A FOURTH-ORDER FINITE DIFFERENCE SCHEME FOR POISSON'S EQUATION IN POLAR COORDINATES ON THE UNIT DISC

by

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ABSTRACT

We solve Poisson's Equation on the unit disc in polar coordinates using a fourth-order finite difference method. We use a half-point shift in the r direction, in order to avoid approximating the solution at $r = 0$. We derive a new fourth-order accurate finite difference method from analysis of the truncation error of the well-known second-order scheme. The resulting linear system is solved very efficiently (with cost almost proportional to the number of unknowns) using a combination of a Matrix Diagonalization Algorithm and Fast Fourier Transforms.

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1 INTRODUCTION

We consider the non-homogeneous Poisson's Equation with Dirichlet Boundary Conditions on the unit disc in Cartesian Coordinates. Assume

$$\tilde{\Omega} = \{(x, y) \in \Re^2 \mid x^2 + y^2 < 1\}.$$

We consider the two-dimensional Poisson's Equation

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} = \tilde{f}(x, y), \quad (x, y) \in \tilde{\Omega}, \tag{1.1}$$

with the Dirichlet Boundary Condition

$$\tilde{u}(x, y) = \tilde{g}(x, y), \quad (x, y) \in \partial\tilde{\Omega}.$$

An application of this problem involves finding the temperature distribution on a disc at equilibrium (since it is time-independent), where the source of heat and the temperature on the boundary of the disc are known [7].

We use the polar transformation

$$x = r\cos\theta, \quad y = r\sin\theta$$

to convert the problem into the more natural polar coordinates. So now we take

$$\Omega = \{(r, \theta) \in \Re^2 \mid 0 < r < 1, \quad 0 \leq \theta < 2\pi\}. \tag{1.2}$$

Using the polar coordinate transformation on (1.1), we have

$$u(r, \theta) = \tilde{u}(r\cos\theta, r\sin\theta), \quad (r, \theta) \in \Omega, \tag{1.3}$$

and we get the divergence form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = f(r, \theta), \quad (r, \theta) \in \Omega, \tag{1.4}$$

subject to the periodic boundary conditions on θ

$$u(r, 0) = u(r, 2\pi), \quad u_\theta(r, 0) = u_\theta(r, 2\pi), \quad r \in (0, 1), \tag{1.5}$$

and the Dirichlet Boundary Condition on r

$$u(1, \theta) = g(\theta), \quad \theta \in [0, 2\pi). \tag{1.6}$$

The functions f in (1.4) and g in (1.6) are given by

$$\tilde{f}(\cos\theta, \sin\theta) = f(r, \theta), \quad \tilde{g}(\cos\theta, \sin\theta) = g(r, \theta).$$

Now we develop a fourth-order convergent finite difference scheme for the solution of (1.4)–(1.6). Fourth-order accurate finite difference schemes for this problem have been previously proposed in [2, 8, 9, 10, 11, 12]. The singularity that occurs at $r = 0$ when converting from Cartesian into polar coordinates is handled in a couple of standard ways. Some authors handle the singularity by assuming that the solution at $r = 0$ is known [12] or by imposing and discretizing (with fourth-order accuracy) boundary conditions at $r = 0$ [8, 9]. Other authors avoid the singularity by using a grid with a half-point shift in the r direction [2, 10, 11]. Britt, Tsynkov, and Turkel [2] solve the problem using a compact (nine-point stencil) scheme with the shift, but only on an annulus with a hole at $r = 0$. Lai and Wang [11], using truncated Fourier Series expansions, symmetry constraints, and a half-point shift, develop a non-compact scheme resulting in solving penta-diagonal linear systems in the r direction. Our non-compact approach leading to penta-diagonal linear systems in the r direction is more simple and direct because it is derived from analysis of the truncation error of the second-order scheme. Lai and Wang [11] use complex arithmetic throughout their process, whereas our approach involves only real-valued vectors and matrices. In [10], Lai develops a compact scheme with the half-point shift which is fourth-order accurate on an annulus, but admittedly only third-order accurate on a solid disc. Our approach, based on using a half-point shift in the r direction, is fourth-order accurate on a solid disc and does not involve any artificial conditions at $r = 0$. Our scheme is also very efficient, with the cost of finding its solution being $\mathcal{O}(NM \log N)$ on an $M \times N$ partition, which matches the efficiency of the schemes in [2, 11].

In the remainder of this thesis, we develop a fourth-order, simple, efficient finite difference scheme with a half-point shift in the r direction for solving Poisson's Equation on a disk in polar coordinates. Section 2 restates the well-known second-order finite difference scheme with a half-point shift, as in [13]. (The half-point shift was originally proposed for a different problem in [6].) In Section 2, we also solve the second-order scheme using the Matrix Diagonalization Algorithm with Fast Fourier Transforms, discuss the efficiency of this algorithm, and present numerical results confirming second-order accuracy. In Section 3, we analyze the truncation error of the second-order scheme and introduce finite difference operators for approximating the third- and fourth-order derivatives. Following the same pattern as in Section 2, we define the fourth-order scheme, use the Matrix Diagonalization Algorithm with Fast Fourier Transforms for computing its solution, discuss the cost of this algorithm, and present numerical results for the fourth-order scheme in Section 4. We make concluding remarks in Section 5, and in the appendix, Section 6, we prove the accuracies of the finite difference operators for approximating the third- and fourth-order derivatives. In Section 6 we also give more detail on Matlab's Fast Fourier Transforms, which are used to efficiently perform multiplication of any real vector by a real matrix with particular sinusoidal entries.

2 SECOND-ORDER SCHEME

We use a finite difference (FD) scheme to approximate the problem (1.4)–(1.6). To this end, we consider $M+1$ subintervals in r and N subintervals in θ, corresponding to the domain (1.2). With

$$h_r = \frac{1}{M+\frac{1}{2}}, \quad h_\theta = \frac{2\pi}{N}, \tag{2.1}$$

we use a half-point shift in r, so that

$$r_i = \left(i - \frac{1}{2}\right) h_r, \quad i = 1, \dots, M+1, \tag{2.2}$$

and as usual

$$\theta_j = j h_\theta, \quad j = 0, \dots, N.$$

For $i = 1, \dots, M+1$ and $j = 1, \dots, N$, we use $U_{i,j}$ to denote the approximate solution at (r_i, θ_j).

We define the operator L_r for any discrete function $v_{i,j}$ and for $j = 1, \dots, N$ by

$$L_r v_{i,j} = \begin{cases} \dfrac{1}{r_1 h_r}\left(r_1 + \dfrac{h_r}{2}\right)\dfrac{v_{2,j} - v_{1,j}}{h_r}, & i = 1, \\ \dfrac{1}{r_i h_r}\left[\left(r_i + \dfrac{h_r}{2}\right)\dfrac{v_{i+1,j} - v_{i,j}}{h_r} - \left(r_i - \dfrac{h_r}{2}\right)\dfrac{v_{i,j} - v_{i-1,j}}{h_r}\right], & i = 2, \dots, M. \end{cases} \tag{2.3}$$

Since $r_1 = \frac{h_r}{2}$ from (2.2), we can simplify the first part of (2.3) to

$$L_r v_{1,j} = \frac{1}{r_1}\frac{v_{2,j} - v_{1,j}}{h_r}, \quad i = 1, \quad j = 1, \dots, N. \tag{2.4}$$

We also define the operator L_θ for any discrete function $v_{i,j}$ by

$$L_\theta v_{i,j} = \frac{v_{i,j-1} - 2v_{i,j} + v_{i,j+1}}{h_\theta^2}, \quad i = 1, \dots, M, \quad j = 1, \dots, N, \tag{2.5}$$

with the convention that $v_{i,0}$ (occurring when $j = 1$) and $v_{i,N+1}$ (occurring when $j = N$) are replaced by $v_{i,N}$ and $v_{i,1}$ respectively, due to (1.5).

Then the standard second-order FD scheme for (1.4)–(1.6) is defined by [13] as

$$L_r U_{i,j} + \frac{1}{r_i^2} L_\theta U_{i,j} = f_{i,j}, \quad i = 1, \dots, M, \quad j = 1, \dots, N, \tag{2.6}$$

where the right-hand side is $f_{i,j} = f(r_i, \theta_j)$, for $i = 1, \dots, M$ and $j = 1, \dots, N$. Due to the known values from the Dirichlet Boundary Conditions at $r = 1$ in (1.6), we get

$$U_{M+1,j} = g(\theta_j), \quad j = 1, \dots, N. \tag{2.7}$$

2.1 Matrix-Vector Form of the Second-Order Scheme

We look for the values $U_{i,j}$ for $i = 1, \dots, M$ and $j = 1, \dots, N$. Therefore our number of unknowns, which is also our number of linear equations, is MN. So for $i = 2, \dots, M-1$ and for $j = 1, \dots, N$, using (2.6), (2.3), and (2.5), we get

$$\frac{1}{r_i h_r}\left[\left(r_i + \frac{h_r}{2}\right)\frac{U_{i+1,j} - U_{i,j}}{h_r} - \left(r_i - \frac{h_r}{2}\right)\frac{U_{i,j} - U_{i-1,j}}{h_r}\right] + \frac{1}{r_i^2}\frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{h_\theta^2} = f_{i,j},$$

which simplifies to

$$\mu_i U_{i-1,j} + \gamma_i \left[U_{i,j-1} - 2U_{i,j} + U_{i,j+1}\right] + \delta U_{i,j} + \nu_i U_{i+1,j} = f_{i,j}, \tag{2.8}$$

where

$$\delta = -\frac{2}{h_r^2}, \tag{2.9}$$

and for $i = 2, \dots, M-1$,

$$\mu_i = \frac{2r_i - h_r}{2r_i h_r^2}, \quad \gamma_i = \frac{1}{r_i^2 h_\theta^2}, \quad \nu_i = \frac{2r_i + h_r}{2r_i h_r^2}. \tag{2.10}$$

When $i = 1$ and $j = 1, \dots, N$, from (2.6), (2.4), (2.5), and the fact that $r_1 = \frac{h_r}{2}$, we have

$$\gamma_1 \left[U_{1,j-1} - 2U_{1,j} + U_{1,j+1}\right] + \delta U_{1,j} + \nu_1 U_{2,j} = f_{1,j}, \tag{2.11}$$

where δ is defined as in (2.9) and

$$\gamma_1 = \frac{1}{r_1^2 h_\theta^2}, \quad \nu_1 = \frac{2}{h_r^2}. \tag{2.12}$$

And when $i = M$ and $j = 1, \dots, N$, using (2.6), (2.7), (2.3), and (2.5), we have

$$\mu_M U_{M-1,j} + \gamma_M \left[U_{M,j-1} - 2U_{M,j} + U_{M,j+1}\right] + \delta U_{M,j} = f_{M,j} - \nu_M g(\theta_j), \tag{2.13}$$

where δ is defined as in (2.9) and

$$\mu_M = \frac{2r_M - h_r}{2r_M h_r^2}, \quad \gamma_M = \frac{1}{r_M^2 h_\theta^2}, \quad \nu_M = \frac{2r_M + h_r}{2r_M h_r^2}. \tag{2.14}$$

As in (2.5), for (2.8), (2.11), and (2.13), for $i = 2, \ldots, M-1$, $i = 1$, and $i = M$ respectively, we assume replacements of $U_{i,0}$ by $U_{i,N}$ and of $U_{i,N+1}$ by $U_{i,1}$. We define sub-vectors of our unknowns as

$$\mathbf{U}_i = [U_{i,1} \quad U_{i,2} \quad \ldots \quad U_{i,N}]^T, \quad i = 1, \ldots, M, \tag{2.15}$$

and our right-hand side vector as

$$\mathbf{F}_i = [f_{i,1} \quad f_{i,2} \quad \ldots \quad f_{i,N}]^T, \quad i = 1, \ldots, M. \tag{2.16}$$

We also introduce the $N \times N$ matrix

$$P = \begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{pmatrix} \tag{2.17}$$

which has ones in the corners because of the replacement of $U_{i,0}$ and $U_{i,N+1}$ by $U_{i,N}$ and $U_{i,1}$, respectively in (2.8), (2.11), and (2.13).

Using (2.8) and the coefficients from (2.10), we have

$$\mu_i \mathbf{U}_{i-1} + [\gamma_i P + \delta I]\mathbf{U}_i + \nu_i \mathbf{U}_{i+1} = \mathbf{F}_i, \quad i = 2, \ldots, M-1, \tag{2.18}$$

where here and in what follows, I is the $N \times N$ identity matrix.

For $i = 1$, from (2.11) with coefficients defined in (2.12) and δ defined in (2.9), we have

$$[\gamma_1 P + \delta I]\mathbf{U}_1 + \nu_1 \mathbf{U}_2 = \mathbf{F}_1. \tag{2.19}$$

For $i = M$, from (2.13) with coefficients defined in (2.14) and δ defined in (2.9), we have

$$\mu_M \mathbf{U}_{M-1} + [\gamma_M P + \delta I]\mathbf{U}_M = \mathbf{F}_M - \nu_M \mathbf{G}, \tag{2.20}$$

where

$$\mathbf{G} = [g(\theta_1) \quad g(\theta_2) \quad \ldots \quad g(\theta_N)]^T. \tag{2.21}$$

2.2 Matrix Diagonalization Algorithm

We begin with a known, real $N \times N$ matrix Q such that

$$QQ^T = I, \tag{2.22}$$

and

$$Q^T P Q = \Lambda, \tag{2.23}$$

where Λ is a known diagonal matrix.

