FRACTIONAL DIFFUSION IN NATURALLY FRACTURED UNCONVENTIONAL RESERVOIRS

by

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ABSTRACT

The objective of the research presented in this Master of Science thesis is to examine the concept of anomalous diffusion as an alternative to the conventional dual-porosity idealizations used to model the stimulated reservoir volume (SRV) around fractured horizontal wells in tight, unconventional reservoirs. The motivation is the recent skepticism about the applicability of dual-porosity models to fractured reservoirs due to the scale and discontinuity of the fractures and the complexity of the heterogeneous, nano-porous matrix and the increased awareness of the promises of fractional derivatives in representing anomalous diffusion in highly heterogeneous porous media.

A trilinear, anomalous-diffusion (TAD) model has been developed for fractured horizontal wells surrounded by an SRV in this work and compared with the existing trilinear, dual-porosity (TDP) model. The trilinear flow model is used because of its relative simplicity and availability for the dual-porosity idealization, which allows a direct comparison. The work includes the analytical solution of a general, 1D time-fractional diffusion equation for a bounded system and the implementation of the new solution in the trilinear model formulation for the fractured inner reservoir. Numerical evaluation of the solution has been performed by a computational code in Matlab. The differences, shortcomings, and advantages of both models are discussed. The application of the models to field data is also demonstrated.

A discussion of the characteristics of the pressure and derivative responses obtained from the TAD model is also provided and related to the fractal nature of fractured media. Physical interpretations are also assigned to fractional derivatives and the phenomenological coefficient of the fractional flux law. It is shown that the anomalous diffusion formulation does not require
explicit references to the intrinsic properties of the matrix and fracture media and thus relaxes the stringent requirements used in dual-porosity idealizations to couple matrix and fracture flows. The trilinear anomalous-diffusion model should be useful for performance predictions and pressure- and rate-transient analysis of fractured horizontal wells in tight unconventional reservoirs.
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CHAPTER 1

INTRODUCTION

This thesis has been prepared and submitted for the partial fulfillment of the requirements for a Master of Science degree in Petroleum Engineering at the Colorado School of Mines. The thesis research has been conducted under the Unconventional Reservoir Engineering Project (UREP). The objective of the research is to investigate the potential of the anomalous diffusion concept as an alternative to the dual-porosity idealizations of the stimulated reservoir volume (SRV) around fractured horizontal wells in tight, unconventional reservoirs. The general skepticism around the applicability of dual-porosity models to fractured reservoirs, and the increased complexity of the fractures and the nano-porous matrix in tight, unconventional reservoirs have provided the motivations of this study.

In this chapter, the problem statement and the motivation of the research are provided. The methodology of the study and the hypotheses of the research are also documented. Chapter 2 presents the background and the relevant literature review for the subject of the research. A trilinear, anomalous-diffusion model is developed and verified in Chapter 3. Asymptotic approximations of the trilinear, anomalous-diffusion solution are derived in Chapter 4. Chapter 5 presents the discussion of the results and documents the findings of the thesis research. Finally, in Chapter 6, the conclusions and the recommendations of the thesis are presented.

1.1 Problem Statement and Motivation

Shale oil and shale gas resources are important sources of domestic oil and gas production. Tight oil production in the US has increased from virtually non-existent (late 1990s)
to about 1/4 of the net domestic production (2011). The production of shale gas has also increased significantly in the same period, following a similar trend. It is projected that these resources will become increasingly important in the domestic oil/gas production portfolio, and will help narrow the gap between energy and more specifically liquid fuel consumption and domestic production on the road toward energy independence.

However, the development and management of these unconventional resources are subject to significant uncertainties. These resources are termed “unconventional” due to their very low permeability ($10^{-16}$ m$^2$ to $10^{-20}$ m$^2$) and severe heterogeneity caused by various scales of connected and unconnected fractures, fissures, micro, macro, and inter-aggregate pores, and the conglomerations of organic matter. The heterogeneity of the microscopic structures also causes preferential flow at the macroscopic level by creating a highly non-uniform velocity field. Preferential flow in the non-uniform velocity field leads to non-equilibrium conditions with respect to pressure and concentrations of hydrocarbon components, which further complicate our ability to model and predict flow and transport in such heterogeneous media. What separates these characterization and modeling challenges from those in conventional reservoirs is the lack of a clear scale separation in unconventional reservoirs.

In order to produce from unconventional reservoirs at economic rates, we need to stimulate these reservoirs. Hydraulic fracturing is the most efficient way to produce from tight formations. However, because of extremely tight matrix and the existence of a complex network of natural fractures in most currently producing nano-porous unconventional reservoirs (Figure 1.1), it has been commonly agreed that the role of hydraulic fractures is more associated with creating (or rejuvenating) a network of natural fractures in tight matrix. The fractures in the network usually display large variations of scale, connectivity, and conductivity.
The two common approaches to model naturally fractured media are the dual-porosity idealizations and the discrete fracture network models. Dual-porosity models (Barenblatt et al., 1960; Warren and Root, 1963; Kazemi, 1969; Kazemi et al., 1976; de Swaan O., 1976) are based on the continuity assumption and effective averaging of the naturally fractured medium properties (Figure 1.2). They are appropriate for systems where a repeated pattern of continuous fractures can be distributed to the entire flow domain. Considering the large variations of scale, connectivity, and conductivity of fractures in shale, the dual-porosity assumption is only a first order approximation (Kuchuk and Biryukov, 2013 and 2014).

In discrete fracture network (DFN) models (Figure 1.3), it is possible to consider the details and distribution of individual fracture properties. However, DFN models require extensive characterization studies and also lead to computationally inefficient models. In general,
the level of detail that can be incorporated by the DFN model is limited by the capabilities of the flow model, which will use the DFN model. From a practical perspective, DFN models are not well suited for routine engineering applications.

![Discrete Fracture Model](image)

Figure 1.3 – Representation of fractures in DFN models.

Considering the shortcomings of the traditional modeling options for fractures in unconventional reservoirs, there is a need to search for alternatives. In the last two decades, non-local, memory-dependent descriptions of flow and transport have gained notable popularity among scientists, engineers, and mathematicians focusing on applications in various forms of nano-porous systems (e.g., Gefen et al., 1983; Le Mehaute and Crepy, 1983; Nigmatullin, 1984; Chang and Yortsos, 1990; Dassas and Duby, 1995; Caputo, 1998; Molz III et al., 2002; Raghavan, 2011; Fomin et al., 2011). These efforts have not attracted much attention in the oil-field applications due to the dominance of advective (Darcy) flow in conventional reservoirs. In unconventional shale-gas reservoirs, on the other hand, diffusive flow mechanisms have been recently incorporated into flow models due to their considerable contribution to flow in shale matrix (Javadpour et al., 2007; Ozkan et al., 2010; Apaydin, 2012). In these works, the advective and diffusive mechanisms were assumed to be independent of each other and locally defined based on the corresponding gradients of the process variables (pressure and concentration); an assumption that presumes linearly additive fluxes and permits the use of the classical diffusion
equation. There has been enough evidence in the literature (Fomin et al., 2011) that these are crude assumptions and classical diffusion is a special case, not a norm, for flow and transport in heterogeneous nano-porous medium.

In statistical physics, diffusion is the result of the random, Brownian motion of individual particles. Classical diffusion is usually associated with homogeneous porous media. It is a special case where the random, Brownian motion of the diffusing particles is governed by a Gaussian probability density whose variance is proportional to the first power of time; that is, the mean square displacement of a particle is a linear function of time:

$$\sigma_r^2 \sim Dt$$  \hspace{1cm} (1.1)

However, a convincing number of works have indicated anomalous diffusion in which the mean square variance grows faster (superdiffusion) or slower (subdiffusion) than that in a Gaussian diffusion process. Thus, a general relationship between the mean square variance and time is given by

$$\sigma_r^2 \sim Dt^\alpha$$ \hspace{1cm} (1.2)

where

$$\begin{cases} 
\alpha = 1 \text{ Normal Diffusion} \\
\alpha \neq 1 \text{ Anomalous Diffusion} \\
\alpha > 1 \text{ Superdiffusion} \\
\alpha < 1 \text{ Subdiffusion}
\end{cases}$$

One of the most popular statistical models of anomalous diffusion is the continuous time random walk model, which corresponds to the fractional diffusion equation underlying the Lévy diffusion process (Fomin et al. 2011). It is worthwhile to note that the parameters of the fractional derivative models have clear physical significance and are easy to obtain from a data fitting of experimental or field measurements. In addition, the models of this type are also
mathematically easy to analyze. However, like nonlinear equation models, these models are not
computationally cheap and are also of a phenomenological description, which does not
necessarily reflect the physical mechanism behind the scenes. Ensuing ramifications of the latter
point to characterization of unconventional reservoirs are non-trivial and may require a complete
overhaul of the conventional approaches.

Comparing the classical and anomalous diffusion formulations in one dimension can
demonstrate the physical and mathematical basis of non-local anomalous diffusion (Fomin et al.,
2011). Classical diffusion model is based on Fick’s first (diffusive flux) and second (continuity
equation) laws given, respectively, by

\[
J_c = -D \frac{\partial C}{\partial x} \tag{1.3}
\]

\[
\frac{\partial C}{\partial t} = -\frac{\partial}{\partial x} (J_c) = \frac{\partial}{\partial x} \left( D \frac{\partial C}{\partial x} \right) \tag{1.4}
\]

Let us define the following similarity variables (non-dimensional):

\[
x_D = \frac{x}{x_0} \tag{1.5}
\]

\[
t_D = \frac{t}{t_0} \tag{1.6}
\]

\[
C_D = \frac{C}{C_0} \tag{1.7}
\]

where \(x_0\), \(t_0\), and \(C_0\) are the characteristics scales. Then, we have
Equations 1.4 and 1.8 indicate that the diffusion equation is self-similar if the spatial and temporal scales are related by

\[
x_0^2 = t_0
\]  

(1.9)

which is typical of normal (Fickian) diffusion.

However, not all diffusive processes follow the normal diffusion. For fractal objects, the mean-square displacement of a random walker depends on time as follows:

\[
\left\langle x^2 \right\rangle = t^{2/(2+\theta)}
\]  

(1.10)

where \( \theta \) is the index of anomalous diffusion and \( \theta = 0 \) corresponds to normal diffusion, where

\[
\left\langle x^2 \right\rangle = t
\]  

(1.11)

As a summary statement, the existing models of production from unconventional reservoirs, including those with the dual-porosity formulations, rely on the assumptions of conventional (normal) diffusion. Incorporation of diffusion and desorption from the organic material does not improve these models as they are the minor contributors of flow in these systems. As described above, alternative representations of flow in heterogeneous media may be available within the scope of anomalous (fractional) diffusion. The objective of the proposed research is, therefore, to explore the viability of these models. To permit direct comparisons, the trilinear flow model of fractured horizontal wells in shale will be modified by replacing the dual-porosity idealization in the original formulation with the fractional diffusion assumption.
1.2 Methodology

The general methodology of the proposed research is analytical, for the derivation, and semi-analytical, for the numerical evaluation, of the solutions. The analytical procedure leads to the solution of the 1D, bounded, time-fractional diffusion problem for variable pressure and flux condition at the fracture face (the inner boundary). This solution is incorporated in the trilinear flow formulation by substituting it for the dual-porosity solution for the inner reservoir region (SRV) fractional diffusion. As suggested in the literature (Raghavan, 2011, and Raghavan and Chen, 2013a, b), the solution is derived in the Laplace transform domain and inverted back to the time domain by using the Stehfest (1970) numerical inversion algorithm. The computational code is in MATLAB technical computing language.

The trilinear, anomalous-diffusion (TAD) model is verified against the trilinear, normal-diffusion model for homogeneous system. The new TAD model is also compared with the trilinear, dual-porosity (TDP) model to demonstrate the differences between the normal and anomalous trilinear-flow behaviors. The comparison delineates the significance of the proposed approach. A field example is also presented.

1.3 Hypotheses

There are four hypotheses for this research. They can be listed as follows;

1. Flow in unconventional reservoirs follows anomalous diffusion.

2. The anomalous diffusion models have a larger range of conditions for applicability than the dual-porosity models for fractured reservoirs.

3. Under certain conditions, dual-porosity results approach those of the anomalous diffusion approach.

4. Anomalous diffusion concept can be incorporated into the trilinear flow model.
CHAPTER 2

BACKGROUND AND LITERATURE REVIEW

This chapter presents the background and a summary of the previous works related to this thesis. The previous studies presented here are pertinent to flow from hydraulically fractured wells, fractals, and anomalous diffusion concepts.

2.1 Trilinear Dual-Porosity (TDP) Flow Model

Modeling flow in a multiply fractured horizontal well in a shale formation is of great interest to the industry as production from unconventional reservoirs is starting to dominate field activities. However, because of the limitations of the resources and data, relatively simpler, approximate, practical models are in greater demand. One such model, called trilinear flow model for fractured horizontal wells in shale, has been developed at the Colorado School of Mines (Brown, 2009) and implemented in the common performance prediction and data analysis tools for unconventional reservoirs.

Trilinear flow model is an analytical model, which considers three regions; the outer reservoir beyond the hydraulic fracture tips, the inner reservoir zone between the hydraulic fractures (SRV) and the hydraulic fracture itself. The linear flows in these contiguous regions are coupled by using the continuity of flux and the equality of pressure at the boundaries between the regions. The schematic of the trilinear model is shown in Figure 2.1.
2.1.1 TDP Model Assumptions

The most important assumption made in the trilinear model is the dominance of linear flow regimes. The model presumes linear flows in all three flow-regions, which are coupled at the mutual boundaries of the regions by the continuity of pressure and flux. The flow regions may have different properties.

Hydraulic fractures in the model are identical and have equal distances between them along the horizontal well. This assumption allows us to apply symmetry in the calculations. Because of symmetry, there is a no-flow boundary between two hydraulic fractures. Hydraulic fractures are also assumed to have finite conductivity. The matrix permeability is extremely low and the outer reservoir is assumed not to have significant contribution to production. Flow in the outer reservoir is perpendicular to flow in the inner reservoir.

