DYNAMIC MODELING AND COMPARISON OF CONTROL ALGORITHMS FOR BUILDING CONDITIONING SYSTEMS

by
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ABSTRACT

The energy cost of monitoring and controlling a building environment (usually temperature, humidity and pressure) is a serious consideration for the Heating, Ventilation, and Air Conditioning (HVAC) engineering. The amount of energy consumed in a building is dependent on the design, control and maintenance of the energy management and control system (EMCS). Proper control and maintenance can result in significant savings in building energy use, but is complicated due to the non-linear time varying characteristics of system and typically unknown load and disturbance.

Here my work is focused on the Variable Air Volume (VAV) systems. All the experimental work was performed on the Slater Wind Tunnel at the Division of Engineering, Colorado School of Mines. The Slater Wind Tunnel was modified to simulate a simple VAV system. To design control algorithms for this system, a dynamic model of system and analysis of energy characteristics was performed. The Proportional, Integral, and Differential (PID) control, adaptive control, optimal control and hybrid control schemes were developed. The adaptive control was applied to the duct pressure control and the results show that the adaptive control performs very well under different operational conditions. The optimal control was also applied to the duct pressure control and the result shows that the optimal control can save 15-30% energy compared with conventional PI control.
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GLOSSARY

$C_{pa}$-specific heat of air
$C_{pw}$-specific heat of water
e(k)-error between the set point and the actual value
$m_a$-mass flow rate of air
$m_w$-mass flow rate of chilled water
$M$-mass of system
$Q$-heat added to system
$P(k)$-pressure at k time
$P_m(k)$-pressure set point
$p_{fan}$-power of fan
$T_{ai}$-inlet temperature of air
$T_{ao}$-outlet temperature of air
$T_{wi}$-inlet temperature of chilled water
$T_{wo}$-outlet temperature of chilled water
$T_1$-zone temperature
$T_2$-supply air temperature
t_p-differentiation time
t_t-integration time
t_s-sampling time
$V_a$-velocity of air
$u_{damper}$-control signal to damper
$u_{fan}$-control signal to fan
$u_{\text{valve}}$-control signal to valve

$z^{-1}$-backward shift operator

$\Delta P$-pressure difference across damper

$\varepsilon$-zero mean white noise

$\rho$-density of air

$\gamma$-forget factor

$\theta(k)$-parameter vector at time $k$

$\varphi(k)$-observation vector

Subscripts

$T1$-parameters related to zone temperature

$T2$-parameters related to supply air temperature

Superscripts

$T$-Transpose of matrix or vector
CHAPTER 1

INTRODUCTION

1.1 Background

The energy cost of controlling building environments (usually temperature, humidity and pressure) is a serious consideration for the Heating, Ventilating and Air Conditioning (HVAC) industry. The amount of energy consumed in building depends on the design, control and maintenance. Proper control and maintenance can result in a significant savings in building energy use (Hittle 1985). The proper control of an HVAC system can be defined in terms of the following specifications:

A. Control of the system to have desired physical variables i.e. temperature, humidity and pressure.

B. The overall system approaches the minimum energy consumption.

The proper control of HVAC system is very difficult because the control plant models of the HVAC system are non-linear and time-varying. The parameters of the model are usually unknown and the time constants are usually large because of the characteristic large thermal mass. For these reasons, it is difficult to optimally tune a controller for HVAC systems. Another problem of control of HVAC system is the fact that the load and disturbance are usually unknown. For example, humans dissipate heat and therefore comprise a significant portion of a cooling load. The numbers of people that occupy a room are variable. Furthermore the weather conditions are usually unknown.
During the past few years, many research results in the modeling and control of HVAC systems have been published. Pape et al. (1991) proposed an optimal control and fault detection method for HVAC systems. Their work was based on the assumption of static power consumption. However, since most of HVAC systems have a large time constant, the transient state should be as important as static state. Shoureshi et al. (1986) made use of Kalman filter to reconstruct a nonlinear thermo fluid system that gave new approach for modeling of thermal systems. Arora (1991) employed a "discrete time method" for optimal control of the Variable Air Volume (VAV) systems. Their work depends on the exact model of VAV system, but the exact model information is usually not available in actual applications.

Athieities et al. (1990) proposed a methodology for building thermal dynamic studies and control applications. They applied the concept of a transfer function to HVAC modeling and developed a scheme for building transfer functions. The development of an HVAC transfer function system is tedious work, because of the coupling among different components controlling the conditions within occupied spaces and other elements of the HVAC systems. Behker et al. (1990) proposed a tuning method for a PI controller of an HVAC system. But in some cases, their assumption of steady-state conditions may not be satisfied. Wallenborg (1990) proposed a self-tuning method for controller tuning of the building thermal process. The non-linearity of the actuator is a limitation for the self-tuning controller. Kreider and Wong (1991) made use of the artificial neural networks to predict the energy use for a commercial building. Many researchers and HVAC engineers are applying late results of control theory and applications, but most of them are in an early stage of development.
1.2 VAV systems

Here my purpose is not to develop a perfect controller for HVAC systems. But to develop control schemes that can be used for HVAC systems and compare the results based on energy consumption. My work has focused on VAV systems. The VAV system was developed during the energy shortage in the early 1970s in response to the need for simple air conditioning systems capable of meeting the minimum demands for heating and cooling in a building while reducing the energy consumption. Since their introduction, VAV systems have been widely embraced by the building industry because they cost less to install compared with other systems and help reduce overall cost of building operations. A single duct VAV system is shown in Fig. 1.1.

Fig. 1.1 Variable Air Volume System
where OA is outside air, RA is return air, EA is exhausted air, T2 is the temperature of supply air and T1 is the temperature of occupied space. ATU1 and ATU2 are air terminal units to distribute the conditioned air into occupied spaces. The purpose of duct pressure control is to maintain the minimum amount of static pressure in the duct system to support proper air distribution into the occupied space or room in the building.

In VAV systems, the supply air is maintained at a specific pressure by modulating the speed of the fan. The ATU modulates the air volume flow rate to provide only enough conditioned air at a specified temperature to meet the load demands of the occupied space. If the space temperature changes, the VAV system modulates terminal units, i.e. the damper position, in order to maintain the occupied space temperature or zone temperature. When this happens, there will be a change in supply air duct pressure. In other words, there is coupling between duct pressure and occupied space temperature. For conventional control methods, control of the ATU will be compensated based on the change of duct pressure. But most ATUs are dampers and are duct pressure dependent (Coffin 1991).

The power and pressure performance of the fan is given as (Lorenzetti 1992)

$$\frac{P_1}{P_2} = \left(\frac{HP_1}{HP_2}\right)^{\frac{2}{3}}$$

A change of load will lead to a change of temperature of occupied space. A change temperature of occupied space will lead to a change of volume flow rate of conditioned air. A change of volume flow rate will lead to a change of duct pressure. So the duct pressure is affected by the building load. If the load is very great, the damper will open to
allow more conditioned air to flow into the occupied space, thus it will cause the duct pressure to decrease. This will cause the fan speed to oscillate and consume an unnecessarily large amount of energy. From the model for performance of fan (equation 1.1), we can see that doubling the horse power of fan will only increase the pressure by 1.587 times. The power consumed by fan is a significant portion of the power consumed by a VAV system (Levermore 1992). From an energy efficiency point of view, the fan should run only if there is a requirement of modulating the fan speed to maintain the duct pressure. Since the load is unknown and the model of HVAC systems is usually time-varying, the conventional PID control has significant limitations for maintaining efficient control.

1.3 Experimental Apparatus

The experimental work was performed on the Slater Wind Tunnel at the Division of Engineering, Colorado School of Mines. The Slater Wind Tunnel was used to simulate a single zone, single duct VAV system. It was built with funds donated by Willard and Emma Slater. Mr. Slater is an alumnus (B.S. Geophysics, 1940) of Colorado School of Mines (CSM). Will and his wife Emma have developed an interest in supporting research at CSM related to energy conservation and alternative energy systems for large residential and small commercial buildings.

A schematic of the Slater Wind Tunnel is shown in Fig. 1.2. Temperature upstream, $T_1$, simulates the zone temperature, that is the occupied space temperature. $T_1$ is controlled by modulating the position of damper which simulates an air terminal unit (ATU) to allow enough conditioned air to flow into the occupied space. The damper is modulated by a KREUTER® MCP 1040-520 pneumatic actuator which is controlled by a
Pneutronics® VIP-Flex electrical pneumatical (E/P) converter which converts the electrical control signal into proportional displacement of the pneumatic actuator.