The matrices Q and Λ are derived using the eigenvalues and eigenvectors of P, from section 1.5.4 of [14] and as used in [1]. They are defined as

$$\Lambda = \operatorname{diag}(\lambda_j)_{j=1}^N, \quad \lambda_j = -4\sin^2\frac{(j-1)\pi}{N},$$

and

$$Q = \left(\frac{2}{N}\right)^{1/2} [q_{i,j}]_{i,j=1}^N,$$

where for N odd,

$$q_{i,1} = \frac{1}{\sqrt{2}}, \quad q_{i,j} = \cos\frac{2(i-1)(j-1)\pi}{N}, \quad q_{i,N+2-j} = \sin\frac{2(i-1)(j-1)\pi}{N},$$

$$i = 1, \ldots, N, \quad j = 2, \ldots, \frac{N+1}{2},$$

and for N even,

$$\begin{gathered} q_{i,1} = \frac{1}{\sqrt{2}}, \quad q_{i,(N+2)/2} = \frac{1}{\sqrt{2}}(-1)^{i-1}, \\ q_{i,j} = \cos\frac{2(i-1)(j-1)\pi}{N}, \quad q_{i,N+2-j} = \sin\frac{2(i-1)(j-1)\pi}{N}, \\ i = 1, \ldots, N, \quad j = 2, \ldots, \frac{N}{2}. \end{gathered} \tag{2.24}$$

We multiply (2.18) on the left by Q^T and use (2.22) to obtain

$$\mu_i Q^T \mathbf{U}_{i-1} + [\gamma_i Q^T P Q Q^T + \delta Q^T]\mathbf{U}_i + \nu_i Q^T \mathbf{U}_{i+1} = Q^T \mathbf{F}_i, \quad i = 2, \ldots, M-1.$$

Next, we introduce

$$\begin{aligned} \mathbf{V}_i &= [V_{i,1} \quad V_{i,2} \quad \ldots \quad V_{i,N}]^T = Q^T \mathbf{U}_i, \quad i = 1, \ldots, M, \\ \mathbf{E}_i &= [E_{i,1} \quad E_{i,2} \quad \ldots \quad E_{i,N}]^T = Q^T \mathbf{F}_i, \quad i = 1, \ldots, M-1. \end{aligned} \tag{2.25}$$

We use (2.23) to arrive at

$$\mu_i \mathbf{V}_{i-1} + [\gamma_i \Lambda + \delta I]\mathbf{V}_i + \nu_i \mathbf{V}_{i+1} = \mathbf{E}_i, \quad i = 2, \ldots, M-1. \tag{2.26}$$

We take a similar approach with (2.19) and (2.20) to get

$$[\gamma_1 \Lambda + \delta I]\mathbf{V}_1 + \nu_1 \mathbf{V}_2 = \mathbf{E}_1, \tag{2.27}$$

and

$$\mu_M \mathbf{V}_{M-1} + [\gamma_M \Lambda + \delta I]\mathbf{V}_M = \mathbf{E}_M, \tag{2.28}$$

where $\mathbf{E}_M = Q^T(\mathbf{F}_M - \nu_M \mathbf{G})$.

Next, for fixed $j = 1, \ldots, N$, we create new vectors

$$\begin{aligned} \tilde{\mathbf{V}}_j &= \begin{bmatrix} V_{1,j} & V_{2,j} & \ldots & V_{M,j} \end{bmatrix}^T, \\ \tilde{\mathbf{E}}_j &= \begin{bmatrix} E_{1,j} & E_{2,j} & \ldots & E_{M,j} \end{bmatrix}^T. \end{aligned} \tag{2.29}$$

Now for fixed $j = 1, \ldots, N$, we take equation j from each of (2.26), (2.27), and (2.28), and we use the fact that Λ is diagonal to form a collection of N independent linear $M \times M$ systems

$$R_j \tilde{\mathbf{V}}_j = \tilde{\mathbf{E}}_j, \quad j = 1, \ldots, N,$$

where we introduce the tri-diagonal $M \times M$ matrix

$$R_j = \begin{pmatrix} \gamma_1\lambda_j + \delta & \nu_1 & & & & \\ \mu_2 & \gamma_2\lambda_j + \delta & \nu_2 & & & \\ \ddots & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{M-1} & \gamma_{M-1}\lambda_j + \delta & \nu_{M-1} \\ & & & & \mu_M & \gamma_M\lambda_j + \delta \end{pmatrix}.$$

We can reconstruct the vectors $\mathbf{V}_i$ from the elements of the vectors $\tilde{\mathbf{V}}_j$. Then we find the vectors of our approximate solutions by using the property in (2.22) applied to (2.25) to compute $\mathbf{U}_i = Q\mathbf{V}_i$ for $i = 1, \ldots, M$.

So the basic steps of our algorithm are:

1. Perform M left-multiplications by Q^T to find the vectors $\mathbf{E}_i = Q^T\mathbf{F}_i$ for $i = 1, \ldots, M$.

2. Solve the collection of tri-diagonal systems $R_j\tilde{\mathbf{V}}_j = \tilde{\mathbf{E}}_j$ for $j = 1, \ldots, N$.

3. Perform M left-multiplications by Q to find the vectors $\mathbf{U}_i = Q\mathbf{V}_i$ for $i = 1, \ldots, M$.

2.3 Cost Analysis

The cost of an algorithm is defined as the number of required arithmetic operations. The cost of computing the entries of the M vectors $\mathbf{F}_i$ depends on the function f, but will be around $\mathcal{O}(MN)$ since each vector $\mathbf{F}_i$ has N entries. Also, the cost of computing the entries of the N tri-diagonal $M \times M$ matrices R_j will be $\mathcal{O}(MN)$. We use the Matlab function "fft" to perform the multiplications by Q and Q^T. See Section 6.3 of the appendix for details. So the cost of steps (1) and (3) in our algorithm is $\mathcal{O}(MN \log N)$, as stated in [4]. Also, since the systems in step (2) are tri-diagonal, the cost of solving the collection of N systems in step (2) is $\mathcal{O}(MN)$, as stated in [3]. Hence the overall cost of solving the problem is $\mathcal{O}(MN \log N)$.

2.4 Numerical Examples

We test the second-order scheme in Matlab on four test functions for which we know the exact solutions. We directly compute f and g for each of our known test functions to obtain numerical results. The test functions we use are:

$$u_1 = e^{r(\cos\theta + \sin\theta)},$$

$$u_2 = e^r,$$

$$u_3 = r \cdot z_1\left(\frac{\theta}{2\pi}\right),$$

$$u_4 = r \cdot z_2\left(\frac{\theta}{2\pi}\right),$$

where

$$z_1(x) = x - 3x^2 + 2x^3,$$

$$z_2(x) = 8 + 2x^2 - 4x^3 + 2x^4.$$

Then $z_1(x)$ is such that

$$z_1'(x) = 1 - 6x + 6x^2,$$

and

$$z_1''(x) = -6 + 12x.$$

Hence

$$z_1(0) = z_1(1) = 0, \quad z_1'(0) = z_1'(1) = 1, \quad z_1''(0) = -6 \neq 6 = z_1''(1).$$

Therefore, for $r \in (0, 1)$

$$u_3(r, 0) = u_3(r, 2\pi),$$

$$\frac{\partial u_3}{\partial \theta}(r, 0) = \frac{\partial u_3}{\partial \theta}(r, 2\pi),$$

but

$$\frac{\partial^2 u_3}{\partial \theta^2}(r, 0) \neq \frac{\partial^2 u_3}{\partial \theta^2}(r, 2\pi).$$

Similarly, $z_2(x)$ is such that

$$z_2'(x) = 4x - 12x^2 + 8x^3,$$

and

$$z_2''(x) = 4 - 24x + 24x^2,$$

and

$$z_2'''(x) = -24 + 48x.$$

Hence

$$z_2(0) = z_2(1) = 8, \quad z_2'(0) = z_2'(1) = 0, \quad z_2''(0) = z_2''(1) = 4, \quad z_2'''(0) = -24 \neq 24 = z_2'''(1).$$

Therefore, for $r \in (0, 1)$

$$\begin{aligned} u_4(r,0) &= u_4(r,2\pi), \\ \frac{\partial u_4}{\partial \theta}(r,0) &= \frac{\partial u_4}{\partial \theta}(r,2\pi), \\ \frac{\partial^2 u_4}{\partial \theta^2}(r,0) &= \frac{\partial^2 u_4}{\partial \theta^2}(r,2\pi), \end{aligned}$$

but

$$\frac{\partial^3 u_4}{\partial \theta^3}(r,0) \neq \frac{\partial^3 u_4}{\partial \theta^3}(r,2\pi).$$

For $k = 1, \ldots, 10$, we take $M = M_k$ and $N = N_k$. For each test function u, we define the maximum absolute error for each M_k as the largest absolute error

$$E_k = \max_{\substack{i=1,\ldots,M_k \\ j=1,\ldots,N_k}} |u(r_i, \theta_j) - U_{i,j}|, \quad k = 1, \ldots, 10. \tag{2.30}$$

Then we define the convergence rate as a number p for which $E_k = C_1 h_r^p + C_2 h_\theta^p$, where C_1 and C_2 are constants independent of h_r and h_θ. We use $N_k = 6M_k$ in our tests, so from (2.1),

$$h_r = \frac{1}{M_k + \frac{1}{2}}, \quad h_\theta = \frac{2\pi}{N_k} = \frac{\pi}{3M_k}.$$

Since $h_r = \dfrac{1}{M_k + \frac{1}{2}} \approx \dfrac{1}{M_k}$ for large M_k, we have

$$\begin{aligned} E_k &= C_1 \left(\frac{1}{M_k + \frac{1}{2}}\right)^p + C_2 \left(\frac{\pi}{3M_k}\right)^p \\ &\approx C_1 \left(\frac{1}{M_k}\right)^p + C_2 \left(\frac{\pi}{3}\right)^p \left(\frac{1}{M_k}\right)^p \\ &= \left[C_1 + C_2 \left(\frac{\pi}{3}\right)^p\right] \left(\frac{1}{M_k}\right)^p. \end{aligned}$$

Hence

$$\frac{E_{k-1}}{E_k} \approx \left(\frac{M_k}{M_{k-1}}\right)^p, \quad k = 2, \ldots, 10.$$

So the convergence rate can be expressed as

$$p \approx \frac{\log(E_{k-1}/E_k)}{\log(M_k/M_{k-1})}, \quad k = 2, \ldots, 10. \tag{2.31}$$

Table 2.1: Numerical Results for u_1

M	Error	Rate
10	2.6532×10^{-3}	
20	6.8945×10^{-4}	1.9442
30	3.0997×10^{-4}	1.9716
40	1.7547×10^{-4}	1.9779
50	1.1269×10^{-4}	1.9845
60	7.8457×10^{-5}	1.9861
70	5.7738×10^{-5}	1.9892
80	4.4269×10^{-5}	1.9892
90	3.5014×10^{-5}	1.9914
100	2.8386×10^{-5}	1.9917

Table 2.2: Numerical Results for u_2

M	Error	Rate
10	2.2204×10^{-3}	
20	5.9221×10^{-4}	1.9066
30	2.6901×10^{-4}	1.9462
40	1.5299×10^{-4}	1.9619
50	9.8560×10^{-5}	1.9705
60	6.8747×10^{-5}	1.9759
70	5.0667×10^{-5}	1.9796
80	3.8884×10^{-5}	1.9823
90	3.0779×10^{-5}	1.9844
100	2.4968×10^{-5}	1.9860

Table 2.3: Numerical Results for u_3

M	Error	Rate
10	3.6691×10^{-3}	
20	1.8319×10^{-3}	1.0021
30	1.2224×10^{-3}	.99773
40	9.1665×10^{-4}	1.0005
50	7.3337×10^{-4}	.99968
60	6.1115×10^{-4}	.99993
70	5.2382×10^{-4}	1.0003
80	4.5837×10^{-4}	.99962
90	4.0742×10^{-4}	1.0004
100	3.6669×10^{-4}	.99964

Table 2.4: Numerical Results for u_4

M	Error	Rate
10	6.4337×10^{-5}	
20	1.6082×10^{-5}	2.0002
30	7.1541×10^{-6}	1.9977
40	4.0232×10^{-6}	2.0009
50	2.5756×10^{-6}	1.9987
60	1.7885×10^{-6}	2.0003
70	1.3141×10^{-6}	1.9996
80	1.0061×10^{-6}	2.0000
90	7.9492×10^{-7}	2.0001
100	6.4390×10^{-7}	1.9997

From the results in Tables 2.1-2.4, we see that the scheme is roughly second-order convergent as expected, for most test functions. The results in Table 2.1 show that for a continuous, smooth test function u_1 that depends on both r and θ and has periodicity of all higher-order θ derivatives, the scheme converges at a rate of 2. Similarly, in Table 2.2, we see that even though the test function u_2 has no θ dependency, since u_2 is smooth the method still approaches a convergence rate of almost 2. Using the special test function u_3, we see from Table 2.3 that a lack of periodicity of the second θ derivative causes to the scheme to converge at a rate of only 1. Assuming, however, that the solution u is periodic in θ up to (but excluding) the third θ derivative, we can see from the results of testing function u_4 in Table 2.4 that the method will still be roughly second-order convergent.

We conclude that if this method were used to find a solution u that was not smooth, had no dependency on r, or lacked periodicity of some higher-order θ derivatives, the method might converge at a rate slower than 2. Since the original problem is converted from Cartesian to polar coordinates, assuming the solution in Cartesian would be smooth, the solution in polar will probably be smooth, dependent on r, and periodic in higher-order θ derivatives; hence we conclude that the second-order scheme we are discussing will converge at a rate of 2 under most circumstances.

3 DERIVATION OF THE FOURTH-ORDER SCHEME

Let

$$u_{i,j} = u(r_i, \theta_j), \quad i = 1, \dots, M+1, \quad j = 0, \dots, N,$$

where u is an exact solution of the problem (1.4)–(1.6).

3.1 Truncation Error in the r-Direction

First we analyze the truncation error of the second-order scheme in the r direction. We seek to perturb the term $L_r U_{i,j}$ from (2.6) to make it fourth-order accurate.