2.1.2 Dimensionless Definitions in TDP Model

In this section, definitions of the dimensionless variables are presented. The dimensionless pressure, time, and distances are defined, respectively, as follows
\[ p_D = \frac{k_i h_i}{141.2 q B \mu} \Delta p = \frac{k_i h_i}{141.2 q B \mu} (p_i - p) \]  

(2.1)

where;

\( p_D \) – Dimensionless pressure

\( k_i \) – Inner reservoir permeability, md

\( h_i \) – Formation thickness, ft

\( q \) – Hydraulic fracture flow rate, STB/d

\( B \) – Formation volume factor, RB/STB

\( \mu \) – Viscosity of oil, cp

\( p_i \) – Initial reservoir pressure, psi

\[ t_D = 2.637 \times 10^{-4} \frac{\eta_i}{x_F^2} t \]  

(2.2)

\[ \eta_i = \frac{k_i}{(\phi c_i)} \mu \]  

(2.3)

where;

\( t_D \) – Dimensionless time

\( t \) – Time, hours

\( \eta_i \) – Inner reservoir diffusivity, ft^2/hr

\( x_F \) – Hydraulic fracture half length, ft

\( \phi \) – Porosity

\( c_i \) – Total compressibility, psi^{-1}

\[ x_D = \frac{x}{x_F} \]  

(2.4)
where;

\( x_D \) – Dimensionless distance in the x-direction

\( x \) – Distance in the x-direction, ft

\[
y_D = \frac{y}{x_F}
\]  
(2.5)

where;

\( y_D \) – Dimensionless distance in the y-direction

\( y \) – Distance in the y-direction, ft

\[
w_D = \frac{w_F}{x_F}
\]  
(2.6)

where;

\( w_D \) – Dimensionless width of the hydraulic fracture

\( w_F \) – Width of the hydraulic fracture, ft

Dimensionless fracture and reservoir conductivities are defined by

\[
C_{FD} = \frac{k_F w_F}{k_i x_F}
\]  
(2.7)

and

\[
C_{RD} = \frac{k_i x_F}{k_O y_e}
\]  
(2.8)

where;

\( k_F \) – Hydraulic fracture permeability, md

\( k_O \) – Outer reservoir permeability, md

\( y_e \) – Distance to reservoir boundary in y-direction, ft
Also, the diffusivity ratios used in this model are defined by

\[ \eta_{FD} = \frac{\eta_F}{\eta_l} \]  
(2.9)

and

\[ \eta_{OD} = \frac{\eta_O}{\eta_l} \]  
(2.10)

where;

\( \eta_F \) – Diffusivity of the hydraulic fracture

\( \eta_O \) – Diffusivity of the outer reservoir.

### 2.1.3 Derivation of the TDP Model

In this section, the formulation of the TDP model is explained. The model is derived in the Laplace domain in terms of the dimensionless variables defined in the previous section. Solutions are developed for the three flow regions, outer reservoir, inner reservoir, and hydraulic fracture, and then they are coupled by the pressure and flux continuity. In order to convert the results into time domain, Stehfest’s (1970) numerical Laplace inversion algorithm is used.

The outer reservoir is the region beyond the hydraulic fracture tips and assumed to be homogeneous and of low permeability. The flow occurs in the x-direction and it is linear; that is, the pressure is not a function of the y-direction.

Diffusivity equation for the outer reservoir in dimensionless form in Laplace domain is given by Eq. 2.11.

\[
\frac{\partial^2 \bar{p}_{OD}}{\partial x_D^2} - \frac{s}{\eta_{OD}} \bar{p}_{OD} = 0
\]  
(2.11)
where $\bar{p}_{OD}$ is the dimensionless outer reservoir pressure in Laplace domain. The overbar symbol is used to indicate dimensionless pressure in the Laplace domain and $s$ is the Laplace transform parameter with respect to dimensionless time, $t_D$.

The outer and the inner boundary conditions for the outer reservoir are given by

$$
\left( \frac{\partial \bar{p}_{OD}}{\partial x_D} \right)_{x_D=x_{od}} = 0
$$

(2.12)

and

$$
(\bar{p}_{OD})_{x_D=1} = (\bar{p}_{ID})_{x_D=1}
$$

(2.13)

The outer reservoir solution in terms of the inner reservoir pressure at the interface of the inner and outer reservoirs is as follows:

$$
\bar{p}_{OD} = (\bar{p}_{ID})_{x_D=1} \cosh \left[ \sqrt{\frac{s}{\eta_{OD}}} \left( x_{od} - x_D \right) \right] \cosh \left[ \sqrt{\frac{s}{\eta_{OD}}} (x_D - 1) \right]
$$

(2.14)

Inner reservoir is the region between two hydraulic fractures and can be assumed either homogeneous or naturally fractured. For naturally fractured inner reservoirs, trilinear model uses the dual-porosity approach. In inner reservoir, the flow occurs in the $y$-direction. Thus, pressure is not a function of the $x$-direction.

The diffusivity equation associated with the inner reservoir is given by

$$
\frac{\partial^2 \bar{p}_{ID}}{\partial y_D^2} + \left( \frac{1}{y_{ID} C_{RD}} \right) \left( \frac{\partial \bar{p}_{OD}}{\partial x_D} \right)_{x_D=1} - u\bar{p}_{ID} = 0
$$

(2.15)

where

$$
u = sf(s)
$$

(2.16)
$f(s)$ is the transfer function between matrix and natural fractures. In Table 2.1, the formulations of transfer function for homogeneous and dual-porosity models are shown.

Table 2.1 – Transfer function formulations

<table>
<thead>
<tr>
<th>$f(s)$</th>
<th>for homogeneous inner reservoir</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(s) = 1$</td>
<td>for pseudo-steady dual-porosity inner reservoir</td>
</tr>
<tr>
<td>$f(s) = \frac{s\omega(1 - \omega) + \lambda}{s(1 - \omega) + \lambda}$</td>
<td></td>
</tr>
<tr>
<td>$f(s) = 1 + \sqrt{\lambda \omega / (3s)} \tan\left(\sqrt{3\omega s / \lambda}\right)$</td>
<td>for transient dual-porosity inner reservoir</td>
</tr>
</tbody>
</table>

For transient dual porosity model, storativity and transmissivity ratios are defined by

$$\omega = \frac{(\phi c_i)_m h_m}{(\phi c_i)_f h_f}$$

and

$$\lambda = 12 \left( \frac{x_F^2}{h_m^2} \right) \left( \frac{k_m h_m}{k_f h_f} \right)$$

respectively.

Taking the derivative of the outer reservoir solution, Eq. 2.14, and substituting it in the Eq. 2.15 yields

$$\frac{\partial^2 \bar{p}_{ID}}{\partial y_D^2} - \frac{\beta_o}{\gamma_o C_{RD}} (\bar{p}_{ID})_{x_0=1} - \mu \bar{p}_{ID} = 0$$

where
\[
\beta_O = \sqrt{s / \eta_{OD}} \tan \left[ \sqrt{s / \eta_{RD}} (x_{eD} - 1) \right]
\] (2.20)

Since pressure of the inner reservoir is not a function of the distance in the x-direction, Eq. 2.19 can be reduced to

\[
\frac{\partial^2 \bar{p}_{ID}}{\partial y_D^2} - \alpha_O \bar{p}_{ID} = 0
\] (2.21)

where

\[
\alpha_O = \left[ \frac{\beta_O}{y_{eD} C_{RD}} + u \right]
\] (2.22)

The outer and the inner boundary conditions for the inner reservoir are given by

\[
\left( \frac{\partial \bar{p}_{ID}}{\partial y_D} \right)_{y_D = y_{ID}} = 0
\] (2.23)

and

\[
(\bar{p}_{ID})_{y_D = y_{wD}/2} = (\bar{p}_{FD})_{y_D = y_{wD}/2}
\] (2.24)

The inner reservoir solution in terms of the hydraulic fracture pressure at the interface is as follows:

\[
\bar{p}_{ID} = (\bar{p}_{FD})_{y_D = y_{wD}/2} \frac{\cosh \left[ \sqrt{\alpha_O} (y_{eD} - y_D) \right]}{\cosh \left[ \sqrt{\alpha_O} (y_{eD} - w_D / 2) \right]}
\] (2.25)

It must be noted that the \( \alpha_O \) term includes the properties of both outer and inner regions.

Hydraulic fracture is the last region to solve in the system. Finally the solutions will be coupled and the dimensionless wellbore pressure solution will be obtained. Flow in hydraulic fracture is in the x-direction. Diffusivity equation for hydraulic fracture is as follows:
\[
\frac{\partial^2 \tilde{p}_{FD}}{\partial x_D^2} + \left( \frac{2}{C_{FD}} \right) \left( \frac{\partial \tilde{p}_{FD}}{\partial y_D} \right)_{y_D = w_D/2} - \frac{s}{\eta_{FD}} \tilde{p}_{FD} = 0
\]  
(2.26)

Taking the derivative of the inner reservoir solution, Eq. 2.25,

\[
\left( \frac{\partial \bar{p}_{FD}}{\partial y_D} \right)_{y_D = w_D/2} = -\beta_F \left( \bar{p}_{FD} \right)_{y_D = w_D/2}
\]  
(2.27)

where

\[
\beta_F = \sqrt{\alpha_O} \tanh \left[ \sqrt{\alpha_O} \left( y_D - w_D/2 \right) \right]
\]  
(2.28)

When we substitute Eq. 2.27 into Eq. 2.26, we have

\[
\frac{\partial^2 \bar{p}_{FD}}{\partial x_D^2} - \alpha_F \bar{p}_{FD} = 0
\]  
(2.29)

where

\[
\alpha_F = \frac{2\beta_F}{C_{FD}} + \frac{s}{\eta_{FD}}
\]  
(2.30)

The outer and the inner boundary conditions for the hydraulic fracture are given by

\[
\left( \frac{\partial \bar{p}_{FD}}{\partial x_D} \right)_{y_D = 1} = 0
\]  
(2.31)

and

\[
\left( \frac{\partial \bar{p}_{FD}}{\partial x_D} \right)_{y_D = 0} = -\frac{\pi}{sC_{FD}}
\]  
(2.32)

Then, the solution of the dimensionless hydraulic fracture pressure is as follows;
Finally, the hydraulic fracture solution will give us the wellbore pressure solution at the interface between the hydraulic fracture and the wellbore (at $x_D = 0$):

$$
\bar{p}_{FD} = \frac{\pi}{sC_{FD} \sqrt{\alpha_f}} \cosh \left( \sqrt{\alpha_f} \left( 1 - x_D \right) \right) \frac{\sinh \left( \sqrt{\alpha_f} \right)}{\sinh \left( \sqrt{\alpha_f} \right)}
$$

(2.33)

The $\alpha_f$ term in Eq. 2.34 carries the properties of all three flow-regions, the outer reservoir, the inner reservoir, and the hydraulic fracture.

### 2.2 Anomalous Diffusion

The disordered structure of unconventional nano-porous media is more in line with the anomalous diffusion models where the variance of the evolution equations is proportional to the fractional power of time. In addition, transport pathways created by the natural and induced fractures have been shown to be fractals. The cascade of scales characteristic of a network of fractures in a rock is suggestive of fractal geometry and has been so documented on a field scale (Sahimi and Yortsos, 1990). Moreover, transport in disordered systems often involves long-range correlations, another fundamental characteristic of fractal systems. Furthermore, the local gradients of the mean diffusion process variables depend on the global pressure field and lead to a non-local anomalous diffusion process.

Because nonlinear modeling of anomalous diffusion is computationally expensive, fractal and fractional derivatives have been introduced. In petroleum engineering, fractal diffusion has
been used to account for the stochastic nature of heterogeneity, mostly natural fractures. The phenomenological descriptions used in fractional diffusion models are different from the conventional ones and not commonly used in petroleum engineering. The limited applications by Chang and Yortsos (1990), Beier (1994), Flamenco-Lopez and Camacho-Velazquez (2003), and Camacho-Velazquez et al., (2011) focused on fractal modeling of production from fractured media using vertical wells. It is worthwhile to note that the essential difference of the fractal and the fractional derivatives lies in the former being a local operator, while the latter is a global operator.

Fractals have been used to account for the non-uniform properties of fractured reservoirs, but, to our knowledge, have not been applied to unconventional reservoirs. Mandelbrot (1982) introduced fractals as self-similar patterns. The dimension of a fractal exceeds its topological dimension and is not an integer. One of the common examples of fractal geometry is the Koch curve (Figure 2.2).

Figure 2.1 – The triadic Koch curve (Feder, 1988)
Chang and Yortsos (1990) applied fractals to petroleum reservoirs. They developed a general formulation for single-phase flow in a system consisting of a fractal object (fracture network) embedded in a Euclidean object (matrix) (Figure 2.3). They defined the fracture properties as fractals as follows:

\[ k(r) \propto r^{d_f - d - \theta} \]  

\[ \phi(r) \propto r^{d_f - d} \]  

where, \( d_f \) is the fractal dimension, \( d \) is the Euclidean dimension, and \( \theta \) is the conductivity index, which characterizes the diffusion process. In general, their approach uses fractals to define the relation of the fracture properties to the space variable and implements it in the diffusion equation as follows

\[ D(r) = D_f r^{-v} \]  

\[ \frac{\partial C}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( D_f r^{-v} \frac{\partial C}{\partial r} \right) \]  

Figure 2.3 – Schematic of a fractal fracture network embedded in a Euclidean matrix (Chang and Yortsos, 1990)

Beier (1994) extended the work of Chang and Yortsos (1990), which assumed a line source well with radial symmetry, to a hydraulically fractured well in a fractal reservoir. He
noted the limited applicability of the approach proposed by Chang and Yortsos to cylindrical wellbore geometries because of the symmetry assumption inherent in the definition of the fractal properties (Eqs. 2.35 and 2.36). Regardless, he applied the same approach to fractured wells with $d_f = 2$ and the following probability density function suggested by O’Shaughnessy and Procaccia (1985 a, b), which is a stretched Gaussian:

$$p(r,t) = t^{-d_f/d_w} \exp \left[ -c \left( \frac{r}{t^{1/d_w}} \right)^{d_w} \right]; d_w = \theta + 2, \quad \theta > 0$$

(2.39)

Acuna et al. (1995) states that $0.6 < d_f/d_w < 0.86$ based on their evaluation of a number of fractured reservoirs. Similarly, Flamenco-Lopez and Camacho-Velazquez (2001) report the range of $d_f/d_w$ as $0.47 < d_f/d_w < 0.67$ for the fractured reservoirs in Mexico.

Flamenco-Lopez and Camacho-Velazquez (2003) investigated the transient pressure behavior of naturally fractured reservoirs with fractal characteristics. They presented an analytical solution during the pseudo steady-state flow period and showed that it is possible to get the values of all four parameters of the fractal model by using their solution and the transient response.

Camacho-Velazquez et al. (2011) investigated the interference tests in naturally fractured reservoirs exhibiting single-porosity behavior with a fractal network of fractures. They included a temporal fractional derivative in their diffusion equation to account for the history effects of flow in fractal reservoirs. However, as presented, their fundamental flux relation based on the Chang and Yortsos model is not consistent with their spatial and temporal fractional diffusion equation (Raghavan and Chen, 2013a, b).

