$

If we add the work done for all of the segments, then the sum gives an approximation for the work done,

$$W = \sum \vec{F} \cdot \Delta\vec{\ell}.$$

Taking the limit as $\Delta\vec{\ell} \rightarrow 0$ produces the line integral

$$W = \int_a^b \vec{F} \cdot d\vec{\ell}$$

As an object moves along a path from a to b, the work done by a force exerted on the object is equal to the line integral of the force along the path. Keep in mind that both the force and the displacement are vectors. The components of $\vec{F}$ can be written as F_x, F_y and F_z with the subscripts indicating the direction of F and the components of $d\vec{\ell}$ coan be written as dx, dy and dz with x, y and z indicating direction. Hence, work can be expressed as

$$W = \int_a^b \vec{F} \cdot d\vec{\ell} = \int_a^b (F_x dx + F_y dy + F_z dz)$$

Often force components are expressed as F_f for frictional force which is parallel to the plane of movement and F_n for normal force which is perpendicular to the plane of movement.

Troublesome Notation: In this problem, vectors are denoted by $\vec{F}$ with the magnitude of $\vec{F}$ denoted as F and the components of $\vec{F}$ as F_x, F_y and F_z. In most calculus texts, a boldface **a** is used for vector notation, $|a|$ would designate its magnitude and (a_1, a_2, a_3) would designate its components. Also, in this problem, $\vec{F} \cdot d\vec{\ell}$ denotes a dot product between two vectors. There are two ways to evaluate a dot product. In calculus, the dot product is written as $\mathbf{a} \cdot \mathbf{b}$ and is either $a_1b_1 + a_2b_2 + a_3b_3$ for three component vectors or $|\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the vectors. This problem uses the second form.

Each of the following problems uses the line integral equation to calculate the amount of work done.

1. A puck with a mass $m = 0.30$ kg moves in a circular path which has a radius $r = 0.80$ m on a horizontal surface of ice as shown in figure 7.43. The coefficient

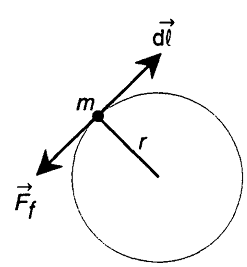

Figure 7.43: Puck's path.

of friction of the ice is $\mu = 0.11$. We want to find the work done by the frictional force as the puck moves through one-quarter of a revolution. The frictional force F_f on the puck is found by multiplying the coefficient of friction by the normal force F_n where $F_n = mg$ with m being the mass of the puck and g being the acceleration due to gravity ($9.81 \frac{\text{m}}{\text{s}^2}$). F_n is not shown in figure 7.43, but points straight down from the mass m. In this problem, take the unit for work is the joule, denoted as J ($1\ \text{J} = 1 \frac{\text{kg}\cdot\text{m}}{\text{s}^2}$).

2. A mass lying on a flat table is attached to a spring whose other end is fastened to the wall as shown in figure 7.44. The spring is extended 40 cm beyond its rest position and released. If the axes are as shown in the figure, when the spring is extended by a distance of x, the force exerted by the spring on the mass is given by

$$\vec{F} = -kx\hat{\imath}$$

where k is a positive constant that depends on the strength of the spring. Suppose the mass moves back to the rest postion. How much work is done by the force exerted by the spring?

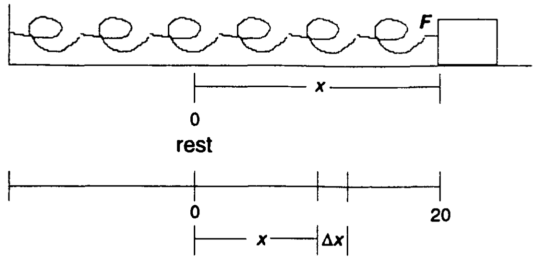

Figure 7.44: Diagram of a spring

3. A particle moving along the x-axis is subjected to a force given by $F_x(x) = F_o\left(e^{x/c} - 1\right)\hat{\imath}$, where F_o and c are constants.

   (a) Determine an expression for the work done by this force as the particle moves form the origin to the point x_1.

   (b) Let $F_0 = 2.5$ N and $a = 0.20$ m. Evaluate the work done if $x_1 = 0.50$ m.

4. Suppose that an object moving along the z-axis is acted on by a force given by $F_z(z) = \frac{-c}{z^3}\hat{k}$, where c is a constant. Taking both z_i and z_f as positive with i indicating initial position and f indicating final position, obtain the expression for the work done by this force as the object moves from z_i to z_f.