3.1.1 Boundedness of Higher Order r Derivatives

Following the proof shown in [13], we have

$$\begin{aligned} &x = r\cos\theta, \quad y = r\sin\theta, \\ &\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta. \end{aligned} \tag{3.1}$$

From (1.3), as in [13],

$$\frac{\partial u}{\partial r} = \frac{\partial \tilde{u}}{\partial x}\cos\theta + \frac{\partial \tilde{u}}{\partial y}\sin\theta,$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 \tilde{u}}{\partial x^2}\cos^2\theta + \frac{\partial^2 \tilde{u}}{\partial x \partial y}\sin 2\theta + \frac{\partial^2 \tilde{u}}{\partial y^2}\sin^2\theta.$$

Similarly, $\frac{\partial^k u}{\partial r^k}$ for $k = 3, 4, 5, 6$ will be linear combinations of partial derivatives of $\tilde{u}$ of order k with bounded sinusoidal coefficients. So assuming boundedness of all higher-order partial derivatives of $\tilde{u}$, we have for some constant $M > 0$

$$\left|\frac{\partial^6 u}{\partial r^6}\right| \leq M. \tag{3.2}$$

3.1.2 Case $i = 2, \ldots, M$

We consider first the operator L_r applied to the exact solution. Using (3.2) and Taylor's Theorem to expand around (r_i, θ_j), and assuming $u(r, \theta)$ is sufficiently smooth with respect to r on Ω, we have

$$u_{i+1,j} = u_{i,j} + h_r\left[\frac{\partial u}{\partial r}\right]_{i,j} + \frac{h_r^2}{2!}\left[\frac{\partial^2 u}{\partial r^2}\right]_{i,j} + \frac{h_r^3}{3!}\left[\frac{\partial^3 u}{\partial r^3}\right]_{i,j} + \frac{h_r^4}{4!}\left[\frac{\partial^4 u}{\partial r^4}\right]_{i,j} + \frac{h_r^5}{5!}\left[\frac{\partial^5 u}{\partial r^5}\right]_{i,j} + \mathcal{O}(h_r^6) \tag{3.3}$$

for $i = 1, \ldots, M$ and $j = 1, \ldots, N$. Also

$$u_{i-1,j} = u_{i,j} - h_r\left[\frac{\partial u}{\partial r}\right]_{i,j} + \frac{h_r^2}{2!}\left[\frac{\partial^2 u}{\partial r^2}\right]_{i,j} - \frac{h_r^3}{3!}\left[\frac{\partial^3 u}{\partial r^3}\right]_{i,j} + \frac{h_r^4}{4!}\left[\frac{\partial^4 u}{\partial r^4}\right]_{i,j} - \frac{h_r^5}{5!}\left[\frac{\partial^5 u}{\partial r^5}\right]_{i,j} + \mathcal{O}(h_r^6), \tag{3.4}$$

for $i = 2, \ldots, M$ and $j = 1, \ldots, N$.

Then using (3.3), we have

$$\frac{u_{i+1,j} - u_{i,j}}{h_r} = \left[\frac{\partial u}{\partial r}\right]_{i,j} + \frac{h_r}{2!}\left[\frac{\partial^2 u}{\partial r^2}\right]_{i,j} + \frac{h_r^2}{3!}\left[\frac{\partial^3 u}{\partial r^3}\right]_{i,j} + \frac{h_r^3}{4!}\left[\frac{\partial^4 u}{\partial r^4}\right]_{i,j} + \frac{h_r^4}{5!}\left[\frac{\partial^5 u}{\partial r^5}\right]_{i,j} + \mathcal{O}(h_r^5), \tag{3.5}$$

for $i = 1, \ldots, M$ and $j = 1, \ldots, N$.

And using (3.4), we have

$$\frac{u_{i,j} - u_{i-1,j}}{h_r} = \left[\frac{\partial u}{\partial r}\right]_{i,j} - \frac{h_r}{2!}\left[\frac{\partial^2 u}{\partial r^2}\right]_{i,j} + \frac{h_r^2}{3!}\left[\frac{\partial^3 u}{\partial r^3}\right]_{i,j} - \frac{h_r^3}{4!}\left[\frac{\partial^4 u}{\partial r^4}\right]_{i,j} + \frac{h_r^4}{5!}\left[\frac{\partial^5 u}{\partial r^5}\right]_{i,j} + \mathcal{O}(h_r^5), \tag{3.6}$$

for $i = 2, \ldots, M$ and $j = 1, \ldots, N$.

Then by the definition of our L_r operator (2.3), and by (3.5), (3.6), for $i = 2, \ldots, M$, and $j = 1, \ldots, N$, we have

$$L_r u_{i,j} = \frac{1}{r_i h_r}\left[\left(r_i + \frac{h_r}{2}\right)\frac{u_{i+1,j} - u_{i,j}}{h_r} - \left(r_i - \frac{h_r}{2}\right)\frac{u_{i,j} - u_{i-1,j}}{h_r}\right]$$

$$= \frac{1}{h_r}\left[\frac{u_{i+1,j} - u_{i,j}}{h_r} - \frac{u_{i,j} - u_{i-1,j}}{h_r}\right] + \frac{1}{2r_i}\left[\frac{u_{i+1,j} - u_{i,j}}{h_r} + \frac{u_{i,j} - u_{i-1,j}}{h_r}\right]$$

$$= \left(\left[\frac{\partial^2 u}{\partial r^2}\right]_{i,j} + \frac{h_r^2}{12}\left[\frac{\partial^4 u}{\partial r^4}\right]_{i,j} + \mathcal{O}(h_r^4) \right) + \frac{1}{r_i}\left(\left[\frac{\partial u}{\partial r}\right]_{i,j} + \frac{h_r^2}{6}\left[\frac{\partial^3 u}{\partial r^3}\right]_{i,j} + \mathcal{O}(h_r^4) \right)$$

$$= \left[\frac{\partial^2 u}{\partial r^2}\right]_{i,j} + \frac{1}{r_i}\left[\frac{\partial u}{\partial r}\right]_{i,j} + \frac{h_r^2}{6r_i}\left(\left[\frac{\partial^3 u}{\partial r^3}\right]_{i,j} + \frac{r_i}{2}\left[\frac{\partial^4 u}{\partial r^4}\right]_{i,j} \right) + \frac{1}{r_i}\mathcal{O}(h_r^4) + \mathcal{O}(h_r^4)$$

$$= \frac{1}{r_i}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]_{i,j} + \alpha_{i,j} + \frac{1}{r_i}\mathcal{O}(h_r^4), \tag{3.7}$$

where $\alpha_{i,j}$ are defined by

$$\alpha_{i,j} \quad \equiv \quad \frac{h_r^2}{6r_i}\left(\left[\frac{\partial^3 u}{\partial r^3}\right]_{i,j} + \frac{r_i}{2}\left[\frac{\partial^4 u}{\partial r^4}\right]_{i,j} \right), \quad i = 2, \ldots, M, \quad j = 1, \ldots, N. \tag{3.8}$$

3.1.3 Case $i = 1$

It follows from (3.5) with $i = 1$ that

$$\frac{u_{2,j} - u_{1,j}}{h_r} = \left[\frac{\partial u}{\partial r}\right]_{1,j} + \frac{h_r}{2!}\left[\frac{\partial^2 u}{\partial r^2}\right]_{1,j} + \frac{h_r^2}{3!}\left[\frac{\partial^3 u}{\partial r^3}\right]_{1,j} + \frac{h_r^3}{4!}\left[\frac{\partial^4 u}{\partial r^4}\right]_{1,j} + \mathcal{O}(h_r^4), \tag{3.9}$$

for $j = 1, \ldots, N$.

Since $r_1 = \dfrac{h_r}{2}$ from (2.2), using (2.4) and (3.9), for $j = 1, \ldots, N$, we have

$$L_r u_{1,j} = \frac{1}{r_1}\frac{u_{2,j} - u_{1,j}}{h_r}$$

$$= \frac{1}{r_1}\left(\left[\frac{\partial u}{\partial r}\right]_{1,j} + r_1\left[\frac{\partial^2 u}{\partial r^2}\right]_{1,j} + \frac{h_r^2}{6}\left[\frac{\partial^3 u}{\partial r^3}\right]_{1,j} + \frac{h_r^3}{24}\left[\frac{\partial^4 u}{\partial r^4}\right]_{1,j} + \mathcal{O}(h_r^4) \right)$$

$$= \left[\frac{\partial^2 u}{\partial r^2}\right]_{1,j} + \frac{1}{r_1}\left[\frac{\partial u}{\partial r}\right]_{1,j} + \frac{h_r^2}{6r_1}\left(\left[\frac{\partial^3 u}{\partial r^3}\right]_{1,j} + \frac{r_1}{2}\left[\frac{\partial^4 u}{\partial r^4}\right]_{1,j} \right) + \frac{1}{r_1}\mathcal{O}(h_r^4)$$

$$= \frac{1}{r_1}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]_{1,j} + \alpha_{1,j} + \frac{1}{r_1}\mathcal{O}(h_r^4), \tag{3.10}$$

where we define

$$\alpha_{1,j} \quad \equiv \quad \frac{h_r^2}{6r_1}\left(\left[\frac{\partial^3 u}{\partial r^3}\right]_{1,j} + \frac{r_1}{2}\left[\frac{\partial^4 u}{\partial r^4}\right]_{1,j} \right), \quad j = 1, \ldots, N, \tag{3.11}$$

which is (3.8) for $i = 1$.

3.2 Truncation Error in the θ-Direction

Next we analyze the truncation error of the second-order scheme in the θ direction. We seek to perturb the term $\dfrac{1}{r_i^2}L_\theta U_{i,j}$ from (2.6) to make it fourth-order accurate.

3.2.1 Boundedness of Higher Order θ Derivatives

From the first line of (3.1) we also have

$$\frac{\partial x}{\partial \theta} = -r\sin\theta = -y, \quad \frac{\partial y}{\partial \theta} = r\cos\theta = x.$$

Following the proof from [13], we see that

$$\frac{\partial u}{\partial \theta} = -y\frac{\partial \tilde{u}}{\partial x} + x\frac{\partial \tilde{u}}{\partial y},$$

$$\frac{\partial^2 u}{\partial \theta^2} = y^2\frac{\partial^2 \tilde{u}}{\partial x^2} + x^2\frac{\partial^2 \tilde{u}}{\partial y^2} - 2xy\frac{\partial^2 \tilde{u}}{\partial x \partial y} - x\frac{\partial \tilde{u}}{\partial x} - y\frac{\partial \tilde{u}}{\partial y}.$$

Similarly, as in [13], for $k = 3, 4, 5, 6$, $\dfrac{\partial^k u}{\partial \theta^k}$ will be a polynomial of degree k in the variables x and y, and will not have any terms of degree zero. The coefficients of each term of degree $l \leq k$ are derivatives of x and/or y, such that the derivatives are also of degree l. Since there will never be any terms of degree zero, but there will always be terms of order one, each term can have only one r factored out of either the x or y in that term. Again we assume boundedness of all partial derivatives of $\tilde{u}$. Hence for a constant $M > 0$,

$$\left|\frac{\partial^6 u}{\partial \theta^6}\right| \leq rM. \tag{3.12}$$

3.2.2 Case $j = 2, \ldots, N-1$

Now we consider the operator L_θ applied to the exact solution. Using Taylor's Theorem (with a remainder term) to expand around (r_i, θ_j), and assuming $u(r,\theta)$ is sufficiently smooth with respect to θ on Ω, we have

$$u_{i,j+1} = u_{i,j} + h_\theta\left[\frac{\partial u}{\partial \theta}\right]_{i,j} + \frac{h_\theta^2}{2!}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,j} + \frac{h_\theta^3}{3!}\left[\frac{\partial^3 u}{\partial \theta^3}\right]_{i,j} + \frac{h_\theta^4}{4!}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,j} + \frac{h_\theta^5}{5!}\left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,j} + \frac{h_\theta^6}{6!}\left[\frac{\partial^6 u}{\partial \theta^6}\right](r_i, \xi_j^{(1)})$$

for $i = 1, \ldots, M$ and $j = 0, \ldots, N-1$, where $\theta_j \leq \xi_j^{(1)} \leq \theta_{j+1}$. Then using (3.12), we have

$$u_{i,j+1} = u_{i,j} + h_\theta\left[\frac{\partial u}{\partial \theta}\right]_{i,j} + \frac{h_\theta^2}{2!}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,j} + \frac{h_\theta^3}{3!}\left[\frac{\partial^3 u}{\partial \theta^3}\right]_{i,j} + \frac{h_\theta^4}{4!}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,j} + \frac{h_\theta^5}{5!}\left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,j} + r_i\mathcal{O}(h_\theta^6) \tag{3.13}$$

for $i = 1, \ldots, M$, $j = 0, \ldots, N-1$.

Also

$$u_{i,j-1} = u_{i,j} - h_\theta \left[\frac{\partial u}{\partial \theta}\right]_{i,j} + \frac{h_\theta^2}{2!}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,j} - \frac{h_\theta^3}{3!}\left[\frac{\partial^3 u}{\partial \theta^3}\right]_{i,j} + \frac{h_\theta^4}{4!}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,j} - \frac{h_\theta^5}{5!}\left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,j} + \frac{h_\theta^6}{6!}\left[\frac{\partial^6 u}{\partial \theta^6}\right](r_i, \xi_j^{(2)})$$

for $i = 1, \ldots, M$ and $j = 1, \ldots, N$, where $\theta_{j-1} \le \xi_j^{(2)} \le \theta_j$. Again using (3.12), we have

$$u_{i,j-1} = u_{i,j} - h_\theta \left[\frac{\partial u}{\partial \theta}\right]_{i,j} + \frac{h_\theta^2}{2!}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,j} - \frac{h_\theta^3}{3!}\left[\frac{\partial^3 u}{\partial \theta^3}\right]_{i,j} + \frac{h_\theta^4}{4!}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,j} - \frac{h_\theta^5}{5!}\left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,j} + r_i\mathcal{O}(h_\theta^6) \tag{3.14}$$

for $i = 1, \ldots, M$, $j = 1, \ldots, N$.

Then by the definition of our L_θ operator in (2.5), and by (3.13) and (3.14), for $i = 1, \ldots, M$ and $j = 2, \ldots, N-1$,

$$\begin{aligned}
\frac{1}{r_i^2} L_\theta u_{i,j} &= \frac{1}{r_i^2}\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_\theta^2} \\
&= \frac{1}{r_i^2}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,j} + \frac{h_\theta^2}{12 r_i^2}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,j} + \frac{1}{r_i h_\theta^2}\mathcal{O}(h_\theta^6) \\
&= \frac{1}{r_i^2}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,j} + \beta_{i,j} + \frac{1}{r_i}\mathcal{O}(h_\theta^4),
\end{aligned} \tag{3.15}$$

where

$$\beta_{i,j} \quad \equiv \quad \frac{h_\theta^2}{12 r_i^2}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,j}, \quad i = 1, \ldots, M, \quad j = 2, \ldots, N-1. \tag{3.16}$$

3.2.3 Case $j = 1$

At $j = 1$, using the definition of $L_\theta v_{i,1}$ in (2.5), the first equation of (1.5), and (3.13) and (3.14) with $j = 1$, and following the derivation of (3.15), for $i = 1, \ldots, M$, we have

$$\begin{aligned}
\frac{1}{r_i^2} L_\theta u_{i,1} &= \frac{1}{r_i^2}\frac{u_{i,N} - 2u_{i,1} + u_{i,2}}{h_\theta^2} \\
&= \frac{1}{r_i^2}\frac{u_{i,0} - 2u_{i,1} + u_{i,2}}{h_\theta^2} \\
&= \frac{1}{r_i^2}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,1} + \beta_{i,1} + \frac{1}{r_i}\mathcal{O}(h_\theta^4),
\end{aligned} \tag{3.17}$$

where

$$\beta_{i,1} \quad \equiv \quad \frac{h_\theta^2}{12 r_i^2}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,1}, \quad i = 1, \ldots, M. \tag{3.18}$$

3.2.4 Case $j = N$

It follows from (3.13) with $j = 0$ that

$$u_{i,1} = u_{i,0} + h_\theta \left[\frac{\partial u}{\partial \theta}\right]_{i,0} + \frac{h_\theta^2}{2!}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,0} + \frac{h_\theta^3}{3!}\left[\frac{\partial^3 u}{\partial \theta^3}\right]_{i,0} + \frac{h_\theta^4}{4!}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,0} + \frac{h_\theta^5}{5!}\left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,0} + r_i\mathcal{O}(h_\theta^6), \tag{3.19}$$

for $i = 1, \ldots, M$.