The efforts of Camacho-Velazquez et al. (2011) to incorporate the memory dependency of flow in fractal reservoirs are in line with the general efforts in mathematical physics and
stochastic fluid mechanics to define anomalous diffusion by a fractional diffusion equation. An example of the anomalous diffusion equation, which includes spatial and temporal fractional derivatives, is given by (Metzler et al., 1994)

\[
\frac{\partial^\gamma P_{Df}}{\partial t^\gamma} = \frac{1}{r_D^{d_{mf} - 1}} \frac{\partial}{\partial r_D} \left( r_D^\beta \frac{\partial P_{Df}}{\partial r_D} \right)
\]  

(2.40)

where

\[
\beta = d_{mf} - \theta - 1
\]  

(2.41)

and

\[
\gamma = 2 / (\theta + 2)
\]  

(2.42)

Recently, Raghavan (2011) and Raghavan and Chen (2013a, b) extended the fractional diffusion models to semi-infinite, fractal media produced through a hydraulic fracture. We are not aware of a bounded linear-system solution, which is amenable to practical numerical calculations. Deriving such a solution is the main problem addressed by this thesis.
CHAPTER 3

TRILINEAR ANOMALOUS DIFFUSION (TAD) MODEL

In this chapter, the derivation of the trilinear model with anomalous diffusion in the SRV of the fractured horizontal well is presented. First, the flux law and temporal-fractional diffusion equation is introduced and then the derivation of the solution is presented. Verification of the solution and application to field data follows.

3.1 The Flux Law and the Fractional Diffusion Equation

The flux law used in the derivation of the TAD model includes the fractional time derivative. As suggested by Raghavan and Chen (2013a and b), in this work, we use the following constitutive relation (flux law) to describe flow in naturally fractured nano-porous media:

\[ v_I = -\lambda_\alpha \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left( \frac{\partial \Delta p_I}{\partial x} \right) \] (3.1)

where \( 0 \leq \alpha \leq 1 \) and \( \lambda_\alpha \) is a phenomenological coefficient. In this work, we assume that the anomalous diffusion is related to the petrophysical heterogeneity of the medium and express the phenomenological coefficient in the following form:

\[ \lambda_\alpha = \frac{k_\alpha}{\mu} \] (3.2)

The temporal fractional derivative in Eq. 3.1 is defined in the Caputo (1967) sense:

\[ \frac{\partial^\beta}{\partial t^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial}{\partial t'} f(t') \frac{dt'}{(t-t')^\beta} \] (3.3)

The convolution integral in Eq. 3.3 signifies the hereditary nature of anomalous diffusion on a
heterogeneous velocity field. In addition, \( k_\alpha \) in Eq. 3.2 is a dynamic property and different from
the conventional Darcy permeability (it has the units of \( L^2T^{1-\alpha} \)). Physical interpretation of \( k_\alpha \) is
not straightforward and, based on Eqs. 3.1 through 3.3, static measurements are not suitable to
determine \( k_\alpha \). Currently, the only viable technique to determine \( k_\alpha \) is to match the dynamic
(transient pressure or flow rate) data with an appropriate model.

Using the flux law (Eq. 3.1) with the mass conservation equation yields the following 2D,
temporal-fractional (anomalous) diffusion equation:

\[
\frac{\partial^2 \Delta p}{\partial x^2} + \frac{\partial^2 \Delta p}{\partial y^2} = \frac{1}{\eta_\alpha} \frac{\partial^{\alpha}}{\partial t^{\alpha}} \Delta p, \quad (3.4)
\]

where

\[
\eta_\alpha = \frac{\lambda_\alpha}{(\phi_c_\alpha)} \quad (3.5)
\]

Subject to the appropriate boundary conditions, the temporal-fractional-diffusion equation given
by Eq. 3.4 is solved in the next section to obtain the solution for a tight, naturally fractured,
unconventional reservoir.

### 3.2 Derivation of the TAD Solution

The difference of the TAD model from the original, TDP model (Brown, 2009) is in the
inner reservoir solution. The outer reservoir and the hydraulic fracture solutions are the same
since we used the normal diffusion equation for these regions. By including the fractional
derivative of time in the inner reservoir solution derivation, we aim to incorporate the hereditary
effects of anomalous diffusion. The continuity of flux and pressure is used for the coupling of the
three regions. For the inner reservoir, the flux relation including the fractional time derivative given in Eq. 3.1 is used.

The inner reservoir is assumed to be a naturally fractured medium. Since fractured unconventional reservoirs are extremely disordered because of the complexity and the discontinuity of the natural fractures, we may define these reservoirs as fractal media and use the fractional diffusion approach to model flow. The formulation only includes the fractional derivative of time, which corresponds to sub-diffusion.

The solutions are given in terms of the scaled variables defined in Section 2.1.2. Note that the scaled variables correspond to the conventional dimensionless variables for the TDP model. For the TAD model, however, $p_D$ and $t_D$ (Eqs 2.1 and 2.2, respectively) are defined based on the phenomenological coefficient, $k_\alpha$, in the constitutive fractional flux law, which is not in the units of permeability. Therefore, although for simplicity, we use the same notation for both models, $p_D$ and $t_D$ are not dimensionless quantities for the trilinear anomalous-diffusion model.

The derivation for the outer reservoir solution is demonstrated in section 2.1.3 and it will be the same for the new derivation. Therefore, we started the derivation with the inner reservoir solution. In the following discussions, we will use the subscripts, $I$ and $O$ for the inner and outer reservoir properties, respectively, $i$ to indicate the initial value of the property, $F$ and $f$ for the hydraulic and natural fracture properties, respectively, $m$ to refer to the property of the matrix, $t$ to indicate total property, $e$ to refer to the external boundary, and $D$ for scaled variables. Although $I$ stands for $f$ in TDP model, for the TAD model, on the other hand, $I$ will correspond to $\alpha$.

The derivation of the trilinear model with anomalous diffusion in the inner reservoir follows the lines similar to those in the TDP model derivation. The main difference is the use of
the fractional diffusion equation for the solution of the inner reservoir problem. Fractional diffusion equation for the inner reservoir is

\[
\frac{\partial}{\partial x} \left( \lambda_a \frac{\partial \Delta p_I}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda_a \frac{\partial \Delta p_I}{\partial y} \right) = (\phi c_i)_a \frac{\partial^\alpha}{\partial t^\alpha} \Delta p_I
\]  

(3.6)

where

\[
\lambda_a = \frac{k_a}{\mu}
\]  

(3.7)

Integrating both sides of Eq. 3.6 yields:

\[
\int_0^x \frac{\partial}{\partial x} \left( \lambda_a \frac{\partial \Delta p_I}{\partial x} \right) \, dx + \int_0^y \frac{\partial}{\partial y} \left( \lambda_a \frac{\partial \Delta p_I}{\partial y} \right) \, dy = (\phi c_i)_a \int_0^x \frac{\partial^\alpha}{\partial t^\alpha} \Delta p_I \, dx
\]  

(3.8)

Recalling that:

\[
\frac{\partial p_I}{\partial y} \neq f(x)
\]  

(3.9)

and

\[
\frac{\partial p_I}{\partial t} \neq f(x)
\]  

(3.10)

Eq. 3.8 becomes,

\[
\lambda_a \frac{\partial \Delta p_I}{\partial x} + \lambda_a \frac{\partial \Delta p_I}{\partial y} \bigg|_{x_F} = (\phi c_i)_a x_F \frac{\partial^\alpha}{\partial t^\alpha} \Delta p_I
\]  

(3.11)

or

\[
\frac{1}{x_F} \frac{\partial \Delta p_I}{\partial x} + \frac{\partial^2 \Delta p_I}{\partial y^2} = \frac{(\phi c_i)_a}{\lambda_a} \frac{\partial^\alpha}{\partial t^\alpha} \Delta p_I
\]  

(3.12)

Assuming \( \lambda_a \) is independent of \( x \) and \( y \), it can be defined that

\[
\eta_a = \frac{k_a}{(\phi c_i)_a \mu} = \frac{\lambda_a}{(\phi c_i)_a}
\]  

(3.13)

Therefore, Eq. 3.12 can be written in the following form:
\[
\frac{1}{x_F} \frac{\partial \Delta p_I}{\partial x} + \frac{\partial^2 \Delta p_I}{\partial y^2} = \frac{1}{\eta_a} \frac{\partial^a}{\partial t^a} \Delta p_I. \tag{3.14}
\]

On the other hand, the continuity of flux at the boundary of the inner and outer reservoirs are given as follows;
\[
\lambda_a \frac{\partial^{1-a}}{\partial t^{1-a}} \left( \frac{\partial \Delta p_I}{\partial x} \right)_{x=x_F} = \frac{k_o}{\mu} \frac{\partial \Delta p_a}{\partial x} \left( \frac{\partial \Delta p_a}{\partial t^a} \right)_{x=x_F}. \tag{3.15}
\]

Converting Eq. 3.14 and Eq. 3.15 into dimensionless form using the definitions given in Section 2.1.2 yields, respectively;
\[
\frac{\partial p_{ID}}{\partial x_D} + \frac{\partial^2 p_{ID}}{\partial y_D^2} = \frac{x_F^2}{\eta_a} \left( \frac{\eta_a}{x_F^2} \right)^{1-a} \frac{\partial^a p_{ID}}{\partial t_D^a} \tag{3.16}
\]
and
\[
\frac{\partial^{1-a}}{\partial t_D^{1-a}} \left( \frac{\partial p_{ID}}{\partial x_D} \right)_{x_D=1} = \frac{k_o}{\mu} \frac{1}{\lambda_a} \left( \frac{x_F^2}{\eta_a} \right)^{1-a} \left( \frac{\partial p_{OD}}{\partial x_D} \right)_{x_D=1}. \tag{3.17}
\]

Applying the Laplace transformation to Eq. 3.16 and Eq. 3.17 with respect to \( t_D \) yields, respectively;
\[
\frac{d\tilde{p}_{ID}}{dx_D} + \frac{d^2\tilde{p}_{ID}}{dy_D^2} = \left( \frac{x_F^2}{\eta_a} \right)^{1-a} s^a \tilde{p}_{ID} \tag{3.18}
\]
and
\[
\left( \frac{d\tilde{p}_{ID}}{dx_D} \right)_{x_D=1} = s^{-1-a} \frac{k_o}{\mu} \frac{1}{\lambda_a} \left( \frac{x_F^2}{\eta_a} \right)^{1-a} \left( \frac{d\tilde{p}_{OD}}{dx_D} \right)_{x_D=1}. \tag{3.19}
\]

Substituting Eq. 3.19 into Eq. 3.18 gives;
\[
s^{-1-a} \frac{k_o}{\mu} \frac{1}{\lambda_a} \left( \frac{x_F^2}{\eta_a} \right)^{1-a} \left( \frac{d\tilde{p}_{OD}}{dx_D} \right)_{x_D=1} + \frac{d^2\tilde{p}_{ID}}{dy_D^2} = \left( \frac{x_F^2}{\eta_a} \right)^{1-a} s^a \tilde{p}_{ID} \tag{3.20}
\]

It may be defined that;
\[
\frac{k_2}{\mu} = \lambda_o \quad (3.21)
\]

Then, Eq. 3.20 can be modified as,
\[
\frac{d^2 \bar{p}_{ID}}{dy_D^2} + s^{(\alpha-1)} \frac{\lambda_o}{\lambda_\alpha} \left( \frac{x_F^2}{\eta_a} \right)^{1-\alpha} \left( \frac{d\bar{p}_{OD}}{dx_D} \right)_{x_o=1} = \left( \frac{x_F^2}{\eta_a} \right)^{1-\alpha} s^\alpha \bar{p}_{ID} \quad (3.22)
\]

Recalling the derivative of the outer reservoir solution from Section 2.1.3,
\[
\left( \frac{d\bar{p}_{OD}}{dx_D} \right)_{x_o=1} = -\beta_o \left( \bar{p}_{ID} \right)_{x_o=1} \quad (3.23)
\]

where
\[
\beta_o = \sqrt{s/\eta_{OD}} \tanh \left[ \sqrt{s/\eta_{OD}} \left( x_{cD} - 1 \right) \right] \quad (3.24)
\]

Substituting Eq. 3.23 back into Eq. 3.22;
\[
\frac{d^2 \bar{p}_{ID}}{dy_D^2} + s^{(\alpha-1)} \frac{\lambda_o}{\lambda_\alpha} \left( \frac{x_F^2}{\eta_a} \right)^{1-\alpha} (-\beta_o) \left( \bar{p}_{ID} \right)_{x_o=1} = \left( \frac{x_F^2}{\eta_a} \right)^{1-\alpha} s^\alpha \bar{p}_{ID} \quad (3.25)
\]

Pressure of the inner reservoir is not a function of distance in the x direction. So,
\[
\left( \bar{p}_{ID} \right)_{x_o=1} = \bar{p}_{ID} \quad (3.26)
\]

Then, Eq. 3.25 becomes
\[
\frac{d^2 \bar{p}_{ID}}{dy_D^2} - \left\{ \left[ \frac{\lambda_o}{\lambda_\alpha} \left( \frac{x_F^2}{\eta_a} \right)^{1-\alpha} \beta_o \right] s^{\alpha-1} + \left[ \left( \frac{x_F^2}{\eta_a} \right)^{1-\alpha} s^\alpha \right] \right\} \bar{p}_{ID} = 0 \quad (3.27)
\]

Defining,
\[
\alpha_o = \left( \frac{x_F^2}{\eta_a} \right)^{1-\alpha} s^\alpha \left[ 1 + \left( \frac{\lambda_o}{\lambda_\alpha} s^{\alpha-1} \beta_o \right) \right] \quad (3.28)
\]

Eq. 3.27 becomes,
\[
\frac{d^2 \bar{p}_{ID}}{dy_D^2} - \alpha_o \bar{p}_{ID} = 0
\]  
(3.29)

The general solution of Eq. 3.29 is;
\[
\bar{p}_{ID} = A \exp\left(-\sqrt{\alpha_o y_D}\right) + B \exp\left(\sqrt{\alpha_o y_D}\right)
\]  
(3.30)

The outer boundary condition for the inner reservoir solution is;
\[
\left( \frac{d \bar{p}_{ID}}{dy_D} \right)_{y_D = y_{oD}} = 0
\]  
(3.31)

Taking the derivative of Eq. 3.30 and using the outer boundary condition;
\[
\left( \frac{d \bar{p}_{ID}}{dy_D} \right)_{y_D = y_{oD}} = -\sqrt{\alpha_o} A \exp\left(-\sqrt{\alpha_o y_{oD}}\right) + \sqrt{\alpha_o} B \exp\left(\sqrt{\alpha_o y_{oD}}\right) = 0
\]  
(3.32)

Then,
\[
B = A \exp\left(-2\sqrt{\alpha_o y_{oD}}\right)
\]  
(3.33)