Solutions to: Work as a variable force

1. $W = \int \vec{F}_f \cdot d\vec{\ell} = \int F_f d\ell \cos\pi = \int -\mu m g d\ell$

$$= -\mu m g \int_0^{\frac{\pi}{2}R} dr = -\mu m g \left(\frac{\pi}{2}R - 0\right)$$

$$= -(0.11)(0.30 \text{ kg})\left(9.8 \frac{\text{m}}{\text{s}^2}\right)\left(\frac{\pi}{2}\right)(0.80 \text{ m}) = -0.41 \text{ J}$$

2. $W = \int_{40}^{0} \vec{F} \cdot d\vec{\ell} = \int_{40}^{0} \left(-kx\hat{i}\right) \cdot \left(dx\hat{i}\right) = \int_{40}^{0} -kx dx$

$$= \left.\frac{-kx^2}{2}\right|_{40}^{0} = 800k \textbf{ N}$$

3. $W = \int \vec{F} \cdot d\vec{\ell} = \int_0^{x_1} F_x(x) dx = \int_0^{x_1} F_o \left(e^{x/a} - 1\right) dx$

$$= F_o \left(\int_0^{x_1} e^{x/a} dx - \int_0^{x_1} dx\right) = F_o \left(\left. a e^{x/a}\right|_0^{x_1} - \left. x\right|_0^{x_1}\right) = F_o a \left(e^{x/a} - 1 - \frac{x_1}{a}\right)$$

4. $W = \int \vec{F} \cdot d\vec{\ell} = \int_{z_i}^{z_f} F_z(z) dz = -C \int_{z_i}^{z_f} \frac{dz}{z^2}$

$$= \left.\frac{C}{z}\right|_{z_i}^{z_f} = C\left(\frac{1}{z_f} - \frac{1}{z_i}\right)$$

7.30 Electric Fields

Calculus Topic:	Integration, Vector Fields
Department:	Physics
Subject Area:	Electricity and Magnetism
Time Needed:	1 hour
Reference:	[17] [23]

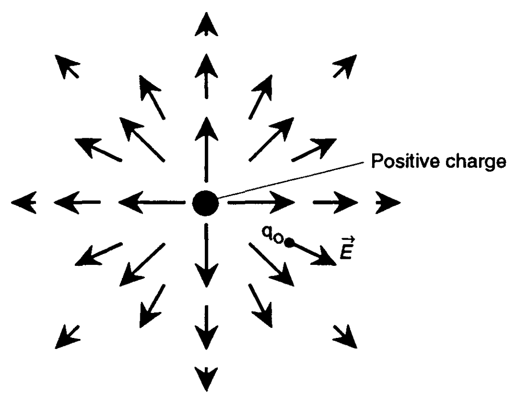

Figure 7.45: An electric field for a positive charge shown as a vector field.

An electric field is a type of vector field of which an example is shown in figure 7.45. The electric field vector $\vec{E}$ at some point in space is defined as the electric force $\vec{F}$ acting on a positive test charge q_o placed at that point divided by the magnitude of the test charge, or

$$\vec{E} = \frac{\vec{F}}{q_o}. \tag{7.50}$$

Finding an electric field vector in an electric field created by a point source of charge is relatively easy as can be seen in figure 7.45. However, when the source of the electric field is a continuous charge distribution such as a charged rod or a charged ring, determining the magnitude and direction of a electric field vector requires a more complex process. To find an electric field vector for a continuous charge distribution, we will make use of Coulomb's Law which finds the force $\vec{F}$ between two charges by the relationship

$$\vec{F} = k\frac{qq_o}{r^2}\,\hat{r} \tag{7.51}$$

Substituting (7.51) into (7.50) gives

$$\vec{E} = k\frac{q}{r^2}\,\hat{r}$$

where q is the charge of the source of the electric field, r is the distance between the source and the point where the electric field is being measured, $\hat{\mathbf{r}}$ is a unit vector which points from q toward the point where the electric field is being measured and $k = 8.9875 \times 10^9 \frac{\mathrm{Nm}^2}{\mathrm{C}^2}$ is a constant (C stands for Coulomb, the unit for charge). If there are several charged particles grouped closely together, the elecric field at a point is a result of the vector sum of each charge acting on that point.

$$\vec{E} = k\sum_i \frac{q}{r_i^2}\,\hat{r}_i$$

This leads to the idea of an electric field for a continuous charge. We will treat the charge as a continuous distribution of infintesimal charge elements Δq. Therefore

$$\Delta\vec{E} = k\frac{\Delta q}{r^2}\,\hat{r} \qquad \text{or} \qquad \vec{E} \approx k\sum_i \frac{\Delta q}{r_i^2}\,\hat{r}_i.$$

Now by letting $\Delta q \rightarrow 0$, we can find an electric field vector for a continuous charge distribution.

$$\vec{E} = k\int_a^b \frac{dq}{r^2}\,\hat{r}$$

Troublesome Notation: In this problem, vectors are denoted by $\vec{E}$ with the magnitude of $\vec{E}$ denoted as E and the components of $\vec{E}$ as E_x, E_y and E_z. In calculus, a boldface $\mathbf{E}$ is used for vector notation, $|E|$ would designate its magnitude and (E_1, E_2, E_3) would designate its components. Here, unit vectors are indicated by the hat symbol, $\hat{r}$. Unit vectors in the x-, y- and z-directions are denoted by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$. Be aware of the difference between r which is a measure of distance and $\hat{r}$ which is a unit vector along the line which r is measured. In the following problems, it will be necessary to define $\hat{r}$ by looking at what is going on in the figure. The symbols r, d, x, y, z and ℓ are all used to represent length or distance in this problem. Be aware that they are often substituted freely into equations and can cause some confusion.