Also from (3.14) with $j = N$, we have

$$u_{i,N-1} = u_{i,N} - h_\theta \left[\frac{\partial u}{\partial \theta}\right]_{i,N} + \frac{h_\theta^2}{2!}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,N} - \frac{h_\theta^3}{3!}\left[\frac{\partial^3 u}{\partial \theta^3}\right]_{i,N} + \frac{h_\theta^4}{4!}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,N} - \frac{h_\theta^5}{5!}\left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,N} + r_i\mathcal{O}(h_\theta^6), \tag{3.20}$$

for $i = 1, \ldots, M$.

So at $j = N$, using the definition of $L_\theta v_{i,N}$ in (2.5), using (3.19) and (3.20) and our periodic boundary conditions from (1.5), and additionally assuming periodicity of the second, third, fourth, and fifth θ derivatives, for $i = 1, \ldots, M$, we have

$$\begin{aligned}
\frac{1}{r_i^2}L_\theta u_{i,N} &= \frac{1}{r_i^2}\frac{u_{i,N-1} - 2u_{i,N} + u_{i,1}}{h_\theta^2} \\
&= \frac{1}{r_i^2}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,N} + \frac{h_\theta^2}{24 r_i^2}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,N} + \frac{1}{r_i h_\theta^2}\mathcal{O}(h_\theta^6) \\
&= \frac{1}{r_i^2}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,N} + \beta_{i,N} + \frac{1}{r_i}\mathcal{O}(h_\theta^4),
\end{aligned} \tag{3.21}$$

where

$$\beta_{i,N} \equiv \frac{h_\theta^2}{24 r_i^2}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,N}, \quad i = 1, \ldots, M. \tag{3.22}$$

3.3 Truncation Error in Both r and θ Directions

It follows from (3.7), (3.10), (3.15), (3.17), (3.21), and (1.4) that

$$\begin{aligned}
L_r u_{i,j} + \frac{1}{r_i^2}L_\theta u_{i,j} &= \frac{1}{r_i}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]_{i,j} + \frac{1}{r_i^2}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,j} + \alpha_{i,j} + \beta_{i,j} + \frac{1}{r_i}\mathcal{O}(h_r^4) + \frac{1}{r_i}\mathcal{O}(h_\theta^4) \\
&= f_{i,j} + \alpha_{i,j} + \beta_{i,j} + \frac{1}{r_i}\mathcal{O}(h_r^4) + \frac{1}{r_i}\mathcal{O}(h_\theta^4), \\
&\qquad i = 1, \ldots, M, \quad j = 1, \ldots, N,
\end{aligned} \tag{3.23}$$

where $\alpha_{i,j}$ and $\beta_{i,j}$ are defined in (3.8), (3.11), (3.16), (3.18), and (3.22) as

$$\alpha_{i,j} = \frac{h_r^2}{6r_i}\left(\left[\frac{\partial^3 u}{\partial r^3}\right]_{i,j} + \frac{r_i}{2}\left[\frac{\partial^4 u}{\partial r^4}\right]_{i,j}\right), \quad i = 1, \ldots, M, \quad j = 1, \ldots, N \tag{3.24}$$

$$\beta_{i,j} = \frac{h_\theta^2}{12r_i^2}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,j}, \quad i = 1,\ldots,M, \quad j = 1,\ldots,N. \tag{3.25}$$

3.4 Higher-Order Derivative Approximations

Next we define finite difference operators for approximating the necessary third- and fourth-order derivatives in $\alpha_{i,j}$ and $\beta_{i,j}$ of (3.24) and (3.25), respectively. All operators are either first- or second-order accurate, as necessary. Proof of accuracy is provided in Sections 6.1 and 6.2 of the appendix.

3.4.1 Derivatives With Respect to r

We approximate the third r derivative with second-order accuracy (see Section 6.1 of the appendix) for $j = 1,\ldots,N$ by

$$L_r^{(3)} v_{i,j} = \frac{1}{2h_r^3}\begin{cases} -5v_{1,j} + 18v_{2,j} - 24v_{3,j} + 14v_{4,j} - 3v_{5,j}, & i = 1, \\ -3v_{1,j} + 10v_{2,j} - 12v_{3,j} + 6v_{4,j} - v_{5,j}, & i = 2, \\ -v_{i-2,j} + 2v_{i-1,j} - 2v_{i+1,j} + v_{i+2,j}, & i = 3,\ldots,M-1, \\ v_{M-3,j} - 6v_{M-2,j} + 12v_{M-1,j} - 10v_{M,j} + 3v_{M+1,j}, & i = M. \end{cases} \tag{3.26}$$

And we approximate the fourth r derivative with first-order accuracy for $i = 1, 2$ and second-order accuracy for $i = 3,\ldots,M$ (see Section 6.1 of the appendix) for $j = 1,\ldots,N$ by

$$L_r^{(4)} v_{i,j} = \frac{1}{h_r^4}\begin{cases} v_{1,j} - 4v_{2,j} + 6v_{3,j} - 4v_{4,j} + v_{5,j}, & i = 1, \\ v_{1,j} - 4v_{2,j} + 6v_{3,j} - 4v_{4,j} + v_{5,j}, & i = 2, \\ v_{i-2,j} - 4v_{i-1,j} + 6v_{i,j} - 4v_{i+1,j} + v_{i+2,j}, & i = 3,\ldots,M-1, \\ -v_{M-4,j} + 6v_{M-3,j} - 14v_{M-2,j} + 16v_{M-1,j} - 9v_{M,j} + 2v_{M+1,j}, & i = M. \end{cases} \tag{3.27}$$

Hence we rewrite (3.24) as

$$\alpha_{i,j} = \frac{h_r^2}{6r_i}\begin{cases} L_r^{(3)} u_{i,j} + \mathcal{O}(h_r^2) + \frac{r_i}{2}\left[L_r^{(4)} u_{i,j} + \mathcal{O}(h_r)\right], & i = 1, 2, \quad j = 1,\ldots,N, \\ L_r^{(3)} u_{i,j} + \mathcal{O}(h_r^2) + \frac{r_i}{2}\left[L_r^{(4)} u_{i,j} + \mathcal{O}(h_r^2)\right], & i = 3,\ldots,M, \quad j = 1,\ldots,N. \end{cases} \tag{3.28}$$

3.4.2 Derivatives With Respect to θ

Assuming periodicity of the second, third, fourth, and fifth-order θ derivatives, we approximate the fourth θ derivative with accuracy $r_i\mathcal{O}(h_\theta^2)$ using

$$L_\theta^{(4)} v_{i,j} = \frac{1}{h_\theta^4}(v_{i,j-2} - 4v_{i,j-1} + 6v_{i,j} - 4v_{i,j+1} + v_{i,j+2}) \tag{3.29}$$

for $i = 1, \ldots, M$ and $j = 1, \ldots, N$ with the convention that $v_{i,-1}$ (occurring when $j = 1$) and $v_{i,0}$ (occurring when $j = 1, 2$) are replaced by $v_{i,N-1}$ and $v_{i,N}$ respectively, and $v_{i,N+1}$ (occurring when $j = N - 1, N$) and $v_{i,N+2}$ (occurring when $j = N$) are replaced by $v_{i,1}$ and $v_{i,2}$ respectively. See Section 6.2 of the appendix.

Hence we rewrite (3.25) as

$$\beta_{i,j} = \frac{h_\theta^2}{12r_i^2}\left[L_\theta^{(4)} u_{i,j} + r_i\mathcal{O}(h_\theta^2)\right], \quad i = 1, \ldots, M, \quad j = 1, \ldots, N. \tag{3.30}$$

4 FOURTH-ORDER SCHEME

In (3.23) we had

$$L_r u_{i,j} + \frac{1}{r_i^2} L_\theta u_{i,j} = f_{i,j} + \alpha_{i,j} + \beta_{i,j} + \frac{1}{r_i}\mathcal{O}(h_r^4) + \frac{1}{r_i}\mathcal{O}(h_\theta^4),$$
$$i = 1, \ldots, M, \quad j = 1, \ldots, N.$$

We substitute into this equation our new expressions for $\alpha_{i,j}$ in (3.28) and $\beta_{i,j}$ in (3.30). Since $r_1 = \frac{h_r}{2}$ and $r_2 = \frac{3h_r}{2}$, we obtain

$$L_r u_{i,j} + \frac{1}{r_i^2} L_\theta u_{i,j} = f_{i,j} + \frac{h_r^2}{6r_i} L_r^{(3)} u_{i,j} + \frac{h_r^2}{12} L_r^{(4)} u_{i,j} + \frac{h_\theta^2}{12r_i^2} L_\theta^{(4)} u_{i,j} + \frac{1}{r_i}\mathcal{O}(h_r^4) + \frac{1}{r_i}\mathcal{O}(h_\theta^4) \tag{4.1}$$

for $i = 1, \ldots, M, \quad j = 1, \ldots, N$.

We obtain our fourth-order scheme by dropping the $\mathcal{O}$ terms, replacing all exact $u_{i,j}$ with the approximations $U_{i,j}$, and moving all terms involving $U_{i,j}$ to the left-hand side:

$$\left(L_r - \frac{h_r^2}{6r_i} L_r^{(3)} - \frac{h_r^2}{12} L_r^{(4)}\right) U_{i,j} + \frac{1}{r_i^2}\left(L_\theta - \frac{h_\theta^2}{12} L_\theta^{(4)}\right) U_{i,j} = f_{i,j},$$
$$i = 1, \ldots, M, \quad j = 1, \ldots, N, \tag{4.2}$$

where L_r and L_θ are defined by (2.3), (2.4), and (2.5), $L_r^{(3)}$, $L_r^{(4)}$, and $L_\theta^{(4)}$ are defined in (3.26), (3.27), and (3.29), and the $U_{M+1,j}$ are defined in (2.7) for $j = 1, \ldots, N$.

4.1 Matrix-Vector Form of the Fourth-Order Scheme

In the development of the matrix-vector form of our scheme (4.2), we again use the understanding that for $i = 1, \ldots, M$, $U_{i,-1}$ and $U_{i,0}$ are replaced by $U_{i,N-1}$ and $U_{i,N}$ respectively, and $U_{i,N+1}$ and $U_{i,N+2}$ are replaced by $U_{i,1}$ and $U_{i,2}$ respectively. We recognize that (4.2) is a perturbation of (2.6), so we modify (2.8), (2.11), and (2.13) accordingly, using (3.26), (3.27), and (3.29). Therefore, in what follows we use δ from (2.9) and μ_i, γ_i, and ν_i from (2.10), (2.12), and (2.14). Also, in all resulting equations, the term $\frac{\gamma_i}{12}[U_{i,j-2} - 4U_{i,j-1} + 6U_{i,j} - 4U_{i,j+1} + U_{i,j+2}]$ comes from the term $\frac{h_\theta^2}{12r_i^2} L_\theta^{(4)} U_{i,j}$ in (4.2). Similarly, the terms involving σ_i and ρ_i, along with the perturbations applied to δ_i, μ_i, and ν_i to obtain $\tilde{\delta}_i$, $\tilde{\mu}_i$, and $\tilde{\nu}_i$ come from the terms $-\frac{h_r^2}{6r_i} L_r^{(3)} U_{i,j}$ and $-\frac{h_r^2}{12} L_r^{(4)} U_{i,j}$ in (4.2). We still consider separately the cases $i = 1, M$, but now we also consider separately $i = 2, M-1$ because of the definitions (3.26) and (3.27), so that our general case is $i = 3, \ldots, M-2$.

From (2.8), for $i = 3, \ldots, M-2$ and $j = 1, \ldots, N$,

$$\begin{aligned}
&\sigma_i U_{i-2,j} + \tilde{\mu}_i U_{i-1,j} + \\
&\quad \gamma_i \left[U_{i,j-1} - 2U_{i,j} + U_{i,j+1}\right] - \frac{\gamma_i}{12}\left[U_{i,j-2} - 4U_{i,j-1} + 6U_{i,j} - 4U_{i,j+1} + U_{i,j+2}\right] + \tilde{\delta} U_{i,j} + \\
&\qquad \tilde{\nu}_i U_{i+1,j} + \rho_i U_{i+2,j} = f_{i,j}, \qquad (4.3)
\end{aligned}$$

where

$$\begin{aligned}
\sigma_i &= -\frac{h_r^2}{6r_i}\left(-\frac{1}{2h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{1}{h_r^4}\right) = \frac{h_r - r_i}{12 r_i h_r^2}, \\
\tilde{\mu}_i &= \mu_i - \frac{h_r^2}{6r_i}\left(\frac{1}{h_r^3}\right) - \frac{h_r^2}{12}\left(-\frac{4}{h_r^4}\right) = \mu_i + \frac{2r_i - h_r}{6r_i h_r^2} = \frac{4r_i - 2h_r}{3r_i h_r^2}, \\
\tilde{\delta} &= \delta - \frac{h_r^2}{6r_i}\left(\frac{0}{2h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{6}{h_r^4}\right) = \delta - \frac{1}{2h_r^2} = -\frac{5}{2h_r^2}, \\
\tilde{\nu}_i &= \nu_i - \frac{h_r^2}{6r_i}\left(-\frac{1}{h_r^3}\right) - \frac{h_r^2}{12}\left(-\frac{4}{h_r^4}\right) = \nu_i + \frac{2r_i + h_r}{6r_i h_r^2} = \frac{4r_i + 2h_r}{3r_i h_r^2}, \\
\rho_i &= -\frac{h_r^2}{6r_i}\left(\frac{1}{2h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{1}{h_r^4}\right) = -\frac{h_r + r_i}{12 r_i h_r^2}.
\end{aligned} \qquad (4.4)$$