The temperature downstream, $T_2$, simulates the supply air temperature. $T_2$ is controlled by modulating the flow rate of chilled water. The flow rate of chilled water is measured by an OMEGA® FP-5820 flow transmitter. The flow rate of chilled water is controlled by a KMC® VCP-43 pneumatically actuated 3-way valve. The pneumatic valve is controlled by a Pneutronics® VIP-Flex electrical pneumatical (E/P) converter. The chilled water is supplied by an APEC® AC-5 portable water chiller. An Ohio Semitronics® PC5-062D power transducer was used to measure the power consumed by the chiller. The heat $Q$ simulates the load and disturbance to the occupied space.

The pressure of supply air duct is maintained at specific value to maintain proper operation of ATUs. The duct pressure $\Delta P$ is measure by a Modus Instrument® T-30-010 pressure transducer. It is controlled by modulating the speed of fan. The speed of the fan is controlled by a Reliance Electric A-C V*S® variable speed drive (VSD). An Ohio Semitronics® PC5-014D power transducer was used to measure the power consumed by the supply fan.

The I/O interface between the computer and control devices, A/D and D/A converters, are accomplished by a Fluke Helios Plus 2273A data acquisition system. The data acquisition system acquires the measured data from measurement devices and converts the analog data into digital data and stores the digital data in computer. A model PS2/30 IBM personal computer is used to input digital data through the data acquisition system. Based on the control software, the computer will output desired control signals through the data acquisition system to the specific actuators.
Fig. 1.2 Slater Wind Tunnel
1.4 Purpose

The main objective is to model the dynamic behavior of the system, develop some control schemes for VAV systems and determine the energy performance. In order to do this, the system was modeled by the experimental data. For the purpose of controlling the transient process, the dynamic behavior of system is very important. Therefore, from the view point of actual applications, on-line methods were applied here. Different control schemes such as PI control, optimal control and adaptive control were developed and are discussed in the discussion that follows. Optimal control was designed based on the energy consumed by system. Adaptive control was designed based on the set point following situations of unknown load, disturbance or change of dynamic model. Finally, the results were compared based on the energy consumption.
CHAPTER 2
DYNAMIC MODELING

In order to design a controller for a dynamic system, it is necessary to have a model of the system which adequately describes the system's dynamic behavior. There are typically two kinds of information available to control designers. First, there is the knowledge of physics, chemistry, thermodynamics, and other basic sciences in the form of equations that describe the system to be controlled. However, because of complex physical phenomena involving properties such as pressure, fluid flow, temperature, velocity etc., and complicated geometries within HVAC systems, theoretical model predictions using only the laws of science are difficult to practically implement for achieving a satisfactory description of the control system dynamic behavior.

The general nature of the theoretical equations use to model thermal systems are well known, only the constants in the coefficient are unknown. Therefore, the designer may turn to the second source of information about the dynamic behavior of system which is the experimental approach. In this approach, a control signal is applied to the system and the response is measured. The process of constructing models and estimating the best values of unknown parameters for the experiment is called system identification (Astrom 1984). With the modern digital computer and data acquisition systems, this approach is available in many situations today. Since most HVAC systems have computerized control, the most useful model is discrete. For dynamic modeling, we first assume the structure of system, that is the form of linear or non-linear difference equations that represent the essential system character. Then a control signal is applied to the system and the response is measured and the unknown parameters of equations are
estimated. Here the model of the system is assumed to be an Auto Regressive Moving Average (ARMA) model (Astrom 1989). For example, the model of system is given in a difference equation form:

\[ A(z^{-1})x(k + 1) = B(z^{-1})u(k) + C(z^{-1})\epsilon(k) \quad (2.1) \]

where \( x(k) \) is the output, \( u(k) \) is the input, \( \epsilon(t) \) is the zero mean white noise, and \( A(z^{-1}), B(z^{-1}) \) and \( C(z^{-1}) \) are polynomials of \( z^{-1} \), the backward shift operator. \( A(z^{-1}), B(z^{-1}) \) and \( C(z^{-1}) \) are given as

\[ A(z^{-1}) = 1 + a_1z^{-1} + \ldots + a_nz^{-n} \quad (2.2.a) \]
\[ B(z^{-1}) = b_0 + b_1z^{-1} + \ldots + b_{n-1}z^{-(n-1)} \quad (2.2.b) \]
\[ C(z^{-1}) = 1 + c_1z^{-1} + \ldots + c_nz^{-n} \quad (2.2.c) \]

For the determined system, i.e. \( C(z^{-1})=0 \), the system can be represented as

\[ x(k + 1) = -a_1x(k) - \ldots - a_nx(k - n + 1) + b_0u(k) + \ldots + b_{n-1}u(k - n + 1) \quad (2.3) \]

Let

\[ \theta' = [-a_1, -a_2, \ldots, -a_n, b_0, b_1, \ldots, b_{n-1}] \quad (2.4) \]

and

\[ \varphi(k + 1) = [x(k)x(k - 1)\ldots x(k - n + 1)u(k)u(k - 1)\ldots u(k - n + 1)] \quad (2.5) \]
If we assume $C(z^{-1})=1$, the model of system is represented as

$$x(k+1) = \varphi(k)\theta + \varepsilon(k)$$  \hfill (2.6)

The criterion function is chosen as:

$$J(\theta) = \sum_{k=1}^{n} \varepsilon^T(k)\varepsilon(k)$$  \hfill (2.7)

My objective in this operation is to find the optimal $\theta^*$ which minimizes the criterion function. Here the Least Square (LS) method (Astrom 1984) is applied to estimate the parameter vector $\theta$.

2.1 Least Square Estimation

The objective here is to determine the values of parameters of $a_i$ and $b_j$ which best describe the system dynamic behavior. In order to do this, $N-n$ values $u(n+1)$, $u(n+2)$,..., $u(N+1)$ are applied to the system and $N-n$ outputs $x(n+1), x(n+2),..., x(N+1)$ are measured. Let

$$\varepsilon = [\varepsilon(n+1)\varepsilon(n+2),...,\varepsilon(N+1)]^T$$  \hfill (2.8)

and
Then we have

\[ X = \Phi \theta + \varepsilon \]  \hspace{1cm} (2.10)

where \( X = [x(n+1)x(n+2), \ldots, x(N+1)]^T \). Substituting \( e(k) \) into criterion function, we will have

\[ J(\theta) = \frac{1}{2} e^T e = \frac{1}{2} (X - \Phi \theta)^T (X - \Phi \theta) \]  \hspace{1cm} (2.11)

Differentiating the criterion function \( J(\theta) \) with respect to parameter vector, yields

\[ \frac{dJ(\theta)}{d\theta} = -X^T \Phi + \theta^T \Phi^T \Phi = 0 \]  \hspace{1cm} (2.12)

The optimal estimation of the LS method is given as

\[ \theta^* = [\Phi^T \Phi]^{-1} \Phi^T X \]  \hspace{1cm} (2.13)

As we can see, the least square method requires a batch of data from which the matrices of \( \Phi \) and \( X \) are composed. For dynamic modeling, especially for on-line
applications, data are acquired sequentially rather than in a batch. At each time when we discard old data and acquire new data we may wish to update the solution. Improvement in the parameter estimation is made as new data are continuously acquired. For this purpose, we rearrange the matrices and the vectors as:

\[ X(N) = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \]  \hspace{1cm} (2.14)

and

\[ X(N+1) = \begin{bmatrix} X(N) \\ x(N+1) \end{bmatrix} \]  \hspace{1cm} (2.15)

The same is done to \( \Phi \).

\[ \Phi(N+1) = \begin{bmatrix} \Phi(N) \\ \varphi(N+1) \end{bmatrix} \]  \hspace{1cm} (2.16)

The Recursive Least Square method (Astrom 1984) is given by

\[ \theta(N+1) = \theta(N) + K(N)[x(N+1) - \varphi^T(N)\theta(N)] \]  \hspace{1cm} (2.17.a)

\[ K(N) = P(N)\varphi^T(N)[1 + \varphi(N)P(N)\varphi^T(N)]^{-1} \]  \hspace{1cm} (2.17.b)

\[ P(N+1) = [I - K(N)\varphi(N)P(N)] \]  \hspace{1cm} (2.17.c)
When we apply the criterion function (2.11), all the data are given the same weight. If we apply the LS estimation to dynamic modeling, parameters are time varying and we may give the data different weights. That is, we have to rewrite the criterion function as:

\[ J(\theta) = \sum_{k=1}^{N} \gamma^{N-k} (x(k) - \theta^T \varphi(k))^2 \]  

(2.18)

where \(0 < \gamma < 1\), \(\gamma\) is the forget factor. If we use (2.18) as criterion function, the least square method is given as:

\[
\theta(N + 1) = \theta(N) + K(N)[x(N + 1) - \varphi^T(N)\theta(N)] \quad (2.19.a)
\]

\[
K(N) = P(N)\varphi^T(N)[\gamma + \varphi(N)P(N)\varphi^T(N)]^{-1} \quad (2.19.b)
\]

\[
P(N + 1) = [I - K(N)\varphi(N)P(N)]/\gamma \quad (2.19.c)
\]

Throughout the remainder of this study, the recursive least square method is applied to the modeling of different HVAC systems.