The following problems allow you to calculate an electric field vector at a point for electric fields created by different types of sources. These problems will look at specific points where symmetry in the figures will simplify the calculations.

1. First we are going to consider a charged rod of length ℓ and total charge Q as the source of a continuous charge distribution. We will assume that the rod has a uniform positive charge per unit length λ, where $\lambda = \dfrac{Q}{\ell}$ and that the $x-$axis of the rod runs through the length of the rod as shown in figure 7.46. We want to find the electric field vector at a point P on the $x-$axis, a distance d away from one end.

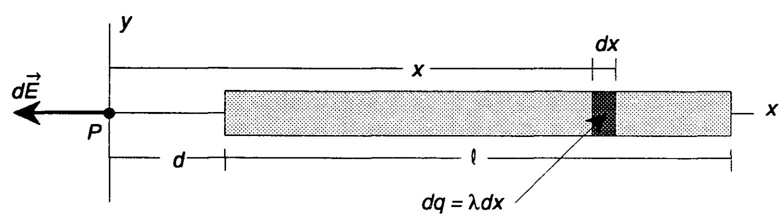

Figure 7.46: An electric field vector generated from a uniformly charged rod.

   (a) To set up the integral, note that in figure 7.46, the differential element is $dq = \lambda dx$. We will choose the variable of integration to be x, the distance from the electric field vector to the differential element. The limits of integration go from d to $d + \ell$ (Why?). $\hat{r}$ is $-\hat{\imath}$. Set up the integral using this information.

   (b) Find the electric field vector in terms of Q. Your answer should be written as a vector.

2. Consider a long thin wire with a uniform line charge as the source of a continuous charge distribution. The charge along a long thin wire is called a line charge and is characterized by its linear charge density, λ. Since we have a uniform line charge, $\lambda = \frac{Q}{2\ell}$ where Q is the total charge and 2ℓ is the length of the wire (2ℓ is chosen for convenience). We want to determine the electric field vector $\vec{E}$ in the perpendicular bisector plane of a long, staight uniformly charged wire as shown in figure 7.47.

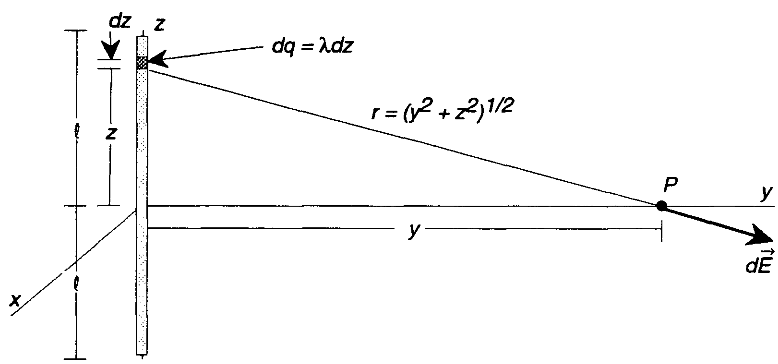

Figure 7.47: An electric field vector generated from a long thin wire.

(a) To set up this problem, first break the vector $d\vec{E}$ into y- and z-components. Because of the symmetry of the long thin wire, $\sum_i dE_{z_i} = 0$. Explain why this is so.

(b) Set up the integral to find the electric field vector at point P. Follow the procedure used in problem 1 with the variable of integration as z. What are the limits of integration?

(c) Write the electric field vector in terms of Q. Your answer should be written as a vector.

3. Next consider a ring of radius a which has a uniform positive charge per unit length with a total charge Q. We want to determine the electric field vector at a point P lying a distance x from the center of the ring along the axis of the ring as shown in figure 7.48.

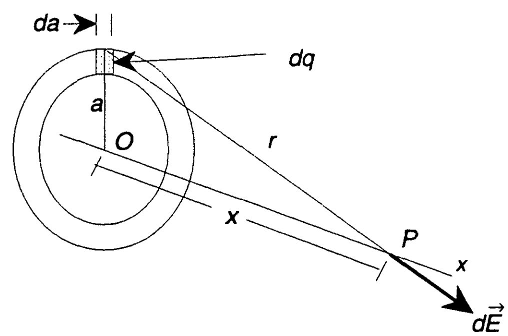

Figure 7.48: An electric field vector generated from a charged ring.

(a) Divide the vector $d\vec{E}$ into three components and evaluate each component by considering the symmetry of the object before setting up the integral.

(b) Use what you have seen in problems (1) and (2) to set up and evaluate the integral.