Substituting Eq. 3.33 back into Eq. 3.30 gives;
\[
\bar{p}_{ID} = A \exp\left(-\sqrt{\alpha_o y_D}\right) + A \exp\left(-2\sqrt{\alpha_o y_{oD}}\right) \exp\left(\sqrt{\alpha_o y_D}\right)
\]  
(3.34)

Eq. 3.34 can be rearranged as,
\[
\bar{p}_{ID} = A \exp\left(-\sqrt{\alpha_o y_{oD}}\right) \left\{ \exp\left[\sqrt{\alpha_o} (y_{oD} - y_D)\right] + \exp\left[-\sqrt{\alpha_o} (y_{oD} - y_D)\right] \right\}
\]  
(3.35)

At the inner boundary of the inner reservoir, the pressure of the hydraulic fracture and inner reservoir must be equal.
\[
\left( \bar{p}_{ID} \right)_{y_D = y_{oD}/2} = \left( \bar{p}_{FB} \right)_{y_D = y_{oD}/2}
\]  
(3.36)

Substituting Eq. 3.35 into the inner boundary condition (Eq. 3.36), we have
\[
\left( \bar{p}_{ID} \right)_{y_D = y_{oD}/2} = A \exp\left(-\sqrt{\alpha_o y_{oD}}\right) \left\{ \exp\left[\sqrt{\alpha_o} (y_{oD} - w_b/2)\right] + \exp\left[-\sqrt{\alpha_o} (y_{oD} - w_b/2)\right] \right\} = \left( \bar{p}_{FB} \right)_{y_D = y_{oD}/2}
\]  
(3.37)
Solving Eq. 3.37 for $A$ yields

$$A = \exp\left(-\sqrt{\alpha_o y_{eD}}\right) \left\{ \exp\left[\sqrt{\alpha_o (y_{eD} - w_D / 2)}\right] + \exp\left[-\sqrt{\alpha_o (y_{eD} - w_D / 2)}\right]\right\}$$  \hspace{1cm} (3.38)

$A$ in Eq. 3.38 can be substituted back into Eq. 3.35 to yield

$$\left(\bar{p}_{ID}\right)_{y_D = w_D / 2} = \left(\bar{p}_{FD}\right)_{y_D = w_D / 2} \frac{\cosh\left[\sqrt{\alpha_o (y_{eD} - y_D)}\right]}{\cosh\left[\sqrt{\alpha_o (y_{eD} - w_D / 2)}\right]}$$  \hspace{1cm} (3.39)

Eq. 3.39 is the inner reservoir solution in terms of the hydraulic fracture pressure and $\alpha_o$ term carries the properties of both the outer and the inner reservoirs.

The inner reservoir solution is ready to be coupled with the hydraulic fracture solution.

The diffusivity equation for the hydraulic fracture is as follows;

$$\frac{\partial^2 \Delta p_F}{\partial x^2} + \frac{2}{w_F} \left(\frac{\partial \Delta p_F}{\partial y}\right)_{y = w_F / 2} = \frac{\left(\phi c_i\right)_F \mu}{k_F} \frac{\partial \Delta p_F}{\partial t}$$  \hspace{1cm} (3.40)

The continuity of flux at the boundary of the hydraulic fracture and the inner reservoir can be expressed as follows;

$$\lambda_\alpha \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left(\frac{\partial \Delta p_I}{\partial y}\right)_{y = w_F / 2} = \lambda_F \left(\frac{\partial \Delta p_F}{\partial y}\right)_{y = w_F / 2}$$  \hspace{1cm} (3.41)

Eq. 3.40 and 3.41 may be represented in dimensionless form, respectively, as follows;

$$\frac{\partial^2 p_{FD}}{\partial x_D^2} + \frac{2}{w_D} \left(\frac{\partial p_{FD}}{\partial y_D}\right)_{y_D = w_D / 2} = \frac{1}{\eta_{FD}} \frac{\partial P_{FD}}{\partial t_D}$$  \hspace{1cm} (3.42)

and

$$\left(\frac{\partial p_{FD}}{\partial y_D}\right)_{y_D = w_D / 2} = \frac{\lambda_\alpha}{\lambda_F} \left(\frac{\eta_\alpha}{\eta_F}\right)^{1-\alpha} \frac{1}{x_F} \left(\frac{\partial p_{ID}}{\partial t_D}\right)_{y_D = w_D / 2}$$  \hspace{1cm} (3.43)

Applying Laplace transformation to Eq. 3.42 and Eq. 3.43 gives, respectively;
\[
\frac{d^2 \bar{p}_{FD}}{dx_D^2} + \frac{2\lambda}{w_p} \left( \frac{d\bar{p}_{FD}}{dy_D} \right)_{y_D = w_p/2} = \frac{1}{\eta_{FD}} s\bar{p}_{FD} \tag{3.44}
\]

and
\[
\left( \frac{d\bar{p}_{FD}}{dy_D} \right)_{y_D = w_p/2} = \lambda \frac{\eta_a}{\lambda_F} \left( \frac{\eta_a}{x_F^2} \right)^{1-a} \frac{1}{s} \left( \frac{d\bar{p}_{ID}}{dy_D} \right)_{y_D = w_p/2} \tag{3.45}
\]

When Eq. 3.45 is substituted in Eq. 3.44, we have
\[
\frac{d^2 \bar{p}_{FD}}{dx_D^2} + \frac{2\lambda}{w_p} \left( \frac{\eta_a}{\lambda_F} \right)^{1-a} \frac{1}{s} \left( \frac{d\bar{p}_{ID}}{dy_D} \right)_{y_D = w_p/2} = \frac{1}{\eta_{FD}} s\bar{p}_{FD} \tag{3.46}
\]

Recalling the inner reservoir solution (Eq. 3.39) and taking the derivative of it gives;
\[
\left( \frac{d\bar{p}_{ID}}{dy_D} \right) = -\left( \bar{p}_{FD} \right)_{y_D = w_p/2} \frac{\alpha_o^2}{\sqrt{\alpha_o (y_D - y_o)}} \frac{\sinh \left[ \alpha_o (y_D - y_o) \right]}{\cosh \left[ \alpha_o (y_D - w_D / 2) \right]} \tag{3.47}
\]

For \( y_D = w_p / 2 \),
\[
\left( \frac{d\bar{p}_{ID}}{dy_D} \right)_{y_D = w_p/2} = -\left( \bar{p}_{FD} \right)_{y_D = w_p/2} \frac{\alpha_o^2}{\sqrt{\alpha_o (y_D - w_p / 2)}} \frac{\tanh \left[ \alpha_o (y_D - w_p / 2) \right]}{1} \tag{3.48}
\]

Defining,
\[
\beta_F = \frac{\alpha_o^2}{\sqrt{\alpha_o (y_D - w_p / 2)}} \frac{1}{\tanh \left[ \alpha_o (y_D - w_p / 2) \right]} \tag{3.49}
\]

Eq. 3.48 can be written in the following form;
\[
\left( \frac{d\bar{p}_{ID}}{dy_D} \right)_{y_D = w_p/2} = -\beta_F \left( \bar{p}_{FD} \right)_{y_D = w_p/2} \tag{3.50}
\]

Substituting Eq. 3.50 back into Eq. 3.46 gives;
\[
\frac{d^2 \bar{p}_{FD}}{dx_D^2} + \frac{2\lambda}{w_p} \left( \frac{\eta_a}{\lambda_F} \right)^{1-a} \frac{1}{s} \left( \bar{p}_{FD} \right)_{y_D = w_p/2} = \frac{1}{\eta_{FD}} s\bar{p}_{FD} \tag{3.51}
\]

The hydraulic fracture pressure is not a function of the \( y \) direction. Therefore,
\[(\tilde{p}_{FD})_{y_o=w_o/2} = \tilde{p}_{FD} \quad (3.52)\]

Then, Eq. 3.51 becomes;
\[
d^2\tilde{p}_{FD} \over dx_D^2 = \left[ \frac{2 \lambda_a}{w_D \lambda_f} \left( \frac{\eta_a}{x_f^2} \right)^{1-a} s^{1-a} \beta_f + \frac{1}{\eta_{FD}} s \right] \tilde{p}_{FD} = 0 \quad (3.53)
\]

Defining,
\[
\alpha_f = \left[ \frac{2 \lambda_a}{w_D \lambda_f} \left( \frac{\eta_a}{x_f^2} \right)^{1-a} s^{1-a} \beta_f + \frac{1}{\eta_{FD}} s \right]
\quad (3.54)
\]

Eq. 3.53 can be simply written as
\[
d^2\tilde{p}_{FD} \over dx_D^2 - \alpha_f \tilde{p}_{FD} = 0 \quad (3.55)
\]

The general solution of Eq. 3.55 is as follows;
\[
\tilde{p}_{FD} = A \exp(-\alpha_f x_D) + B \exp(\sqrt{\alpha_f} x_D) \quad (3.56)
\]

The outer boundary of the hydraulic fracture is the tip of the fracture. Assuming no flow across the fracture tip, the outer boundary condition of the hydraulic fracture is
\[
\left( \frac{d\tilde{p}_{FD}}{dx_D} \right)_{x_D=1} = 0 \quad (3.57)
\]

Taking the derivative of Eq. 3.56 and using in the boundary condition (Eq. 3.57) gives
\[
\left( \frac{d\tilde{p}_{FD}}{dx_D} \right)_{x_D=1} = -\sqrt{\alpha_f} A \exp(-\sqrt{\alpha_f}) + \sqrt{\alpha_f} B \exp(\sqrt{\alpha_f}) = 0 \quad (3.58)
\]

Then,
\[
A = B \exp(2\sqrt{\alpha_f}) \quad (3.59)
\]

Substituting A (Eq. 3.59) back into Eq. 3.56 yields
\[ \bar{p}_{FD} = B \exp \left( 2 \sqrt{\alpha_f} \right) \exp \left( -\sqrt{\alpha_f} x_D \right) + B \exp \left( \sqrt{\alpha_f} x_D \right) \] (3.60)

or

\[ \bar{p}_{FD} = B \exp \left[ \sqrt{\alpha_f} \left( 2 - x_D \right) \right] + B \exp \left( \sqrt{\alpha_f} x_D \right) \] (3.61)

The inner boundary condition for the hydraulic fracture is

\[
\left( \frac{d\bar{p}_{FD}}{dx_D} \right)_{x_D=0} = -\frac{\pi}{sC_{FD}}
\] (3.62)

Substituting Eq. 3.61 into the inner boundary condition (Eq. 3.62) gives

\[
\left( \frac{d\bar{p}_{FD}}{dx_D} \right)_{x_D=0} = -B \sqrt{\alpha_f} \exp \left[ \left( 2 \sqrt{\alpha_f} \right) \right] + B \sqrt{\alpha_f} = -\frac{\pi}{sC_{FD}}
\] (3.63)

Then,

\[
B = -\frac{\pi}{sC_{FD} \sqrt{\alpha_f} \left[ 1 - \exp \left( 2 \sqrt{\alpha_f} \right) \right]}
\] (3.64)

Substituting B (Eq. 3.64) into Eq. 3.63, yields the following hydraulic fracture solution.

\[
\bar{p}_{FD} = \frac{\pi}{sC_{FD} \sqrt{\alpha_f}} \frac{\cosh \left[ \sqrt{\alpha_f} \left( 1 - x_D \right) \right]}{\sinh \left( \sqrt{\alpha_f} \right)}
\] (3.65)

At the interface between the hydraulic fracture and the wellbore, the pressure of the hydraulic fracture and the wellbore is the same. Therefore,

\[
\bar{p}_{wD} = (\bar{p}_{FD})_{x_D=0} = \frac{\pi}{sC_{FD} \sqrt{\alpha_f} \tanh \left( \sqrt{\alpha_f} \right)}
\] (3.66)

Eq. 3.66 is the solution of the scaled wellbore pressure.
3.3 Verification of the TAD Solution

In this section, the TAD model is verified against the TDP model (Brown, 2009, Brown et al., 2011, and Ozkan et al., 2011). Table 3.1 shows the general data used in the verification of the model and Chapter 5. Any differences from the data in Table 3.1 are stated specifically in the discussion and noted in the figures.

Table 3.1 – General data used for the model verification cases

<table>
<thead>
<tr>
<th>WELL, RESERVOIR, AND FLUID DATA (Intrinsic Properties)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Formation thickness, $h$, ft</td>
<td>250</td>
</tr>
<tr>
<td>Wellbore radius, $r_w$, ft</td>
<td>0.25</td>
</tr>
<tr>
<td>Horizontal well length, $L_h$, ft</td>
<td>2800</td>
</tr>
<tr>
<td>Number of hydraulic fractures, $n_F$</td>
<td>15</td>
</tr>
<tr>
<td>Distance between hydraulic fractures, $d_F$, ft</td>
<td>200</td>
</tr>
<tr>
<td>Distance to boundary parallel to well (1/2 well spacing), $x_e$, ft</td>
<td>250</td>
</tr>
<tr>
<td>Inner reservoir size, $y_e$, ft</td>
<td>100</td>
</tr>
<tr>
<td>Viscosity, $\mu$, cp</td>
<td>0.3</td>
</tr>
<tr>
<td>Hydraulic fracture porosity, $\phi_F$, fraction</td>
<td>0.38</td>
</tr>
<tr>
<td>Hydraulic fracture permeability, $k_F$, md</td>
<td>5.0E+04</td>
</tr>
<tr>
<td>Hydraulic fracture total compressibility, $c_{tF}$, psi^{-1}</td>
<td>1.0E-04</td>
</tr>
<tr>
<td>Hydraulic fracture half-length, $x_F$, ft</td>
<td>250</td>
</tr>
<tr>
<td>Hydraulic fracture width, $w_F$, ft</td>
<td>0.01</td>
</tr>
<tr>
<td>Outer reservoir permeability, $k_O$, md</td>
<td>1.0E-04</td>
</tr>
<tr>
<td>Outer reservoir porosity, $\phi_O$</td>
<td>0.05</td>
</tr>
<tr>
<td>Outer reservoir compressibility, $c_{tO}$, psi^{-1}</td>
<td>1.0E-05</td>
</tr>
<tr>
<td>Constant flow rate, $q$, stb/day</td>
<td>150</td>
</tr>
</tbody>
</table>

INNER RESERVOIR DATA

<table>
<thead>
<tr>
<th>TDP (Intrinsic Properties)</th>
<th>TAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matrix permeability, $k_m$, md</td>
<td>1.0E-4</td>
</tr>
<tr>
<td>Matrix porosity, $\phi_m$</td>
<td>0.05</td>
</tr>
<tr>
<td>Matrix total compressibility, $c_{t_m}$, psi^{-1}</td>
<td>1.0E-5</td>
</tr>
<tr>
<td>Natural fracture permeability, $k_f$, md</td>
<td>1.0E+3</td>
</tr>
<tr>
<td>Natural fracture porosity, $\phi_f$</td>
<td>0.7</td>
</tr>
<tr>
<td>Natural fracture total compressibility, $c_{t_f}$, psi^{-1}</td>
<td>5.5E-1</td>
</tr>
<tr>
<td>Natural fracture width, $h_f$, ft</td>
<td>3.0E-3</td>
</tr>
</tbody>
</table>
To verify the TAD model, we first use the asymptotic case of $\alpha=1$. This case corresponds to normal diffusion in a homogeneous reservoir, which can be obtained from the TDP solution for $f(s) = 1$ when $k_f$ and $(\phi_c)_f$ are chosen equal to $k_{\alpha}$ and $(\phi_c)_{\alpha}$. The results in Fig. 3.1 show excellent agreement between the TAD and TDP solutions.