From (2.11), for $i = 1$ and $j = 1, \ldots, N$,

$$\begin{aligned}
&\gamma_1 \left[U_{1,j-1} - 2U_{1,j} + U_{1,j+1}\right] - \frac{\gamma_1}{12}\left[U_{1,j-2} - 4U_{1,j-1} + 6U_{1,j} - 4U_{1,j+1} + U_{1,j+2}\right] + \\
&\qquad \tilde{\delta}_1 U_{1,j} + \tilde{\nu}_1 U_{2,j} + \rho_1 U_{3,j} - \frac{2}{h_r^2} U_{4,j} + \frac{5}{12h_r^2} U_{5,j} = f_{1,j}, \qquad (4.5)
\end{aligned}$$

where

$$\tilde{\delta}_1 = \delta - \frac{h_r^2}{6r_1}\left(-\frac{5}{2h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{1}{h_r^4}\right) = \delta + \frac{3}{4h_r^2} = -\frac{5}{4h_r^2},$$
$$\tilde{\nu}_1 = \nu_1 - \frac{h_r^2}{6r_1}\left(\frac{9}{h_r^3}\right) - \frac{h_r^2}{12}\left(-\frac{4}{h_r^4}\right) = \nu_1 - \frac{8}{3h_r^2} = -\frac{2}{3h_r^2}, \quad (4.6)$$
$$\rho_1 = -\frac{h_r^2}{6r_1}\left(-\frac{12}{h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{6}{h_r^4}\right) = \frac{7}{2h_r^2},$$

and the final expressions for the coefficients of $U_{4,j}$ and $U_{5,j}$ are obtained from

$$-\frac{h_r^2}{6r_1}\left(\frac{7}{h_r^3}\right) - \frac{h_r^2}{12}\left(-\frac{4}{h_r^4}\right) = -\frac{2}{h_r^2},$$

$$-\frac{h_r^2}{6r_1}\left(-\frac{3}{2h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{1}{h_r^4}\right) = \frac{5}{12h_r^2}.$$

From (2.8), for $i = 2$ and $j = 1, \ldots, N$,

$$\tilde{\mu}_2 U_{1,j} + \gamma_2 \left[U_{2,j-1} - 2U_{2,j} + U_{2,j+1}\right] - \frac{\gamma_2}{12}\left[U_{2,j-2} - 4U_{2,j-1} + 6U_{2,j} - 4U_{2,j+1} + U_{2,j+2}\right] + \tilde{\delta}_2 U_{2,j} + \tilde{\nu}_2 U_{3,j} - \frac{1}{36h_r^2} U_{5,j} = f_{2,j}, \quad (4.7)$$

where

$$\tilde{\mu}_2 = \mu_2 - \frac{h_r^2}{6r_2}\left(-\frac{3}{2h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{1}{h_r^4}\right) = \mu_2 + \frac{1}{12h_r^2} = \frac{3}{4h_r^2},$$
$$\tilde{\delta}_2 = \delta - \frac{h_r^2}{6r_2}\left(\frac{5}{h_r^3}\right) - \frac{h_r^2}{12}\left(-\frac{4}{h_r^4}\right) = \delta - \frac{2}{9h_r^2} = -\frac{20}{9h_r^2}, \quad (4.8)$$
$$\tilde{\nu}_2 = \nu_2 - \frac{h_r^2}{6r_2}\left(-\frac{6}{h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{6}{h_r^4}\right) = \nu_2 + \frac{1}{6h_r^2} = \frac{3}{2h_r^2},$$

and the final expression for the coefficients of $U_{4,j}$ and $U_{5,j}$ are obtained from

$$-\frac{h_r^2}{6r_2}\left(\frac{3}{h_r^3}\right) - \frac{h_r^2}{12}\left(-\frac{4}{h_r^4}\right) = 0,$$

$$-\frac{h_r^2}{6r_2}\left(-\frac{1}{2h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{1}{h_r^4}\right) = -\frac{1}{36h_r^2}.$$

From (2.8), for $i = M-1$ and $j = 1, \ldots, N$,

$$\sigma_{M-1} U_{M-3,j} + \tilde{\mu}_{M-1} U_{M-2,j} + \gamma_{M-1}\left[U_{M-1,j-1} - 2U_{M-1,j} + U_{M-1,j+1}\right] - \frac{\gamma_{M-1}}{12}\left[U_{M-1,j-2} - 4U_{M-1,j-1} + 6U_{M-1,j} - 4U_{M-1,j+1} + U_{M-1,j+2}\right] + \tilde{\delta} U_{M-1,j} + \tilde{\nu}_{M-1} U_{M,j} = f_{M-1,j} - \rho_{M-1} g(\theta_j), \quad (4.9)$$

where

$$\begin{aligned}
\sigma_{M-1} &= -\frac{h_r^2}{6r_{M-1}}\left(-\frac{1}{2h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{1}{h_r^4}\right) = \frac{h_r - r_{M-1}}{12r_{M-1}h_r^2}, \\
\tilde{\mu}_{M-1} &= \mu_{M-1} - \frac{h_r^2}{6r_{M-1}}\left(\frac{1}{h_r^3}\right) - \frac{h_r^2}{12}\left(-\frac{4}{h_r^4}\right) = \mu_{M-1} + \frac{2r_{M-1} - h_r}{6r_{M-1}h_r^2} = \frac{4r_{M-1} - 2h_r}{3r_{M-1}h_r^2}, \\
\tilde{\delta} &= \delta - \frac{h_r^2}{6r_{M-1}}\left(\frac{0}{2h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{6}{h_r^4}\right) = \delta - \frac{1}{2h_r^2} = -\frac{5}{2h_r^2}, \\
\tilde{\nu}_{M-1} &= \nu_{M-1} - \frac{h_r^2}{6r_{M-1}}\left(-\frac{1}{h_r^3}\right) - \frac{h_r^2}{12}\left(-\frac{4}{h_r^4}\right) = \nu_{M-1} + \frac{2r_{M-1} + h_r}{6r_{M-1}h_r^2} = \frac{4r_{M-1} + 2h_r}{3r_{M-1}h_r^2}, \\
\rho_{M-1} &= -\frac{h_r^2}{6r_{M-1}}\left(\frac{1}{2h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{1}{h_r^4}\right) = -\frac{h_r + r_{M-1}}{12r_{M-1}h_r^2}.
\end{aligned} \tag{4.10}$$

From (2.13), for $i = M$ and $j = 1, \ldots, N$,

$$\begin{aligned}
&\frac{1}{12h_r^2}U_{M-4,j} - \frac{h_r + 6r_M}{12r_M h_r^2}U_{M-3,j} + \sigma_M U_{M-2,j} + \tilde{\mu}_M U_{M-1,j} + \\
&\gamma_M\left[U_{M,j-1} - 2U_{M,j} + U_{M,j+1}\right] - \frac{\gamma_M}{12}\left[U_{M,j-2} - 4U_{M,j-1} + 6U_{M,j} - 4U_{M,j+1} + U_{M,j+2}\right] + \\
&\qquad \tilde{\delta}_M U_{M,j} = f_{M,j} - \tilde{\nu}_M g(\theta_j),
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
\sigma_M &= -\frac{h_r^2}{6r_M}\left(-\frac{3}{h_r^3}\right) - \frac{h_r^2}{12}\left(-\frac{14}{h_r^4}\right) = \frac{7r_M + 3h_r}{6r_M h_r^2}, \\
\tilde{\mu}_M &= \mu_M - \frac{h_r^2}{6r_M}\left(\frac{6}{h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{16}{h_r^4}\right) = \mu_M - \frac{3h_r + 4r_M}{3r_M h_r^2} = -\frac{2r_M + 9h_r}{6r_M h_r^2}, \\
\tilde{\delta}_M &= \delta - \frac{h_r^2}{6r_M}\left(-\frac{5}{h_r^3}\right) - \frac{h_r^2}{12}\left(-\frac{9}{h_r^4}\right) = \delta + \frac{10h_r + 9r_M}{12r_M h_r^2} = \frac{10h_r - 15r_M}{12r_M h_r^2}, \\
\tilde{\nu}_M &= \nu_M - \frac{h_r^2}{6r_M}\left(\frac{3}{2h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{2}{h_r^4}\right) = \nu_M - \frac{3h_r + 2r_M}{12r_M h_r^2} = \frac{10r_M + 3h_r}{12r_M h_r^2},
\end{aligned} \tag{4.12}$$

and the final expressions for the coefficients of $U_{M-4,j}$ and $U_{M-3,j}$ are obtained from

$$-\frac{h_r^2}{12}\left(-\frac{1}{h_r^4}\right) = \frac{1}{12h_r^2},$$

$$-\frac{h_r^2}{6r_M}\left(\frac{1}{2h_r^3}\right) - \frac{h_r^2}{12}\left(\frac{6}{h_r^4}\right) = -\frac{h_r + 6r_M}{12r_M h_r^2}.$$

We notice that, using the definition of our P matrix from (2.17),

$$P^2 = \begin{pmatrix} 6 & -4 & 1 & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & 1 \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ 1 & & & 1 & -4 & 6 & -4 \\ -4 & 1 & & & 1 & -4 & 6 \end{pmatrix} \tag{4.13}$$

which has ones and negative fours in the corners because of the convention used in (3.29).

Therefore, using our definitions from (2.15), (2.16), and (2.17), we have the following cases for $i = 1, 2, (3, \ldots, M-2), M-1$, and M. In what follows, γ_i, σ_i, $\tilde{\mu}_i$, $\tilde{\delta}_i$, $\tilde{\nu}_i$, and ρ_i are defined in (2.10), (4.4), (2.12), (4.6), (4.8), (4.10), (2.14), and (4.12) accordingly.

For $i = 3, \ldots, M-2$, from (4.3) we have

$$\sigma_i \mathbf{U}_{i-2} + \tilde{\mu}_i \mathbf{U}_{i-1} + \left[\gamma_i P - \frac{\gamma_i}{12} P^2 + \tilde{\delta} I\right] \mathbf{U}_i + \tilde{\nu}_i \mathbf{U}_{i+1} + \rho_i \mathbf{U}_{i+2} = \mathbf{F}_i. \tag{4.14}$$

For $i = 1$, from (4.5) we have

$$\left[\gamma_1 P - \frac{\gamma_1}{12} P^2 + \tilde{\delta}_1 I\right] \mathbf{U}_1 + \tilde{\nu}_1 \mathbf{U}_2 + \rho_1 \mathbf{U}_3 - \frac{2}{h_r^2} \mathbf{U}_4 + \frac{5}{12 h_r^2} \mathbf{U}_5 = \mathbf{F}_1. \tag{4.15}$$

For $i = 2$, from (4.7) we have

$$\tilde{\mu}_2 \mathbf{U}_1 + \left[\gamma_2 P - \frac{\gamma_2}{12} P^2 + \tilde{\delta}_2 I\right] \mathbf{U}_2 + \tilde{\nu}_2 \mathbf{U}_3 - \frac{1}{36 h_r^2} \mathbf{U}_5 = \mathbf{F}_2. \tag{4.16}$$

For $i = M-1$ from (4.9) we have

$$\sigma_{M-1} \mathbf{U}_{M-3} + \tilde{\mu}_{M-1} \mathbf{U}_{M-2} + \left[\gamma_{M-1} P - \frac{\gamma_{M-1}}{12} P^2 + \tilde{\delta} I\right] \mathbf{U}_{M-1} + \tilde{\nu}_{M-1} \mathbf{U}_M = \mathbf{F}_{M-1} - \rho_{M-1} \mathbf{G}, \tag{4.17}$$

where $\mathbf{G}$ is defined in (2.21).

For $i = M$ from (4.11) we have

$$\frac{1}{12 h_r^2} \mathbf{U}_{M-4} - \frac{h_r + 6 r_M}{12 r_M h_r^2} \mathbf{U}_{M-3} + \sigma_M \mathbf{U}_{M-2} + \tilde{\mu}_M \mathbf{U}_{M-1} + \left[\gamma_M P - \frac{\gamma_M}{12} P^2 + \tilde{\delta}_M I\right] \mathbf{U}_M = \mathbf{F}_M - \tilde{\nu}_M \mathbf{G}, \tag{4.18}$$

where $\mathbf{G}$ is defined in (2.21).

4.2 Matrix Diagonalization Algorithm

We begin with the same real, orthogonal Q and diagonal Λ as in (2.24) and (2.23). Then we left-multiply (4.14) by Q^T and use (2.22) to get

$$\sigma_i Q^T \mathbf{U}_{i-2} + \tilde{\mu}_i Q^T \mathbf{U}_{i-1} + [\gamma_1 Q^T PQQ^T - \frac{\gamma_1}{12} Q^T PQQ^T PQQ^T + \tilde{\delta} Q^T QQ^T]\mathbf{U}_i + \tilde{\nu}_1 Q^T \mathbf{U}_{i+1} + \rho_1 Q^T \mathbf{U}_{i+2} = Q^T \mathbf{F}_i$$

Next we introduce $\mathbf{V}_i$ and $\mathbf{E}_i$ as in (2.25) and use (2.23) to arrive at

$$\sigma_i \mathbf{V}_{i-2} + \tilde{\mu}_i \mathbf{V}_{i-1} + \left[\gamma_i \Lambda - \frac{\gamma_i}{12}\Lambda^2 + \tilde{\delta} I\right] \mathbf{V}_i + \tilde{\nu}_i \mathbf{V}_{i+1} + \rho_i \mathbf{V}_{i+2} = \mathbf{E}_i, \qquad (4.19)$$

$$i = 3, \ldots, M-2.$$

Similarly, (4.15) becomes

$$[\gamma_1 \Lambda - \frac{\gamma_1}{12}\Lambda^2 + \tilde{\delta} I]\mathbf{V}_1 + \tilde{\nu}_1 \mathbf{V}_2 + \rho_1 \mathbf{V}_3 - \frac{2}{h_r^2}\mathbf{V}_4 + \frac{5}{12h_r^2}\mathbf{V}_5 = \mathbf{E}_1, \qquad (4.20)$$

(4.16) becomes

$$\tilde{\mu}_2 \mathbf{V}_1 + \left[\gamma_2 \Lambda - \frac{\gamma_2}{12}\Lambda^2 + \tilde{\delta} I\right] \mathbf{V}_2 + \tilde{\nu}_2 \mathbf{V}_3 - \frac{1}{36h_r^2}\mathbf{V}_5 = \mathbf{E}_2, \qquad (4.21)$$

(4.17) becomes

$$\sigma_{M-1} \mathbf{V}_{M-3} + \tilde{\mu}_{M-1} \mathbf{V}_{M-2} + \left[\gamma_{M-1} \Lambda - \frac{\gamma_{M-1}}{12}\Lambda^2 + \tilde{\delta} I\right] \mathbf{V}_{M-1} + \tilde{\nu}_{M-1} \mathbf{V}_M = \mathbf{E}_{M-1}, \qquad (4.22)$$

where $\mathbf{E}_{M-1} = Q^T(\mathbf{F}_{M-1} - \rho_{M-1}\mathbf{G})$, and (4.18) becomes

$$\frac{1}{12h_r^2}\mathbf{V}_{M-4} - \frac{h_r + 6r_M}{12r_M h_r^2}\mathbf{V}_{M-3} + \sigma_M \mathbf{V}_{M-2} + \tilde{\mu}_M \mathbf{V}_{M-1} + \left[\gamma_M \Lambda - \frac{\gamma_M}{12}\Lambda^2 + \tilde{\delta} I\right] \mathbf{V}_M = \mathbf{E}_M, \qquad (4.23)$$

where $\mathbf{E}_M = Q^T(\mathbf{F}_M - \tilde{\nu}_M \mathbf{G})$.