4. A disk of radius R has a uniform charge per unit area σ. Find the electric field vector at point P which is located along the axis of the disk, a distance x from

its center as shown in figure 7.49. Use what you have found in problems (1), (2) and (3) to help you.

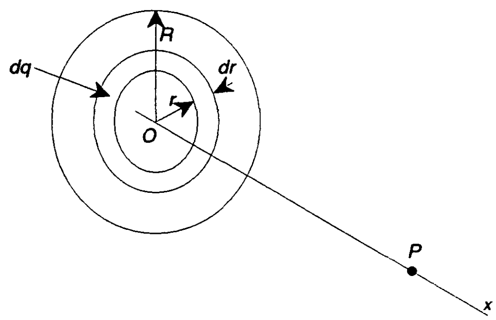

Figure 7.49: An electric field vector generated from a charged plate.

Solutions to: Electric Fields

1. $\vec{E} = k\int \frac{dq}{r^2}\hat{r} = k\int \frac{dq}{r^2}\left(-\hat{\imath}\right) = k\int_d^{\ell+d} \frac{\lambda dx}{x^2}\left(-\hat{\imath}\right)$

$$= k\lambda\left[-\frac{1}{x}\right]_d^{\ell+d}\left(-\hat{\imath}\right) = \frac{k\lambda\ell}{d\left(\ell+d\right)}\left(-\hat{\imath}\right)$$

$$= -\frac{kQ}{d\left(\ell+d\right)}\ \hat{\imath}$$

2. $\vec{E} = k\int \frac{dq}{r^2}\hat{r} = \int\left(dE_y\hat{\jmath} + dE_z\hat{k}\right)$

Using symmetry to take all differential elements dz into account, $\sum dE_z = 0$.

$$\vec{E} = \int_{-\ell}^{\ell} dE_y\hat{\jmath} = \int_{-\ell}^{\ell} dE\cos\theta\ \hat{\jmath} = \int_{-\ell}^{\ell}\frac{\lambda k y dz}{\left(y^2+z^2\right)^{3/2}}\ \hat{\jmath}$$

$$= \lambda k y\left[\frac{z}{y^2\left(y^2+z^2\right)^{1/2}}\right]_{-\ell}^{\ell}\ \hat{\jmath} = \frac{2\ell\lambda k}{y\left(y^2+\ell^2\right)^{1/2}}\ \hat{\jmath}$$

3. $\vec{E} = k\int \frac{dq}{r^2}\hat{r} = k\int \frac{dq}{r^2}\cos\theta\ \hat{\imath}$

Using symmetry to take all differential elements dy and dz into account,

$\sum dE_y = 0$ and $\sum dE_z = 0$.

$$\vec{E} = k\int_0^{2\pi}\frac{dq}{r^2}\frac{x}{r}\ \hat{\imath} = k\int_0^{2\pi}\frac{x dq}{\left(x^2+a^2\right)^{3/2}}\ \hat{\imath} \qquad \left(r = \sqrt{x^2+a^2}\right)$$

$$= \frac{kx}{\left(x^2+a^2\right)^{3/2}}\int_0^{2\pi} dq\ \hat{\imath} = \frac{kx}{\left(x^2+a^2\right)^{3/2}}2\pi q\ \hat{\imath} = \frac{kx}{\left(x^2+a^2\right)^{3/2}}Q\ \hat{\imath}$$

4. $$\vec{E} = k \int \frac{dq}{r^2}\hat{r} = k \int_0^R \frac{x}{(x^2+r^2)^{3/2}} 2\pi\sigma r dr \ \hat{i}$$

Since the element is a ring, we have $dQ = 2\pi dq = 2\pi r dr$.

Using symmetry to take all differential elements dy and dz into account,

$\sum dE_y = 0$ and $\sum dE_z = 0$.

$$\vec{E} = \pi\sigma \int_0^R \frac{2rdr}{(x^2+r^2)^{3/2}} \ \hat{i} = kx\pi\sigma \left[\frac{-1}{2\,(x^2+r^2)^{1/2}}\right]_0^R \ \hat{i}$$

$$= 2\pi k\sigma \left(1 - \frac{x}{(x^2+R^2)^{1/2}}\right) \ \hat{i}$$

7.31 Magnetic Fields

Calculus Topic: Vector Fields, Vectors

Department: Physics

Subject Area: Magnetism

Time Needed: 1 hour

Reference: [23]

A magnetic field vector $\vec{B}$ at some point in space is the magnetic force that would be exerted on an appropriate test object at that point. Magnetic fields can be created by electric currents. Here we will consider magnetic fields created by currents moving through wires or objects that can be treated as wires. The direction of the magnetic field $\vec{B}$ created by a current is determined by the right-hand rule. If the

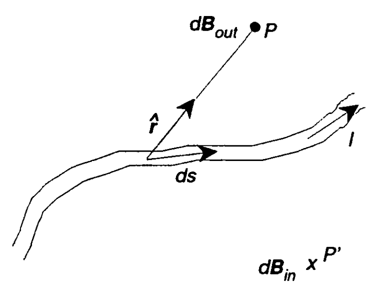

Figure 7.50: The magnetic field at a point P due to a current element $d\vec{s}$. The magnetic field circles the wire, coming out of the page at P and entering the page at P'.