As another means of verification, in Fig. 3.2, we match the results obtained from the TDP model for $k_f = 10^6$ md and $k_m = 10^{-4}$ md with the TAD model. As shown by Fig. 3.2, the TAD model for $\alpha=0.8$ and $k_{\alpha}=1200$ provides a reasonable match with the TDP model. This example has been provided to show that the TAD model captures the naturally fractured reservoir behavior idealized by the TDP models. It does not, however, imply a general correspondence of the TAD and TDP models. As it will be discussed in the next sections, the TAD model displays flow characteristics not observed for the TDP model.
3.4 Field Example

This example is provided to demonstrate the viability of field data analysis by the TAD model. We consider the Barnett field data analyzed by Brown et al. (2011) by using the TDP model. The details of the data are given in Brown et al. (2011). Figure 3.3 shows the matching of the rate-normalized pseudopressure $[\Delta m(p)/q]$ data by the TAD model and the results obtained from the match are given in Table 3.2. For comparison, the TDP-model match obtained by Brown et al. (2011) is also shown in Fig. 3.3. Both the TAD and TDP models yield a reasonable match and it is not possible to choose one over the other. Because the TAD model does not require explicit references to the intrinsic properties of the matrix and natural fractures, the TAD model requires fewer regression parameters than the TDP model.
Figure 3.3 – Matching the Barnett field data with the TAD and TDP models

Table 3.2 – Data used for matching field data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Formation thickness, $h$, ft</td>
<td>300</td>
</tr>
<tr>
<td>Reservoir temperature, $T$, R</td>
<td>565.67</td>
</tr>
<tr>
<td>Distance to boundary parallel to well (1/2 well spacing), $x_e$, ft</td>
<td>275</td>
</tr>
<tr>
<td>Inner reservoir size, $y_e$, ft</td>
<td>90.3</td>
</tr>
<tr>
<td>Viscosity, $\mu$, cp</td>
<td>0.02</td>
</tr>
<tr>
<td>The order of fractional derivative of time, $\alpha$</td>
<td>0.8</td>
</tr>
<tr>
<td>Phenomenological coefficient of anomalous diffusion, $k_{\alpha}$, md-$hr^{1-\alpha}$</td>
<td>0.13</td>
</tr>
<tr>
<td>Porosity – compressibility product of inner reservoir, $(\phi c_t)_{\alpha}$, psi$^{-1}$</td>
<td>2.00E-04</td>
</tr>
<tr>
<td>Hydraulic fracture porosity, $\phi_F$</td>
<td>0.38</td>
</tr>
<tr>
<td>Hydraulic fracture permeability, $k_F$, md</td>
<td>1.00E+03</td>
</tr>
<tr>
<td>Hydraulic fracture total compressibility, $c_{tF}$, psi$^{-1}$</td>
<td>1.00E-04</td>
</tr>
<tr>
<td>Hydraulic fracture half-length, $x_F$, ft</td>
<td>275</td>
</tr>
<tr>
<td>Hydraulic fracture width, $w_F$, ft</td>
<td>0.01</td>
</tr>
<tr>
<td>Outer reservoir permeability, $k_O$, md</td>
<td>1.00E-06</td>
</tr>
<tr>
<td>Outer reservoir porosity, $\phi_O$</td>
<td>0.04</td>
</tr>
<tr>
<td>Outer reservoir compressibility, $c_{tO}$, psi$^{-1}$</td>
<td>3.00E-04</td>
</tr>
</tbody>
</table>
CHAPTER 4

ASYMPTOTIC APPROXIMATIONS OF THE TAD MODEL

Obtaining asymptotic approximations of the TAD model are useful to understand the flow regime characteristics and to determine diagnostic features of the pressure-transient and production data of fractured horizontal wells in shale. In this work, the intermediate- and late-time asymptotic behaviors are of interest. To obtain the asymptotic approximations of the Laplace-domain solution of the TAD model for large values of time, the limiting forms of the solution as the Laplace transform parameter, \( s \), approaches zero are evaluated and the resulting expressions are analytically inverted to the time domain.

The scaled wellbore pressure solution obtained from the TAD model in Chapter 3 is given by

\[
\bar{p}_{wD} = \frac{\pi}{sC_F D \sqrt{\alpha_F} \tanh\left(\sqrt{\alpha_F}\right)}.
\]  

(4.1)

where

\[
\alpha_F = \left[ \frac{2 \lambda_a}{w_D \lambda_F} \left( \frac{\eta_a}{x_F^2} \right)^{1-a} s^{1-a} \beta_F + \frac{1}{\eta_{FD}} s \right].
\]  

(4.2)

\[
\beta_F = \sqrt{\alpha_o} \tanh\left[ \sqrt{\alpha_o} \left( y_{oD} - w_D / 2 \right) \right]
\]  

(4.3)

\[
\alpha_o = \left( \frac{x_F^2}{\eta_a} \right)^{1-a} s^a \left[ 1 + \left( \frac{\lambda_o}{\beta_o} s^{1-a} \right) \right]
\]  

(4.4)

and

\[
\beta_o = \sqrt{s / \eta_{oD}} \tanh\left[ \sqrt{s / \eta_{oD}} \left( x_{oD} - 1 \right) \right]
\]  

(4.5)
The limiting forms of the TAD solution will be considered as follows:

$$\lim_{s \to 0} p_{uD} = \lim_{s \to 0} \frac{\pi}{s C_{FD} \sqrt{\alpha_f} \tanh \left( \sqrt{\alpha_f} \right)} \cdot (4.6)$$

To evaluate the limiting forms, we can use the following limits of \( \tanh(x) \):

$$\tanh(x) = \begin{cases} 
    x & \text{for } x \to 0 \\
    1 & \text{for } x \to \infty 
\end{cases} \quad (4.7)$$

Different combinations of the limiting forms of \( \tanh(x) \) used in the evaluations correspond to different physical conditions and lead to different asymptotic behaviors. Below, several of the possible asymptotic approximations will be considered.

### 4.1 Case 1

First, we consider the following limit:

$$\lim_{s \to 0} \tanh \left[ \sqrt{s / \eta_{OD}} \left( x_{eD} - 1 \right) \right] = \sqrt{s / \eta_{OD}} \left( x_{eD} - 1 \right) \text{ and } x_{eD} \neq 1 \quad (4.8)$$

Then,

$$\lim_{s \to 0} \beta = \sqrt{s / \eta_{OD}} \tanh \left[ \sqrt{s / \eta_{OD}} \left( x_{eD} - 1 \right) \right] = \left( s / \eta_{OD} \right) \left( x_{eD} - 1 \right) \quad (4.9)$$

and

$$\lim_{s \to 0} \alpha = \left( \frac{x_{eD}^2}{\eta_a} \right)^{1-a} \left[ 1 + \left( \frac{\lambda_{OD}}{\lambda_a} s \beta \right) \right] = \left( \frac{x_{eD}^2}{\eta_a} \right)^{1-a} \left[ 1 + \left( \frac{\lambda_{OD}}{\lambda_a} \left( x_{eD} - 1 \right) / \eta_{OD} \right) \right] s^a = A_{OD} s^a \quad (4.10)$$

where

$$A_{OD} = \left( \frac{x_{eD}^2}{\eta_a} \right)^{1-a} \left[ 1 + \left( \frac{\lambda_{OD}}{\lambda_a} \left( x_{eD} - 1 \right) / \eta_{OD} \right) \right] \quad (4.11)$$
We can consider two conditions of $\alpha; \alpha \neq 0$ and $\alpha = 0$.

**Case 1.1:**

If $\alpha \neq 0$, we have

$$\lim_{s \to 0} \beta_F = \sqrt{\alpha_0} \tanh \left[ \sqrt{\alpha_0} \left( y_{eD} - w_D / 2 \right) \right] = \sqrt{A_{01}s^\alpha} \tanh \left[ \sqrt{A_{01}s^\alpha} \left( y_{eD} - w_D / 2 \right) \right]$$

(4.12)

**Case 1.1.1:**

If $\tanh (y) = y$, then

$$\lim_{s \to 0} \tanh \left[ \sqrt{A_{01}s^\alpha} \left( y_{eD} - w_D / 2 \right) \right] = \sqrt{A_{01}s^\alpha} \left( y_{eD} - w_D / 2 \right)$$

(4.13)

We have

$$\lim_{s \to 0} \beta_F = B_{F1,\alpha}s^\alpha$$

(4.14)

where

$$B_{F1,\alpha} = A_{01} \left( y_{eD} - w_D / 2 \right)$$

(4.15)

Then

$$\lim_{s \to 0} \alpha_F = \frac{2}{w_D} \frac{\lambda_a}{\lambda_F} \left( \frac{\eta_a}{\sigma_F} \right)^{1-\alpha} B_{F1,\alpha}s = A_{F1,\alpha}s$$

(4.16)

where

$$A_{F1,\alpha} = \frac{2}{w_D} \frac{\lambda_a}{\lambda_F} \left( \frac{\eta_a}{\sigma_F} \right)^{1-\alpha} B_{F1,\alpha}$$

(4.17)

Then
\[
\lim_{s \to 0} \bar{p}_{wD} = \frac{\pi}{s C_{FD} \sqrt{A_{F1,\alpha}} s \tanh \left( \sqrt{A_{F1,\alpha}} s \right)} .
\] 

(4.18)

a. If

\[
\lim_{s \to 0} \tanh \left[ \sqrt{A_{F1,\alpha}} s \right] = \sqrt{A_{F1,\alpha}} s
\] 

(4.19)

then

\[
\lim_{s \to 0} \bar{p}_{wD} = \frac{\pi}{s^2 C_{FD} A_{F1,\alpha}} ,
\] 

(4.20)

which yields

\[
\lim_{t_0 \to \infty} \bar{p}_{wD} = \frac{\pi I_D}{C_{FD} A_{F1,\alpha}} .
\] 

(4.21)

b. If

\[
\lim_{s \to 0} \tanh \left[ \sqrt{A_{F1,\alpha}} s \right] = 1 \text{ (only for very low conductivity fractures)} ,
\] 

(4.22)

then

\[
\lim_{s \to 0} \bar{p}_{wD} = \frac{\pi}{s^{3/2} C_{FD} \sqrt{A_{F1,\alpha}}} ,
\] 

(4.23)

which yields

\[
\lim_{t_0 \to \infty} \bar{p}_{wD} = \frac{2 \sqrt{\pi I_D}}{C_{FD} \sqrt{A_{F1,\alpha}}} .
\] 

(4.24)

Case 1.1.2:

If \( \tanh(y) = 1 \), then
\[
\lim_{s \to 0} \left[ \sqrt{A_{01}} s^{\alpha} \left( y_{\epsilon D} - w_D / 2 \right) \right] = 1
\]  
(4.25)

and we have
\[
\lim_{s \to 0} \beta_F = B_{F2,\alpha} s^{\alpha/2}
\]  
(4.26)

where
\[
B_{F2,\alpha} = \sqrt{A_{01}}
\]  
(4.27)

Then
\[
\lim_{s \to 0} \alpha_F = \frac{2}{w_D} \frac{\lambda_F}{\lambda_F} \left( \frac{\eta_{\alpha}}{x_F^2} \right)^{1-\alpha} B_{F2,\alpha} s^{1-\alpha} = A_{F2,\alpha} s^{1-\alpha},
\]  
(4.28)

where
\[
A_{F2,\alpha} = \frac{2}{w_D} \frac{\lambda_F}{\lambda_F} \left( \frac{\eta_{\alpha}}{x_F^2} \right)^{1-\alpha} B_{F2,\alpha}
\]  
(4.29)

Then
\[
\lim_{s \to 0} \bar{p}_{wD} = \frac{\pi}{s C_{FD} \sqrt{A_{F2,\alpha}} s^{1-\alpha/2} \tanh \left( \sqrt{A_{F2,\alpha}} s^{1-\alpha/2} \right)}.
\]  
(4.30)

a. If
\[
\lim_{s \to 0} \left[ \sqrt{A_{F2,\alpha}} s^{1-\alpha/2} \right] = \sqrt{A_{F2,\alpha}} s^{1-\alpha/2},
\]  
(4.31)

then
\[
\lim_{\alpha \to 0} \bar{p}_{w_D} = \frac{\pi}{s^{1+\frac{2-\alpha}{2}} C_{FD} A_{F2,\alpha}}, \\
\]
which yields
\[
\lim_{\alpha \to 0} p_{w_D} = \frac{2^{2-\alpha} \pi t_D^2}{(2-\alpha) \Gamma\left(\frac{2-\alpha}{2}\right) C_{FD} A_{F2,\alpha}}. \\
\]

b. If
\[
\lim_{\alpha \to 0} \tanh\left[\sqrt{A_{F2,\alpha}} s\right] = 1 \text{ (only for very low conductivity fractures)}, \\
\]
then
\[
\lim_{s \to 0} \bar{p}_{w_D} = \frac{\pi}{s^{1+\frac{2-\alpha}{4}} C_{FD} \sqrt{A_{F2,\alpha}}}, \\
\]
which yields
\[
\lim_{\alpha \to 0} p_{w_D} = \frac{4^{2-\alpha} \pi t_D^4}{(2-\alpha) \Gamma\left(\frac{2-\alpha}{4}\right) C_{FD} \sqrt{A_{F2,\alpha}}}. \\
\]

Case 1.2:

If \( \alpha = 0 \), we have
\[
\lim_{\alpha \to 0} \beta_F = \sqrt{\alpha} \tanh\left[\sqrt{\alpha} \left( y_{c_D} - w_D / 2 \right) \right] = B_{F,0}, \\
\]
where
\[
B_{F,0} = \sqrt{A_{c1}} \tanh\left[\sqrt{A_{c1}} \left( y_{c_D} - w_D / 2 \right) \right] \\
\]
then

$$\lim_{s \to 0, \alpha = 0} \alpha_F = \left[ \frac{2}{w_D} \frac{\lambda_F}{x_F^2} \left( \eta_F \right)^{1-\alpha} B_{F,0} + \frac{1}{\eta_{FD}} \right] s = A_{F,0}s, \quad (4.39)$$

where

$$A_{F,0} = \frac{2}{w_D} \frac{\lambda_F}{x_F^2} B_{F,0} + \frac{1}{\eta_{FD}}. \quad (4.40)$$