Next, for each fixed $j = 1, \ldots, N$, we create new vectors $\tilde{\mathbf{V}}_j$ and $\tilde{\mathbf{E}}_j$ as in (2.29). Then for fixed $j = 1, \ldots, N$, we take equation j from each of (4.19)–(4.23) and use the fact that Λ is diagonal to form a collection of N independent linear $M \times M$ systems

$$R_j \tilde{\mathbf{V}}_j = \tilde{\mathbf{E}}_j, \quad j = 1, \ldots, N,$$

with an almost penta-diagonal $M \times M$ matrix

$$R_j = \begin{pmatrix} D_1 & \tilde{\nu}_1 & \rho_1 & -\dfrac{2}{h_r^2} & \dfrac{5}{12h_r^2} & & & \\ \tilde{\mu}_2 & D_2 & \tilde{\nu}_2 & 0 & -\dfrac{1}{36h_r^2} & & & \\ \sigma_3 & \tilde{\mu}_3 & D_3 & \tilde{\nu}_3 & \rho_3 & & & \\ & \sigma_4 & \tilde{\mu}_4 & D_4 & \tilde{\nu}_4 & \rho_4 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & \sigma_{M-2} & \tilde{\mu}_{M-2} & D_{M-2} & \tilde{\nu}_{M-2} & \rho_{M-2} \\ & & & & \sigma_{M-1} & \tilde{\mu}_{M-1} & D_{M-1} & \tilde{\nu}_{M-1} \\ & & & \dfrac{1}{12h_r^2} & -\dfrac{h_r + 6r_M}{12r_M h_r^2} & \sigma_M & \tilde{\mu}_M & D_M \end{pmatrix},$$

where for fixed j and $i = 1, \dots, M$

$$D_i = \gamma_i \lambda_j - \frac{\gamma_i}{12}\lambda_j^2 + \tilde{\delta}.$$

We create exactly penta-diagonal system by:

1. Subtracting from the second equation the third equation multiplied by $\dfrac{(R_j)_{2,5}}{(R_j)_{3,5}}$,
2. Subtracting from the first equation the third equation multiplied by $\dfrac{(R_j)_{1,5}}{(R_j)_{3,5}}$,
3. Subtracting from the first equation the new second equation multiplied by $\dfrac{(R_j)_{1,4}}{(R_j)_{2,4}}$,
4. Subtracting from the M equation the $M-2$ equation multiplied by $\dfrac{(R_j)_{M,M-4}}{(R_j)_{M-2,M-4}}$,
5. Subtracting from the M equation the $M-1$ equation multiplied by $\dfrac{(R_j)_{M,M-3}}{(R_j)_{M-1,M-3}}$.

All of this leads to a collection of N independent linear penta-diagonal $M \times M$ systems

$$\hat{R}_j \tilde{\mathbf{V}}_j = \hat{\mathbf{E}}_j, \quad j = 1, \dots, N.$$

We can reconstruct the vectors $\mathbf{V}_i$ from the elements of the vectors $\tilde{\mathbf{V}}_j$. Then we find the vectors of our approximate solutions by computing $\mathbf{U}_i = Q\mathbf{V}_i$ for $i = 1, \dots, M$.

So the basic steps of our algorithm are:

1. Perform M left-multiplications by Q^T to find the vectors $\mathbf{E}_i = Q^T F_i$ for $i = 1, \ldots, M$.

2. For each $j = 1, \ldots, N$, perform 5 operations to obtain penta-diagonal linear systems.

3. Solve the collection of penta-diagonal systems $\hat{R}_j \tilde{\mathbf{V}}_j = \hat{\mathbf{E}}_j$ for $j = 1, \ldots, N$.

4. Perform M left-multiplications by Q to find the vectors $\mathbf{U}_i = QV_i$ for $i = 1, \ldots, M$.

4.3 Cost Analysis

As with the second-order scheme, the cost of computing the entries of $\mathbf{F}_i$ and R_j will each be around $\mathcal{O}(MN)$. We again use the Matlab function "fft" to perform the multiplications by Q and Q^T. Please see Section 6.3 of the appendix for details. So the cost of steps (1) and (4) in our algorithm is again $\mathcal{O}(MN \log N)$, as stated in [4]. The cost of step (2) for five operations at each $j = 1, \ldots, N$ is $\mathcal{O}(N)$. Also, since the systems in step (3) are penta-diagonal, the cost of solving the collection of N systems in step (3) is $\mathcal{O}(MN)$, as can be seen from the pseudo-code for penta-diagonal systems shown in [3]. Hence the overall cost of solving the problem with fourth-order accuracy is still $\mathcal{O}(MN \log N)$.

4.4 Numerical Examples

We test the fourth-order scheme in Matlab on four test functions for which we know the exact solutions (similarly to Section 2.4). We directly compute f and g for each of our known test functions to obtain numerical results. The test functions we now use are:

$$u_5 = e^{r(\cos\theta + \sin\theta)},$$

$$u_6 = e^r,$$

$$u_7 = r \cdot z_3\left(\frac{\theta}{2\pi}\right),$$

$$u_8 = r \cdot z_4\left(\frac{\theta}{2\pi}\right),$$

where

$$z_3(x) = 8 - x + 10x^3 - 15x^4 + 6x^5,$$

$$z_4(x) = 8 - x^2 + 5x^4 - 6x^5 + 2x^6.$$

Then $z_3(x)$ is such that

$$z_3'(x) = -1 + 30x^2 - 60x^3 + 30x^4,$$

and

$$z_3''(x) = 60x - 180x^2 + 120x^3,$$

and

$$z_3'''(x) = 60 - 360x + 360x^2,$$

and

$$z_3^{(4)}(x) = -360 + 720x.$$

Hence

$$z_3(0) = z_3(1) = 8, \quad z_3'(0) = z_3'(1) = -1, \quad z_3''(0) = z_3''(1) = 0,$$

$$z_3'''(0) = z_3'''(1) = 60, \quad z_3^{(4)}(0) = -360 \neq 360 = z_3^{(4)}(1).$$

Therefore, for $r \in (0,1)$,

$$u_7(r,0) = u_7(r,2\pi),$$

$$\frac{\partial u_7}{\partial \theta}(r,0) = \frac{\partial u_7}{\partial \theta}(r,2\pi),$$

$$\frac{\partial^2 u_7}{\partial \theta^2}(r,0) = \frac{\partial^2 u_7}{\partial \theta^2}(r,2\pi),$$

$$\frac{\partial^3 u_7}{\partial \theta^3}(r,0) = \frac{\partial^3 u_7}{\partial \theta^3}(r,2\pi),$$

but

$$\frac{\partial^4 u_7}{\partial \theta^4}(r,0) \neq \frac{\partial^4 u_7}{\partial \theta^4}(r,2\pi).$$

Similarly, $z_4(x)$ is such that

$$z_4'(x) = -2x + 20x^3 - 30x^4 + 12x^5,$$

and

$$z_4''(x) = -2 + 60x^2 - 120x^3 + 60x^4,$$

and

$$z_4'''(x) = 120x - 360x^2 + 240x^3,$$

and

$$z_4^{(4)}(x) = 120 - 720x + 720x^2,$$

and

$$z_4^{(5)}(x) = -720 + 1440x.$$

Hence

$$z_4(0) = z_4(1) = 8, \quad z_4'(0) = z_4'(1) = 0, \quad z_4''(0) = z_4''(1) = -2,$$

$$z_4'''(0) = z_4'''(1) = 0, \quad z_4^{(4)}(0) = z_4^{(4)}(1) = 120, \quad z_4^{(5)}(0) = -720 \neq 720 = z_4^{(5)}(1).$$

Therefore, for $r \in (0, 1)$,

$$u_8(r, 0) = u_8(r, 2\pi),$$
$$\frac{\partial u_8}{\partial \theta}(r, 0) = \frac{\partial u_8}{\partial \theta}(r, 2\pi),$$
$$\frac{\partial^2 u_8}{\partial \theta^2}(r, 0) = \frac{\partial^2 u_8}{\partial \theta^2}(r, 2\pi),$$
$$\frac{\partial^3 u_8}{\partial \theta^3}(r, 0) = \frac{\partial^3 u_8}{\partial \theta^3}(r, 2\pi),$$
$$\frac{\partial^4 u_8}{\partial \theta^4}(r, 0) = \frac{\partial^4 u_8}{\partial \theta^4}(r, 2\pi),$$

but

$$\frac{\partial^5 u_8}{\partial \theta^5}(r, 0) \neq \frac{\partial^5 u_8}{\partial \theta^5}(r, 2\pi).$$

We take $M = M_k$ and $N = N_k$, and we again use $N_k = 6M_k$ for each $k = 1, \ldots, 10$. We define the maximum absolute error and rate of convergence as in (2.30) and (2.31).

Table 4.1: Numerical Results for u_5

M	Error	Rate
5	1.7849×10^{-4}	
10	5.2020×10^{-6}	5.1007
15	1.0390×10^{-6}	3.9726
20	3.4717×10^{-7}	3.8106
25	1.4664×10^{-7}	3.8621
30	7.1898×10^{-8}	3.9093
35	3.9273×10^{-8}	3.9229
40	2.3195×10^{-8}	3.9436
45	1.4565×10^{-8}	3.9507
50	9.5966×10^{-9}	3.9598

Table 4.2: Numerical Results for u_6

M	Error	Rate
55	3.6008×10^{-9}	
60	2.6141×10^{-9}	3.6804
65	1.9441×10^{-9}	3.6997
70	1.4754×10^{-9}	3.7227
75	1.1396×10^{-9}	3.7422
80	8.9437×10^{-10}	3.7551
85	7.1154×10^{-10}	3.7723
90	5.7250×10^{-10}	3.8036
95	4.6713×10^{-10}	3.7624
100	3.8439×10^{-10}	3.8006

From the results in Tables 4.1-4.4, we see that the scheme is roughly fourth-order convergent as expected, for most test functions. The results in Table 4.1 show that for a continuous, smooth test function u_5 that depends on both r and θ and has periodicity of all higher-order θ derivatives, the scheme converges at a rate of 4. Similarly, in Table 4.2, we see that even though the test function u_6 has no θ dependency, since u_6 is smooth the method still approaches a convergence rate of about 4. Using the special test function u_7, we expect the lack

Table 4.3: Numerical Results for u_7

M	Error	Rate
5	1.1974×10^{-6}	
10	9.7565×10^{-8}	3.6174
15	2.1736×10^{-8}	3.7033
20	7.3753×10^{-9}	3.7570
25	3.1655×10^{-9}	3.7905
30	1.5784×10^{-9}	3.8168
35	8.7570×10^{-10}	3.8219
40	5.2398×10^{-10}	3.8460
45	3.3290×10^{-10}	3.8512
50	2.2187×10^{-10}	3.8511

Table 4.4: Numerical Results for u_8

M	Error	Rate
5	4.8655×10^{-7}	
10	3.1781×10^{-8}	3.9363
15	6.3548×10^{-9}	3.9699
20	2.0294×10^{-9}	3.9678
25	8.3443×10^{-10}	3.9829
30	4.0374×10^{-10}	3.9819
35	2.1838×10^{-10}	3.9865
40	1.2786×10^{-10}	4.0087
45	8.0020×10^{-11}	3.9793
50	5.2120×10^{-11}	4.0691

of periodicity of the fourth θ derivative to cause the convergence rate to be only 3, but we see from Table 4.3 that we still have a convergence rate of 4. This is surprising, but we note that (3.12) is still satisfied for both u_7 and u_8. We will discuss this further in future work. Not surprisingly, though, when the solution u is periodic in θ up to the fifth θ derivative, we can see from the results of testing function u_8 in Table 4.4 that the method will still be roughly fourth-order convergent.

We conclude that if this method were used to find a solution u that was not smooth, had no dependency on r, or lacked periodicity of higher-order θ derivatives, the method might converge at a rate slower than 4. Since the original problem is converted from Cartesian to polar coordinates, assuming the solution in Cartesian would be smooth, the solution in polar will probably be smooth, dependent on r, and periodic in higher-order θ derivatives; hence we conclude that the fourth-order scheme we are discussing will usually converge at a rate of 4.

5 CONCLUSION

In this paper we have derived and implemented a fourth-order accurate finite difference scheme for Poisson's Equation on a solid disc. This scheme uses a half-point shift in the r direction to avoid the singularity at $r = 0$. It is more simple and direct than previously proposed fourth-order schemes, since it is based on analysis of the truncation error of the second-order scheme and uses only real-valued vectors and matrices. Our scheme also matches, to the best of our knowledge, the cost of the most efficient pre-existing schemes.

Our numerical results demonstrate the fourth-order accuracy of our scheme and seem to suggest that the scheme is stable, although we have yet to prove stability theoretically. Stability has been proven for the second-order scheme using the maximum principle; although it might not be possible to extend that proof for the fourth-order scheme, in the future we will strive to theoretically prove the stability that the numerical results suggest. We will also look in the future to refine the periodicity assumptions on the θ derivatives, as the numerical results suggest that our assumptions are more conservative than necessary. Our method could be extended to solve a problem with Neumann Boundary Conditions instead of Dirichlet Boundary Conditions; this would require a fourth-order accurate discretization of the Neumann Boundary Conditions.

REFERENCES

[1] B. Bialecki, G. Fairweather, and A. Karageorghis. "Matrix Decomposition Algorithms for Modified Spline Collocation for Helmholtz Problems." SIAM *Journal of Scientific Computing*, Vol. 24, No. 5, (2003): 1733-1753.