thumb of the right hand is oriented along the direction of current flow in a wire, the fingers curl in the direction of the magnetic field. In order to calculate the strength of a magnetic field acting on some point P, it is necessary to make use of the Biot-Savart Law. Figure 7.50 shows the direction of the vectors acting in the Biot-Savart Law, which is written in the following form,

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \hat{r}}{r^2}$$

where I is the current moving through the wire, $d\vec{s}$ is a small element of the wire, $\hat{r}$ is a unit vector pointing from $d\vec{s}$ to the point P, r is the distance from $d\vec{s}$ to P and $\mu_0 = 4\pi \times 10^{-7} \frac{\text{T}\cdot\text{m}}{\text{A}}$ is a constant ($\text{T} = \frac{\text{N}}{\text{A}\cdot\text{m}}$ stands for tesla, a unit for measuring the strength of a magnetic field and A stands for Ampere, a unit for measuring current). Hence,

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_a^b \frac{d\vec{s} \times \hat{r}}{r^2}.$$

Troublesome Notation: In this problem, vectors are denoted by **B** with the magnitude of **B** denoted as B. In most calculus texts, a boldface **a** is used for vector notation and $|a|$ would designate its magnitude. Also, be careful of the definitions of the vectors $d\vec{s}$ and $\hat{r}$ and the distance r. $d\vec{s}$ is just an element of the wire and $\hat{r}$ is a unit vector pointing in the direction of r.

These two problems can be solved by using the above results.

a. A lightning bolt may carry a current of 10^4 A for a short period of time. What is the resulting magnetic field at points 50 m and 200 m from the bolt?

b. Measurements of the magnetic field of a large tornado were made at the Geophysical Observatory in Tulsa, Oklahoma in 1962. If the tornado's field was $B = 1.5 \times 10^{-8}$ T pointing north when the tornado was 9 km east of the observatory, what current was carried up/down the funnel of the tornado?

To find the solution to each of these problems, we will consider the magnetic field at a particular point resulting from the current as if it were moving through a long thin straight wire. This will be found in problem (1).

1. First we need to find the magnetic field resulting from current moving through a long thin wire. Let the wire carry a constant current I and be placed along the $x-$axis as in figure 7.51. Calculate the total magnetic field at the point P located at a distance a from the wire. First find $d\vec{s} \times \hat{r}$ (What is the direction?).

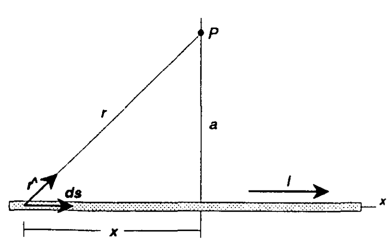

Figure 7.51: A long thin wire aligned with the x-axis.

Note that θ, r and x are all variable, so using the figure, express r and x in terms of θ, then substitute into the equation. Now find the magnetic field vector at point P by integrating over all elements subtending angles ranging from θ_1 to θ_2 as shown in figure 7.52. What happens if the wire is of infinite length, i. e., what values do θ_1 and θ_2 become? Now find the result of the Biot-Savart Law for an infinite wire.

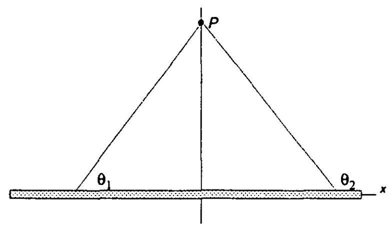

Figure 7.52: A long thin wire aligned with the x-axis showing the angles θ_1 and θ_2.

2. Use this result to calculate the answers to the two problems (a) and (b).

   Other applications of the Biot-Savart Law are as follows.

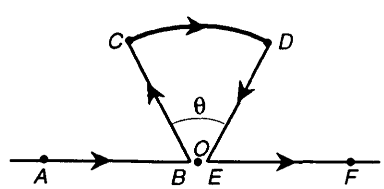

Figure 7.53: A long thin wire forming a loop.

3. For the wire in figure 7.53, calculate the magnetic field at the point O for the nearly closed current loop. The loop consists of two straight portions and a

circular arc of radius R, which subtends an angle θ at the center of the arc. As a hint, find the contribution to the magnetic field for each segment of the loop. It may help to draw the vectors $d\vec{s}$ and $\hat{r}$ on each segment and look at their orientation.

4. Consider the magnetic field produced by a circular loop of radius a carrying a current I as shown in figure 7.54. To set up this problem, first break the vector $d\vec{B}$ into x- and y-components. Because of the symmetry of the long thin wire, $\sum_i dB_{y_i} = 0$. Explain why this is so, then calculate the magnetic field at a point P along the $x-$axis which has been chosen as the axis of the loop.