### 2.2 Modeling of Duct Pressure

The control of duct pressure is shown in Fig. 2.1. In VAV systems, the duct pressure is maintained at a specific set point by modulating the fan speed. The open loop transient response of pressure to the speed of fan is very important for the design of a pressure controller. Proper control of pressure requires having complete information of the transient and steady-state response of pressure. The pressure model is based on the input, i.e. the control signal to fan speed and the output, i.e. the duct pressure.
Here the control signal was selected as a Pseudo Random Binary Signal (PRBS) (Astrom 1984). The output pressure drop was measured. P is used throughout the remainder of this study instead of ΔP for simplification. The assumed model structure of pressure is given as

\[ P(k+1) = a(k)P(k) + b(k)u_{fan}(k) \]  \hspace{1cm} (2.20)

where \( P(k) \) is the pressure at \( k \)-time, \( u(k) \) is the control at \( k \)-time and \( a(k) \) and \( b(k) \) are parameters to be determined.

The open loop response is shown in Fig 2.2. From the open loop response of pressure, we can see that a model of the pressure response to fan speed can be a first order system.
Next we must determine the time varying parameters of $a(k)$ and $b(k)$. Modeling is accomplished by applying the PRBS control signal to system and measuring the response of system. Both the PRBS signal and the pressure response are shown in Fig. 2.3.

The estimated parameters of $a(k)$ and $b(k)$ are shown in Fig. 2.4. We can see that the parameters are stable and approach to certain values. The model and the actual measurement are shown in Fig. 2.5. The result shows that the model can approach the measurement data and it is acceptable.
Fig. 2.3 Pressure vs. PRBS Control Signal

Fig. 2.4 Estimated Parameters for Duct Pressure
2.3 Modeling of Supply Air Temperature

The heat exchanger is used to condition the air upstream of the fan. The control of supply air temperature is shown in Fig. 2.6. Modeling of supply air temperature is accomplished by performing an energy balance about the heat exchanger.

The temperature downstream $T_{ao}$, that is the temperature $T_2$ in Fig. 1.2, simulates temperature of the supply air.

The open loop response of supply air temperature $T_{ao}$ is shown in Fig. 2.7. From this response, we can assume that the dynamic model of the supply air temperature is a first order system. The oscillations on the temperature of supply air are due to the built-in safety control unit on the chiller, limiting the return chilled water temperature to prevent freezing in the pipe.

Fig. 2.5 Comparison of Estimated and Measured Output
Fig. 2.6 Control of the Supply Air Temperature

To simplify the problem, the supply air temperature was modeled based on the following assumptions:

A. Constant density of the chilled water
B. Constant density of the air
C. Constant temperature of the inlet air $T_{ai}$
D. Constant temperature of inlet and outlet chilled water
E. The control valve is linear
Based on the energy balance, we can have the following equation:

\[ m_w C_{pa} (T_{ai} - T_{ao}) + MC_{pa} \frac{dT_{ao}}{dt} + Q = m_w C_{pw} (T_{wo} - T_{wi}) \]  

(2.21)

For this case, equation 2.21 is rewritten as:

\[ a_1 \frac{dT}{dt} + a_2 T = b m_w - Q \]  

(2.22)

where
\[ T = T_{ao} - T_{ai} \]
\[ a_1 = M_a C_{pa} \]
\[ a_2 = -m_a C_{pa} \]
\[ b = C_{pw} (T_{wo} - T_{wi}) \]

Since we assume \( T_{ai} \) i.e. \( T_2 \), is constant, so

\[ \frac{dT_{ao}}{dt} = \frac{dT}{dt} \]

Rewriting the equation (2.22) as:

\[ \frac{dT}{dt} + \frac{a_2}{a_1} T = \frac{b}{a_1} m_w - \frac{l}{a_1} Q \]

(2.23)

Discretizing equation (2.23), we will have

\[ T(k + 1) = a(k) T(k) + b(k) m_w (k) + c(k) \]

(2.24)

Control signal \( m_w (k) \) was applied to the heat exchanger and the temperature \( T(k) \) was measured. By the Least Square Method, parameters were estimated as:

\[ a = 0.8586 \]
\[ b = -32.96 \]
\[ c = 4.396 \]
Fig. 2.8 Estimated Parameters of Supply Air Temperature
The estimated parameters are shown in Fig. 2.8 and they are stable and approaches to certain values. The estimated model is shown in Fig. 2.9. We can see that the estimated model approaches the measured data and it is acceptable.

![Graph showing temperature over time with Measurement and Model lines.

Fig. 2.9 Model of Supply Air Temperature.

2.4 Modeling of Zone Temperature

Zone or occupied space temperature is simulated by temperature $T_1$ upstream of the heat exchanger. The control of the upstream temperature or zone temperature also requires a dynamic model. The control components for the zone temperature is shown in Fig. 2.10.
Based on the energy balance equation, we will have the following equation:

\[ MC_p \frac{dT_1}{dt} = m_a C_p (T_1 - T_2) + Q \] (2.25)

From equation (2.25), we can see that the resulting system model is nonlinear since the control variable \( m_a \) is coupled with the controlled variable \( T_1 \). The mass flow rate of chilled water depends on the fan speed as well as the damper. That is

\[ m_a = V A \rho = a(u_{fan}, u_{damper}) A \rho \]

Fig. 2.10 Control of Zone Temperature.
If we assume that the fan speed is constant, the flow rate of chilled water depends only on the position of damper. So

\[ m_a = a(u_{\text{damper}}) \]

Rewriting the equation yields:

\[ MC_p \frac{dT_i}{dt} + a(u_{\text{damper}})C_p T_i = a(u_{\text{damper}})C_p T_2 + Q \] (2.26)

This is a first order nonlinear system which describes the dynamic behavior of the temperature at the certain operational point. Since we consider the dynamic behavior of the system, by differentiating it with respect to time, we have the equation which describes the temperature variation.

\[ a_1 \frac{d^2 T_i}{dt^2} + a_2(u_{\text{damper}}) \frac{dT_i}{dt} + T_i \frac{da_2(u_{\text{damper}})}{dt} = \frac{db(u_{\text{damper}})}{dt} \] (2.27)

where

\[ a_1 = MC_p \]
\[ a_2(u_{\text{damper}}) = a(u_{\text{damper}})C_p \]
\[ b(u_{\text{damper}}) = a(u_{\text{damper}})T_2 + Q \]

Since the system at hand is a second order system, the following second order difference model was used to predict the zone temperature.
\[ T_2(k + 1) = a_1(k)T_2(k) + a_2(k)T_2(k - 1) + b_1(k)u(k) + b_2(k)u(k - 1) \quad (2.28) \]

Fig. 2.11 Estimated Parameters of Zone Temperature

Fig. 2.12 Model of Zone Temperature
The estimated parameters are shown in Fig. 2.11. The estimated model is shown in Fig. 2.12. The estimated parameters are also stable and approach certain values. The estimated model approaches to the measurement data and it is acceptable for the control purpose.

2.5 Modeling of Zone Temperature and Duct Pressure

In single zone single duct VAV systems, the pressure is controlled by modulating the fan speed. But we have already seen that the pressure is affected by the damper position as well. Therefore strong coupling exists between the pressure and the zone temperature. This situation is exacerbated by the wind tunnel because we have a single damper control. Typical building systems will have multiple dampers or ATUs for a single fan loop. Therefore, the coupling effects in the "real" systems are evident but at a reduced level. If the cooling load is very large, the damper position will open wider, the pressure will be greatly affected by the damper position. For such a strongly coupled system, we cannot simply model each control loop separately. In order to have information of the coupling effects between them, the system was modeled by a two input and two output system. The structure of system is given by:

\[
\begin{bmatrix}
P(k+1) \\
T_i(k+1)
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \begin{bmatrix}
P(k) \\
T_i(k)
\end{bmatrix} + \begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{bmatrix} \begin{bmatrix}
u_{\text{fan}}(k) \\
u_{\text{damper}}(k)
\end{bmatrix}
\]  
(2.29)

Using the Least Square (LS) method, the estimated parameters of pressure and zone temperature are found (as shown in Fig. 2.14). Because the steady magnitudes of
$a_{21}$ and $b_{21}$ are large compared to the uncoupled terms, we can see that there is a serious coupling between duct pressure $P$ and zone temperature $T_1$. The estimated models and measurement data are shown in Fig. 2.13. The pressure set point is 0.4 inch of water. The zone temperature set point is 25 degree C.