Then

$$\lim_{s \to 0, \alpha = 0} \bar{p}_{wD} = \frac{\pi}{sC_{FD} \sqrt{A_{F,0}s \tanh \left( \sqrt{A_{F,0}s} \right)}}. \quad (4.41)$$

a. If

$$\lim_{s \to 0, \alpha = 0} \tanh \left[ \sqrt{A_{F,0}s} \right] = \sqrt{A_{F,0}s} \quad (4.42)$$

then

$$\lim_{s \to 0, \alpha = 0} \bar{p}_{wD} = \frac{\pi}{s^2 C_{FD} A_{F,0}}. \quad (4.43)$$

which yields

$$\lim_{t_0 \to \infty} p_{wD} = \frac{\pi t_0}{C_{FD} A_{F,0}}. \quad (4.44)$$

b. If

$$\lim_{s \to 0, \alpha = 0} \tanh \left[ \sqrt{A_{F,0}s} \right] = 1, \quad (4.45)$$

then
\[
\lim_{s \to 0, \alpha = 0} \bar{p}_{wD} = \frac{\pi}{s^{3/2} C_{FD} \sqrt{A_{F,0}}},
\]  

(4.46)

which yields

\[
\lim_{t_0 \to +\infty, \alpha = 0} p_{wD} = \frac{2\pi \bar{p}_{wD}}{C_{FD} \sqrt{A_{F,0}}}.
\]  

(4.47)

4.2 Case 2

We now assume the following limit:

\[
\lim_{s \to 0} \tanh \left( \sqrt{s / \eta_{OD}} (x_{eD} - 1) \right) = 1 \quad \text{and} \quad x_{eD} \neq 1
\]  

(4.48)

We consider

\[
\lim_{s \to 0} \beta_O = \sqrt{s / \eta_{OD}} \tanh \left( \sqrt{s / \eta_{OD}} (x_{eD} - 1) \right) = \sqrt{s / \eta_{OD}}
\]  

(4.49)

and two conditions of \( \alpha \); \( \alpha \neq 0.5 \), and \( \alpha = 0.5 \).

**Case 2.1:**

If \( \alpha \neq 0.5 \), we have

\[
\lim_{s \to 0, \alpha \neq 0.5} \alpha_O = \left( \frac{x_F}{\eta_x} \right)^{1-a} s^a \left[ 1 + \left( \frac{\lambda_o}{\lambda_x s} \beta_O \right) \right] = \left( \frac{x_F}{\eta_x} \right)^{1-a} \left[ 1 + \left( \frac{\lambda_o}{\lambda_x s} \right) \left( \frac{1}{\sqrt{s \eta_{OD}}} \right) \right] s^a = A_{O,2,a} s^{a - \frac{1}{2}}
\]  

(4.50)

where

\[
A_{O,2,a} = \left( \frac{x_F^2}{\eta_x} \right)^{1-a} \left( \frac{\lambda_o}{\lambda_x} \right) \left( \frac{1}{\sqrt{s \eta_{OD}}} \right)
\]  

(4.51)

Then
\[
\lim_{s \to 0} \beta_F = \sqrt{\alpha_0} \tanh \left[ \sqrt{\alpha_0} \left( y_{eD} - w_D / 2 \right) \right] = \sqrt{A_{O2,a} s^{\alpha - 1 / 2}} \tanh \left[ \sqrt{A_{O2,a} s^{\alpha - 1 / 2}} \left( y_{eD} - w_D / 2 \right) \right]
\] (4.52)

**Case 2.1.1:**

If \( \tanh(y) = y \),

\[
\lim_{s \to 0} \tanh \left[ \sqrt{A_{O2,a} s^{\alpha - 1 / 2}} \left( y_{eD} - w_D / 2 \right) \right] = \left[ \sqrt{A_{O2,a} s^{\alpha - 1 / 2}} \left( y_{eD} - w_D / 2 \right) \right]
\] (4.53)

Then we have

\[
\lim_{s \to 0} \beta_F = B_{F2.1,a} s^{\alpha - 1 / 2}
\] (4.54)

where

\[
B_{F2.1,a} = A_{O2,a} \left( y_{eD} - w_D / 2 \right).
\] (4.55)

Then

\[
\lim_{s \to 0} \alpha_F = \frac{2}{s} \frac{\lambda_F}{w_D} \frac{\eta_a}{x_F} \left( \frac{\eta_a}{x_F} \right)^{1-\alpha} B_{F2.1,a} \sqrt{s} = A_{F2.1,a} \sqrt{s},
\] (4.56)

where

\[
A_{F2.1,a} = \frac{2}{s} \frac{\lambda_F}{w_D} \frac{\eta_a}{x_F} \left( \frac{\eta_a}{x_F} \right)^{1-\alpha} B_{F2.1,a}
\] (4.57)

Then

\[
\lim_{s \to 0} \bar{p}_{WD} = \frac{\pi}{s C_{FD} \sqrt{A_{F2.1,a} s^{1/2}}} \tanh \left( \sqrt{A_{F2.1,a} s^{1/2}} \right).
\] (4.58)
a. If

\[ \lim_{s \to 0} \tanh \left( \sqrt{A_{F,2,1,\alpha} s^{1/2}} \right) = \sqrt{A_{F,2,1,\alpha} s^{1/2}} , \]  

then

\[ \lim_{s \to 0} \overline{p}_{wD} = \frac{\pi}{s^{3/2} C_{FD} A_{F,2,1,\alpha}} , \]  

which yields

\[ \lim_{s \to 0} p_{wD} = \frac{2\sqrt{\pi I_D}}{C_{FD} A_{F,2,1,\alpha}} . \]  

b. If

\[ \lim_{s \to 0} \tanh \left( \sqrt{A_{F,2,1,\alpha} s^{1/2}} \right) = 1 \quad \text{(only for very low conductivity fractures)} , \]  

then

\[ \lim_{s \to 0} \overline{p}_{wD} = \frac{\pi}{s^{1+1/4} C_{FD} \sqrt{A_{F,2,1,\alpha}}} , \]  

which yields

\[ \lim_{\theta \to \pi} p_{wD} = \frac{4\pi t_D^{1/4}}{\Gamma(1/4) C_{FD} \sqrt{A_{F,2,1,\alpha}}} . \]  

Case 2.1.2:

If \( \tanh(y) = 1 \),

\[ \]
\[
\lim_{s \to 0} \tanh \left[ \sqrt{A_{0,2,a} s^{-\alpha/2}} \left( y_{eb} - w_D / 2 \right) \right] = 1
\] (4.65)

and we have

\[
\lim_{s \to 0} \beta_F = B_{F,2,2,a} s^{(2\alpha - 1)/4}
\] (4.66)

where

\[
B_{F,2,2,a} = \sqrt{A_{0,2,a}}.
\] (4.67)

Then

\[
\lim_{s \to 0} \alpha_F = \frac{2}{w_D} \frac{\lambda_a}{\bar{\lambda}_F} \left( \frac{\eta_a}{x_F^2} \right)^{1-\alpha} B_{F,2,2,a} s^{(3-2\alpha)/4} = A_{F,2,2,a} s^{(3-2\alpha)/4}
\] (4.68)

where

\[
A_{F,2,2,a} = \frac{2}{w_D} \frac{\lambda_a}{\bar{\lambda}_F} \left( \frac{\eta_a}{x_F^2} \right)^{1-\alpha} B_{F,2,2,a}.
\] (4.69)

Then

\[
\lim_{s \to 0} \bar{p}_{WD} = \frac{\pi}{s C_{FD} \sqrt{A_{F,2,2,a} s^{(3-2\alpha)/4} \tanh \left( \sqrt{A_{F,2,2,a} s^{(3-2\alpha)/4}} \right) \sqrt{A_{F,2,2,a} s^{(3-2\alpha)/4}}}}.
\] (4.70)

a. If

\[
\lim_{s \to 0} \tanh \left[ \sqrt{A_{F,2,2,a} s^{(3-2\alpha)/4}} \right] = \sqrt{A_{F,2,2,a} s^{(3-2\alpha)/4}},
\] (4.71)

then
\[
\lim_{\alpha \to 0.5} P_{WD} = \frac{\pi}{s^{1+(3-2\alpha)/4} C_{FD} A_{F2,2,\alpha}},
\]  

which yields

\[
\lim_{\alpha \to 0.5} p_{WD} = \frac{4\pi t_D^{(3-2\alpha)/4}}{(3-2\alpha) \Gamma\left(\frac{3-2\alpha}{4}\right) C_{FD} A_{F2,2,\alpha}}.
\]  

b. If

\[
\lim_{s \to 0} \tanh\left[\sqrt{A_{F2,2,\alpha}} s^{(3-2\alpha)/4}\right] = 1 \quad \text{(only for very low conductivity fractures)},
\]

then

\[
\lim_{s \to 0} \bar{P}_{WD} = \frac{\pi}{s^{1+(3-2\alpha)/8} C_{FD} \sqrt{A_{F2,2,\alpha}}},
\]

which yields

\[
\lim_{\alpha \to 0.5} p_{WD} = \frac{8\pi t_D^{(3-2\alpha)/8}}{(3-2\alpha) \Gamma\left(\frac{3-2\alpha}{8}\right) C_{FD} \sqrt{A_{F2,2,\alpha}}}.
\]

**Case 2.2:**

If \( \alpha = 0.5 \), we have

\[
\lim_{s \to 0} a_{O} = \left(\frac{x_F^2}{\eta_a}\right)^{1-\alpha} \left[1 + \left(\frac{\lambda_0}{\lambda_a} \beta_{O}\right)\right] = \left(\frac{x_F^2}{\eta_a}\right)^{1-\alpha} \left[1 + \left(\frac{\lambda_0}{\lambda_a}\right)\left(\frac{1}{\sqrt{s_{OD}}}\right)\right] s^\alpha = A_{O2,0.5}
\]  

where

\[
A_{O2,0.5} = \left(\frac{x_F^2}{\eta_a}\right)^{1-\alpha} \left(\frac{\lambda_0}{\lambda_a}\right)\left(\frac{1}{\sqrt{s_{OD}}}\right)
\]  

(4.78)
Then
\[
\lim_{s \to 0} \beta_F = \sqrt{\alpha_o} \tanh \left( \sqrt{\alpha_o} \left( y_{ed} - w_d / 2 \right) \right) = B_{F2.05} \tag{4.79}
\]

where
\[
B_{F2.05} = \sqrt{A_{Q2.05}} \tanh \left( \sqrt{A_{Q2.05}} \left( y_{ed} - w_d / 2 \right) \right). \tag{4.80}
\]

Then
\[
\lim_{t \to 0} \alpha_F = \frac{2}{\lambda_{e_D}} \left( \eta_x \right)^{1/2} B_{F2.05} \sqrt{s} = A_{F2.05} \sqrt{s}, \tag{4.81}
\]

where
\[
A_{F2.05} = \frac{2}{\lambda_{e_D}} \left( \eta_x \right)^{1/2} B_{F2.05}. \tag{4.82}
\]

Then
\[
\lim_{t \to 0} \frac{p_{w D}}{s} = \frac{\pi}{s C_{FD} \sqrt{A_{F2.05} s^{3/2}} \tanh \left( \sqrt{A_{F2.05} s^{3/2}} \right)}, \tag{4.83}
\]
a. If
\[
\lim_{t \to 0} \tanh \left( \sqrt{A_{F2.05} s^{3/2}} \right) = \sqrt{A_{F2.05} s^{1/2}}, \tag{4.84}
\]

then
\[
\lim_{t \to 0} \frac{p_{w D}}{s} = \frac{\pi}{s^{3/2} C_{FD} A_{F2.05}}, \tag{4.85}
\]

which yields
b. If
\[
\lim_{t_0 \to \infty} \frac{\pi}{\alpha = 0.5} = \frac{2\sqrt{\pi I_D}}{C_{FD} A_{F2.0.5}}.
\]

\[
\lim_{s \to 0} \tanh \left( s^{1/2} \right) = 1 \quad \text{(only for very low conductivity fractures)},
\]
then
\[
\lim_{s \to 0} \frac{\pi}{\alpha = 0.5} = \frac{\pi}{s^{1/4} C_{FD} \sqrt{A_{F2.0.5}}}.
\]

which yields
\[
\lim_{t_0 \to \infty} \frac{\pi}{\alpha = 0.5} = \frac{4\pi t_D^{1/4}}{\Gamma \left( 1/4 \right) C_{FD} \sqrt{A_{F2.0.5}}}.
\]

4.3 Case 3

If \( x_{cD} = 1 \), we have
\[
\tanh \left( s / \eta_{OD} \right) \left( x_{cD} - 1 \right) = 0
\]
Then
\[
\beta_0 = \sqrt{s / \eta_{OD}} \tanh \left( s / \eta_{OD} \right) \left( x_{cD} - 1 \right) = 0
\]
and
\[
\lim_{s \to 0} \alpha_0 = A_{o3} s^\alpha
\]
where
\[
A_{o3} = \frac{x_{o3}^{1-\alpha}}{\eta_\alpha}
\]
We can consider two conditions of $\alpha$; $\alpha \neq 0$ and $\alpha = 0$

**Case 3.1:**

If $\alpha \neq 0$, we have

$$\lim_{s \to 0} \beta_F = \sqrt{\alpha_0} \tanh \left[ \sqrt{\alpha_0} \left( y_{ed} - w_D / 2 \right) \right] = \sqrt{A_{03}^\alpha} \tanh \left[ \sqrt{A_{03}^\alpha} \left( y_{ed} - w_D / 2 \right) \right]$$

(4.94)

**Case 3.1.1:**

If $\tanh(y) = y$

$$\lim_{s \to 0} \tanh \left[ \sqrt{A_{03}^\alpha} \left( y_{ed} - w_D / 2 \right) \right] = \sqrt{A_{03}^\alpha} \left( y_{ed} - w_D / 2 \right)$$

(4.95)

Then, we have

$$\lim_{s \to 0} \beta_F = B_{F3,\alpha,1}^s$$

(4.96)

where

$$B_{F3,\alpha,1} = A_{03} \left( y_{ed} - w_D / 2 \right).$$

(4.97)

Then

$$\lim_{s \to 0} \alpha_F = \frac{2}{w_D} \frac{\lambda_a}{\lambda_F} \left( \frac{\eta_a}{\lambda_F^2} \right)^{1-\alpha} B_{F3,\alpha,1}^s A_{F3,\alpha,1}^s,$$