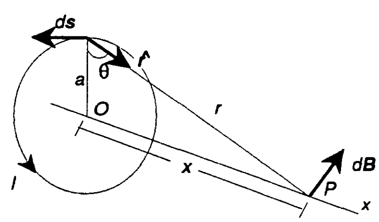

Figure 7.54: A magnetic field produced by a circular loop.

Solutions to: Magnetic Fields

1. $d\mathbf{s} \times \hat{r} = |d\mathbf{s} \times \hat{r}|\,\hat{k} = dx \sin\theta \; \hat{k}$ and $d\mathbf{B} = dB\hat{k}$

hence $dB = \dfrac{\mu_o I}{4\pi} \dfrac{dx \sin\theta}{r^2}$ (x, r and θ are all variables)

$r = \dfrac{a}{\sin\theta} = a \csc\theta$. $\tan\theta = \dfrac{-a}{x} \Rightarrow x = -a\cot\theta$ and $dx = a\csc^2\theta d\theta$.

$$d\mathbf{B} = \frac{\mu_o I}{4\pi} \frac{a \csc^2\theta \sin\theta}{a^2 \csc^2\theta} d\theta = \frac{\mu_o I}{4\pi a} \sin\theta d\theta$$

$$\mathbf{B} = \frac{\mu_o I}{4\pi a} \int_{\theta_1}^{\theta_2} \sin\theta d\theta = \frac{\mu_o I}{4\pi} (\cos\theta_2 - \cos\theta_1)$$

For an infinitely long wire, $\theta_1 \rightarrow 0$ and $\theta_2 \rightarrow \pi \Rightarrow \mathbf{B} = \dfrac{\mu_o I}{2\pi a}$

2*a*. $$\mathbf{B} = \frac{\mu_o I}{2\pi a} = \frac{\left(4\pi \times 10^{-7} \frac{\text{T}\cdot\text{m}}{\text{A}}\right)(10^4 \text{ A})}{2\pi\,(100\text{m})} = 2.0 \times 10^{-5} \text{ T}$$

2*b*. $$I = \frac{2\pi a\mathbf{B}}{\mu_o} = \frac{2\pi\,(9000 \text{ m})\,(1.5 \times 10^{-8} \text{ T})}{4\pi \times 10^{-7} \frac{\text{T}\cdot\text{m}}{\text{A}}} = 675 \text{ A}$$

3. $$\mathbf{B} = \frac{\mu_o I}{4\pi} \int_A^F \frac{d\mathbf{s} \times \hat{r}}{r^2}$$

$$= \frac{\mu_o I}{4\pi}\left(\int_A^B \frac{\mathbf{ds}\times\hat{r}}{r^2} + \int_B^C \frac{\mathbf{ds}\times\hat{r}}{r^2} + \int_C^D \frac{\mathbf{ds}\times\hat{r}}{r^2} + \int_D^E \frac{\mathbf{ds}\times\hat{r}}{r^2} + \int_E^F \frac{\mathbf{ds}\times\hat{r}}{r^2}\right)$$

Note that for segments AB, BC, DE and EF, $\mathbf{ds}\times\hat{r} = ds\sin 0 = 0$.

For segment CD, $\mathbf{ds}\times\hat{r} = ds\sin\frac{\pi}{2} = ds$.

Hence, $\mathbf{B} = \frac{\mu_o I}{4\pi}\int_C^D \frac{ds}{r^2}$ r is a constant and $ds = rd\theta$, so

$$\mathbf{B} = \frac{\mu_o I}{4\pi r^2}\int_0^\theta = \frac{\mu_o I\theta}{4\pi r}$$

4. $d\mathbf{B} = \frac{\mu_o I\mathbf{ds}\times\hat{r}}{4\pi r^2}$ Note that $\mathbf{ds}\times\hat{r} = ds\sin\frac{\pi}{2} = ds$ and that when all dB_y are considered, symmetry dictates that $\int dB_y = 0$. Hence $d\mathbf{B} = dB_x$.

$$B_x = \int \frac{\mu_o I ds\cos\theta}{4\pi\left(x^2+a^2\right)}$$

Note that $\int ds = s = 2\pi a$ and $cos\theta = \frac{a}{r} = \frac{a}{\left(x^2+a^2\right)^{1/2}}$

$$B_x = \frac{\mu_o I 2\pi a^2}{4\pi\left(x^2+a^2\right)^{3/2}}.$$

7.32 Electric Flux

Calculus Topic: Surface Integrals, Dot Products, Vectors

Department: Physics

Subject Area: Electricity

Time Needed: 1 hour

Reference: [23]

Electric flux is a measure of the number of electric field lines penetrating a surface. In other words, it is defined as a surface integral of the electric field $\vec{E}$ over a surface S. In a calculus book, the flux of a three-dimensional vector field $\vec{F}$ across an oriented surface S in the direction $\vec{n}$ is defined as Flux $= \int\int_{surface} \vec{F} \cdot \vec{n} d\sigma$. Electric flux, $\vec{\Phi}$, is the dot product of the electric field strength, $\vec{E}$, and a surface area, $\vec{A}$, as shown in figure 7.55. If we consider a general situation, choose a small element $\Delta\vec{A}_i$