Fig 2.13 Model of Pressure and Zone Temperature
Fig. 2.14 Parameters of Pressure and Zone Temperature
2.6 Energy Performance Analysis

As mentioned previously, the energy consumed by an HVAC system depends strongly on the building environmental control system. In order to have an efficient control, we need to know the energy related system response to various inputs.

For instance, the energy consumed by the fan is a significant portion of energy consumed by an HVAC system. The fan is controlled to maintain a specific duct pressure within the system. This pressure is very important for the operation of air terminal units (ATU) since the ATU air flow control depends on the pressure produced by the fan. In order to control the fan speed to maintain the desired duct pressure while minimizing the energy consumption, we need to know the power performance of fan.

The power consumed by fan was measured while the control signal to the variable speed fan motor ranged from 4 mA to 18 mA. The relationship between the control signal to the fan motor drive and the measured power is shown in Fig. 2.15. A regressive method was applied to estimate the power performance of fan. A brief introduction of regressive method is given in the appendix.

Since the power of fan is not a linear function of the control signal to the fan motor drive, it was modeled with a quadratic equation.

\[ p_{\text{fan}} = a \cdot u_{\text{fan}}^2 + b \cdot u_{\text{fan}} + c \]

By regressive method, the parameters \( a, b, \) and \( c \) were estimated as:

\[ a = 3041.84 \]
\[ b = -25.3615 \]
\[ c = 0.211623 \]
The estimated model and the measured power of fan is shown in Fig. 2.15.

For optimal energy control, the cost function can be selected as a quadratic form which gives the appropriate description of the fan power consumed. In the next chapter, the quadratic model of fan power and the dynamic model of duct pressure are used to develop the duct pressure optimal control.

![Graph showing quadratic model and measured power comparison](image)

Fig. 2.15 Quadratic Model of Fan Power Compared with Measured Fan Power
CHAPTER 3

CONTROL OF HVAC SYSTEM

From our efforts in dynamic modeling, we have learned about the nature of the system dynamic response to specific control signals. The final objective is to control the system and efficiently meet the set points. Based on the models that we have developed, we can design the control algorithm to make the output of system follow the desired set point. First, a conventional PI control algorithm was applied to the three different systems previously described. Then an adaptive and an optimal control algorithm was developed and applied to only the duct pressure control. Finally the results were compared based on energy consumption.

3.1 PID Control

Proportional, Integral and Differential (PID) control (Astrom 1984) is widely used in process control. A typical PID control scheme is shown in Fig. 3.1.

![PID Control Diagram]

Fig. 3.1 PID Control
where the output of the system is fed back through a sensor and compared with the input, (set point of system). If there is an error between the output and the set point, the PID controller will generate a control signal $u$ to control the process while decreasing the error or difference between the output and the set point. The PID algorithm is

$$u(t) = K_p[e(t) + \frac{1}{t_i} \int_0^t e(\tau)d\tau + t_D \frac{de(t)}{dt}]$$  \hspace{1cm} (3.1)

where

e(t) is the error between the set point and output,
u(t) is the control signal at time $t$
$K_p$ is the system gain
t$_D$ is the differential time constant
t$_I$ is the integral time constant.

For computerized control systems, the control algorithm is more conveniently represented in difference form. That is

$$u(k) = K_p[e(k) + \frac{t_s}{t_I} \sum_{i=0}^{k} e(i) + \frac{t_D}{t_s}(e(k) - e(k-1))]$$  \hspace{1cm} (3.2)

where $t_s$ is the sampling time. Here the integration is based on the whole time interval. Direct solution requires a large amount of computer memory. In order to save computation time and computer memory, the recursive PID control method (Astrom 1984) was applied. Rewriting equation (3.2) at time k-1, yields
\[ u(k - 1) = K_p [e(k - 1) + \frac{t_i}{t} \sum_{i=0}^{k-1} e(i - 1) + \frac{t_o}{t_s} (e(k - 1) - e(k - 2))] \]  

(3.3)

Subtracting (3.3) from (3.2), yields

\[ u(k) - u(k - 1) = q_0 e(k) + q_1 e(k - 1) + q_2 e(k - 2) \]

(3.4)

where

\[ q_0 = K_p \left( 1 + \frac{t_o}{t_s} \right) \]

\[ q_1 = -K_p \left( 1 + 2 \frac{t_o}{t_s} - \frac{t_i}{t_f} \right) \]

\[ q_2 = K_p \frac{t_o}{t_s} \]

With equation (3.4), we can see that the control signal at time \( k \) is related to the control signal at time \( k-1 \) and the errors at time \( k \), \( k-1 \) and \( k-2 \). So we only need to save error values of several previous time steps, rather than all previous time steps. Also we do not need to integrate. Therefore, this technique saves computer memory and time.

### 3.1.1 PI Control of Pressure

Recall that, from the dynamic modeling process, we have developed a model of duct differential pressure in response to the fan speed control signal (Equation 3.5).

\[ P(k + 1) = 0.9856 P(k) + 0.696 u_{\text{fan}}(k) \]

(3.5)

It is a first order system. Using Ziegler Nichols method (Ziegler 1947), the sampling time \( t_s \), the integral time \( t_i \) and the system gain \( K_p \) were selected as follows,
Using the resulting PI control on the experimental apparatus, the data shown in Fig. 3.2 was generated. For the desired pressure is 0.3 inches of water, it takes about 30 seconds for the system to stabilize at this desired set point. Note that there is some overshoot. Therefore, the system will consume some energy unnecessarily. For the control of duct pressure, the damper position is assumed to be fixed, this is the same as assuming the cooling load to be constant. Since there is strong coupling between the duct pressure and the zone temperature, the duct pressure will be affected by damper position control and the zone temperature will be affected by the duct pressure as well.

![Graph of Measured Pressure vs. Desired Pressure](image)

**Fig. 3.2 PI Control of Pressure**
3.1.2 PI Control of Zone Temperature

In the Slater Wind Tunnel, the upstream temperature simulates zone temperature. As in modeling the supply air temperature, the model of zone temperature is estimated as a second order model:

\[ T_j(k+1) - 1.43T_j(k) + 0.26T_j(k-1) = -1.36u_{\text{damper}}(k) - 350u_{\text{damper}}(k-1) \]  

(3.6)

A PI control algorithm was applied to the zone temperature. Again, by Ziegler Nichols method, the sampling time \( t_s \), the integration time \( t_i \) and the system gain \( K_p \) were selected as follows,

\[
\begin{align*}
  t_s &= 3 \text{ sec} \\
  t_i &= 9 \text{ sec} \\
  K_p &= .0062
\end{align*}
\]

Fig. 3. 3 PI Control of Zone Temperature
The system response is shown in Fig. 3.3. with a set point of 28 degrees C. It takes about 3 minutes for the control system to follow the set point to within 2% error.

3.1.3 PI Control of Supply Air Temperature

The downstream temperature $T_2$ in the Slater Wind Tunnel simulates the supply air temperature. The model of supply air temperature is

$$T_2(k+1) = 0.8586T_2(k) + 32.96u_{valve}(k) + 4.396$$

As before, by Ziegler Nichols method, the sampling time $t_s$, the integral time $t_i$ and the system gain $K_p$ were selected as follows,

$$t_s = 3 \text{ sec}$$
$$t_i = 6 \text{ sec}$$
$$K_p = 0.0065$$

The system response is shown Fig. 3.4. with a set point of the supply air temperature $T_2$, i.e. the output air temperature of the heat exchanger is 20 degrees C. It takes about 5 minutes for the system to follow the set point within 2% error. It takes a longer time than the control of $T_1$. This is due to the control of $T_2$ is indirect. As mentioned before, $T_2$ is controlled by modulating the flow rate of the chilled water. It will take more time for the heat to transfer from air to the chilled water. This means that the time constant of supply air temperature is larger than the zone temperature. In fact the large time constant is due to the large thermal mass.
3.1.4 Simultaneous PI Control

In the preceding sections, the PI control algorithm was applied to each control loop. The operational conditions were assumed to be unchanged. But in actuality, all the control loops in a building must work simultaneously and in ensemble. Since there is strong coupling between each loop, there must be interactive effects among them. In order to observe the coupling among them, the three loops were first set to work simultaneously, but separately. The simultaneous PI controls are shown in Fig. 3.5. Recall that the set points of zone temperature, supply air temperature and duct pressure are 25 degrees C, 21 degrees C and 0.3 inches of water respectively.

\[
t_s = 2\, \text{sec} \\
K_{T_i} = 0.0062 \quad t_{T_i} = 9\, \text{sec}
\]
\[ K_{T_2} = 0.0065 \quad t_{1,T_2} = 12 \text{ sec} \]
\[ K_p = 0.012 \quad t_{1,p} = 8 \text{ sec} \]

Parameters with subscript \( T_1 \) are the parameters of the zone temperature. Parameters with subscript \( T_2 \) are the parameters of the supply air temperature. Parameters with subscript \( p \) are the parameters of the duct pressure.