(4.98)

where

$$A_{F3,\alpha,1} = \frac{2}{w_D} \frac{\lambda_a}{\lambda_F} \left( \frac{\eta_a}{\lambda_F^2} \right)^{1-\alpha} B_{F3,\alpha,1} + \frac{1}{\eta_{FD}}$$

(4.99)

Then
\[
\lim_{s \to 0} \frac{p_{wD}}{s C_{FD} \sqrt{A_{F3,a,l} s \tanh \left( \sqrt{A_{F3,a,l} s} \right)}} = \pi.
\]  
(4.100)

a. If

\[
\lim_{s \to 0} \tanh \left[ \sqrt{A_{F3,a,l} s} \right] = \sqrt{A_{F3,a,l} s},
\]  
(4.101)

then

\[
\lim_{s \to 0} \frac{p_{wD}}{s C_{FD} A_{F3,a,l}} = \frac{\pi}{s^2 C_{FD} A_{F3,a,l}}.
\]  
(4.102)

which yields

\[
\lim_{t_D \to \infty} p_{wD} = \frac{\pi t_D}{C_{FD} A_{F3,a,l}}.
\]  
(4.103)

b. If

\[
\lim_{s \to 0} \tanh \left[ \sqrt{A_{F3,a,l} s} \right] = 1 \text{ (only for very low conductivity fractures)},
\]  
(4.104)

then

\[
\lim_{s \to 0} \frac{p_{wD}}{s C_{FD} \sqrt{A_{F3,a,l}}} = \frac{\pi}{s^{3/2} C_{FD} \sqrt{A_{F3,a,l}}},
\]  
(4.105)

which yields

\[
\lim_{t_D \to \infty} p_{wD} = \frac{2 \sqrt{\pi t_D}}{C_{FD} \sqrt{A_{F3,a,l}}}.
\]  
(4.106)

**Case 3.1.2:**

If \( \tanh(y) = 1 \),
\[
\lim_{s \to 0} \left[ \sqrt{A_{\alpha s}} \left( y_{eD} - w_D / 2 \right) \right] = 1 \quad (4.107)
\]

and we have

\[
\lim_{s \to 0} \beta_F = B_{F3,\alpha s}^{\alpha / 2} \quad (4.108)
\]

where

\[
B_{F3,\alpha s} = \sqrt{A_{O3}} \quad (4.109)
\]

Then

\[
\lim_{s \to 0} \alpha_F = \frac{2 \lambda}{w_D} \left( \frac{\eta_s}{\lambda_F^2} \right)^{1-\alpha} \frac{B_{F3,\alpha s}}{s^{1-\alpha / 2}} = A_{F3,\alpha s} s^{1-\alpha / 2} \quad (4.110)
\]

where

\[
A_{F3,\alpha s} = \frac{2 \lambda}{w_D} \left( \frac{\eta_s}{\lambda_F^2} \right)^{1-\alpha} B_{F3,\alpha s} \quad (4.111)
\]

Then

\[
\lim_{s \to 0} \bar{\rho}_{wD} = \frac{\pi}{s C_{TD} \sqrt{A_{F3,\alpha s} s^{1-\alpha / 2}}} \tan \left( \sqrt{A_{F3,\alpha s} s^{1-\alpha / 2}} \right) \quad (4.112)
\]

a. If

\[
\lim_{s \to 0} \sqrt{A_{F3,\alpha s} s^{1-\alpha / 2}} = \sqrt{A_{F3,\alpha s} s^{1-\alpha / 2}} \quad (4.113)
\]

then
\[
\lim_{s \to 0} \frac{\bar{p}_{wD} \pi}{s^{1-\frac{2-\alpha}{2}} C_{FD} A_{F3,\alpha,2}} \!,
\]

which yields

\[
\lim_{t_{D} \to \infty} p_{wD} = \frac{2\pi t_{D}^{\frac{2-\alpha}{2}}}{(2-\alpha) \Gamma \left( \frac{2-\alpha}{2} \right) C_{FD} A_{F3,\alpha,2}}.
\]

(4.115)

b. If

\[
\lim_{s \to 0} \tanh \left[ \sqrt{A_{F3,\alpha,2}} s \right] = 1 \text{ (only for very low conductivity fractures)},
\]

(4.116)

then

\[
\lim_{s \to 0} \frac{\bar{p}_{wD} \pi}{s^{1-\frac{2-\alpha}{4}} C_{FD} \sqrt{A_{F3,\alpha,2}}} \!,
\]

which yields

\[
\lim_{t_{D} \to \infty} p_{wD} = \frac{4\pi t_{D}^{\frac{2-\alpha}{4}}}{(2-\alpha) \Gamma \left( \frac{2-\alpha}{4} \right) C_{FD} \sqrt{A_{F3,\alpha,2}}}.
\]

(4.118)

**Case 3.2:**

If \( \alpha = 0 \), we have

\[
\lim_{s \to 0} \beta_{F} = \sqrt{\alpha_{x}} \tanh \left[ \sqrt{\alpha_{x}} \left( y_{x} - w_{D} / 2 \right) \right] = B_{F3,0}
\]

(4.119)

where

\[
B_{F3,0} = \sqrt{A_{03}} \tanh \left[ \sqrt{A_{03}} \left( y_{x} - w_{D} / 2 \right) \right]
\]

(4.120)
Then

\[
\lim_{\substack{s \to 0 \\ \alpha = 0}} \alpha_F = \left[ \frac{2}{\alpha F w_D} \cdot \frac{\eta_F^2}{\eta_F^2 + 1} \right] B_{F3,0}^{\gamma} \cdot \frac{1}{\eta_{FD}} \right] s = A_{F3,0}^{\gamma}. \tag{4.121}
\]

where

\[
A_{F3,0}^{\gamma} = \frac{2}{\alpha F w_D} \cdot \frac{\eta_F^2}{\eta_F^2 + 1} \cdot B_{F3,0}^{\gamma} + \frac{1}{\eta_{FD}}. \tag{4.122}
\]

Then

\[
\lim_{\substack{s \to 0 \\ \alpha = 0}} \vec{p}_{wD} = \frac{\pi}{s C_{FD} \sqrt{A_{F3,0}^{\gamma}} \cdot \tanh \left( \sqrt{A_{F3,0}^{\gamma}} \right)}. \tag{4.123}
\]

a. If

\[
\lim_{\substack{s \to 0 \\ \alpha = 0}} \tanh \left( \sqrt{A_{F3,0}^{\gamma}} \right) = \sqrt{A_{F3,0}^{\gamma}}. \tag{4.124}
\]

then

\[
\lim_{\substack{s \to 0 \\ \alpha = 0}} \vec{p}_{wD} = \frac{\pi}{s^2 C_{FD} A_{F3,0}^{\gamma}}. \tag{4.125}
\]

which yields

\[
\lim_{\substack{s \to 0 \\ \alpha = 0}} P_{wD} = \frac{\pi n_D}{C_{FD} A_{F3,0}^{\gamma}}. \tag{4.126}
\]

b. If

\[
\lim_{\substack{s \to 0 \\ \alpha = 0}} \tanh \left( \sqrt{A_{F3,0}^{\gamma}} \right) = 1. \tag{4.127}
\]

then
\[ \lim_{\alpha \to 0} \bar{p}_{wd} = \frac{\pi}{s^{3/2} C_{FD} \sqrt{A_{F3,0}}} . \]  

(4.128)

which yields

\[ \lim_{t_0 \to \infty} p_{wd} = \frac{2 \sqrt{\pi t_D}}{C_{FD} \sqrt{A_{F3,0}}} . \]  

(4.129)

### 4.4 Summary of Asymptotic Approximations

A summary of the asymptotic solutions developed in this chapter is presented in Table 4.1. The constants used in Table 4.1 are provided in Table 4.2.
<table>
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<th>Time Range</th>
<th>Conditions</th>
<th>Pressure</th>
<th>Log-log Slope</th>
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</thead>
<tbody>
<tr>
<td>Late Time</td>
<td>( \alpha \neq 0, x_{\omega D} \neq 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{\pi t_\omega}{C_{PD}A_{I,0}} ]</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0, x_{\omega D} \neq 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{\pi t_\omega}{C_{PD}A_{F,0}} ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \alpha \neq 0, x_{\omega D} = 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{\pi t_\omega}{C_{PD}A_{F,3,0}} ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0, x_{\omega D} = 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{\pi t_\omega}{C_{PD}A_{F,3,0}} ]</td>
<td></td>
</tr>
<tr>
<td>Late-Intermediate Time</td>
<td>( \alpha \neq 0, x_{\omega D} \neq 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{2\pi t_\omega}{C_{PD}A_{I,0}} ]</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0, x_{\omega D} \neq 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{2\pi t_\omega}{C_{PD}A_{F,0}} ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \alpha \neq 0.5, x_{\omega D} \neq 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{2\sqrt{\pi t_\omega}}{C_{PD}A_{F,0.5}} ]</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0.5, x_{\omega D} \neq 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{2\sqrt{\pi t_\omega}}{C_{PD}A_{F,0.5}} ]</td>
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<td></td>
<td>( \alpha \neq 0, x_{\omega D} = 1 )</td>
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</tr>
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<td></td>
<td>( \alpha = 0, x_{\omega D} = 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{2\sqrt{\pi t_\omega}}{C_{PD}A_{F,3,0}} ]</td>
<td></td>
</tr>
<tr>
<td>Early-Intermediate Time</td>
<td>( \alpha \neq 0.5, x_{\omega D} \neq 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{4\pi t_\omega^{1/4}}{\Gamma(1/4)C_{PD}A_{F,1,0}} ]</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0.5, x_{\omega D} \neq 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{4\pi t_\omega^{1/4}}{\Gamma(1/4)C_{PD}A_{F,0.5}} ]</td>
<td></td>
</tr>
<tr>
<td>Late-Intermediate (( \alpha \to 1 )) to Late (( \alpha \to 0 )) Times</td>
<td>( \alpha \neq 0, x_{\omega D} \neq 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{2\pi t_\omega}{(2-\alpha)t^{3/2}C_{PD}A_{F,2,0}} ]</td>
<td>( \frac{2-\alpha}{2} )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0, x_{\omega D} = 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{2\pi t_\omega}{(2-\alpha)t^{3/2}C_{PD}A_{F,2,0}} ]</td>
<td></td>
</tr>
<tr>
<td>Early-Intermediate (( \alpha \to 1 )) to Late-Intermediate (( \alpha \to 0 )) Times</td>
<td>( \alpha \neq 0, x_{\omega D} \neq 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{4\pi t_\omega^{3/4}}{(2-\alpha)t^{3/2}C_{PD}A_{F,2,0}} ]</td>
<td>( \frac{2-\alpha}{4} )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0, x_{\omega D} = 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{4\pi t_\omega^{3/4}}{(2-\alpha)t^{3/2}C_{PD}A_{F,2,0}} ]</td>
<td></td>
</tr>
<tr>
<td>Intermediate to Late Times</td>
<td>( \alpha \neq 0.5, x_{\omega D} \neq 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{4\pi t_\omega^{3/4}}{(3-2\alpha)t^{3/2}C_{PD}A_{F,2,0}} ]</td>
<td>( \frac{3-2\alpha}{4} )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0.5, x_{\omega D} \neq 1 )</td>
<td>[ \lim_{t_\omega \to 0} p_{\omega D} = \frac{8\pi t_\omega^{3/4}}{(3-2\alpha)t^{3/2}C_{PD}A_{F,2,0}} ]</td>
<td>( \frac{3-2\alpha}{8} )</td>
</tr>
</tbody>
</table>

*Table 4.1 – Intermediate- and Late-Time Approximations of the TAD Solution*
Table 4.2 – Constants Used in Asymptotic Approximations (Table 4.1)

| $A_{01}$ | $\left(\frac{\lambda_2}{\lambda_0}\right)^{1-a}\left[1 + \left(\frac{\lambda_0}{\lambda_2}\right)\frac{(\log e)^2}{\eta_0}\right]$ |
| $A_{02,\alpha}$ | $\left(\frac{\lambda_2}{\lambda_0}\right)^{1-a}\left(\frac{\lambda_0}{\lambda_2}\right)\left(\frac{1}{\eta_0}\right)$ |
| $A_{02,0.5}$ | $\left(\frac{\lambda_2}{\lambda_0}\right)^{0.5}\left(\frac{\lambda_0}{\lambda_2}\right)\left(\frac{1}{\eta_0}\right)$ |
| $A_{03}$ | $\left(\frac{\lambda_2}{\lambda_0}\right)^{1-a}$ |
| $A_{F,0}$ | $\frac{2}{w_0\lambda_0} \left(\frac{\eta_0}{\eta_2}\right)^{1-a} B_{F,0} + \frac{1}{\eta_{FD}}$ | $B_{F,0}$ | $\sqrt{A_{01}} \tanh\left[\sqrt{A_{01}}(y_eD - w_D/2)\right]$ |
| $A_{F,1,\alpha}$ | $\frac{2}{w_0\lambda_0} \left(\frac{\eta_0}{\eta_2}\right)^{1-a} B_{F,1,\alpha}$ | $B_{F,1,\alpha}$ | $A_{01}(y_eD - w_D/2)$ |
| $A_{F,2,\alpha}$ | $\frac{2}{w_0\lambda_0} \left(\frac{\eta_0}{\eta_2}\right)^{1-a} B_{F,2,\alpha}$ | $B_{F,2,\alpha}$ | $\sqrt{A_{01}}$ |
| $A_{F,2,1,\alpha}$ | $\frac{2}{w_0\lambda_0} \left(\frac{\eta_0}{\eta_2}\right)^{1-a} B_{F,2,1,\alpha}$ | $B_{F,2,1,\alpha}$ | $A_{02}(y_eD - w_D/2)$ |
| $A_{F,2,2,\alpha}$ | $\frac{2}{w_0\lambda_0} \left(\frac{\eta_0}{\eta_2}\right)^{1-a} B_{F,2,2,\alpha}$ | $B_{F,2,2,\alpha}$ | $\sqrt{A_{02,0.5}}$ |
| $A_{F,2,0.5}$ | $\frac{2}{w_0\lambda_0} \left(\frac{\eta_0}{\eta_2}\right)^{1/2} B_{F,2,0.5}$ | $B_{F,2,0.5}$ | $\sqrt{A_{02,0.5}} \tanh\left[\sqrt{A_{02,0.5}}(y_eD - w_D/2)\right]$ |
| $A_{F,3,0}$ | $\frac{2}{w_0\lambda_0} \left(\frac{\eta_0}{\eta_2}\right)^{1-a} B_{F,3,0} + \frac{1}{\eta_{FD}}$ | $B_{F,3,0}$ | $\sqrt{A_{03}} \tanh\left[\sqrt{A_{03}}(y_eD - w_D/2)\right]$ |
| $A_{F,3,1,\alpha}$ | $\frac{2}{w_0\lambda_0} \left(\frac{\eta_0}{\eta_2}\right)^{1-a} B_{F,3,1,\alpha} + \frac{1}{\eta_{FD}}$ | $B_{F,3,1,\alpha}$ | $A_{03}(y_eD - w_D/2)$ |
| $A_{F,3,2,\alpha}$ | $\frac{2}{w_0\lambda_0} \left(\frac{\eta_0}{\eta_2}\right)^{1-a} B_{F,3,2,\alpha}$ | $B_{F,3,2,\alpha}$ | $\sqrt{A_{03}}$ |
CHAPTER 5

DISCUSSION OF RESULTS

This chapter presents the results obtained from the TAD model and discusses the indications of the results. The objective here is to document some of the diagnostic features of the model and their implications. Moreover, interpretation of the results are expected to shed light on the physical meaning of the anomalous diffusion parameters, such as the phenomenological coefficient of the flux law and the fractional order of the time derivative.