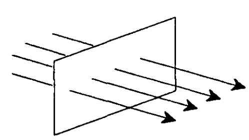

Figure 7.55: An electric field moving through a surface.

on the surface area which has a corresponding $\vec{E}_i$ and define θ as the angle between them. Then

$$\Delta\Phi_i = E_i \Delta A_i \cos\theta = \vec{E}_i \cdot \Delta\vec{A}_i.$$

By letting the area element $\Delta\vec{A}_i \to 0$, we get the general definition of flux:

$$\vec{\Phi} \equiv \lim_{\Delta A_i \to 0} \sum_i \vec{E}_i \cdot \Delta\vec{A}_i = \int_S \vec{E} \cdot d\vec{A}$$

Troublesome notation: The symbol $\int\int_{surface}$ is a general method of indicating integration over a surface which is two-dimensional. When setting up the problem, it will be necessary to define the limits for each part of the integral. In this problem, vectors are denoted by $\vec{E}$ with the magnitude of $\vec{E}$ denoted as E and the components of $\vec{E}$ as E_x, E_y and E_z. In calculus, a boldface $\mathbf{E}$ is used for vector notation, $|E|$ would designate its magnitude and (E_1, E_2, E_3) would designate its components. Also in this problem, $\vec{E} \cdot d\vec{A}$ denotes a dot product between two vectors. There are two ways to evaluate a dot product. In calculus, the dot product is written as $\mathbf{a} \cdot \mathbf{b}$ and is either $a_1b_1 + a_2b_2 + a_3b_3$ for three component vectors or $|\mathbf{a}||\mathbf{b}|\cos\theta$ where θ is the angle between the vectors. This problem uses the second form.

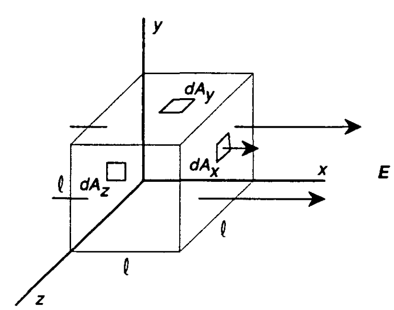

Figure 7.56: An electric field moving through a cube.

1. This problem demonstrates movement of flux through a cube. Consider a uniform electric field, $\vec{E}$ oriented in the x direction as shown in figure 7.56. Find the net electric flux through a cube with edges of length ℓ by evaluating $\vec{\Phi} = \int_S \vec{E} \cdot d\vec{A}$ for each surface. Remember to consider the direction of $\vec{E}$ when evaluating $\vec{\Phi}$.

2. A nonuniform electric field is given by the expression $\vec{E} = ay\hat{\imath} + bz\hat{\jmath} + cx\hat{k}$ where a, b and c are constants. Determine the electric flux through a rectangular surface in the xy plane, extending from $x = 0$ to $x = w$ and $y = 0$ to $y = h$. Draw a diagram of the surface to help define the limits of integration.

3. An electric field is given by $\vec{E} = az\hat{\imath} + bx\hat{k}$, where a and b are constants. Determine the electric flux through the triangular surface shown in figure 7.57.

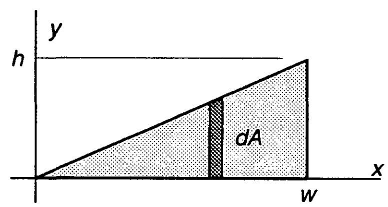

Figure 7.57: A triangular surface in the xy plane.

The definition of flux associated with a magnetic field is similar to that of an electric field. Consider an element of area dA on an arbitrarily shaped surface. If the magnetic field at this element is $\vec{B}$, then the magnetic flux through the element is $\vec{B} \cdot d\vec{A}$ where

$d\vec{A}$ is a vector perpendicular to the surface whose magnitude is equal to the area $d\vec{A}$. The total magnetic flux through the surface is defined as

$$\Phi_m = \int \vec{B} \cdot d\vec{A}.$$

4. A rectangular loop of width w and length ℓ is located a distance d from a long wire carrying a current I as shown in figure 7.58. The wire is parallel to the long side of the loop. We want to find the total magnetic flux through the loop.

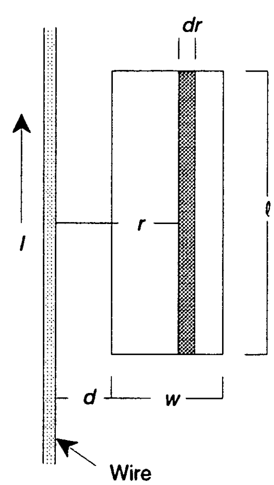

Figure 7.58: A rectangular loop near long wire.