[2] S. Britt, S. Tsynkov, and E. Turkel. "A Compact Fourth Order Scheme for the Helmholtz Equation in Polar Coordinates." *Journal of Scientific Computing*, 45, Nos. 1-3, (2010): 26-47.

[3] W. Cheney and D. Kincaid. *Numerical Mathematics and Computing.* 6 Ed. Thomson Learning, Inc: Belmont, CA, 2008.

[4] "Documentation Center: Fast Fourier Transform (FFT)." *The MathWorks, Inc.* (2013) `http://www.mathworks.com/help/matlab/math/fast-fourier-transform-fft.html`.

[5] "Documentation Center: fft." *The MathWorks, Inc.* (2013) `http://www.mathworks.com/help/matlab/ref/fft.html`.

[6] D. Eisen. "On the Numerical Solution of $u_t = u_{rr} + \frac{2}{r}u_r$." *Numerische Mathematik*, 10, (1967): 397-409.

[7] R. Haberman. *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems.* 4 Ed. Pearson Prentice Hall: Upper Saddle River, New Jersey, 2004.

[8] R.K. Iyengar and R. Manohar. "High Order Difference Methods for Heat Equation in Polar Cylindrical Coordinates." *Journal of Computational Physics*, 77, (1988): 425–438.

[9] M.K. Jain, R.K. Jain, and M. Krishna. "A Fourth-Order Difference Scheme for Quasi-linear Poisson Equation in Polar Co-ordinates." *Communications in Numerical Methods in Engineering*, 10, (1994): 791–797.

[10] M.C. Lai. "A Simple Compact Fourth-Order Poisson Solver on Polar Geometry." *Journal of Computational Physics*, 182, (2002): 337–345.

[11] M.C. Lai and W.C. Wang. "Fast Direct Solvers for Poisson Equation on 2D Polar and Spherical Geometries." *Numer. Methods Partial Differential Eq.*, 18, (2002): 56-68.

[12] R.C. Mittal and S. Gahlaut. "High-Order Finite-Difference Schemes to Solve Poisson's Equation in Polar Coordinates." IMA *Journal of Numerical Analysis*, 11, (1991): 261–270.

[13] A.A. Samarskii and W.B. Andreev. *Difference Methods for Elliptic Equations.* Nauka, Moskow, 1976. (In Russian.)

[14] A.A. Samarskii and E.S. Nikolaev. *Numerical Methods for Grid Equations.* Vol. I, "Direct Methods". Birkhäuser Verlag: Basel, Boston, Berlin, 1989.

APPENDIX

6.1 r Direction

The formulas for $L_r^{(3)}$ and $L_r^{(4)}$ from (3.26) and (3.27) were derived using the method of undetermined coefficients, the boundedness from (3.2), and the Taylor Expansions on the exact solution $u(r,\theta)$, with the convention that $u_{i,j} = u(r_i, \theta_j)$. Proofs of their accuracy are provided below.

6.1.1 $i = 1$

We use Taylor's Theorem to expand the exact solution around (r_1, θ_j) to get

$$u_{k,j} = u_{1,j} + (k-1)h_r \left[\frac{\partial u}{\partial r}\right]_{1,j} + \frac{(k-1)^2 h_r^2}{2!}\left[\frac{\partial^2 u}{\partial r^2}\right]_{1,j} + \frac{(k-1)^3 h_r^3}{3!}\left[\frac{\partial^3 u}{\partial r^3}\right]_{1,j} + \frac{(k-1)^4 h_r^4}{4!}\left[\frac{\partial^4 u}{\partial r^4}\right]_{1,j} + \mathcal{O}(h_r^5), \quad k = 2,3,4,5. \quad (6.1)$$

From (3.26) at $i = 1$, we have

$$L_r^{(3)} u_{1,j} = \frac{1}{2h_r^3}\left(-5u_{1,j} + 18u_{2,j} - 24u_{3,j} + 14u_{4,j} - 3u_{5,j}\right),$$

and we substitute in $u_{k,j}$ from (6.1). Then the coefficient of $u_{1,j}$ is

$$\frac{1}{2h_r^3}\left(-5 + 18 - 24 + 14 - 3\right) = \frac{1}{2h_r^3}(0) = 0.$$

The coefficient of $\left[\frac{\partial^l u}{\partial r^l}\right]_{1,j}$ for $l = 1,2,3,4$ is

$$\frac{h_r^{l-3}}{2l!}\left[18 - 24\cdot 2^l + 14\cdot 3^l - 3\cdot 4^l\right],$$

which is 0 for $l = 1,2,4$ and is 1 for $l = 3$. This was verified using Matlab. Hence

$$L_r^{(3)} u_{1,j} = \left[\frac{\partial^3 u}{\partial r^3}\right]_{1,j} + \frac{1}{h_r^3}\mathcal{O}(h_r^5) = \left[\frac{\partial^3 u}{\partial r^3}\right]_{1,j} + \mathcal{O}(h_r^2).$$

Also, from (3.27) at $i = 1$, we have

$$L_r^{(4)} u_{1,j} = \frac{1}{h_r^4}\left(u_{1,j} - 4u_{2,j} + 6u_{3,j} - 4u_{4,j} + u_{5,j}\right),$$

and we substitute in $u_{k,j}$ from (6.1). Then the coefficient of $u_{1,j}$ is

$$\frac{1}{h_r^4}(1-4+6-4+1) = \frac{1}{h_r^4}(0) = 0.$$

The coefficient of $\left[\frac{\partial^l u}{\partial r^l}\right]_{1,j}$ for $l = 1, 2, 3, 4$ is

$$\frac{h_r^{l-4}}{l!}[-4 + 6 \cdot 2^l - 4 \cdot 3^l + 4^l],$$

which is 0 for $l = 1, 2, 3$ and is 1 for $l = 4$. Hence

$$L_r^{(4)} u_{1,j} = \left[\frac{\partial^4 u}{\partial r^4}\right]_{1,j} + \frac{1}{h_r^4}\mathcal{O}(h_r^5) = \left[\frac{\partial^4 u}{\partial r^4}\right]_{1,j} + \mathcal{O}(h_r).$$

6.1.2 $i = 2$

We use Taylor's Theorem to expand the exact solution around (r_2, θ_j) to get

$$u_{k,j} = u_{2,j} + (k-2)h_r \left[\frac{\partial u}{\partial r}\right]_{2,j} + \frac{(k-2)^2 h_r^2}{2!}\left[\frac{\partial^2 u}{\partial r^2}\right]_{2,j} + \frac{(k-2)^3 h_r^3}{3!}\left[\frac{\partial^3 u}{\partial r^3}\right]_{2,j} + \frac{(k-2)^4 h_r^4}{4!}\left[\frac{\partial^4 u}{\partial r^4}\right]_{2,j} + \mathcal{O}(h_r^5), \quad k = 1, 3, 4, 5. \tag{6.2}$$

From (3.26) at $i = 2$, we have

$$L_r^{(3)} u_{2,j} = \frac{1}{2h_r^3}(-3u_{1,j} + 10u_{2,j} - 12u_{3,j} + 6u_{4,j} - u_{5,j}),$$

and we substitute in $u_{k,j}$ from (6.2). Then the coefficient of $u_{2,j}$ is

$$\frac{1}{2h_r^3}(-3 + 10 - 12 + 6 - 1) = \frac{1}{2h_r^3}(0) = 0.$$

The coefficient of $\left[\frac{\partial^l u}{\partial r^l}\right]_{2,j}$ for $l = 1, 2, 3, 4$ is

$$\frac{h_r^{l-3}}{2l!}\left[-3 \cdot (-1)^l - 12 + 6 \cdot 2^l - 3^l\right],$$

which is 0 for $l = 1, 2, 4$ and is 1 for $l = 3$. Hence

$$L_r^{(3)} u_{2,j} = \left[\frac{\partial^3 u}{\partial r^3}\right]_{2,j} + \frac{1}{h_r^3}\mathcal{O}(h_r^5) = \left[\frac{\partial^3 u}{\partial r^3}\right]_{2,j} + \mathcal{O}(h_r^2).$$

Also, from (3.27) at $i = 2$, we have

$$L_r^{(4)} u_{2,j} = \frac{1}{h_r^4}(u_{1,j} - 4u_{2,j} + 6u_{3,j} - 4u_{4,j} + u_{5,j}),$$

and we substitute in $u_{k,j}$ from (6.2). Then the coefficient of $u_{2,j}$ is

$$\frac{1}{h_r^4}\left(1-4+6-4+1\right)=\frac{1}{h_r^4}(0)=0.$$

The coefficient of $\left[\frac{\partial^l u}{\partial r^l}\right]_{2,j}$ for $l=1,2,3,4$ is

$$\frac{h_r^{l-4}}{l!}[(-1)^l+6-4\cdot 2^l+3^l],$$

which is 0 for $l=1,2,3$ and is 1 for $l=4$. Hence

$$L_r^{(4)}u_{2,j}=\left[\frac{\partial^4 u}{\partial r^4}\right]_{2,j}+\frac{1}{h_r^4}\mathcal{O}(h_r^5)=\left[\frac{\partial^4 u}{\partial r^4}\right]_{2,j}+\mathcal{O}(h_r).$$

6.1.3 $i=M$

We use Taylor's Theorem to expand the exact solution around (r_M,θ_j) to get

$$u_{M+k,j}=u_{M,j}+kh_r\left[\frac{\partial u}{\partial r}\right]_{M,j}+\frac{k^2h_r^2}{2!}\left[\frac{\partial^2 u}{\partial r^2}\right]_{M,j}+\frac{k^3h_r^3}{3!}\left[\frac{\partial^3 u}{\partial r^3}\right]_{M,j}+$$
$$\frac{k^4h_r^4}{4!}\left[\frac{\partial^4 u}{\partial r^4}\right]_{M,j}+\frac{k^5h_r^5}{5!}\left[\frac{\partial^5 u}{\partial r^5}\right]_{M,j}+\mathcal{O}(h_r^6),\quad k=-4,-3,-2,-1,1. \quad (6.3)$$

From (3.26) at $i=M$, we have

$$L_r^{(3)}u_{M,j}=\frac{1}{2h_r^3}\left(u_{M-3,j}-6u_{M-2,j}+12u_{M-1,j}-10u_{M,j}+3u_{M+1,j}\right),$$

and we substitute in $u_{k,j}$ from (6.3). Then the coefficient of $u_{M,j}$ is

$$\frac{1}{2h_r^3}\left(1-6+12-10+3\right)=\frac{1}{2h_r^3}(0)=0.$$

The coefficient of $\left[\frac{\partial^l u}{\partial r^l}\right]_{M,j}$ for $l=1,2,3,4$ is

$$\frac{h_r^{l-3}}{2l!}\left[(-3)^l-6\cdot(-2)^l+12\cdot(-1)^l+3\right],$$

which is 0 for $l=1,2,4$ and is 1 for $l=3$. Hence

$$L_r^{(3)}u_{M,j}=\left[\frac{\partial^3 u}{\partial r^3}\right]_{M,j}+\frac{1}{h_r^3}\mathcal{O}(h_r^5)=\left[\frac{\partial^3 u}{\partial r^3}\right]_{M,j}+\mathcal{O}(h_r^2).$$

Also, from (3.27) at $i=M$, we have

$$L_r^{(4)}u_{M,j}=\frac{1}{h_r^4}\left(-u_{M-4,j}+6u_{M-3,j}-14u_{M-2,j}+16u_{M-1,j}-9u_{M,j}+2u_{M+1,j}\right),$$

and we substitute in $u_{k,j}$ from (6.3). Then the coefficient of $u_{M,j}$ is

$$\frac{1}{h_r^4}\left(-1+6-14+16-9+2\right) = \frac{1}{h_r^4}(0) = 0.$$

The coefficient of $\left[\frac{\partial^l u}{\partial r^l}\right]_{M,j}$ for $l = 1, 2, 3, 4, 5$ is

$$\frac{h_r^{l-4}}{l!}[-(-4)^l + 6\cdot(-3)^l - 14\cdot(-2)^l + 16\cdot(-1)^l + 2],$$

which is 0 for $l = 1, 2, 3, 5$ and is 1 for $l = 4$. Hence

$$L_r^{(4)} u_{M,j} = \left[\frac{\partial^4 u}{\partial r^4}\right]_{M,j} + \frac{1}{h_r^4}\mathcal{O}(h_r^6) = \left[\frac{\partial^4 u}{\partial r^4}\right]_{M,j} + \mathcal{O}(h_r^2).$$

6.2 θ Direction

The formula for $L_\theta^{(4)}$ from (3.29) was derived using the method of undetermined coefficients, the boundedness from (3.12), and Taylor Expansions on the exact solution $u(r,\theta)$, with the convention that $u_{i,j} = u(r_i, \theta_j)$. Proof of its accuracy is provided below.