5.1 Effect of Fractional Order of the Time Derivative

Figure 5.1 shows the pressure and derivative responses of the TAD model for $0 < \alpha \leq 1$ and a fixed value of $k_a=1.2$. All pressure and derivative responses in Figure 5.1 display straight-line trends at early, intermediate, and late times.

![Figure 5.1 - Pressure and derivative responses obtained from the TAD solution for various $\alpha$](image)

Figure 5.1 – Pressure and derivative responses obtained from the TAD solution for various $\alpha$
In Fig. 5.1, the pressure responses for all $\alpha$ intersect at an early time. Similarly, another intersection point exists at early times for the derivative responses. For times larger than the intersection time, the pressure drop increases as the value of $\alpha$ decreases. This is consistent with the expectation that $\alpha < 1$ corresponds to subdiffusion; in other words, as $\alpha$ becomes smaller, the velocity field becomes more heterogeneous and the movement of the fluid is interrupted more often.

At early times, two straight lines with the slopes of $1/4$, corresponding to $\alpha=1$, and $1/2$, corresponding to $\alpha \to 0$, bound the pressure and derivative responses from below and above, respectively. Similarly, at late times, two unit-slope straight lines for $\alpha=1$ and $\alpha \to 0$ bound the pressure and derivative responses from below and above, respectively. Based on the trends of the data observed in Figure 5.1, all pressure and derivative responses, except for $\alpha=0$, collapse into the same unit-slope straight line at late times. For the time ranges used in Figure 5.1, this behavior is evident for $\alpha \geq 0.6$, but only implied by the trends of the data for $0 < \alpha < 0.6$. Because $\Delta p > 1E+04$ psi for all $\alpha$ after $t > 1E+08$ hr, from a practical perspective, the late-time, unit-slope trend will not be observed for $\alpha < 0.6$ for the cases in Figure 5.1.

To further comment on the flow-regime characteristics, we scrutinize the derivative responses as a function of $\alpha$ in Figure 5.2. As expected from the asymptotic relations in Chapter 4, the derivative responses in Figure 5.2 display straight lines with a variety of slopes at early, intermediate, and late times. As also noted in Figure 5.1, the late-time derivative responses are bounded by two unit-slope straight lines for $\alpha=1$ and $\alpha \to 0$, which indicate the depletion of the system (boundary-dominated flow). Theoretically, all derivative responses for $0 < \alpha < 1$ are expected to merge with $\alpha=1$ (normal diffusion) case and display a unit-slope line at late times. Although the results in Figure 5.2 indicate this trend, the merger does not happen in practical
times for $\alpha < 0.6$. This is because of slowing diffusion in matrix as $\alpha$ approaches zero. In general, $\alpha \to 0$ indicates longer interruptions of the fracture flow by the matrix elements, which can be physically caused by a sparsely fractured, tight-matrix and loosely connected fractures. As a consequence, effective depletion of the matrix, and thus the total system, takes longer when $\alpha$ becomes smaller. At the limit of $\alpha = 0$, the delay approaches infinity and the matrix is never depleted. On the other hand, as $\alpha \to 1$, flow in the fracture network is not much hindered by the interruptions of the matrix; that is, the system is densely fractured and the fractures are effectively connected. Therefore, the system is depleted faster and more efficiently.

Figure 5.2 – Slopes of the straight lines observed from the results of the TAD model

Slowing diffusion in matrix causes the slopes of the derivative responses decrease from 1 for $\alpha \to 0$ to $3/4$ for $\alpha = 0.5$ at intermediate times and an approximate straight-line may be fitted through the derivative responses for $0 < \alpha \leq 0.5$ at late-intermediate times. For $\alpha > 0.5$, the derivative responses display a transitional behavior at intermediate times with a shallower slope.
than 3/4, which is not reasonably constant to fit a straight line through the data, before merging with the derivative responses for $\alpha=1$ and displaying a unit-slope straight line at late times.

Straight lines with slopes from 1/4 for $\alpha=1$ to 1/2 for $\alpha\to0$ characterize the early-time derivative responses in Figure 5.2. These results are consistent with the interpretation that for $\alpha\to1$, natural fractures dominate the flow in the reservoir and the flow from the reservoir to finite-conductivity hydraulic fractures begin early (while there is still linear flow in hydraulic fractures) to cause a bilinear flow behavior. If the fracture storativity were large, infinite acting linear flow in the fracture system would be expected to continue after the depletion of the hydraulic fracture and reservoir linear flow would prevail (that is, 1/2-slope behavior would follow the 1/4-slope behavior). In the case of $\alpha\to0$, the reservoir response is very weak due to the dominance of the matrix and the early-time behavior is governed by the flow in hydraulic fractures.

To interpret the observation from Figure 5.2 within the context of dual-porosity idealization, in Figure 5.3, we consider the TDP responses for the data in Table 5.1 (we have obtained the $\omega$ and $\lambda$ values in Table 5.1 by changing the thickness of the matrix elements).

Figure 5.3 indicates that when $\lambda$ is smaller; that is, when matrix blocks are larger (fewer fractures), the bilinear flow (1/4 slope) is not followed by the reservoir linear (1/2-slope). On the other hand, reservoir linear flow (1/2-slope) follows the bilinear flow (1/4 slope) period for the larger values of $\lambda$, which correspond to smaller matrix blocks (more fractures). These results are consistent with the observations made from Figure 5.2 for the TAD model.
Table 5.1 – Data used in Figure 5.3

<table>
<thead>
<tr>
<th>Property</th>
<th>$\lambda=20$</th>
<th>$\lambda=50$</th>
<th>$\lambda=250$</th>
<th>$\lambda=500$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\omega=5.4E-4$</td>
<td>$\omega=2.2E-4$</td>
<td>$\omega=4.3E-5$</td>
<td>$\omega=2.2E-5$</td>
</tr>
<tr>
<td>Matrix block dimension, $h_m$, ft</td>
<td>1.25</td>
<td>0.5</td>
<td>0.1</td>
<td>0.05</td>
</tr>
<tr>
<td>Natural fracture density, $\rho_f$, n/f</td>
<td>0.8</td>
<td>2</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Number of natural fractures, $n_f$</td>
<td>200</td>
<td>500</td>
<td>2500</td>
<td>5000</td>
</tr>
</tbody>
</table>

Figure 5.3 – Bilinear and linear flow behaviors observed from the results of the TDP model

5.2 Combined Effect of Anomalous Diffusion Parameters

In the results discussed thus far, only the effect of the fractional order of the time derivative, $\alpha$, on pressure and derivative characteristics have been considered. In Figure 5.4, we consider the combined effects of the phenomenological coefficient, $k_\alpha$, and the anomalous diffusion exponent, $\alpha$, on pressure and derivative characteristics. The results in Figure 5.4 indicate that an increase in $k_\alpha$ for constant $\alpha$ decreases both the pressure drop and the derivative values. Variations of $\alpha$ for constant $k_\alpha$, on the other hand, cause a change in both the magnitude
of the pressure drop and the flow regime characteristics (indicated by the changing slopes of the derivative responses). It should be also noted that the variation of $k_\alpha$ for constant $\alpha$ causes a parallel shift in the pressure and derivative responses for all practical times for $\alpha \leq 0.5$ and at early and intermediate times for $\alpha > 0.5$. The pressure and derivative responses become independent of $\alpha$ at late times for $\alpha > 0.5$.

![Graph showing the combined effect of permeability, $k$, and $\alpha$ on pressure and derivative characteristics of TAD model](image)

**Figure 5.4** – Combined effect of the permeability, $k$, and $\alpha$ on pressure and derivative characteristics of TAD model

### 5.2 Rate Decline Characteristics of the TAD Model

For completeness, we also present the rate decline characteristics of the TAD model as a function of $\alpha$ in Figure 5.5. The physical interpretations presented for the pressure and derivative responses in Figs. 5.1 and 5.2 are also applicable to the rate-transient responses shown in Figure 5.5. The early-time rate responses, after the intersection time, display straight lines with slopes
between 1/4 for $\alpha = 1$ and 1/2 for $\alpha \to 0$. The early-time straight lines for $\alpha = 1$ and 0 are followed by sharp exponential-decline periods, which are the terminal flow regimes for these cases. For $\alpha \to 0$, exponential decline corresponds to the depletion of a system consisting mostly of a tight-matrix. On the other hand, for $\alpha = 1$, the system is dominated by natural-fractures and their depletion causes the exponential decline behavior.

For $0 < \alpha < 1$, the flow rates in Figure 5.5 display straight lines with slopes less than or equal to 1 ($\alpha = 0.1$) and greater than or equal to 1/2 ($\alpha = 0.9$) at intermediate times for $\alpha > 0.5$ and late times for $\alpha \leq 0.5$. For $\alpha \leq 0.5$, the delay of flow by the tight matrix causes a sharper drop in the flow rates at intermediate times before the display of the late-time straight lines. For $\alpha > 0.5$, the higher decline rates follow the intermediate-time straight lines.

Figure 5.5 – Rate declines obtained from the TAD solution for various $\alpha$. 
CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

In this thesis, an analytical anomalous-diffusion model has been developed for fractured horizontal wells in tight, unconventional reservoirs and shown to be a viable alternative to the dual-porosity based models. The versatility of the anomalous diffusion model to account for the complex forms of the reservoir and velocity-field heterogeneity can be inferred from the wide variety of the flow behaviors described by the asymptotic approximations of the model. Physical interpretations of the key parameters of the anomalous diffusion model, such as the fractional diffusion exponent and the coefficient of the flux law, can be deduced from comparisons with the models depending explicitly on the intrinsic properties of the heterogeneous features of the reservoir, such as the dual-porosity models. The interpretations of the pressure and flow rate behaviors predicted by the anomalous diffusion model are consistent with the physical expectations and the results of the alternate models. The fact that the anomalous diffusion formulation does not require explicit references to the intrinsic properties of the matrix and fracture media relaxes the stringent requirements used in dual-porosity idealizations to couple matrix and fracture flows. The trilinear anomalous-diffusion model presented in this thesis is useful for performance predictions and pressure- and rate-transient analysis of fractured horizontal wells in tight unconventional reservoirs.

For further research, inclusion of the space-fractional anomalous diffusion in the model is recommended. More detailed investigation of the phenomenological coefficient of the flux law should also provide more physical insight to anomalous diffusion model.
## NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>Formation volume factor, rb/stb</td>
</tr>
<tr>
<td>C</td>
<td>Concentration</td>
</tr>
<tr>
<td>C_{FD}</td>
<td>Hydraulic fracture conductivity, dimensionless</td>
</tr>
<tr>
<td>C_{RD}</td>
<td>Reservoir conductivity, dimensionless</td>
</tr>
<tr>
<td>c_t</td>
<td>Total compressibility, psi^{-1}</td>
</tr>
<tr>
<td>D</td>
<td>Flux coefficient</td>
</tr>
<tr>
<td>d</td>
<td>Euclidean dimension</td>
</tr>
<tr>
<td>d_f</td>
<td>Fractal dimension</td>
</tr>
<tr>
<td>d_F</td>
<td>Distance between two adjacent hydraulic fractures, ft</td>
</tr>
<tr>
<td>f</td>
<td>Dual porosity transfer function</td>
</tr>
<tr>
<td>h</td>
<td>Reservoir thickness, ft</td>
</tr>
<tr>
<td>h_m</td>
<td>Matrix block dimension, ft</td>
</tr>
<tr>
<td>J_C</td>
<td>Diffusive flux</td>
</tr>
<tr>
<td>k</td>
<td>Permeability, md</td>
</tr>
<tr>
<td>k_{I}</td>
<td>Permeability of the inner reservoir, md</td>
</tr>
<tr>
<td>k_{f}</td>
<td>Natural fracture permeability, md</td>
</tr>
<tr>
<td>k_F</td>
<td>Hydraulic fracture permeability, md</td>
</tr>
<tr>
<td>k_{O}</td>
<td>Permeability of the outer reservoir, md</td>
</tr>
<tr>
<td>k_m</td>
<td>Matrix permeability, md</td>
</tr>
<tr>
<td>k_{\alpha}</td>
<td>Phenomenological coefficient of anomalous diffusion, md-hr^{1-\alpha}</td>
</tr>
<tr>
<td>L_h</td>
<td>Horizontal well length, ft</td>
</tr>
<tr>
<td>m</td>
<td>Pseudopressure, psi^2/cp</td>
</tr>
</tbody>
</table>
nf  Number of natural fractures
nF  Number of hydraulic fractures
p   Pressure, psi
q   Volumetric rate, STB/day
r_w Wellbore radius, ft
s   Laplace parameter
t   Time, hrs
T   Temperature, °R
w_F Hydraulic fracture width, ft
x   Distance in x-direction, ft
x_e Reservoir size, x-direction, ft
x_F Hydraulic fracture half-length, ft
y   Distance in y-direction, ft
y_e Reservoir size, y-direction, ft

GREEK
α   Order of fractional derivative of time
α_{O,F} Parameter defined in the model
β_{O,F} Parameter defined in the model
Γ   Gamma function
Δ   Difference operator
η   Diffusivity, ft²/hr
θ   Conductivity index
λ   Flow capacity ratio (in TDP model)
\( \lambda_F \)  Permeability viscosity ratio of hydraulic fracture (\( k_F/\mu \), in TAD model)

\( \lambda_O \)  Permeability viscosity ratio of outer reservoir (\( k_O/\mu \), in TAD model)

\( \lambda_\alpha \)  Phenomenological coefficient (\( k_\alpha/\mu \), in TAD model)

\( \mu \)  Viscosity, cp

\( \pi \)  Pi constant

\( \sigma_r^2 \)  Mean square variance

\( \phi \)  Porosity

\( \omega \)  Storativity ratio
REFERENCES CITED