(a) Describe the magnetic field in the rectangular loop. What direction is the magnetic field? Is the magnetic field uniform through the loop?

(b) Ampere's Law states that the magnitude of the magnetic field can be expressed as $B = \dfrac{\mu_o I}{2\pi r}$ where μ_o is a constant, I represents the current, and r is the distance from the wire to the point where the magnetic field is being measured. If teslas are used as units for the magnetic field, the constant μ_o in Ampere's Law is $4\pi \times 10^{-7}\ \frac{\text{T}\cdot\text{m}}{\text{A}}$. Write dA in terms of the measurements of the rectangle and evaluate Φ_m.

Solutions to: Electric Flux

1. For the sides of the cube in the xy and xz planes,

$$\vec{\Phi} = \int \vec{E} \cdot d\vec{A} = \int E dA \cos\frac{\pi}{2} = 0$$

In the yz plane, $\vec{\Phi} = \int_1 \vec{E} \cdot d\vec{A} + \int_2 \vec{E} \cdot d\vec{A}$

$$= \int_1 E dA \cos 0 + \int_2 E dA \cos\pi = -\int_1 E dA + \int_2 E dA$$

$$= -E\ell^2 + E\ell^2 = 0.$$

2. $$\vec{\Phi} = \int \vec{E} \cdot d\vec{A} = \int_0^w \left(ay\hat{\imath} + bz\hat{\jmath} + cx\hat{k}\right) \cdot \left(hdx\hat{k}\right) = \int_0^w cx\,(hdx)$$

$$= ch\int_0^w xdx = \frac{chw^2}{2}$$

3. $$\vec{\Phi} = \int \vec{E} \cdot d\vec{A} = \int_0^w \left(az\hat{\imath} + bx\hat{k}\right) \cdot \left(ydx\hat{k}\right) = \int_0^w bx\,(ydx) = \int_0^w bx\left(\frac{h}{w}x\right)dx$$

$$= \frac{bh}{w}\int_0^w x^2 dx = \frac{bhw^2}{3}$$

4. $$\Phi_m = \int \vec{B} \cdot d\vec{A} = \int_d^{d+w} bdA\cos 0 \qquad\qquad dA = \ell dr$$

$$= \int_d^{d+w} \frac{\mu_o I}{2\pi r}\ell dr = \frac{\mu_o I\ell}{2\pi}\int_d^{d+w}\frac{dr}{r}$$

$$= \frac{\mu_o I\ell}{2\pi}\ln r\Big|_d^{d+w} = \frac{\mu_o I\ell}{2\pi}\ln\frac{d+w}{d}$$

Chapter 8

CONCLUSION

It is hoped that these problems will be a useful aid to students learning calculus. Comments made by faculty members show the importance of students learning this subject well. Evaluation of the eventual effectiveness of this approach remains to be done. Some initial evaluation was done in the Fall, 1996 semester to test the idea behind this thesis, but not all of the problems have been exposed to students yet.

Evaluation of these problems needs to cover three areas.

- Do all of the problems work well, that is, are the students able to follow the procedures and come up with a reasonable solution?
- Do the problems enhance the learning of calculus concepts by extending student knowledge and fostering student interest?
- Do the problems help students learn to translate their knowledge of calculus as they encounter it in other courses?

It will take a few years study to formulate answers to these questions.

As an initial evaluation, in the Fall, 1996 semester, several of these problems were given to students in the Calculus II Honors classes. Students worked in groups to solve the problems. A focus was placed on understanding the concepts and preparing a well written solution or explanation for each problem. Group interaction was generally lively and groups were quite often able to find solutions without much help. Older students visited the class and commented on how exciting they felt this method was. At the end of the semester, a student evaluation form was given to students. When asked what element of the course help them learn calculus the most, twenty-eight of forty-four students listed the problem solving sessions as being very helpful in learning calculus concepts and only one said that they were no help at all. A couple of comments were "The real problem applications (helped) - when you have to answer a question, it makes you really understand what the calculus is telling you" and "The real problems are fun and facilitate the learning of new concepts". The biggest complaint about the problems from students was that they had to sit and think. When asked to list connections of calculus to chemistry, engineering, physics and economics, nearly all students were able to list at least two and many listed three to four.

This same group of students is being tracked through Physics I in the Spring, 1997 semester to see what effect the problem solving had. Early reports indicate the Physics faculty feels they are able to deliver this course with greater mathematical content. Since much more evaluation needs to be done, it will take a few years for the effect of problems like these to be known.

As the projects similar to this one are being developed at Rensselaer Polytechnic Institute, the University of Pennsylvania and Dartmouth College among others, more resources for applications will come available and information about the effectiveness of these types of problems will begin appearing. However, at this point in time their evaluation processes are still in the earliest stages. There is still a wealth of information to use at CSM for the development of more problems, so a much larger resource exists here also. Having faculty involved in providing problems has improved the interaction between the various departments and MCS and has enhanced its image. Real application problems can make a difference in the way students view mathematics.

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