6.2.1 $j = 3, \ldots, N-2$

We use Taylor's Theorem to expand the exact solution around (r_i, θ_j) and (3.12) to get

$$u_{i,j+k} = u_{i,j} + kh_\theta\left[\frac{\partial u}{\partial \theta}\right]_{i,j} + \frac{k^2h_\theta^2}{2!}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,j} + \frac{k^3h_\theta^3}{3!}\left[\frac{\partial^3 u}{\partial \theta^3}\right]_{i,j} + \frac{k^4h_\theta^4}{4!}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,j} + \frac{k^5h_\theta^5}{5!}\left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,j} + r_i\mathcal{O}(h_\theta^6), \quad k = -2, -1, 1, 2. \quad (6.4)$$

From (3.29) for $j = 3, \ldots, N-2$, we have

$$L_\theta^{(4)} u_{i,j} = \frac{1}{h_\theta^4}(u_{i,j-2} - 4u_{i,j-1} + 6u_{i,j} - 4u_{i,j+1} + u_{i,j+2}),$$

and we substitute in $u_{i,j+k}$ from (6.4). Then the coefficient of $u_{i,j}$ is

$$\frac{1}{h_\theta^4}\left(1-4+6-4+1\right) = \frac{1}{h_\theta^4}(0) = 0.$$

The coefficient of $\left[\frac{\partial^l u}{\partial \theta^l}\right]_{i,j}$ for $l = 1, 2, 3, 4, 5$ is

$$\frac{h_\theta^{l-4}}{l!}\left[(-2)^l - 4\cdot(-1)^l - 4 + 2^l\right],$$

which is 0 for $l = 1, 2, 3, 5$ and is 1 for $l = 4$. Hence

$$L_\theta^{(4)} u_{i,j} = \left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,j} + \frac{1}{h_\theta^4} r_i\mathcal{O}(h_\theta^6) = \left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,j} + r_i\mathcal{O}(h_\theta^2), \quad j = 3, \ldots, N-2. \quad (6.5)$$

6.2.2 $j = 1$

We use Taylor's Theorem to expand the exact solution around (r_i, θ_0) and (3.12) to get

$$u_{i,k} = u_{i,0} + kh_\theta \left[\frac{\partial u}{\partial \theta}\right]_{i,0} + \frac{k^2 h_\theta^2}{2!}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,0} + \frac{k^3 h_\theta^3}{3!}\left[\frac{\partial^3 u}{\partial \theta^3}\right]_{i,0} + \frac{k^4 h_\theta^4}{4!}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,0} + \frac{k^5 h_\theta^5}{5!}\left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,0} + r_i\mathcal{O}(h_\theta^6), \quad k = 1, 2, 3. \tag{6.6}$$

Also using (3.20) with (1.5) and additionally assuming periodicity of the second, third, fourth, and fifth θ derivatives, we get

$$u_{i,N-1} = u_{i,0} - h_\theta \left[\frac{\partial u}{\partial \theta}\right]_{i,0} + \frac{h_\theta^2}{2!}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,0} - \frac{h_\theta^3}{3!}\left[\frac{\partial^3 u}{\partial \theta^3}\right]_{i,0} + \frac{h_\theta^4}{4!}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,0} - \frac{h_\theta^5}{5!}\left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,0} + r_i\mathcal{O}(h_\theta^6). \tag{6.7}$$

From (3.29) and with $j = 1$, we have

$$L_\theta^{(4)} u_{i,1} = \frac{1}{h_\theta^4}(u_{i,N-1} - 4u_{i,N} + 6u_{i,1} - 4u_{i,2} + u_{i,3}).$$

So we use the first equation in (1.5) to obtain

$$L_\theta^{(4)} u_{i,1} = \frac{1}{h_\theta^4}(u_{i,N-1} - 4u_{i,0} + 6u_{i,1} - 4u_{i,2} + u_{i,3}),$$

and we substitute in $u_{i,k}$ from (6.6) and $u_{i,N-1}$ from (6.7). Then the coefficient of $u_{i,0}$ is

$$\frac{1}{h_\theta^4}(1 - 4 + 6 - 4 + 1) = \frac{1}{h_\theta^4}(0) = 0.$$

The coefficient of $\left[\frac{\partial^l u}{\partial \theta^l}\right]_{i,0}$ for $l = 1, 2, 3, 4, 5$ is

$$\frac{h_\theta^{l-4}}{l!}\left[(-1)^l + 6 - 4 \cdot 2^l + 3^l\right],$$

which is 0 for $l = 1, 2, 3, 5$ and is 1 for $l = 4$.

So we have

$$\begin{aligned} L_\theta^{(4)} u_{i,1} &= \left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,0} + h_\theta \left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,0} + \frac{1}{h_\theta^4} r_i\mathcal{O}(h_\theta^6) \\ &= \left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,0} + h_\theta \left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,0} + r_i\mathcal{O}(h_\theta^2). \end{aligned}$$

Also,

$$\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,1} = \left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,0} + h_\theta \left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,0} + r_i\mathcal{O}(h_\theta^2).$$

Hence

$$L_\theta^{(4)} u_{i,1} = \left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,1} + r_i\mathcal{O}(h_\theta^2).$$

6.2.3 $j = 2$

From (3.29) we have

$$L_\theta^{(4)} u_{i,2} = \frac{1}{h_\theta^4}(u_{i,N} - 4u_{i,1} + 6u_{i,2} - 4u_{i,3} + u_{i,4}).$$

We use the first equation of (1.5) to obtain

$$L_\theta^{(4)} u_{i,2} = \frac{1}{h_\theta^4}(u_{i,0} - 4u_{i,1} + 6u_{i,2} - 4u_{i,3} + u_{i,4}).$$

Since $\theta_0 = 0$ is within our domain, this is similar to the case considered in Section 6.2.1 and reduces to (6.5).

6.2.4 $j = N - 1$

We use Taylor's Theorem to expand the exact solution around (r_i, θ_N) and (3.12) to get

$$u_{i,N-k} = u_{i,N} - kh_\theta \left[\frac{\partial u}{\partial \theta}\right]_{i,N} + \frac{k^2 h_\theta^2}{2!}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,N} - \frac{k^3 h_\theta^3}{3!}\left[\frac{\partial^3 u}{\partial \theta^3}\right]_{i,N} + \frac{k^4 h_\theta^4}{4!}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,N} - \frac{k^5 h_\theta^5}{5!}\left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,N} + r_i \mathcal{O}(h_\theta^6), \quad k = 1, 2, 3. \tag{6.8}$$

Also using (3.19) with (1.5) and additionally assuming periodicity of the second, third, fourth, and fifth θ derivatives, we get

$$u_{i,1} = u_{i,N} + h_\theta \left[\frac{\partial u}{\partial \theta}\right]_{i,N} + \frac{h_\theta^2}{2!}\left[\frac{\partial^2 u}{\partial \theta^2}\right]_{i,N} + \frac{h_\theta^3}{3!}\left[\frac{\partial^3 u}{\partial \theta^3}\right]_{i,N} + \frac{h_\theta^4}{4!}\left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,N} + \frac{h_\theta^5}{5!}\left[\frac{\partial^5 u}{\partial \theta^5}\right]_{i,N} + r_i \mathcal{O}(h_\theta^6). \tag{6.9}$$

From (3.29) and with $j = N - 1$, we have

$$L_\theta^{(4)} u_{i,N-1} = \frac{1}{h_\theta^4}(u_{i,N-3} - 4u_{i,N-2} + 6u_{i,N-1} - 4u_{i,N} + u_{i,1}),$$

and we substitute in $u_{i,N-k}$ from (6.8) and $u_{i,1}$ from (6.9). Then the coefficient of $u_{i,N}$ is

$$\frac{1}{h_\theta^4}(1 - 4 + 6 - 4 + 1) = \frac{1}{h_\theta^4}(0) = 0.$$

The coefficient of $\left[\frac{\partial^l u}{\partial \theta^l}\right]_{i,N}$ for $l = 1, 2, 3, 4, 5$ is

$$\frac{h_\theta^{l-4}}{l!}\left[(-3)^l - 4 \cdot (-2)^l + 6 \cdot (-1)^l + 1\right],$$

which is 0 for $l = 1, 2, 3, 5$ and is 1 for $l = 4$.

So we have

$$L_\theta^{(4)} u_{i,N-1} = \left[\frac{\partial^4 u}{\partial\theta^4}\right]_{i,N} - h_\theta \left[\frac{\partial^5 u}{\partial\theta^5}\right]_{i,N} + \frac{1}{h_\theta^4} r_i \mathcal{O}(h_\theta^6)$$
$$= \left[\frac{\partial^4 u}{\partial\theta^4}\right]_{i,N} - h_\theta \left[\frac{\partial^5 u}{\partial\theta^5}\right]_{i,N} + r_i \mathcal{O}(h_\theta^2).$$

Also,

$$\left[\frac{\partial^4 u}{\partial\theta^4}\right]_{i,N-1} = \left[\frac{\partial^4 u}{\partial\theta^4}\right]_{i,N} - h_\theta \left[\frac{\partial^5 u}{\partial\theta^5}\right]_{i,N} + r_i \mathcal{O}(h_\theta^2).$$

Hence

$$L_\theta^{(4)} u_{i,N-1} = \left[\frac{\partial^4 u}{\partial\theta^4}\right]_{i,N-1} + r_i \mathcal{O}(h_\theta^2).$$

6.2.5 $j = N$

We use Taylor's Theorem to expand the exact solution around (r_i, θ_N) and (3.12) to get

$$u_{i,N-k} = u_{i,N} - kh_\theta \left[\frac{\partial u}{\partial\theta}\right]_{i,N} + \frac{k^2 h_\theta^2}{2!} \left[\frac{\partial^2 u}{\partial\theta^2}\right]_{i,N} - \frac{k^3 h_\theta^3}{3!} \left[\frac{\partial^3 u}{\partial\theta^3}\right]_{i,N} + \frac{k^4 h_\theta^4}{4!} \left[\frac{\partial^4 u}{\partial\theta^4}\right]_{i,N} - \frac{k^5 h_\theta^5}{5!} \left[\frac{\partial^5 u}{\partial\theta^5}\right]_{i,N} + r_i \mathcal{O}(h_\theta^6), \quad k = 1, 2. \quad (6.10)$$

Also in (6.9) we have

$$u_{i,1} = u_{i,N} + h_\theta \left[\frac{\partial u}{\partial\theta}\right]_{i,N} + \frac{h_\theta^2}{2!} \left[\frac{\partial^2 u}{\partial\theta^2}\right]_{i,N} + \frac{h_\theta^3}{3!} \left[\frac{\partial^3 u}{\partial\theta^3}\right]_{i,N} + \frac{h_\theta^4}{4!} \left[\frac{\partial^4 u}{\partial\theta^4}\right]_{i,N} + \frac{h_\theta^5}{5!} \left[\frac{\partial^5 u}{\partial\theta^5}\right]_{i,N} + r_i \mathcal{O}(h_\theta^6).$$

Also, by Taylor Expansion and using (3.12), we have

$$u_{i,2} = u_{i,0} + 2h_\theta \left[\frac{\partial u}{\partial\theta}\right]_{i,0} + \frac{2^2 h_\theta^2}{2!} \left[\frac{\partial^2 u}{\partial\theta^2}\right]_{i,0} + \frac{2^3 h_\theta^3}{3!} \left[\frac{\partial^3 u}{\partial\theta^3}\right]_{i,0} + \frac{2^4 h_\theta^4}{4!} \left[\frac{\partial^4 u}{\partial\theta^4}\right]_{i,0} + \frac{2^5 h_\theta^5}{5!} \left[\frac{\partial^5 u}{\partial\theta^5}\right]_{i,0} + r_i \mathcal{O}(h_\theta^6),$$

and we use (1.5) and assume periodicity of the second, third, fourth, and fifth θ derivatives to get

$$u_{i,2} = u_{i,N} + 2h_\theta \left[\frac{\partial u}{\partial\theta}\right]_{i,N} + \frac{2^2 h_\theta^2}{2!} \left[\frac{\partial^2 u}{\partial\theta^2}\right]_{i,N} + \frac{2^3 h_\theta^3}{3!} \left[\frac{\partial^3 u}{\partial\theta^3}\right]_{i,N} + \frac{2^4 h_\theta^4}{4!} \left[\frac{\partial^4 u}{\partial\theta^4}\right]_{i,N} + \frac{2^5 h_\theta^5}{5!} \left[\frac{\partial^5 u}{\partial\theta^5}\right]_{i,N} + r_i \mathcal{O}(h_\theta^6). \quad (6.11)$$

From (3.29) and with $j = N$, we have

$$L_\theta^{(4)} u_{i,N} = \frac{1}{h_\theta^4}(u_{i,N-2} - 4u_{i,N-1} + 6u_{i,N} - 4u_{i,1} + u_{i,2}),$$

and we substitute in $u_{i,N-k}$ from (6.10), $u_{i,1}$ from (6.9), and $u_{i,2}$ from (6.11). Then the coefficient of $u_{i,N}$ is

$$\frac{1}{h_\theta^4}(1-4+6-4+1) = \frac{1}{h_\theta^4}(0) = 0.$$

The coefficient of $\left[\frac{\partial^l u}{\partial \theta^l}\right]_{i,N}$ for $l = 1,2,3,4,5$ is

$$\frac{h_\theta^{l-4}}{l!}\left[(-2)^l - 4\cdot(-1)^l - 4 + 2^l\right],$$

which is 0 for $l = 1,2,3,5$ and is 1 for $l = 4$. Hence

$$L_\theta^{(4)} u_{i,N} = \left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,N} + \frac{1}{h_\theta^4} r_i \mathcal{O}(h_\theta^6) = \left[\frac{\partial^4 u}{\partial \theta^4}\right]_{i,N} + r_i \mathcal{O}(h_\theta^2).$$

6.3 Use of Fast Fourier Transforms

Since we test our method using only even values of N, we use the definition of Q given in (2.24). Also, from [5] we know that the use of the command $\mathbf{X} = \text{fft}(\mathbf{x})$ in Matlab yields

$$(\mathbf{X})_k = \sum_{j=1}^{N} (\mathbf{x})_j \omega_N^{(j-1)(k-1)},$$

where

$$\omega_N = e^{-2\pi i/N}.$$

To multiply a real vector $\mathbf{v}$ by Q^T, we take

$$\mathbf{x} = \sqrt{\frac{2}{N}} \cdot \mathbf{v},$$

compute $\mathbf{X} = \text{fft}(\mathbf{x})$, find

$$\begin{cases} (\mathbf{V})_k = \mathrm{R}[(\mathbf{X})_k], & k = 1,\ldots,\frac{N}{2}+1, \\ (\mathbf{V})_k = \mathrm{Im}[(\mathbf{X})_k], & k = \frac{N}{2}+2,\ldots,N, \end{cases}$$

and take $(\mathbf{V})_k = (\mathbf{V})_k \cdot \frac{1}{\sqrt{2}}$ for $k = 1, \frac{N}{2}+1$. Then

$$\mathbf{V} = Q^T \mathbf{v}.$$

To multiply a real vector $\mathbf{v}$ by Q, we take

$$\mathbf{a} = \begin{cases} (\mathbf{a})_j = \sqrt{\frac{1}{N}} \cdot (\mathbf{v})_j, & j = 1, \frac{N}{2}+1, \\ (\mathbf{a})_j = \sqrt{\frac{2}{N}} \cdot (\mathbf{v})_j, & j = 2,\ldots,\frac{N}{2}, \\ (\mathbf{a})_j = 0, & j = \frac{N}{2}+2,\ldots,N. \end{cases}$$

and

$$\mathbf{b} = \begin{cases} (\mathbf{b})_j = 0, & j = 1, \ldots, \frac{N}{2}+1, \\ \\ (\mathbf{b})_j = -\sqrt{\frac{2}{N}} \cdot (\mathbf{v})_j, & j = \frac{N}{2}+2, \ldots, N. \end{cases}$$

Then we use $\mathbf{x} = \mathbf{a} + i\mathbf{b}$, and compute $\mathbf{X} = \text{fft}(\mathbf{x})$, and take $\mathbf{V} = \text{real}(\mathbf{X})$. So

$$\mathbf{V} = Q\mathbf{v}.$$