From Fig. 3.5, we can see that there is a coupling between zone temperature and duct pressure. At the beginning, as determined by the load, the actual zone temperature is higher than the set point. The damper is initially fully open to allow as much conditioned air flow as possible into the zone to decrease the actual zone temperature. This means that the control signal is at its upper limit. When the temperature approaches the set point, the difference between the set point and the actual temperature is quite small, while the damper remains at a fixed position. If there is a disturbance to the temperature, the damper is modulated to drive the temperature back to the set point. For example, if the temperature is higher than the set point, the damper will open wider and this will affect the duct pressure and the duct pressure will decrease. Otherwise the duct pressure will increase. We can see this effect in the box portion of Fig. 3.5.

Just as we mentioned in the modeling procedure, the coupling is exacerbated by the wind tunnel model since we have a single damper control. Typical building systems will have multiple dampers or ATUs for a single fan loop. The coupling effects in the "real" systems are evident but at a reduced level. In "real" systems, the duct pressure control, the zone temperature control and the supply air temperature control operate simultaneously but separately. For this reason we did not apply the two input, two output coupled model for this study. But we can observe the coupling effects between the zone temperature and the duct pressure from the two input and two output model.
Fig. 3.5 Simultaneous PI Control
3.2 Adaptive Control of Duct Pressure

For a VAV system, the duct pressure is very important for maintaining a comfortable environment and assuring proper operation of the air terminal units. As we have seen there is coupling between duct pressure and zone temperature. Although duct pressure is controlled by the fan, it is affected by the zone temperature as well as the load. In fact, load is usually variable and unpredictable. Since the conventional PID controller is tuned for a specific operational condition, it cannot yield satisfactory control if the load change is too large. On the other hand, as we have found from the modeling process, the parameters of system are time varying, so it is difficult to tune a PID controller for a specific condition and have a system that works well when the load changes. Therefore, two different adaptive control schemes were developed to the control of duct pressure, namely the self tuning regulator (STR) (Astrom 1989) adaptive control and minimum variance (MV) (Clarke 1979) adaptive control.

3.2.1 STR Adaptive Control

The STR adaptive control is shown in Fig. 3.6. In this control system, parameter estimation is accomplished with an on-line recursive least square method. Estimates of the system parameters and the controller coefficients were adjusted to generate the appropriate control signal $u$ based on the previously selected design criteria. The successful process yields the desired output. The model of duct pressure is

$$A(k, z^{-1})P(k + 1) = B(k, z^{-1})u_{fan}(k) + \varepsilon(k)$$

(3.8)
where $P(k)$ is the pressure at time $k$, $u(k)$ is the control signal of the fan speed at time $k$ and $\varepsilon(k)$ is the zero mean white noise. Furthermore, the model of the duct pressure is assumed to be second order. $A(k,z^{-1})$ and $B(k,z^{-1})$ are polynomials of the shift operator $z^{-1}$ at time $k$.

\begin{align}
A(k,z^{-1}) &= 1 + a_1(k)z^{-1} + a_2(k)z^{-2} \\
B(k,z^{-1}) &= b_0(k) + b_1(k)z^{-1}
\end{align}  \tag{3.9}\label{eq:3.9}

The discrepancy between the pressure set point $P_m(k)$ and the actual pressure $P(k)$ is eliminated by the following regulation dynamics.

\begin{equation}
C_r(z^{-1})e^*(k) = 0 \tag{3.10}
\end{equation}

where

\begin{equation}
e^*(k) = P_m(k) - P(k) \tag{3.11}
\end{equation}
\( C_r(z^{-1}) \) is an asymptotically stable polynomial of shift factor \( z^{-1} \) which specifies the desired closed-loop roots for the controlled process. \( \theta(k) \) is the parameter vector to be estimated and is defined as:

\[
\theta(k) = [a_1(k) a_2(k) b_0(k) b_1(k)]^T
\]  

(3.12)

\( \phi(k) \) is the measurement vector and is defined as:

\[
\phi(k) = [-P(k) - P(k - 1)u(k)u(k - 1)]
\]  

(3.13)

By the recursive least square method, we have:

\[
\theta(k + 1) = \theta(k) + K(k) C_r(z^{-1})[P(k + 1) - P_m(k)]
\]  

(3.14.a)

\[
K(k) = F(k + 1)\phi^T(k + 1)[\gamma + \phi(k + 1)F(k)\phi^T(k + 1)]
\]  

(3.14.b)

\[
F(k + 1) = [I - K(k)\phi(k + 1)F(k)]/\gamma
\]  

(3.14.c)

where \( 0 < \gamma < 1 \), is the forget factor. The adaptive control algorithm is therefore

\[
u(k) = \frac{C_r(z^{-1})P_m(k + 1) - R(z^{-1})P(k) - b_1(k)u(k - 1)}{b_0(k)}
\]  

(3.15)

where

\[
R(z^{-1}) = (c_1 - a_1(k))z^{-1} + (c_2 - a_2(k))z^{-2}
\]  

(3.16)
The set point $P_m(k)$ is a constant value and $C_1(z^{-1})$ is assumed to be 1. In order to save computation time and memory, $b_0$ was selected from the open loop response of duct pressure. By experiment, $b_0$ and $\gamma$ were selected as follows

\[
\begin{align*}
    b_0 &= 88 \\
    \gamma &= 0.89
\end{align*}
\]

$\gamma$ is a weighting factor of the observation. If $\gamma=1$, all of the observation will have the same weights.

The results are shown in Fig. 3.7 to Fig. 3.10. For the damper position changes from 6.8% to 37%, the adaptive control scheme yields a satisfactory result. This is because the adaptive control can adapt its control parameters to fit different operational and load conditions.

Recall that, the Slater wind Tunnel was used to simulate a single duct, single zone VAV system. In order to have the desired zone temperature for the fixed supply air temperature and fixed duct pressure, the damper position was modulated differently for different loads. In other words, different damper positions in this system are equivalent to different loads. The results also imply that this adaptive control scheme works very well under different loads. Therefore, the adaptive control scheme automatically tunes the controllers for different situations. This is the advantage of adaptive control over conventional PID control. In order to compare the adaptive control with the conventional PID control, it was applied to the pressure control problem together with PI control of zone temperature and PI control of supply air temperature simultaneously. The results are shown in Fig. 3.11. We can see that, even if there is strong coupling between the duct
pressure and zone temperature, the STR adaptive control performs well. The duct pressure can be controlled at set point value.

In Fig. 3.11, the set point of zone temperature is 26 degrees C. The set point of supply air temperature is 19 degrees C. The set point of duct pressure is 0.3 inches of water. As we mentioned before, initially the damper is wide open to decrease the zone temperature. However, in this situation, even if the fan is at its highest speed, it cannot increase the duct pressure. When the zone temperature approaches the set point, the damper is relatively fixed at a certain position, such that an increase in the fan speed can increase the duct pressure. This provides an explanation for the flat line at the beginning of the pressure response.
Fig. 3.7 Adaptive Control of Pressure with 6.8% Open Damper

Fig. 3.8 Adaptive Control of Pressure with 17% Open Damper
Fig. 3.9 Adaptive Control of Pressure with 27% Open Damper

Fig. 3.10 Adaptive Control of Pressure with 37% Open Damper
Fig. 3.11 Adaptive Control and Simultaneous Control
3.2.2 Minimum Variance Control

From the preceding sections, we have seen that the STR adaptive control method performs well under different load conditions. If the difference between the actual output and the set point is very large, the calculated control signal $u$ will be very large and out of the range of hardware instrumentation. The standard control signal $u$ is typically either a 4-20 mA, or 0-10 V signal. The control actuator will be saturated. The minimum variance adaptive control technique was applied to minimize the weighted difference in the output of the system and the weighted control signal to the system. By selecting appropriate weighting factors, we can compromise the discrepancy of output and reduce the variation of the control signal to avoid actuator saturation.

If the system is modeled by:

$$A(z^{-1})y(k) = B(z^{-1})u(k - d) + C(z^{-1})e(k) \quad (3.17)$$

where $d$ is time delay, $e(k)$ is zero mean white noise and

$$A(z^{-1}) = 1 + a_1 z^{-1} + ... + a_n z^{-n}$$

$$B(z^{-1}) = b_0 + b_1 z^{-1} + ... + b_n z^{-n}$$

$$C(z^{-1}) = 1 + c_1 z^{-1} + ... + c_n z^{-n}$$

The cost function is

$$J = E\{ (P(z^{-1})y(k + d) - R(z^{-1})r(k + d))^2 + \mu(Q'(z^{-1})u(k))^2 \} \quad (3.18)$$
where \( r(k) \) is the set point of output and

\[
\mu = \frac{b_o}{q_o}
\]

\( P(z^{-1}) \), \( R(z^{-1}) \) and \( Q(z^{-1}) \) are polynomials of the backward shift factor \( z^{-1} \), yet to be determined. We can change them to add different weighting factors using the difference between the output and the control signal \( u(k) \). As we can see the power of the fan is a monotonic function of fan control signal \( u \). Therefore, minimizing the control signal to fan is equivalent to minimizing the power of fan. The minimum variance adaptive control method was applied to minimize the weighted difference between the output and the weighted power consumed by fan. Let

\[
\psi(k) = P(z^{-1})y(k)
\]  

(3.19)

Then

\[
\psi(k + d) = \frac{P(z^{-1})B(z^{-1})}{A(z^{-1})} u(k) + \frac{P(z^{-1})C(z^{-1})}{A(z^{-1})} e(k + d)
\]  

(3.20)

Let

\[
P(z^{-1})C(z^{-1}) = A(z^{-1})F(z^{-1}) + z^{-d}G(z^{-1})
\]

\[
F(z^{-1}) = 1 + f_1 z^{-1} + \ldots + f_{d-1} z^{-(d-1)}
\]

\[
G(z^{-1}) = g_0 + g_1 z^{-1} + \ldots + g_{n-1} z^{-(n-1)}
\]

Multiplying (3.17) by \( F(z^{-1}) \), we have
\[ F(z^{-1})A(z^{-1})y(k+d) = F(z^{-1})B(z^{-1})u(k) + F(z^{-1})C(z^{-1})\epsilon(k+d) \quad (3.21) \]

Rewriting (3.21) yields

\[ P(z^{-1})C(z^{-1})y(k+d) = G(z^{-1})y(k) + F(z^{-1})B(z^{-1})u(k) + F(z^{-1})C(z^{-1})\epsilon(k+d) \quad (3.22) \]

i.e.

\[ \psi(k+d) = \frac{1}{C(z^{-1})} [G(z^{-1})y(k) + F(z^{-1})B(z^{-1})u(k)] + F(z^{-1})\epsilon(k+d) \quad (3.23) \]

\( F(z^{-1})\epsilon(k+d) \) is independent of other terms, so the optimal prediction of \( \psi(k+d) \) is

\[ \psi^*(k+d/k) = \frac{G(z^{-1})}{C(z^{-1})} y(k) + \frac{B(z^{-1})}{C(z^{-1})} u(k) \quad (3.24) \]

and

\[ \psi(k+d) = \psi^*(k+d/k) + F(z^{-1})\epsilon(k+d) \quad (3.25) \]

Substituting (3.25) into the cost function yields:

\[ J = E\{[\psi^*(k+d/k) - R(z^{-1})r(k)]^2 + \mu[Q'(z^{-1})u(k)]^2\} + E[F(z^{-1})\epsilon(k+d)]^2 \quad (3.26) \]
Differentiating the cost function with respect to \( u(k) \), we will have

\[
\nu^*(k + d / k) - R(z^{-1})r(k) + Q^i(z^{-1})u(k) = 0
\]  

Equation (3.27) is the condition of minimizing the cost function. Since \( \nu^*(k + d / k) \) is not measured, we have to define an auxiliary output of the system to calculate the control algorithm and let

\[
\phi(k + d) = P(z^{-1})y(k + d) + Q(z^{-1})u(k) - R(z^{-1})v(k + d)
\]  

i.e.

\[
\phi(k + d) = \nu(k + d) - R(z^{-1})r(k) + Q^i(z^{-1})u(k)
\]  

Therefore

\[
\phi^*(k + d / k) = \nu^*(k + d / k) - R(z^{-1})r(k) + Q^i(z^{-1})u(k)
\]  

Based on (3.25), we have

\[
\phi(k + d) = \phi^*(k + d) + F(z^{-1})\varepsilon(k + d)
\]  

and an auxiliary cost function is defined as:
\[ J_\phi = E[\phi(k+d)]^2 \]  

Substituting (3.31) into (3.32) and manipulating the equation yields

\[ J_\phi = E[\phi^*(k+d/k)]^2 + E[F(z^{-1})\varepsilon(k+d)]^2 \]  

Differentiating (3.33) with respect to \( \phi^* \), we will have

\[ \psi^*(k+d/k) - R(z^{-1})r(k) + Q'(z^{-1})u(k) = 0 \]

So \( J \) is equivalent to \( J\phi \). In order to minimize the cost function \( J \), we only need to find the control algorithm to minimize the auxiliary cost function \( J_{\phi} \). From (3.28) to (3.30), we have

\[ \phi(k+d) = \psi^*(k+d/k) + Q'(z^{-1})u(k) - R(z^{-1})r(k+d) + F(z^{-1})\varepsilon(k+d) \]  

For \( C(z^{-1}) = 1 \), substituting (3.24) into (3.34) yields

\[ \phi(k+d) = G(z^{-1})y(k) + [Q'(z^{-1}) + F(z^{-1})B(z^{-1})]u(k) - R(z^{-1})r(k) + F(z^{-1})\varepsilon(k+d) \]  

\[ (3.35) \]
Let

\[ N(z^{-1}) = B(z^{-1})F(z^{-1}) + Q'(z^{-1}) \]

\[ \varphi(k) = [y(k)y(k-1),...,u(k)u(k-1),...,r(k)-r(k-1),...] \]

\[ \Theta(k) = [g_0,g_1,...,n_0,n_1,...,r_0,r_1,...]^\top \]

\[ \varepsilon_i(k+d) = F(z^{-1})\varepsilon(k+d) \]

Then

\[ \phi(k+d) = \varphi(k)\Theta(k) + \varepsilon_i(k+d) \quad (3.36) \]

By the recursive least square method, we estimate the parameters \( \Theta(k) \) and we get

\[ \Theta(k+1) = \Theta(k) + K(k)[\phi(k+1) - \varphi^\top(k)\Theta(k)] \quad (3.37.a) \]

\[ K(k) = P(k)\phi(k-d+1)[\gamma + \varphi^\top(k-d+1)P(k)\phi(k-d+1)]^{-1} \quad (3.37.b) \]

\[ P(k+1) = P(k)[I - K(k)\varphi^\top(k-d+1)]/\gamma \quad (3.37.c) \]

The \( u(k) \) can be calculated as

\[ u(k) = -\frac{1}{N(z^{-1})}(G(z^{-1})y(k) + R(z^{-1})r(k+d)) \quad (3.38) \]

The minimum variance adaptive control method was applied for a second model of the duct pressure
\[ A(z^{-1}) = 1 + a_1 z^{-1} + a_2 z^{-1} \]
\[ B(z^{-1}) = b_0 + b_1 z^{-1} \]
\[ C(z^{-1}) = 1 \]

where \( y(k) \) is the duct pressure at time \( k \). To simplify the problem, it is also assumed that \( d=1, \mu=1 \) and

\[ P(z^{-1}) = R(z^{-1}) = 1, \]
\[ Q'(z^{-1}) = q_0^2 \]

The resulting cost function is

\[ J = E[(y(k+d) - r(k+d))^2 + q_0 u_{fan}^2(k)] \]  \hspace{1cm} (3.39)

and

\[ 1 = 1 + a_1 z^{-1} + a_2 z^{-2} + g_1 z^{-1} + g_2 z^{-2} \]
\[ n_0 + n_1 z^{-1} = -a_1 - a_2 z^{-1} \]
\[ r_0 = 1 \]

The auxiliary output is given as:

\[ \phi^T(k) = [y(k) y(k-1) u_{fan}(k) u_{fan}(k-1) r(k+d)] \]
and the parameter vector is

$$\theta^T(k) = [g_0g_1n_0n_1]$$

Then, the control signal $u(k)$ is calculated as:

$$u_{fan}(k) = -\frac{1}{n_0}[g_0y(k) + g_1y(k-1) + n_1u_{fan}(k-1) + r(k+1)]$$

(3.40)

The results are shown in Fig. 3.16. The set point is 0.4 inches of water. It takes the system about 30 seconds to follow the set point with 2% error.
3.3 Optimal Control of Pressure

As identified from modeling process, the model of pressure is

\[ P(k+1) = aP(k) + bu_{\text{fan}}(k) \]  \hspace{1cm} (3.41)

where

\[ a=0.986 \]
\[ b=0.692 \]

The boundary conditions are:

\[ P(0)=P_0 \]
\[ P(N)=P_d \]

where \( N \) is the total sampling number and \( P_d \) is the pressure set point. From the energy performance analysis, we know that the power of fan is

\[ p_{\text{fan}} = c_1 u_{\text{fan}}^2(k) + c_2 u_{\text{fan}}(k) + c_3 \]  \hspace{1cm} (3.42)

where

\[ c_1=3041.16 \]
\[ c_2=-25.3615 \]
\[ c_3=0.562 \]

The cost function is the integration of the fan power function which is obtained from the energy analysis.

\[ J = \varphi(P(N),N) + \frac{1}{N} \sum_{k=0}^{N-1} (c_1 u_{\text{fan}}^2(k) + c_2 u_{\text{fan}}(k) + c_3) \]  \hspace{1cm} (3.43)

where \( \varphi(P(N),N)=P(N)-P_d \)
For the given system equation and the performance function, a Hamiltonian is defined as (Bryson 1975):

\[
H(k) = \frac{1}{N} (c_1 u_{fan}(k) + c_2 u_{fan}(k) + c_3) + \lambda(k + 1)(aP(k) + bu_{fan}(k)) + \nu \eta(P(N), N)
\]

(3.44)

The costate equations are:

\[
P(k + 1) = \frac{\partial H(k)}{\partial \lambda(k + 1)} = aP(k) + bu_{fan}(k)
\]

(3.45.a)

\[
\lambda(k) = \frac{\partial H(k)}{\partial P(k)} = \lambda(k + 1)a
\]

(3.45.b)

\[
\lambda(N) = \frac{\partial H(k)}{\partial P(N)} = \nu
\]

(3.45.c)

In order to have optimal control, we must solve the costate equations (3.45).

\[
\frac{\partial H(k)}{\partial u(k)} = \frac{2c_1}{N} u_{fan}(k) + \frac{c_2}{N} + \lambda(k + 1)b = 0
\]

(3.46)

\[
\lambda(k) = a\lambda(k + 1)
\]

(3.47)

\[
P(k + 1) = aP(k) + bu_{fan}(k)
\]

(3.48)

From equation (3.46), we have
\[ u_{\text{fan}}(k) = (-\lambda(k + 1)b + \frac{c_2}{N}) \cdot \frac{N}{2c_1} \]  

(3.49)

From (3.41), (3.46) and boundary conditions, we have:

\[ \lambda(k) = a\lambda(k + 1) \]  

(3.50.a)

\[ P(k + 1) = aP(k) + bu_{\text{fan}}(k) \]  

(3.50.b)

\[ \lambda(N) = \nu \]  

(3.50.c)

Solving the costate equations, we have

\[ \dot{\lambda}(k) = a^{N-k} \nu \]  

(3.51)

Substituting (3.49) into (3.50.b), we have

\[ P(1) = aP(0) - (\lambda(1)b^2 + \frac{bc_2}{N}) \cdot \frac{N}{2c_1} = aP_0 - a^{N-1} \nu b^2 \cdot \frac{N}{2c_1} - \frac{bc_2}{2c_1} \]  

(3.52)

and boundary condition:

\[ P(0) = P_0 \]

Solving equation (3.52), we will have
\[ P(1) = aP(0) - \left( \lambda(1)b^2 + \frac{bc_2}{N} \right) \frac{N}{2c_1} = aP_0 - a^{N-1}vb^2 \cdot \frac{N}{2c_1} - \frac{bc_2}{2c_1} \]  (3.53.1)

\[ P(2) = (a^2P_0 - \frac{abc_2}{2c_1} - \frac{bc_2}{2c_1}) - (a^N + a^{N-2}) \frac{b^2Nv}{2c_1} \]  (3.53.2)

\[ P_d = P(N) = a^N P_0 - \left( \sum_{k=0}^{N-1} \frac{bc_2}{2c_1} \right)a^k - \left( \sum_{k=0}^{N-1} a^{2(N-k-1)} \frac{b^2N}{2c_1} \right)v \]  (3.53.N)

Let

\[ \alpha_1 = \sum_{k=0}^{N-1} \frac{bc_2}{2c_1} a^k \]

\[ \alpha_2 = \sum_{k=0}^{N-1} a^{2(N-k-1)} \frac{b^2N}{2c_1} \]

Then we will have,

\[ P_d = a^N P_0 - \alpha_1 - \alpha_2 v \]  (3.54)

Solving for the constant \( v \), we have

\[ v = \frac{a^N P_0 - \alpha_1 - P_d}{\alpha_2} \]  (3.55)

and

\[ \lambda(k) = a^{N-k} v \]  (3.56)
Finally, the optimal control $u(k)$ is calculated as:

$$u(k) = (-\lambda(k + 1)b - \frac{c_2}{N} \frac{N}{2c_1})$$

(3.57)

The results are shown in Fig. 3.13. Where $N=140$, the set point is 0.4 inches of water. Note that the pressure is increased slowly to save energy. There is no overshoot. Compared with PI control and adaptive control, the optimal control requires less energy than the other two control schemes. The fan operates only when it is necessary to run to increase the duct pressure.
3.4 PID and Optimal Control

From the preceding section, I have shown that the design of optimal control algorithm requires an accurate model of the system. If there is much difference between the model and the actual dynamic system or if there is some disturbance to system, the system stability may be compromised. The optimal control is an open loop controller and the stability is not guaranteed. In order to solve these problems, the conventional PI control and the optimal control were combined and applied to the duct pressure control. The combination of the PI and optimal control system is shown in Fig. 3.14. At first, the optimal control is applied to the system. If the difference between the output and the set point is small enough, the optimal control is disconnected and the conventional PI control is initiated. Therefore, the transient process is controlled by optimal control and the steady state is controlled by PI control. By such a hybrid control scheme, the user can take advantage of the benefits of both optimal and PI control while avoiding some of the disadvantages. The results are shown in Fig. 3.15.

![Fig. 3.14 PID and Optimal Control](image-url)

**Fig. 3.14 PID and Optimal Control**
Fig. 3.15 PI Control and Optimal Control of Pressure
CHAPTER 4

RESULTS AND DISCUSSION

PID control is widely used in HVAC systems. All PID controllers must be tuned or adjusted to accommodate the specific characteristics of the process that they control (Coffin 1991). There are several approaches to achieve the tuning task. Some of the techniques are mathematical. However, the most widely used approach combines both experience and trial-and-error on the part of the building maintenance worker responsible for building environmental control. The performance of a PID control system depends strongly on its tuning. Tuning the PID controller is time-consuming and requires information about the system and experience.

4.1 Tuning of PID Controllers

In HVAC systems, the specific responsibility of tuning the existing PID controllers lies with the operator or building maintenance manager. A properly tuned controller will have the expected performance for the specific operational condition. In HVAC systems, the tuning of PID controllers occurs in two different approaches, Ziegler Nichols method and an approach that I will call empirical method.

4.1.1 Ziegler-Nichols Method

The Ziegler-Nichols method is based on the observation that the controller parameters can be determined from the open loop transient response of system. For example, a typical open loop transient response is shown in Fig. 4.1. The x axis is time and the y axis is temperature or pressure. Where L is the time delay and R is the
maximum ratio of amplitude to time. The PID controller parameters can be determined as shown in Table 4.1.

Fig. 4.1 Ziegler Nichols Method

Table 4.1 Ziegler Nichols Method

<table>
<thead>
<tr>
<th></th>
<th>$K_D$</th>
<th>$t_I$</th>
<th>$t_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>$1/RL$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PI</td>
<td>$0.9/RL$</td>
<td>$3L$</td>
<td></td>
</tr>
<tr>
<td>PID</td>
<td>$1.2/RL$</td>
<td>$2L$</td>
<td>$0.5L$</td>
</tr>
</tbody>
</table>
The sampling time \( t_s \) is selected

\[
\frac{t_s}{L} = 0.2 - 1.0
\]  

(4.1)

4.1.2 Empirical Methods

In order to maintain a fast response as well as a small steady state error, the PID controller is tuned to have only one over-shoot or no over-shoot. This is the method presently implemented by building maintenance workers (Ferrell 1993).

After the controller is tuned for the specific conditions, the control function is fixed. Even a PID controller that is well tuned at its commission may perform poorly as the operating conditions change due to seasonal and other load variation. Such situations frequently occur in building systems. Engineers have tried different ways to handle this problem. Some have used an auto tuning method to the PID controller (Polonyi 1991; Wallenborg 1989). This approach was motivated by a desire to develop a simple and robust tuning scheme which requires very little prior system information. It is also based on a special technique for system identification which automatically generates an appropriate test signal and a variation of the classical Ziegler-Nichols method for control design. The auto tuner is based on the idea that the system gain and frequency can be determined by introducing a relay feedback. By observing the zero-crossing points and the peak to peak values, the periodic value and the amplitude value can be determined and then control parameters can be determined. If the operating condition changes, a new tuning is performed on the demand from operator. The resulting parameters are stored in a table together with the variable which characterizes the operating condition. New
tuning is then required only when the conditions change. A system of this type is semi-
automatic because the decision to retune rests with the operator.

Since the conventional PID control has a fixed control function for a given control
system and tuning of a controller is time-consuming, researchers have developed adaptive
control to handle unknown model parameters and changing operating conditions. There
are two parts in a conventional adaptive control, system identification and controller
design. If we assume the structure of system, system identification becomes parameter
estimation. Parameter estimation is an on-line system identification technique used to
estimate unknown parameters of a system. After the parameters of system have been
found, the controller parameters can be determined based on prior information.

Here STR adaptive control and minimum variance adaptive control were applied to
duct pressure control. The STR adaptive control works very well for different operating
conditions. The minimum variance adaptive control is to minimize the expectancy of
weighted difference of the system output and control signals. This compromises the
increased deviation of output and reduces the control variations to avoid actuator
saturation. Since the designers of the adaptive control have only consider an objective of
control precision and its affects on control actuator saturation, the energy consumption is
not minimized.

4.2 Comparison of Different Control Algorithm

The study of energy optimization in HVAC systems has received wide spread
Optimization typically involves modeling a process to have a dynamic and energy
performance model. Using a optimal control scheme to find the optimal control signal,
the cost function is minimized. For instance, the cost function is given as the power consumed by a control system. Optimal energy control results in minimum energy consumption. Since optimal control is an open loop control, relatively large parameter differences in the model or disturbances may render the system unstable. Therefore, in order to apply optimal control to actual situation, it is prudent to combine both optimal control and conventional PID control methods. For this study this was done to duct pressure.

The results were compared based on the power consumed by different control schemes. Here different methods were applied to tune a PID controller and the results are shown in Fig. 4.1. The two over shoot controller stands for the controller designed by Ziegler-Nichols method.

The total energy consumed by different controllers were obtained by integrating power over time. The sampling time is 1.5 second. The energy comparison is shown in Table 4.2.

<table>
<thead>
<tr>
<th></th>
<th>Adaptive</th>
<th>2 shoot</th>
<th>1 Shoot</th>
<th>0 Shoot</th>
<th>Optimal</th>
</tr>
</thead>
</table>

From Table 4.2, we can see that by applying optimal control, up to 30% of the energy can be saved. Since design of adaptive controller is based on the difference of the output of and control signal for system. The energy consumed is not considered in the design of adaptive control system. The energy consumed by the adaptive control is as
much as the PID controller. By tuning a controller to have no over-shoot, the energy consumption can be less than that for one over-shoot or two over-shoots. Although less energy is consumed by a system, more time needed to get the set point. Therefore a trade-off exists between the time (possibly human comfort) and energy.
Fig. 4.2. Comparison of Responses of Controllers
Fig 4.3. Power Consumed by Different Controllers
CHAPTER 5
CONCLUSION

In this thesis, by applying the least square method, I have developed various dynamic models for an HVAC system. The dynamic models include duct pressure related to fan speed control signal, the zone temperature related to the damper position control signal and the supply air temperature related to the chilled water valve control signal. Since there is strong coupling between duct pressure and zone temperature, I have also modeled the duct pressure and zone temperature related them to the control signals to the fan speed and damper position as a two input, two output system. For actual applications, all the modeling was done on-line with the Slater Wind Tunnel which simulates a single duct single zone VAV system.

Secondly, I have also performed the analysis of energy performance of supply fan which reveals that the energy performance of fan can be described as a linear quadratic function of the fan speed control signal.

By dynamic modeling the system and energy performance analysis, I have developed PI control schemes for duct pressure zone temperature and supply air temperature within the single VAV system both separately and simultaneously. I have found that if it is well tuned, the conventional PI control system works very well for specific operational conditions. But if the operational conditions are different from the conditions for which it tuned, the system may not have the desired performance. There is strong coupling between duct pressure and zone temperature. Although the duct pressure is controlled by the fan speed, it is affected by the damper position as well as the load. In order to control the duct pressure for different damper positions, i.e. different loads, I
have applied self tuning regulator adaptive control scheme to duct pressure control. The results show that the adaptive control scheme works well for different operational conditions. Since the actuator has its own limited range on the control signal, I have also applied the minimum variance adaptive control method to the duct pressure control to compromise the large discrepancy between the output and the set point and large variation of the control signal to avoid actuator saturation.

Finally, by using the dynamic models and the energy performance function, an optimal control scheme for the duct pressure control was found. I compared the energy consumed by different control schemes and found that the optimal control scheme can save up to 15-30 percent of the energy consumed by the well-tuned conventional PI controller with a 150 second transient process.
REFERENCES

8. Ferrell, Thomas 1993 Personal Communication with the Author.


Regressive Method

Regressive method is used to find the function between two sets of data. If the structure of function is assumed, the parameters can be determined based on the minimum square error between the estimated value and the actual value. Let $X$ and $Y$ be arbitrary variables and

$$X = \{x_1, x_2, \ldots, x_n\}$$

$$Y = \{y_1, y_2, \ldots, y_n\}$$

In order to find the relationship between the variables, we assume the relationship is linear.

$$y_i = ax_i + b \quad (i = 1, 2, \ldots, n)$$

Let the criterion function be

$$Q^* = \sum_{i=1}^{n} \left[ y_i - (ax_i + b) \right]^2$$

So the goal is to find parameters $a$ and $b$ which have the minimum criterion function $Q^*$. Differentiating $Q^*$ with respect to $a$ and $b$ yields:
\[ \frac{\partial Q^*}{\partial a} = 0 \]
\[ \frac{\partial Q^*}{\partial b} = 0 \]

Substituting \( x \) and \( y \), \( a \) and \( b \) will be solved as:

\[ a = \frac{S_{xy}}{S_{xx}} \]
\[ b = \bar{y} - a\bar{x} \]

where

\[ \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \]
\[ \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \]
\[ S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2 \]
\[ S_{xy} = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \]

Furthermore, if we assume the relationship between the two sets of data is a quadratic form, we let

\[ y_i = ax_i^2 + bx_i + c \]

The criterion function is defined as:
\[
Q^* = \sum_{i=1}^{n} \left[ y_i - \left( ax_i^2 + bx_i + c \right) \right]^2
\]

Differentiating \( Q^* \) with respect to \( a \), \( b \) and \( c \) yields:

\[
\frac{\partial Q^*}{\partial a} = 0
\]
\[
\frac{\partial Q^*}{\partial b} = 0
\]
\[
\frac{\partial Q^*}{\partial c} = 0
\]

We will have the following equations:

\[
\left( \sum_{i=1}^{n} y_i x_i \right) - \left( \sum_{i=1}^{n} x_i^3 \right) a - \left( \sum_{i=1}^{n} x_i^2 \right) b - \left( \sum_{i=1}^{n} x_i \right) c = 0
\]

\[
\left( \sum_{i=1}^{n} y_i \right) - \left( \sum_{i=1}^{n} x_i^2 \right) a - \left( \sum_{i=1}^{n} x_i \right) b - \left( \sum_{i=1}^{n} 1 \right) c = 0
\]

\[
\left( \sum_{i=1}^{n} y_i \right) - \left( \sum_{i=1}^{n} x_i \right) a - \left( \sum_{i=1}^{n} x_i \right) b - \left( \sum_{i=1}^{n} 1 \right) c = 0
\]

Substituting values of \( x_i \) and \( y_i \), the above equations can be solved for parameters of \( a \), \( b \) and \( c \).